A Normal Form for Hardy Inequalities

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Abstract:

Let b be a non-negative, non-increasing function on $(0, \infty)$ and let $H_b f(x) = \int_0^{b(x)} f$. The inequality $\|H_b f\|q \le C\|f\|_p$ expresses the boundedness of this operator from unweighted $L^p(0, \infty)$ to unweighted $L^q(0, \infty)$. It is called a *normal form Hardy inequality*.

An abstract formulation of a Hardy inequalities is given and every abstract Hardy inequality is shown to be equivalent, in a strong sense, to one in normal form. This equivalence applies to Hardy operators and their duals of the weighted continuous, weighted discrete, and general measures types, as well as those based on averages over starshaped sets in many dimensions. A straightforward formula relates each Hardy inequality to its normal form parameter b.

Besides giving a uniform treatment of many different types of Hardy operator, the reduction to normal form provides new insights, simple proofs of known theorems, and new results concerning best constants.