SUITABLE WEAK SOLUTIONS TO THE NAVIER–STOKES EQUATIONS AND SOME APPLICATIONS

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1. Introduction

The aim of this course is to present further properties of solutions to the Navier–Stokes equations. It is a continuation of the course devoted to mathematical theory of the Navier–Stokes equations. Therefore all important definitions and results can be found in the Lecture Notes [5]. Therein, we consider mostly the weak and the Leray–Hopf solutions (i.e., weak solutions which additionally fulfil the energy inequality). Here, we shall mostly consider so called *suitable weak solutions*. This notion is well-suited for the study of partial regularity and local results. The second part of the Lecture Notes is devoted to further results which are connected with the regularity of solutions. More precisely, in Chapter 2 we shall introduce the suitable weak solution, show its existence and its basic properties connected the partial regularity of the solutions. This part is mostly based on papers [1], [3] and [4]. The following chapter is based on paper [6]. We shall show that if the pressure connected to the to the given weak solution of the Cauchy problem is bounded from below, then the solution is as regular as the data of the problem allow. The last part, based on paper [2], gives the same result under the assumption that the weak solution belongs additionally to the space $L^\infty(0,T;(L^3(\mathbb{R}^3))^3)$.

First, we shall recall what we know about the Navier–Stokes equations. Let $\Omega \subset \mathbb{R}^N$. We look for functions

$$\mathbf{u}\colon (0,T)\times\Omega \to \mathbb{R}^N,$$

$$p\colon (0,T)\times\Omega \to \mathbb{R}$$

such that

$$\begin{aligned}
&\left.\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla p = \mathbf{f}\\ &\operatorname{div}\mathbf{u} = 0\end{aligned}\right\} && \text{in } (0,T)\times\Omega,\\
&\mathbf{u}(0,x) = \mathbf{u}_0(x) && \text{in } \Omega,\\
&\mathbf{u}(t,x) = 0 && \text{on } (0,T)\times\partial\Omega.
\end{aligned} \tag{1.1}$$

As the existence of classical solutions (at least in the spatial dimension $N \geq 3$) seems to be not obvious, we studied the weak solution:

$$\mathbf{u} \in L^\infty(0,T;(L^2(\Omega))^N) \cap L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))$$

such that

$$\frac{\partial \mathbf{u}}{\partial t} \in L^q(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*),\ q\geq 1$$

and

$$\Big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}\Big\rangle + \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle$$
$$\forall \boldsymbol{\varphi} \in W^{1,2}_{0,\mathrm{div}}(\Omega)$$

and a.e. $t \in (0,T)$. Moreover,

$$\lim_{t\to 0+} \int_\Omega \mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega \mathbf{u}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}x \qquad \forall\boldsymbol{\varphi} \in (L^2(\Omega))^N.$$

Let us recall that

- $\mathbf{u} \in C([0,T);(L^2(\Omega))^N_w) \Rightarrow$ the initial condition is fulfilled in this sense
- the weak formulation does not contain the pressure $\Rightarrow$ it is important to know if we can reconstruct the pressure.

Concerning the weak solution, we proved the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N = 2,3$, further let the right-hand side $\mathbf{f} \in L^2((0,T),(W^{-1,2}(\Omega))^N)$ and the initial condition $\mathbf{u}_0 \in L^2_{0,div}(\Omega)$.*

Then there exists at least one weak solution to (1.1); additionally, this solution fulfils the energy inequality:

$$\frac{1}{2}\int_\Omega |\mathbf{u}(t,\cdot)|^2\,\mathrm{d}x + \int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau \le \frac{1}{2}\int_\Omega |\mathbf{u}_0(t,\cdot)|^2\,\mathrm{d}x + \int_0^t \langle \mathbf{f},\mathbf{u}\rangle\,\mathrm{d}\tau \tag{1.2}$$

for a.e. $t\in(0,T)$ (while by changing the function on a subset of the time interval of measure zero it is possible to obtain the inequality $\forall t\in(0,T)$).

Moreover, if $N=2$, then the solution is unique in the class of all weak solutions[1]*. Further, for $N=2$, if $\Omega\in C^2$, $\mathbf{u}_0\in W^{1,2}_{0,div}(\Omega)$ and $\mathbf{f}\in L^2(0,T;(L^2(\Omega))^2)$, then $\mathbf{u}\in C([0,T];(W^{1,2}(\Omega))^2)\cap L^2(0,T;(W^{2,2}(\Omega))^2)$, $\frac{\partial\mathbf{u}}{\partial t}\in L^2(0,T;(L^2(\Omega))^2)$.*

We did not get a similar result for $N=3$. We only have

- We can construct a solution which fulfils the energy inequality.
- The uniqueness is not known, we can get it only in the class of the Leray–Hopf solutions and moreover, the unique solutiuon must be more regular, i.e.,
$$\mathbf{u}\in L^t(0,T;(L^s(\Omega))^3) \qquad \frac{2}{t}+\frac{3}{s}\le 1$$
(we proved the case $s>3$).
- Higher regularity of the solution is not evident. We only have for $\Omega\in C^2$, $\mathbf{f}\in L^2(0,T;(L^2(\Omega)))^3$, $\mathbf{u}_0\in W^{1,2}_{0,\mathrm{div}}(\Omega)$, and, additionally
$$\mathbf{u}\in L^t(0,T;(L^s(\Omega))^3) \qquad \frac{2}{t}+\frac{3}{s}\le 1,$$
that
$$\mathbf{u}\in C([0,T);W^{1,2}_{0,\mathrm{div}}(\Omega))\cap L^2(0,T;(W^{2,2}(\Omega))^3)$$
(and again, we proved only the case $s>3$; the case $s=3$ will be shown in Chapter 4).

The question of the existence of the pressure is highly non-trivial. We discussed several methods, the most suitable for us was based on the higher regularity of the Stokes problem with integrable right-hand side, i.e., we consider the problem

$$\begin{aligned} &\left.\begin{aligned}&\frac{\partial\mathbf{v}}{\partial t}-\Delta\mathbf{v}+\nabla p=\mathbf{g}\\ &\operatorname{div}\mathbf{v}=0\end{aligned}\right\}\ \text{in } (0,T)\times\Omega,\\ &\mathbf{v}(0,x)=\mathbf{u}_0(x)\ \text{in } \Omega,\\ &\mathbf{v}(t,x)=0\ \text{on } (0,T)\times\partial\Omega.\end{aligned}\tag{1.3}$$

Then, if $\Omega\in C^2$, $\mathbf{g}\in L^t(0,T;(L^s(\Omega))^N)$, $\mathbf{u}_0$ is sufficiently smooth (e.g., it is enough $\mathbf{u}_0\in(W^{1,\infty}(\Omega))^N)$, then $\|\frac{\partial\mathbf{v}}{\partial t},\nabla^2\mathbf{v},\nabla p\|_{L^t(0,T;L^s(\Omega))}\le C(\|\mathbf{g}\|_{L^t(0,T;(L^s(\Omega))^N)},\mathbf{u}_0)$.

Thus, if we set $\mathbf{g}=\mathbf{f}-\mathbf{u}\cdot\nabla\mathbf{u}$, then due to the uniqueness of the solution of the Stokes problem the same also holds for $\mathbf{u}$, solution to (1.1). In particular, for $\mathbf{f}$ and $\mathbf{u}_0$ sufficiently smooth

$$\|\nabla p\|_{L^t(0,T;(L^s(\Omega))^N)}\le C(\mathbf{f},\mathbf{u}_0,\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^t(0,T;(L^s(\Omega))^N)}),$$

i.e., $\nabla p\in L^t(0,T;(L^s(\Omega))^N)$, where $\frac{2}{t}+\frac{N}{s}\le N+1$, $1<s<\frac{N}{N-1}$ a $N=2,3$. Furthermore, taking the pressure such that (additive constant is not fixed, at least in the case of a bounded domain)

$$\int_\Omega p(t,\cdot)\,\mathrm{d}x=0 \qquad \text{for a.e. } t\in(0,T),$$

[1]This solution fulfils the energy equality and $\mathbf{u}\in C([0,T);(L^2(\Omega))^2)$.

then we have for $N = 3$
$$p \in L^t(0,T;(L^s(\Omega))^N)$$
with $\frac{2}{t} + \frac{3}{s} \leq 3$, $\frac{3}{2} < s < 3$; in particular for $s = t = \frac{5}{3}$
$$p \in L^{\frac{5}{3}}((0,T)\times\Omega).$$

2. Suitable weak solution

2.1. **Basic notions.** Before we define the suitable weak solution, we perform a formal calculation. We take $\Phi \in C_0^\infty((0,T)\times\Omega)$, formally multiply equation $(1.1)_1$ by $2\Phi\mathbf{u}$ and integrate over $(0,t)\times\Omega$, $t \leq T$. Then

$$\begin{aligned}
2\int_0^t\int_\Omega \frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{u}\Phi\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^t\int_\Omega \frac{\partial}{\partial t}|\mathbf{u}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&= -\int_0^t\int_\Omega |\mathbf{u}|^2\frac{\partial\Phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega |\mathbf{u}|^2(t,\cdot)\Phi(t,\cdot)\,\mathrm{d}x, \\
2\int_0^t\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\Phi\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^t\int_\Omega \mathbf{u}\cdot\nabla|\mathbf{u}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&= -\int_0^t\int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau, \\
-2\int_0^t\int_\Omega (\Delta\mathbf{u}\cdot\mathbf{u})\Phi\,\mathrm{d}x\,\mathrm{d}\tau &= 2\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad +2\int_0^t\int_\Omega (\nabla\mathbf{u}\cdot\mathbf{u})\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&= 2\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t\int_\Omega |\mathbf{u}|^2\Delta\Phi\,\mathrm{d}x\,\mathrm{d}\tau, \\
2\int_0^t\int_\Omega \nabla p\cdot\mathbf{u}\Phi\,\mathrm{d}x\,\mathrm{d}\tau &= -2\int_0^t\int_\Omega p\mathbf{u}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Altogether we have $\forall\Phi \in C_0^\infty((0,T)\times\Omega)$ and $\forall t \in (0,T)$:

$$\begin{aligned}
\int_\Omega |\mathbf{u}|^2(t,\cdot)\Phi(t,\cdot)\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau &= \int_0^t\int_\Omega |\mathbf{u}|^2\Big(\frac{\partial\Phi}{\partial t} + \Delta\Phi\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\
+\int_0^t\int_\Omega (|\mathbf{u}|^2 + 2p)\mathbf{u}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau &+ \int_0^t\int_\Omega 2\mathbf{f}\cdot\mathbf{u}\Phi\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Similarly as for the weak solution, we should not expect equality, rather inequality. This brings us to the following definition

Definition 2.1. Let $\mathbf{f} \in L^2(0,T;(L^{\frac{6}{5}}(\Omega))^3) + L^1(0,T;(L^2(\Omega))^3)$, $\Omega \subset \mathbb{R}^3$ is a bounded domain. Then the pair $(\mathbf{u},p)$ is called a suitable weak solution to the Navier–Stokes equations, if

- $(\mathbf{u},p)$ fulfils (1.1) in the sense of distributions, i.e., it holds
$$\begin{aligned}
-\int_0^T\int_\Omega \mathbf{u}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^T\int_\Omega (\mathbf{u}\otimes\mathbf{u}):\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^T\int_\Omega \mathbf{u}\cdot\Delta\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau \\
-\int_0^T\int_\Omega p\,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^T\int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau \qquad \forall\boldsymbol{\varphi} \in (C_0^\infty((0,T)\times\Omega))^3
\end{aligned}$$
and
$$\int_\Omega \mathbf{u}\cdot\nabla\psi\,\mathrm{d}x = 0 \qquad \text{pro s.v. } t \in (0,T) \text{ a } \forall\psi \in C_0^\infty(\Omega)$$
- $\mathbf{u} \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega)) \cap L^\infty(0,T;(L^2(\Omega))^3)$, $p \in L^{\frac{3}{2}}((0,T)\times\Omega)$

- $\forall \Phi \in C_0^\infty((0,T)\times\Omega)$, $\Phi \geq 0$ the generalized energy inequality holds:

$$
\begin{aligned}
\int_\Omega |\mathbf{u}|^2(t,\cdot)\Phi(t,\cdot)\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla \mathbf{u}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau \leq \int_0^t\int_\Omega |\mathbf{u}|^2\Big(\frac{\partial \Phi}{\partial t} + \Delta\Phi\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\
+ \int_0^t\int_\Omega (|\mathbf{u}|^2 + 2p)\mathbf{u}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau + 2\int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}\Phi\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
\tag{2.1}
$$

for a.e. $t \in (0,T)$.

Remark 2.2. The definition of the suitable weak solution does not contain the initial condition. Based on the regularity of the velocity field we know that $\mathbf{u} \in C([0,T);(L^2(\Omega))^3_w)$; we can assume that the initial condition is satisfied in this sense. To characterize precisely the space with the least regularity for $\mathbf{u}_0$ to get existence of a suitable weak solution we would have to introduce certain interpolation spaces which we do not want to do. Therefore we shall assume that $\mathbf{u}_0$ is a sufficiently smooth function.

We aim at showing the following two results:

1) Under certain assumptions on $\mathbf{u}_0$, $\mathbf{f}$ and Ω there exists a suitable weak solution to problem (1.1).
2) The set of possible singularities of a suitable weak solution is small.

The conditions under which we shall prove these results will be specified later.

Remark 2.3. We considered up to now only bounded domains. With a little effort all results can be generalized e.g. to $\mathbb{R}^3$, to exterior domains or domains with noncompact boundaries. We will not deal here with such problems as well as we skip possible generalizations to spatial dimensions $N \geq 4$.

2.2. **Existence of suitable weak solution.** We aim at proving the following theorem.

Theorem 2.4. *Let the bounded domain $\Omega \in C^2$, let $\mathbf{u}_0$ be sufficiently smooth and let $\mathbf{f} \in L^2(0,T;(L^{\frac{6}{5}}(\Omega))^3)$. Then there exists at least one suitable weak solution to the Navier–Stokes equations.*

Remark 2.5. The assumptions on $\mathbf{f}$ can be modified, e.g. $\mathbf{f} \in L^{\frac{5}{3}}(0,T;(L^{\frac{15}{14}}(\Omega))^3) \cap L^1(0,T;(L^2(\Omega))^3)$.

Let us consider instead of (1.1) the problem

$$
\begin{aligned}
&\left.\begin{aligned}\frac{\partial \mathbf{u}^\delta}{\partial t} + (\mathbf{u}^\delta)_\delta\cdot\nabla\mathbf{u}^\delta - \Delta\mathbf{u}^\delta + \nabla p^\delta = \mathbf{f} \\ \operatorname{div}\mathbf{u}^\delta = 0\end{aligned}\right\} \text{ in } (0,T)\times\Omega, \\
&\mathbf{u}^\delta(0,x) = \mathbf{u}_0(x) \text{ in } \Omega, \\
&\mathbf{u}^\delta(t,x) = \mathbf{0} \text{ on } (0,T)\times\partial\Omega,
\end{aligned}
\tag{2.2}
$$

where

$$
(\mathbf{u}^\delta)_\delta(t,x) = (\omega_\delta * \widetilde{\mathbf{u}})(t,x) = \frac{1}{\delta^3}\int_{\mathbb{R}^3}\omega\Big(\frac{x-y}{\delta}\Big)\widetilde{\mathbf{u}}(t,y)\,\mathrm{d}y
$$

for

$$
\widetilde{\mathbf{u}}(t,y) = \mathbf{u}^\delta(t,y) \quad \text{for } y \in \Omega, \qquad \widetilde{\mathbf{u}}(t,y) = \mathbf{0} \quad \text{for } y \notin \Omega,
$$

$\omega(\cdot)$ is the standard mollification kernel, $0 < t < T$. As $\mathbf{u}^\delta = \mathbf{0}$ on $\partial\Omega$ in the sense of traces, $\widetilde{\mathbf{u}}(t,\cdot) \in (W^{1,2}(\mathbb{R}^3))^3$. Therefore also

$$\begin{aligned}
\operatorname{div}(\mathbf{u}^\delta)_\delta(t,x) &= \operatorname{div} \frac{1}{\delta^3}\int_{\mathbb{R}^3} \omega\Big(\frac{x-y}{\delta}\Big)\widetilde{\mathbf{u}}(t,y)\,\mathrm{d}x \\
&= \frac{1}{\delta^3}\int_{\mathbb{R}^3} \frac{\partial}{\partial x_i}\omega\Big(\frac{x-y}{\delta}\Big)\widetilde{u}_i(t,y)\,\mathrm{d}x = -\frac{1}{\delta^3}\int_{\mathbb{R}^3} \frac{\partial}{\partial y_i}\omega\Big(\frac{x-y}{\delta}\Big)\widetilde{u}_i(t,y)\,\mathrm{d}x \\
&= \frac{1}{\delta^3}\int_{\mathbb{R}^3} \omega\Big(\frac{x-y}{\delta}\Big)\frac{\partial \widetilde{u}_i(t,y)}{\partial y_i}\,\mathrm{d}x = 0.
\end{aligned}$$

This ensures that the important property which enables us to obtain a priori estimates is kept. Moreover, for

$$\mathbf{u} \in L^\infty(0,T;(L^2(\Omega))^3) \cap L^2(0,T;(W^{1,2}(\Omega))^3)$$

the function

$$(\mathbf{u}^\delta)_\delta \in L^\infty(0,T;(L^\infty(\Omega))^3) \cap L^2(0,T;(W^{1,\infty}(\Omega))^3),$$

and it is even smooth in spatial variables. Thus we have

Lemma 2.6. *Let $\delta > 0$ and let the assumptions of Theorem 2.4 be fulfilled. Then there exists at least one weak solution to problem (2.2) (in a similar sense as the weak solution for the Navier–Stokes equations).*

Proof. We will not perform the proof in detail as we basically only copy the existence proof for the Navier–Stokes equations. We therefore only give a few hints.

***Step 1*:** Galerkin approximation — it is the same as for the Navier–Stokes equations, just the nonlinear term is slightly modified, i.e. the nonlinearity is quadratic, but it has a different form. However, this does not change anything in the proof of local existence of a solution on $(0,T^*)$.

***Step 2*:**
- We use as a test function

$$\mathbf{u}^n(t,x) = \sum_{j=1}^n c_j^n(t)\mathbf{w}^j(x),$$

where $\{\mathbf{w}^i\}_{i=1}^\infty$ is a suitable basis of $W^{1,2}_{0,\mathrm{div}}(\Omega)$. We get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^n\|_2^2 + \|\nabla\mathbf{u}^n\|_2^2 = \int_\Omega \mathbf{f}\cdot\mathbf{u}^n\,\mathrm{d}x \le C\|\mathbf{f}\|_{\frac{6}{5}}\|\nabla\mathbf{u}^n\|_2,$$

the rest is the same as for the Navier–Stokes equations. In particular, the solution is global in time (on $(0,T]$).
- Similarly as for the Navier–Stokes equations we can estimate the time derivative. If we use additionally the higher integrability of the convective term, we get

$$\Big\|\frac{\partial\mathbf{u}^n}{\partial t}\Big\|_{L^2(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} \le C(\delta),$$
$$\Big\|\frac{\partial\mathbf{u}^n}{\partial t}\Big\|_{L^{\frac{4}{3}}(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} \le C.$$

Recall that we have the following δ-independent estimates

$$\|u_\delta\|_q \le \|u\|_q, \qquad 1 \le q \le \infty;$$

they follow by the Hörmander–Young inequality for convolutions

$$\|u*\omega_\delta\|_q \le \|\omega_\delta\|_1\|u\|_q = \|\omega\|_1\|u\|_q,$$

generally then

$$\|u * \omega_\delta\|_r \le \|\omega_\delta\|_p \|u\|_q, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

Therefore, concerning the δ-independent estimates, they are exactly the same as for the Navier–Stokes equations.

***Step 3*:** Limit passage — similarly as for the Navier–Stokes equations we employ the Aubin–Lions lemma and the fact that a priori estimates and strong convergence in the space $L^2((0,T)\times\Omega)$ imply the strong convergence in $L^2(0,T;L^q(\Omega))$ for $1 \le q < 6$ and $L^p(0,T;L^2(\Omega))$ for $1 \le p < \infty$. (The details are left as a useful exercise for the kind reader.)

□

To summarize, the lemma above provides the estimates

$$\|\mathbf{u}^\delta\|_{L^\infty(0,T;(L^2(\Omega))^3)} + \|\nabla \mathbf{u}^\delta\|_{L^2(0,T;(L^2(\Omega))^{3\times 3})} + \Big\|\frac{\partial \mathbf{u}^\delta}{\partial t}\Big\|_{L^{\frac{4}{3}}(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} \le C,$$

where the constant C is independent of δ.

Next step is the passage $\delta \to 0^+$. We have the following result.

Lemma 2.7. *There exists a sequence $\delta_n \to 0^+$ such that*

$$\begin{aligned} \mathbf{u}^{\delta_n} &\rightharpoonup^* \mathbf{u} \quad &&\text{in} \quad L^\infty(0,T;(L^2(\Omega))^3),\\ \mathbf{u}^{\delta_n} &\rightharpoonup \mathbf{u} \quad &&\text{in} \quad L^2(0,T;(W^{1,2}(\Omega))^3),\\ \mathbf{u}^{\delta_n} &\to \mathbf{u} \quad &&\text{in} \quad (L^2((0,T)\times\Omega))^3, \end{aligned}$$

where $\mathbf{u}$ *is a weak solution to the Navier–Stokes equations.*

Proof. We proceed as above. The only term which is not trivial is the convective one. We have to be slightly more careful, due to the presence of a nonlocal term. We know that $\nabla\mathbf{u}^{\delta_n} \rightharpoonup \nabla\mathbf{u}$ in $(L^2((0,T)\times\Omega))^{3\times 3}$. Therefore it remains to show that

$$(\mathbf{u}^{\delta_n})_{\delta_n} \to \mathbf{u} \text{ in } (L^2((0,T)\times\Omega))^3,$$

or

$$\mathbf{u}^{\delta_n} * \omega_{\delta_n} \to \mathbf{u} \text{ in } (L^2((0,T)\times\Omega))^3.$$

Recall that we know $\mathbf{u}_{\delta_n} \to \mathbf{u}$ in $(L^2((0,T)\times\Omega))^3$, hence

$$\begin{gathered} \|(\mathbf{u}^{\delta_n})_{\delta_n} - \mathbf{u}\|_{(L^2((0,T)\times\Omega))^3} \\ \le \|(\mathbf{u}^{\delta_n})_{\delta_n} - \mathbf{u}^{\delta_n}\|_{(L^2((0,T)\times\Omega))^3} + \|\mathbf{u}^{\delta_n} - \mathbf{u}\|_{(L^2((0,T)\times\Omega))^3}, \end{gathered}$$

and it remains to show that the first term goes to 0. Thus

$$\begin{aligned} &\|(\mathbf{u}^{\delta_n})_{\delta_n} - \mathbf{u}^{\delta_n}\|^2_{(L^2((0,T)\times\Omega))^3} \\ &= \int_0^T \int_\Omega \Big(\frac{1}{\delta_n^3}\int_\Omega \omega\Big(\frac{x-y}{\delta_n}\Big)\big(\mathbf{u}^{\delta_n}(t,y) - \mathbf{u}^{\delta_n}(t,x)\big)\,\mathrm{d}y\Big)^2 \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \Big(\int_{B_1(0)} \omega(z)\big(\mathbf{u}^{\delta_n}(t,x-z\delta_n) - \mathbf{u}^{\delta_n}(t,x)\big)\,\mathrm{d}z\Big)^2 \,\mathrm{d}x\,\mathrm{d}t \equiv I_1. \end{aligned}$$

As $\mathbf{u}_{\delta_n} \to \mathbf{u}$ in $(L^2((0,T)\times\Omega))^3$, the functions $\{\mathbf{u}^{\delta_n}\}$ are uniformly 2-mean continuous, i.e.

$$I_1 \le \int_{B_1(0)} \omega^2(z)\,\mathrm{d}z \int_{B_1(0)} \int_0^T \int_\Omega \big(\mathbf{u}^{\delta_n}(t,x-z\delta_n) - \mathbf{u}^{\delta_n}(t,x)\big)^2 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}z \to 0$$

for $\delta_n \to 0^+$. □

It remains to check the existence of the pressure and to verify the generalized energy inequality. If we put the convective term to the right-hand side, we get both for the limit Navier–Stokes equations as well as for (2.2) the following estimates for the pressure:

$$
\begin{aligned}
&\|\nabla p\|_{L^t(0,T;(L^s(\Omega))^3)} \\
&\le C\big(\|\mathbf{f}\|_{L^t(0,T;(L^s(\Omega))^3)} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^t(0,T;(L^s(\Omega))^3)}\big) + C_1(\mathbf{u}_0), \\
&\|\nabla p^{\delta_n}\|_{L^t(0,T;(L^s(\Omega))^3)} \\
&\le C\big(\|\mathbf{f}\|_{L^t(0,T;(L^s(\Omega))^3)} + \|(\mathbf{u}^{\delta_n})_{\delta_n}\cdot\nabla\mathbf{u}^{\delta_n}\|_{L^t(0,T;(L^s(\Omega))^3)}\big) + C_1(\mathbf{u}_0).
\end{aligned}
$$

Choosing $t = \frac{5}{3}$ and $s = \frac{15}{14}$ it yields

$$
\begin{aligned}
&\|(\mathbf{u}^{\delta_n})_{\delta_n}\cdot\nabla\mathbf{u}^{\delta_n}\|_{L^{\frac{5}{3}}(0,T;(L^{\frac{15}{14}}(\Omega))^3)} \\
&\le \|(\mathbf{u}^{\delta_n})_{\delta_n}\|_{L^{10}(0,T;(L^{\frac{30}{13}}(\Omega))^3)}\|\nabla\mathbf{u}^{\delta_n}\|_{L^2(0,T;(L^2(\Omega))^{3\times 3})} \\
&\le C\|(\mathbf{u}^{\delta_n})_{\delta_n}\|^{\frac{1}{5}}_{L^2(0,T;(L^6(\Omega))^3)}\|(\mathbf{u}^{\delta_n})_{\delta_n}\|^{\frac{4}{5}}_{L^\infty(0,T;(L^2(\Omega))^3)}\|\nabla\mathbf{u}^{\delta_n}\|_{L^2(0,T;(L^2(\Omega))^{3\times 3})} \\
&\le \text{ const.}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\nabla p\|_{L^{\frac{5}{3}}(0,T;(L^{\frac{15}{14}}(\Omega))^3)} &\le C, \\
\|\nabla p^{\delta_n}\|_{L^{\frac{5}{3}}(0,T;(L^{\frac{15}{14}}(\Omega))^3)} &\le C.
\end{aligned}
$$

If we normalize the pressure

$$
\int_\Omega p^{\delta_n}\,\mathrm{d}x = \int_\Omega p\,\mathrm{d}x = 0 \qquad \forall t \in (0,T),
$$

we have

$$
\begin{aligned}
\|p\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} &\le C_1 \\
\|p^{\delta_n}\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} &\le C_1,
\end{aligned}
$$

where C_1 is independent of δ_n. Therefore the pressure exists both for (2.2) and the limit problem (1.1). The sequence of pressures is bounded in $L^{\frac{5}{3}}((0,T)\times\Omega)$ and thus also in $L^{\frac{3}{2}}((0,T)\times\Omega)$. Hence the pair $(\mathbf{u}, p)$ is a distributional solution to (1.1) and belongs to the required spaces. It remains to verify the validity of the generalized energy inequality.

Lemma 2.8. *The solution $(\mathbf{u}, p)$ fulfils the generalized energy inequality (2.1).*

Proof. As $(\mathbf{u}^{\delta_n})_{\delta_n}$ and its spatial gradient are bounded functions for fixed δ_n, it is not difficult to see that $\mathbf{u}^{\delta_n}$ can be used as test function in the weak formulation for (2.2) with $\delta = \delta_n$. More precisely, we use rather $2\mathbf{u}^{\delta_n}\Phi$, where Φ is a non-negative smooth compactly supported function in $(0,T)\times\Omega$. Repeating the computations from Section 2.1, we get

$$
\begin{aligned}
&\int_\Omega |\mathbf{u}^{\delta_n}|^2(t,\cdot)\Phi(t,\cdot)\,\mathrm{d}x + 2\int_0^t\int_\Omega |\nabla\mathbf{u}^{\delta_n}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^t\int_\Omega |\mathbf{u}^{\delta_n}|^2\Big(\frac{\partial\Phi}{\partial t} + \Delta\Phi\Big)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega |\mathbf{u}^{\delta_n}|^2(\mathbf{u}^{\delta_n})_{\delta_n}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ 2\int_0^t\int_\Omega p^{\delta_n}\mathbf{u}^{\delta_n}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau + 2\int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}^{\delta_n}\Phi\,\mathrm{d}x\,\mathrm{d}\tau\,.
\end{aligned}
$$

We multiply this inequality by $\psi(t) \geq 0$, $\psi(t) \in C_0^\infty(0,T)$, and integrate over the time interval $(0,T)$:

$$\int_0^T \int_\Omega |\mathbf{u}^{\delta_n}|^2(t,\cdot)\Phi(t,\cdot)\,\mathrm{d}x\psi(t)\,\mathrm{d}t + 2\int_0^T \int_0^t \int_\Omega |\nabla \mathbf{u}^{\delta_n}|^2 \Phi \,\mathrm{d}x\,\mathrm{d}\tau\psi(t)\,\mathrm{d}t$$
$$= \int_0^T \Big[\int_0^t \int_\Omega |\mathbf{u}^{\delta_n}|^2 \Big(\frac{\partial \Phi}{\partial t} + \Delta\Phi\Big)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega |\mathbf{u}^{\delta_n}|^2 (\mathbf{u}^{\delta_n})_{\delta_n} \cdot \nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+2\int_0^t \int_\Omega p^{\delta_n}\mathbf{u}^{\delta_n}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau + 2\int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}^{\delta_n}\Phi\,\mathrm{d}x\,\mathrm{d}\tau\Big]\psi(t)\,\mathrm{d}t.$$

Next we let $\delta_n \to 0^+$. We use in the first term the fact that $\mathbf{u}_n \to \mathbf{u}$ v $(L^2((0,T)\times\Omega))^3$, in the second one the Fatou lemma. In the third term we use the strong convergence (see Lemma 2.7), in the fourth one the strong convergence in $L^3((0,T)\times\Omega)$; it follows from the estimate

$$\|\mathbf{u}\|_{(L^{\frac{10}{3}}((0,T)\times\Omega))^3} \leq \|\mathbf{u}\|^{\frac{2}{5}}_{L^\infty(0,T;(L^2(\Omega))^3)} \|\mathbf{u}\|^{\frac{3}{5}}_{L^2(0,T;(L^6(\Omega))^3)}$$

and the interpolation of L^3 between L^2 and $L^{\frac{10}{3}}$. The computation for $(\mathbf{u}^{\delta_n})_{\delta_n}$ is similar as above, i.e. $\|(\mathbf{u}^{\delta_n})_{\delta_n} - \mathbf{u}^{\delta_n}\|_{(L^3((0,T)\times\Omega))^3} \to 0$. In the fifth term we use the weak convergence of p^{δ_n} in $L^{\frac{5}{3}}((0,T)\times\Omega)$ and the strong convergence of $\mathbf{u}^{\delta_n}$ in $(L^{\frac{5}{2}}((0,T)\times\Omega))^3$, the last term is obvious. Therefore we have

$$\int_0^T \int_\Omega |\mathbf{u}|^2\Phi\,\mathrm{d}x\psi(t)\,\mathrm{d}t + 2\int_0^T \int_0^t \int_\Omega |\nabla\mathbf{u}|^2\Phi\,\mathrm{d}x\,\mathrm{d}\tau\psi(t)\,\mathrm{d}t$$
$$\leq \int_0^T \int_0^t \int_\Omega |\mathbf{u}|^2\Big(\frac{\partial\Phi}{\partial t} + \Delta\Phi\Big)\,\mathrm{d}x\,\mathrm{d}\tau\psi(t)\,\mathrm{d}t + \int_0^T \int_0^t \int_\Omega |\mathbf{u}|^2\mathbf{u}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau\psi(t)\,\mathrm{d}t$$
$$+ 2\int_0^T \int_0^t \int_\Omega p\mathbf{u}\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}\tau\psi(t)\,\mathrm{d}t + 2\int_0^T \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}\Phi\,\mathrm{d}x\,\mathrm{d}\tau\psi(t)\,\mathrm{d}t.$$

As the inequality holds $\forall\psi \in C_0^\infty(0,T)$, $\psi \geq 0$, we get the desired generalized energy inequality (2.1) a.e. in $(0,T)$. □

Lemma 2.8 completes the proof of Theorem 2.4.

2.3. **Partial regularity of the suitable weak solution.** The aim of this section is to characterize the size of possible sets of singular points. Before we start with it, we have to precise several notions as a singular and a regular point and to explain the difference between the k-dimensional parabolic and Hausdorff measure.

In what follows, for the reason of simplicity, we assume $\mathbf{f} = \mathbf{0}$. The case $\mathbf{f} \neq \mathbf{0}$ is studied in [3] (and it requires to deal with Morrey–Campanato spaces).

Definition 2.9. Let $z = (t,x) \in (0,T)\times\Omega$. We say that z is *a regular point* of the suitable weak solution to the Navier–Stokes equations in $(0,T)\times\Omega$, if there $\exists U_\delta(z)$ such that $\mathbf{u} \in C^{0,\alpha}(U_\delta(z))$ for a certain $0 < \alpha \leq 1$. The point z is *a singular point* of the suitable weak solution to the Navier–Stokes equations in $(0,T)\times\Omega$, if it is not a regular point.

Remark 2.10. The Hölder continuity of the velocity in fact implies a certain smoothness of the pressure, but we shall not discuss it here. Just note that in general it is not known whether $\mathbf{u}$ is in the neighbourhood of the regular point continuously differentiable in time and thus the full regularity is not known to hold.

Let us introduce $(z_0 = (t_0,x_0) \in (0,T)\times\Omega)$:

$$Q(z_0,r) = \big\{z = (t,x) \in (0,T)\times\Omega;\ t \in (t_0 - r^2, t_0),\ x \in B_r(x_0)\big\},$$
$$Q^*(z_0,r) = \big\{z = (t,x) \in (0,T)\times\Omega;\ t \in \big(t_0 - \tfrac{7}{8}r^2, t_0 + \tfrac{1}{8}r^2\big),\ x \in B_r(x_0)\big\}.$$

Definition 2.11. Let $X \subset \mathbb{R} \times \mathbb{R}^N$, $k \in \mathbb{R}^+$. Then $\mathcal{P}^k(X)$, defined as

$$\mathcal{P}^k(X) = \lim_{\delta \to 0} \mathcal{P}^k_\delta(X) = \sup_{\delta > 0} \mathcal{P}^k_\delta(X),$$

where

$$\mathcal{P}^k_\delta(X) = \inf \Big\{ \sum_{i=1}^\infty r_i^k;\ X \subset \bigcup_{i=1}^\infty Q(z_i, r_i),\ r_i < \delta \Big\},$$

is called the *k-dimensional parabolic measure* of X (i.e., we cover X by countably many parabolic cylinders) and $\mathcal{H}^k(X)$, defined as

$$\mathcal{H}^k(X) = \lim_{\delta \to 0} \mathcal{H}^k_\delta(X) = \sup_{\delta > 0} \mathcal{H}^k_\delta(X),$$

where

$$\mathcal{H}^k_\delta(X) = \inf \Big\{ \sum_{i=1}^\infty r_i^k;\ X \subset \bigcup_{i=1}^\infty B_{r_i}(z_i),\ r_i < \delta \Big\},$$

is called the *k-dimensional Hausdorff measure* of X (i.e., we cover X by countably many balls in $\mathbb{R}^{N+1}$).

Remark 2.12. It holds $\mathcal{H}^k(X) \le c\mathcal{P}^k(X)$, since for $r < 1$ we have the situation as in Figure 1, i.e., the set covered by a parabolic cylinder can be covered by m balls

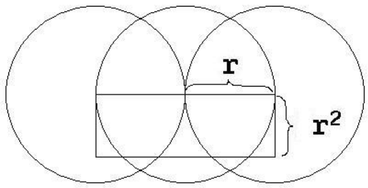

FIGURE 1. Covering of the parabolic cylinder by balls

with the same diameter, m is finite and independent of δ and k, but it depends on N. Thus

$$m \sum_i r_i^k = m \sum_i r_i^k = \sum_j r_j^k$$

(in other words, $m\times$ covering by cylinders = $m\times$ covering by balls = covering by balls) and thus

$$\begin{aligned} &m \inf \big\{ \textstyle\sum_i r_i^k;\ \text{X covered by cylinders } Q(z_i, r_i),\ r_i < \delta \big\} \\ &\ge \inf \big\{ \textstyle\sum_j r_j^k;\ \ \text{X covered by balls with diameters } r_j, r_j < \delta \big\} \\ &\Rightarrow m\mathcal{P}^k_\delta(X) \ge \mathcal{H}^k_\delta(X), \quad \forall 0 < \delta < 1. \end{aligned}$$

We aim at proving the following

Theorem 2.13. *Let $(\mathbf{u}, p)$ be a suitable weak solution to the Navier–Stokes equations in $(0,T) \times \Omega$, bounded. Let $D \subset \overline{D} \subset (0,T) \times \Omega$, $S_D = S \cap D$, where S is the set of all singular points, i.e. of all points from $(0,T) \times \Omega$, which are not regular. Then $\mathcal{P}^1(S_D) = 0$, i.e., the one dimensional parabolic measure of the set of singular points lying inside a compact subset of $(0,T) \times \Omega$ is zero.*

To prove Theorem 2.13 we shall need

Theorem 2.14. *There exists $\epsilon^* > 0$ such that if*

$$\limsup_{r \to 0^+} \frac{1}{r} \int_{Q^*(z_0, r)} |\nabla \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t < \epsilon^*, \tag{2.3}$$

then z_0 is a regular point.

We shall prove Theorem 2.14 in the next section. We need the following covering lemma

Lemma 2.15. *Let $\mathcal{J}$ be a class of parabolic cylinders $Q^*(z,r)$, which are contained in a bounded subset of $\mathbb{R}\times\mathbb{R}^3$. Then there exists an at most countable subclass $\mathcal{J}' = \{Q_i^*(z_i,r_i)\}_{i=1}^\infty$ such that*

$$Q_i^* \cap Q_j^* = \emptyset, \qquad i \neq j, \tag{2.4}$$

$$\forall Q^* \in \mathcal{J} \quad \exists Q_i^*(z_i,r_i) \in \mathcal{J}': \quad Q^* \subset Q_i^*(z_i, 5r_i). \tag{2.5}$$

Proof. We set $\mathcal{J}_0 = \mathcal{J}$ and proceed by induction. Let $\{Q_k^*\}_{k=1}^n$ be chosen and we set $\mathcal{J}_n = \{Q^* \in \mathcal{J},\, Q^* \cap Q_k^* = \emptyset,\, 1 \le k \le n\}$ (i.e., for $n = 0$ we do nothing). If $\mathcal{J}_n \neq \emptyset$, we choose $Q^*_{n+1}(z_{n+1}, r_{n+1}) \in \mathcal{J}_n$ such that $\forall Q^*(z,r) \in \mathcal{J}_n : r \le \frac{3}{2} r_{n+1}$. If $\mathcal{J}_n = \emptyset$, we finish the process and $\mathcal{J}' = \bigcup_{i=1}^n Q_i^*$. If the process is infinite, then necessarily $r_n \to 0$ (otherwise we get contradiction with the boundedness of the set). It follows from the construction that $\mathcal{J}'$ are disjoint. It remains to show the second property. We take arbitrary $\widetilde{Q}^* = \widetilde{Q}^*(z,r) \in \mathcal{J} \setminus \mathcal{J}'$. Then there exists $n \in \mathbb{N}_0$ such that $\widetilde{Q}^* \in \mathcal{J}_i$ for $i = 0, ..., n$ and $\widetilde{Q}^* \notin \mathcal{J}_{n+1}$ (otherwise contradiction with $r_n \to 0$). Thus $\widetilde{Q}^* \cap Q^*_{n+1} \neq \emptyset$ and $r_{n+1} \ge \frac{2}{3} r$. We extend the cylinder x-times and then

$$x \cdot r_{n+1} \ge \Big(1 + 2\cdot\frac{3}{2}\Big) r_{n+1} = 4 r_{n+1} \qquad \Longrightarrow \qquad x \ge 4,$$

$$\frac{1}{8}(x r_{n+1})^2 \ge \Big(\frac{1}{8} + \frac{9}{4}\Big) r_{n+1}^2 = \frac{19}{8} r_{n+1}^2 \qquad \Longrightarrow \qquad x^2 \ge 19.$$

Therefore it is enough to take $x = 5$ and $\widetilde{Q}^* \subset Q^*_{n+1}(z_{n+1}, 5r_{n+1})$. □

Proof. (Theorem 2.13) Let $(\mathbf{u}, p)$ be a suitable weak solution and let S be its singular set, S_D its intersection with a bounded set D lying inside the time-space cylinder. Then due to Theorem 2.14

$$z = (t,x) \in S_D \Rightarrow \limsup_{r\to 0^+} \frac{1}{r}\int_{Q^*(z,r)} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \ge \epsilon^*.$$

Let V be a neighbourhood of S_D in $\mathbb{R}\times\mathbb{R}^3$ and let $\delta > 0$ be sufficiently small. We choose for any $(t,x) \in S_D$ a cylinder $Q^*(z,r)$ with $r < \delta$ such that

$$\frac{1}{r}\int_{Q^*(z,r)} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \ge \frac{\epsilon^*}{2} \qquad \text{and } Q^*(z,r) \subset V.$$

Lemma 2.15 provides existence of a disjoint class $\{Q_i^*(z_i,r_i)\}_{i=1}^\infty$ such that $S_D \subset \bigcup_i Q_i^*(z_i, 5r_i)$ and

$$\sum_{i=1}^\infty r_i \le \frac{2}{\epsilon^*}\sum_i \int_{Q_i^*(z_i,r_i)} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{2}{\epsilon^*}\int_V |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{K}{\epsilon^*}.$$

We have ($\mathcal{L}^4$ denotes the four-dimensional Lebesgue measure)

$$\mathcal{L}^4(S_D) \le C \sum_{i=1}^\infty (5r_i)^5 \le C\delta^4 \sum_{i=1}^\infty r_i \le C\frac{\delta^4}{\epsilon^*},$$

where $\delta > 0$ can be taken arbitrarily small. Thus $\mathcal{L}^4(S_D) = 0$. Furthermore, $\mathcal{P}^1(S_D) \le \sum_{i=1}^\infty 5r_i \le \frac{5}{\epsilon^*}\int_V |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t$ for any neighbourhood V of the set S_D. As the four-dimensional Lebesgue measure of S_D is zero and $\nabla \mathbf{u} \in (L^2((0,T)\times\Omega))^{3\times 3}$, the measure of the set V can be arbitrarily small and due to the absolute continuity of the Lebesgue integral $\mathcal{P}^1(S_D) = 0$. □

Corollary 2.16. *The set of singular times (i.e., time instants τ in $(0,T)$ such that there exists a point $(\tau,x)\in S$) has $\frac{1}{2}$-dimensional Hausdorff measure zero.*

Proof. If $X\subset\mathbb{R}\times\mathbb{R}^3$, Σ_X is the projection of X on $\mathbb{R}$ and $\mathcal{P}^1(X)=0$, then $\mathcal{H}^{\frac{1}{2}}(\Sigma_X)=0$. Namely, if X is covered by countably many cylinders with diameter $r_i\le\delta$ such that $\sum_i r_i=o(1)$ for $\delta\to 0^+$, then the projection on the time axis is covered by countably many intervals of the length $\rho_i=r_i^2\le\delta^2=\Delta$. Then $\sum_i\rho_i^{\frac{1}{2}}=\sum_i r_i=o(1)$ for $\delta\to 0^+$, hence also for $\Delta\to 0^+$. □

Corollary 2.17. *If the solution fulfils $\nabla\mathbf{u}\in L^4(0,T;(L^2(\Omega))^{3\times 3})$ (i.e., we also have $\mathbf{u}\in L^4(0,T;(L^6(\Omega))^3)$), then S is empty.*

Proof. Let $z=(t,x)$. We compute

$$\int_{Q^*(z,r)}|\nabla\mathbf{u}|^2\,\mathrm{d}y\,\mathrm{d}t=\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\Big(\int_{|x-y|<r}|\nabla\mathbf{u}|^2\,\mathrm{d}y\Big)\,\mathrm{d}\tau$$
$$\le Cr\Big(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\Big(\int_{|x-y|<r}|\nabla\mathbf{u}|^2\,\mathrm{d}y\Big)^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}},$$

thus

$$\limsup_{r\to 0^+}\frac{1}{r}\int_{Q^*(z,r)}|\nabla\mathbf{u}|^2\,\mathrm{d}y\,\mathrm{d}t$$
$$\le C\limsup_{r\to 0^+}\Big(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\Big(\int_{|x-y|<r}|\nabla\mathbf{u}|^2\,\mathrm{d}y\Big)^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}}=0$$

due to the absolute continuity of the Lebesgue integral (the integral is finite and the set is getting smaller). □

Remark 2.18. Note that $\mathbf{u}\in L^4(0,T;(L^6(\Omega))^3)$ corresponds exactly to the Prodi–Serrin conditions, as $\frac{2}{4}+\frac{3}{6}=1$. More generally, if $\nabla\mathbf{u}\in L^p(0,T;(L^q(\Omega))^{3\times 3})$, $\frac{2}{p}+\frac{3}{q}=2$, then it can be shown that for arbitrary $\infty\ge q>\frac{3}{2}$, thus $1\le p<\infty$, the solution is regular and unique in the class of all Leray–Hopf weak solutions; the proof is similar to the case of Prodi–Serrin conditions for the velocity itself. Note also that the case $\nabla\mathbf{u}\in L^\infty(0,T;(L^{\frac{3}{2}}(\mathbb{R}^3))^{3\times 3})$ implies the regularity as $\mathbf{u}\in L^\infty(0,T;(L^3(\mathbb{R}^3))^3)$. Moreover, if we assume that $\nabla\mathbf{u}\in L^p(0,T;(L^q(\Omega))^{3\times 3})$ for both p, $q\ge 2$ (hence $q\in[2,3]$), then we get as above

$$\int_{Q^*(z,r)}|\nabla\mathbf{u}|^2\,\mathrm{d}y\,\mathrm{d}t=\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\Big(\int_{|x-y|<r}|\nabla\mathbf{u}|^2\,\mathrm{d}y\Big)\,\mathrm{d}\tau$$
$$\le Cr^{2\frac{p-2}{p}+3\frac{q-2}{q}}\Big(\int_{t-\frac{7}{8}r^2}^{t+\frac{1}{8}r^2}\Big(\int_{|x-y|<r}|\nabla\mathbf{u}|^q\,\mathrm{d}y\Big)^{\frac{p}{q}}\,\mathrm{d}\tau\Big)^{\frac{2}{p}},$$

thus if $2\frac{p-2}{p}+3\frac{q-2}{q}$, i.e., $\frac{2}{p}+\frac{3}{q}=2$, we get that the singular set must be empty. However, in the computation we need that both p, $q\ge 2$.

2.4. **Proof of local regularity criterion.** This section contains the proof of Theorem 2.14, which allowed us to deal with the partial regularity in the previous section. Without loss of generality (to simplify the notation), we take $z=(0,0)$ and instead of Q^* we take Q. The case $z\neq(0,0)$ and Q^*, respectively, can be obtained similarly. In what follows, we write Q_r instead of the correct $Q((0,0),r)$.

We first show

Theorem 2.19. *There exist constants $\epsilon_0>0$, $C_0>0$ such that if it holds for $(\mathbf{u},p)$ a suitable weak solution*

$$\int_{Q_1}\big(|\mathbf{u}|^3+|p|^{\frac{3}{2}}\big)\,\mathrm{d}x\,\mathrm{d}t\le\epsilon_0,\tag{2.6}$$

then $\|\mathbf{u}\|_{(C^{0,\alpha}(\overline{Q_k}))^3} \leq C_0$ *for certain* $0 < \alpha \leq 1$, $0 < k \leq 1$.

To prove the theorem, we need several auxiliary results.

Lemma 2.20. *Let* $(\mathbf{u}_n, p_n)$ *be a sequence of suitable solutions to the Navier–Stokes equations which fulfils*

$$\operatorname{ess}\sup_{t\in(-1,0)} \int_{B_1(0)} |\mathbf{u}_n(t,\cdot)|^2 \,\mathrm{d}x < \infty, \tag{2.7}$$

$$\int_{Q_1} |\nabla \mathbf{u}_n|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{2.8}$$

$$\int_{Q_1} |p_n|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{2.9}$$

If $(\mathbf{u}, p)$ *is the weak (or the weak-***) limit of* $(\mathbf{u}_n, p_n)$ *in the spaces with the norms given above, then* $(\mathbf{u}, p)$ *is a suitable weak solution to the Navier–Stokes equations.*

Proof. The proof is similar to the proof of Theorem 1.1. We need to show the strong convergence of $\mathbf{u}$ in $L^3(Q_1)$ and to this aim we would like to apply the Aubin–Lions Lemma. Therefore we need an estimate of the time derivative. We have in the distributional sense

$$\frac{\partial \mathbf{u}_n}{\partial t} = \Delta \mathbf{u}_n - \mathbf{u}_n \cdot \nabla \mathbf{u}_n - \nabla p_n$$

and due to our assumptions, the following sequences are bounded:

- $\Delta \mathbf{u}_n$ in $L^2(-1,0;(W^{-1,2}(B_1))^3)$
- ∇p_n in $L^{\frac{3}{2}}(-1,0;(W^{-1,\frac{3}{2}}(B_1))^3)$
- $\mathbf{u}_n \cdot \nabla \mathbf{u}_n$ in $L^{\frac{5}{3}}(-1,0;(L^{\frac{15}{14}}(B_1))^3)$,

where $L^{\frac{15}{14}}(B_1) \hookrightarrow W^{-1,\frac{5}{3}}(B_1) \hookrightarrow W^{-1,\frac{3}{2}}(B_1) = (W_0^{1,3}(B_1))^*$. The weakest information comes from the pressure. Therefore we have that $\frac{\partial \mathbf{u}_n}{\partial t}$ is bounded in $L^{\frac{3}{2}}(-1,0;(W^{-1,\frac{3}{2}}(B_1))^3)$. The Aubin–Lions Lemma implies $W^{1,2}(B_1) \hookrightarrow\hookrightarrow L^2(B_1) \hookrightarrow W^{-1,\frac{3}{2}}(B_1)$, thus $\mathbf{u}_n \to \mathbf{u}$ in $(L^2(Q_1))^3$. It implies together with the boundedness of $\mathbf{u}_n$ in $(L^{\frac{10}{3}}(Q_1))^3$ that $\mathbf{u}_n \to \mathbf{u}$ in $(L^q(Q_1))^3$ for $1 \leq q < \frac{10}{3}$, i.e., in particular also for $q = 3$. The rest of the proof is obvious. $\square$

Lemma 2.21. *There exists* $\epsilon_0 > 0$ *such that if* $\int_{Q_1} \big(|\mathbf{u}|^3 + |p|^{\frac{3}{2}}\big) \,\mathrm{d}x\,\mathrm{d}t \leq \epsilon_0$ *for* $(\mathbf{u}, p)$ *a suitable weak solution to the Navier–Stokes equations, then*

$$\begin{aligned} &\Big(\theta^{-5} \int_{Q_\theta} \frac{|\mathbf{u} - \mathbf{u}_\theta|^3}{\theta^{\alpha_0}} \,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{1}{3}} + \theta\Big(\theta^{-5} \int_{Q_\theta} \frac{|p - p_\theta(t)|^{\frac{3}{2}}}{\theta^{\alpha_0}} \,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{2}{3}} \\ &\qquad \leq \frac{1}{2}\Big(\Big(\int_{Q_1} |\mathbf{u}|^3 \,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{1}{3}} + \Big(\int_{Q_1} |p|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{2}{3}}\Big) \end{aligned} \tag{2.10}$$

for a certain $\theta \in (0,1)$ *(it is possible to take* $\theta \in (\underline{\theta}, \overline{\theta})$, $\underline{\theta} = \overline{\theta}^2$, $0 < \underline{\theta} < \overline{\theta} < 1$*) and* $\alpha_0 \in (0, \frac{1}{2})$, *where*

$$\mathbf{u}_\theta = \theta^{-5} \int_{Q_\theta} \mathbf{u}(\tau, y) \,\mathrm{d}y\,\mathrm{d}\tau, \qquad p_\theta(t) = \theta^{-3} \int_{B_\theta} p(t,y)\,\mathrm{d}y, \quad -\theta^{-2} \leq t \leq 0. \tag{2.11}$$

Proof. We prove the result by contradiction. Assume the contrary, i.e. let there exist a sequence $\epsilon_i \to 0^+$ such that $\epsilon_i = \|\mathbf{u}_i\|_{(L^3(Q_1))^3} + \|p_i\|_{L^{\frac{3}{2}}(Q_1)}$, where $(\mathbf{u}_i, p_i)$ is a sequence of suitable weak solutions. Furthermore, let (2.10) do not hold for any

$\theta \in (0,1)$ and any $(\mathbf{u}_i, p_i)$, $i \in \mathbb{N}$. Denote $\mathbf{U}_i = \frac{\mathbf{u}_i}{\epsilon_i}$ and $P_i = \frac{p_i}{\epsilon_i}$. These functions fulfil

$$\frac{\partial \mathbf{U}_i}{\partial t} + \epsilon_i \mathbf{U}_i \cdot \nabla \mathbf{U}_i - \Delta \mathbf{U}_i + \nabla P_i = \mathbf{0},$$
$$\operatorname{div} \mathbf{U}_i = 0$$

in the weak sense; moreover, it is a suitable weak solution to the Navier–Stokes equations and the generalized energy inequality is satisfied for any $\Phi \in C_0^\infty((-1,0] \times B_1)$, $\Phi \geq 0$ in the form

$$\int_{B_1} \Phi(t,\cdot)|\mathbf{U}_i(t,\cdot)|^2 \,\mathrm{d}x + 2\int_{-1}^t \int_{B_1} \Phi |\nabla \mathbf{U}_i|^2 \,\mathrm{d}x \,\mathrm{d}t$$
$$\leq \int_{-1}^t \int_{B_1} |\mathbf{U}_i|^2 \left(\frac{\partial \Phi}{\partial t} + \Delta \Phi\right) \mathrm{d}x \,\mathrm{d}t + \int_{-1}^t \int_{B_1} \left(2P_i + \epsilon_i |\mathbf{U}_i|^2\right) \mathbf{U}_i \cdot \nabla \Phi \,\mathrm{d}x \,\mathrm{d}t.$$

We have $\|\mathbf{U}_i\|_{(L^3(Q_1))^3} \leq 1$, $\|P_i\|_{L^{\frac{3}{2}}(Q_1)} \leq 1$; whence $\mathbf{U}_i$ is also bounded in the spaces $L^\infty_{loc}((-1,0];(L^2_{loc}(B_1))^3)$ and in $L^2_{loc}((-1,0];(W^{1,2}_{loc}(B_1))^3)$, thus also in $(L^{\frac{10}{3}}_{loc}((-1,0] \times B_1))^3$. Using the same procedure as in Lemma 2.20 we get

$$\mathbf{U}_i \rightharpoonup \mathbf{U} \quad \text{in } (L^3(Q_1))^3,$$
$$P_i \rightharpoonup P \quad \text{in } L^{\frac{3}{2}}(Q_1)$$

and

$$\mathbf{U}_i \to \mathbf{U} \quad \text{in } (L^q_{loc}((-1,0] \times B_1))^3 \text{ for any } 1 \leq q < \frac{10}{3},$$

where the pair $(\mathbf{U}, P)$ satisfies in the weak sense

$$\frac{\partial \mathbf{U}}{\partial t} - \Delta \mathbf{U} + \nabla P = \mathbf{0},$$
$$\operatorname{div} \mathbf{U} = 0.$$

The weak lower semicontinuity of a norm implies $\|\mathbf{U}\|_{(L^3(Q_1))^3} \leq 1$ and $\|P\|_{L^{\frac{3}{2}}(Q_1)} \leq 1$. We can now use properties of the Stokes problem, in particular that $\mathbf{U}$ is Hölder continuous in the time variable, say with the exponent $2\alpha_0$, and Lipschitz continuous in the spatial variable. (The proof is technical, but nowadays standard and well known.) Therefore we have

$$\theta^{-5} \int_{Q_\theta} |\mathbf{U} - \mathbf{U}_\theta|^3 \,\mathrm{d}x \,\mathrm{d}t \leq C\theta^{-5} \int_{Q_\theta} \left(\theta^{2\alpha_0} + \theta\right)^3 \mathrm{d}x \,\mathrm{d}t \leq \frac{1}{2}\frac{1}{5^3}\theta^{\alpha_0}$$

if we choose sufficiently small $\theta \leq \frac{1}{2}$. Moreover, we have $\mathbf{U}_i \to \mathbf{U}$ v $(L^3_{loc}((-1,0] \times B_1)^3$, thus

$$\theta^{-5} \int_{Q_\theta} |\mathbf{U}_i - \mathbf{U}_{i,\theta}|^3 \,\mathrm{d}x \,\mathrm{d}t \leq \frac{1}{5^3}\theta^{\alpha_0} \tag{2.12}$$

for a sufficiently large i_0, $i \geq i_0$.

Let us now consider the pressure. We have

$$\Delta P_i = -\epsilon_i \operatorname{div}\operatorname{div}(\mathbf{U}_i \otimes \mathbf{U}_i) = -\epsilon_i \frac{\partial U_i^k}{\partial x_l}\frac{\partial U_i^l}{\partial x_k}.$$

Hence we can write $P_i = h_i + g_i$, where h_i is a harmonic function in $B_{\frac{2}{3}}$ for any $t \in (-1,0)$ and g_i satisfies

$$\Delta g_i = -\epsilon_i \frac{\partial U_i^k}{\partial x_l}\frac{\partial U_i^l}{\partial x_k} \quad \text{in } B_{\frac{2}{3}},$$
$$g_i = 0 \quad \text{on } \partial B_{\frac{2}{3}}.$$

We set

$$h_{i,\theta}(t) = \theta^{-3} \int_{B_\theta} h_i(t,x)\,\mathrm{d}x,$$
$$g_{i,\theta}(t) = \theta^{-3} \int_{B_\theta} g_i(t,x)\,\mathrm{d}x.$$

Then we have

$$\int_{Q_\theta} |P_i - P_{i,\theta}|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t \le C\Big(\int_{Q_\theta} |h_i - h_{i,\theta}|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\theta} |g_i|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\theta} |g_{i,\theta}|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t\Big).$$

Now

$$\int_{Q_\theta} |g_i|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\theta} |g_{i,\theta}|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t \le C\epsilon_i^{\frac{3}{2}} \int_{Q_\theta} |\mathbf{U}_i|^3\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\theta} \theta^{-\frac{9}{2}} \left(\int_{B_\theta} |g_i(t,y)|^{\frac{3}{2}} \theta^{\frac{3}{2}}\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t \le C\epsilon_i^{\frac{3}{2}} \int_{Q_\theta} |\mathbf{U}_i|^3\,\mathrm{d}x\,\mathrm{d}t,$$

where we employed the Hölder inequality with 1 and the Fubini theorem. Furthermore

$$\int_{Q_\theta} |h_i - h_{i,\theta}|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t \le C\theta^3\theta^{\frac{3}{2}},$$

as h_i are harmonic functions, therefore smooth in the spatial variables. Additionally, they are bounded $L^{\frac{3}{2}}(-1,0;L^{\frac{3}{2}}(B_{\frac{2}{3}}))$, as g_i and P_i are so. Altogether

$$\theta\Big(\theta^{-5} \int_{Q_\theta} |P_i - P_{i,\theta}|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{2}{3}} \le C\theta^{\frac{2}{3}} + C\epsilon_i\theta^{-\frac{7}{3}} \le \frac{1}{5}\theta^{\frac{2}{3}\alpha_0}$$

for a sufficiently small ϵ_i and a suitable θ. Summarizing,

$$\Big(\theta^{-5} \int_{Q_\theta} \frac{|U_i - U_{i,\theta}|^3}{\theta^{\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{1}{3}} + \theta\Big(\theta^{-5} \int_{Q_\theta} \frac{|P_i - P_{i,\theta}|^{\frac{3}{2}}}{\theta^{\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{2}{3}} \le \frac{2}{5},$$

which leads to a contradiction. □

We can now start with the proof of Theorem 2.19.

Proof. (Theorem 2.19) Let $\int_{Q_1} \big(|\mathbf{u}|^3 + |p|^{\frac{3}{2}}\big)\,\mathrm{d}x\,\mathrm{d}t \le \epsilon_0$. Then

$$\left(\int_{Q_1} |\mathbf{u}|^3\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{3}} + \left(\int_{Q_1} |p|^{\frac{3}{2}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{3}} \le \widetilde{\epsilon_0}$$

for $\widetilde{\epsilon_0}$ small. We define

$$\mathbf{u}_1(t,x) = \theta^{-\frac{\alpha_0}{3}}\big(\mathbf{u}(\theta^2 t, \theta x) - \mathbf{u}_\theta\big)$$
$$p_1(t,x) = \theta^{1-\frac{\alpha_0}{3}}\big(p(\theta^2 t, \theta x) - p_\theta(\theta^2 t)\big).$$

We recompute the differential operators:

$$\frac{\partial \mathbf{u}_1}{\partial t}(t,x) = \frac{\partial \mathbf{u}}{\partial \tau}(\theta^2 t, \theta x)\theta^{2-\frac{\alpha_0}{3}}$$
$$\mathbf{u}_1 \cdot \nabla \mathbf{u}_1(t,x) = \mathbf{u} \cdot \nabla_y \mathbf{u}(\theta^2 t, \theta x)\theta^{1-\frac{2\alpha_0}{3}} - \mathbf{u}_\theta \cdot \nabla_y \mathbf{u}(\theta^2 t, \theta x)\theta^{1-\frac{2\alpha_0}{3}} \quad \text{or otherwise}$$
$$\mathbf{u}_1 \cdot \nabla \mathbf{u}_1(t,x) = \mathbf{u} \cdot \nabla_y \mathbf{u}(\theta^2 t, \theta x)\theta^{1-\frac{2\alpha_0}{3}} - \mathbf{u}_\theta \cdot \nabla_x \mathbf{u}_1(t,x)\theta^{-\frac{\alpha_0}{3}}$$
$$\Delta \mathbf{u}_1(t,x) = \Delta_y \mathbf{u}(\theta^2 t, \theta x)\theta^{2-\frac{\alpha_0}{3}}$$
$$\nabla p_1(t,x) = \nabla_y p(\theta^2 t, \theta x)\theta^{2-\frac{\alpha_0}{3}}$$

and the Navier–Stokes equations for $(\mathbf{u},p)$ transform to the equation for $(\mathbf{u}_1,p_1)$

$$\frac{\partial \mathbf{u}_1}{\partial t} + \theta\left(\mathbf{u}_\theta + \theta^{\frac{\alpha_0}{3}}\mathbf{u}_1\right)\cdot\nabla\mathbf{u}_1 + \nabla p_1 - \Delta\mathbf{u}_1 = 0 \quad \text{in } Q_1.$$

We use Lemma 2.21 for the pair $(\mathbf{u},p)$:

$$\Big(\theta^{-5}\int_{Q_\theta}\frac{|\mathbf{u}(t,x)-\mathbf{u}_\theta|^3}{\theta^{\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac13} + \theta\Big(\theta^{-5}\int_{Q_\theta}\frac{|p(t,x)-p_\theta(t)|^{\frac32}}{\theta^{\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac23} \le \frac12\widetilde{\epsilon_0}.$$

By virtue of the change of variables $x=\theta y$, $t=\theta^2\tau$ we have $(\tau,y)\in Q_1$ and

$$\Big(\int_{Q_1}|\mathbf{u}_1(\tau,y)|^3\,\mathrm{d}y\,\mathrm{d}\tau\Big)^{\frac13} + \theta\Big(\int_{Q_1}|p_1(\tau,y)|^{\frac32}\theta^{-\frac32+\frac{\alpha_0}{2}-\alpha_0}\,\mathrm{d}y\,\mathrm{d}\tau\Big)^{\frac23} \le \frac12\widetilde{\epsilon_0};$$

as the power at θ is negative and $\theta<1$, we also have

$$\Big(\int_{Q_1}|\mathbf{u}_1(\tau,y)|^3\,\mathrm{d}y\,\mathrm{d}\tau\Big)^{\frac13} + \Big(\int_{Q_1}|p_1(\tau,y)|^{\frac32}\,\mathrm{d}y\,\mathrm{d}\tau\Big)^{\frac23} \le \frac12\widetilde{\epsilon_0}.$$

We now want to use a similar lemma as Lemma 2.21 for $(\mathbf{u}_1,p_1)$ and for $\theta\in(\underline{\theta},\overline{\theta})$, $\underline{\theta}=\overline{\theta}^2$. The limit problem is here

$$\begin{aligned}\frac{\partial \mathbf{U}_2}{\partial t} + \mathbf{b}\cdot\nabla\mathbf{U}_2 - \Delta\mathbf{U}_2 + \nabla P_2 &= \mathbf{0},\\ \operatorname{div}\mathbf{U}_2 &= 0\end{aligned}$$

for $\mathbf{b}=\theta(\mathbf{u})_\theta$ a constant vector. Even though the limit problem is not the Stokes problem, we have similar properties of the solution, in particular the Hölder continuity. Note that we also have $\operatorname{div}(\mathbf{u}_\theta\cdot\nabla\mathbf{u}_i)=0$. Now

$$\begin{aligned}\theta^{-5}\int_{Q_\theta}\frac{|\mathbf{u}_1-\mathbf{u}_{1,\theta}|^3}{\theta^{\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t &= \theta^{-10}\int_{Q_{\theta^2}}\frac{|\mathbf{u}-\mathbf{u}_{\theta^2}|^3}{\theta^{2\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t,\\ \theta\Big(\theta^{-5}\int_{Q_\theta}\frac{|p_1-p_{1,\theta}(t)|^{\frac32}}{\theta^{\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac23} &= \theta^{2+\frac{\alpha_0}{3}}\Big(\theta^{-10}\int_{Q_{\theta^2}}\frac{|p-p_{\theta^2}(t)|^{\frac32}}{\theta^{2\alpha_0}}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac23},\end{aligned}$$

and iterating we get

$$r^{-5}\int_{Q_r}|\mathbf{u}-\mathbf{u}_r|^3\,\mathrm{d}x\,\mathrm{d}t \le C\overline{\epsilon}_0 r^{\alpha_0} \quad \forall r\in(0,\overline{\theta}), \tag{2.13}$$

which implies the Hölder continuity — see below. □

Remark 2.22. The fact that $\mathbf{u}$ is Hölder continuous follows from the theory of Morrey–Campanato spaces. We shall not introduce them, we only show how the Hölder continuity can be obtained. We have

$$\frac{1}{r^{5+\alpha_0}}\int_{Q_{((t_0,x_0);r)}}|\mathbf{u}-\mathbf{u}_{(t_0,x_0);r}|^3\,\mathrm{d}x\,\mathrm{d}t \le C, \quad \alpha_0>0, r\in(0,\overline{\theta}), (t_0,x_0)\in\overline{Q}_\beta,$$
$$0<\beta\le 1, \quad \mathbf{u}_{(t_0,x_0);r} = \frac{1}{|Q_r|}\int_{Q_{((t_0,x_0),r)}}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t.$$

For simplicity, we take $(t_0,x_0)=(0,0)$. For $R_0=\frac{\overline{\theta}}{2}$, $R_{i+1}=\frac{R_i}{2}$ we have

$$|\mathbf{u}_{R_i}-\mathbf{u}_{R_{i+1}}|^3 \le C\left(|\mathbf{u}_{R_i}-\mathbf{u}(t,x)|^3 + |\mathbf{u}(t,x)-\mathbf{u}_{R_{i+1}}|^3\right).$$

Integrating $\int_{Q_{R_{i+1}}}\cdot\,\mathrm{d}x\,\mathrm{d}t$ yields

$$\begin{aligned}&|\mathbf{u}_{R_i}-\mathbf{u}_{R_{i+1}}|^3\\ &\quad\le CR_{i+1}^{-5}\Big(\int_{Q_{R_i}}|\mathbf{u}_{R_i}-\mathbf{u}(t,x)|^3\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_{R_{i+1}}}|\mathbf{u}(t,x)-\mathbf{u}_{R_{i+1}}|^3\,\mathrm{d}x\,\mathrm{d}t\Big)\\ &\quad\le CR_i^{\alpha_0}.\end{aligned}$$

Thus

$$|\mathbf{u}_{R_0} - \mathbf{u}_{R_{n+1}}| \le C \sum_{i=1}^{n} R_0^{\alpha_0/3} 2^{-i\alpha_0/3} \le C R_0^{\alpha_0/3},$$

Similarly

$$|\mathbf{u}_{R_n} - \mathbf{u}_{R_{n+m}}| \le C R_n^{\alpha_0/3}.$$

From here we see that $\mathbf{u}_{R_n}$ is a Cauchy sequence, thus there exists $\lim_{n\to\infty} \mathbf{u}_{R_n} = \overline{\mathbf{u}}$, which equals to $\mathbf{u}(0,0)$.[2] Thus

$$|\mathbf{u}(0,0) - \mathbf{u}_{(0,0);R}| \le C R^{\frac{\alpha_0}{3}}.$$

The whole construction can be performed for a.a. points of the time-space cylinder. The estimate above is uniform with respect to $(t,x) \in \overline{Q}_r$. We therefore have for $R = |z_1 - z_2| = \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$

$$|\mathbf{u}(z_1) - \mathbf{u}(z_2)| \le |\mathbf{u}(z_1) - \mathbf{u}_{z_1;2R}| + |\mathbf{u}_{z_1;2R} - \mathbf{u}_{z_2;2R}| + |\mathbf{u}_{z_2;2R} - \mathbf{u}(z_2)|.$$

The first and the third terms can be estimated by $CR^{\frac{\alpha_0}{3}}$, while the second term we estimate by means of the Hölder inequality as follows

$$\begin{aligned} &|\mathbf{u}_{z_1;2R} - \mathbf{u}_{z_2;2R}| \\ &\quad \le \frac{1}{|S|}\Big(\int_{Q(z_1;2R)} |\mathbf{u}_{z_1;2R} - \mathbf{u}(z)| \, \mathrm{d}z + \int_{Q(z_2,2R)} |\mathbf{u}_{z_2;2R} - \mathbf{u}(z)| \, \mathrm{d}z \Big) \\ &\qquad\qquad\qquad\qquad \le C \frac{1}{R^5} R^{\frac{1}{3}(5+\alpha_0)} R^{5\frac{2}{3}} = C R^{\frac{\alpha_0}{3}}, \end{aligned}$$

where $S = Q(z_1, 2R) \cap Q(z_2, 2R)$. Thus

$$|\mathbf{u}(z_1) - \mathbf{u}(z_2)| \le C R^{\frac{\alpha_0}{3}} \le C |z_1 - z_2|^{\frac{\alpha_0}{3}}.$$

The Hölder continuity is proved.

Let us introduce the following notation:

$$\begin{aligned} A(r) &= \sup_{-r^2 \le t \le 0} \frac{1}{r} \int_{B_r} |\mathbf{u}|^2 \, \mathrm{d}x, \\ B(r) &= \frac{1}{r} \int_{Q_r} |\nabla \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t, \\ C(r) &= \frac{1}{r^2} \int_{Q_r} |\mathbf{u}|^3 \, \mathrm{d}x \, \mathrm{d}t, \\ D(r) &= \frac{1}{r^2} \int_{Q_r} |p|^{\frac{3}{2}} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{2.14}$$

Lemma 2.23. *It holds for* $0 \le r \le \rho$

$$C(r) \le K \Big[\Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \Big]. \tag{2.15}$$

[2] In fact, the limit is for a.e. $(t,x) \in \overline{Q}_r$ equal to $\mathbf{u}(t,x)$. We assume without loss of generality that $(0,0)$ is such a point.

Proof. Denote $\overline{f_\rho} = \frac{1}{|B_\rho|}\int_{B_\rho} f\,\mathrm{d}x$. Then

$$\begin{aligned}\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x &\le \int_{B_r}\big(|\mathbf{u}|^2 - \overline{(|\mathbf{u}|^2)_\rho}\big)\,\mathrm{d}x + \int_{B_r}\overline{(|\mathbf{u}|^2)_\rho}\,\mathrm{d}x\\ &\le \int_{B_\rho}\big||\mathbf{u}|^2 - \overline{(|\mathbf{u}|^2)_\rho}\big|\,\mathrm{d}x + \int_{B_r}\overline{(|\mathbf{u}|^2)_\rho}\,\mathrm{d}x\\ &\le K\rho\int_{B_\rho}|\nabla(|\mathbf{u}|^2)|\,\mathrm{d}x + \Big(\frac{r}{\rho}\Big)^3\int_{B_\rho}|\mathbf{u}|^2\,\mathrm{d}x\\ &\le K\rho\int_{B_\rho}|\mathbf{u}||\nabla\mathbf{u}|\,\mathrm{d}x + \Big(\frac{r}{\rho}\Big)^3\int_{B_\rho}|\mathbf{u}|^2\,\mathrm{d}x.\end{aligned}$$

Due to the Poincaré inequality

$$\int_\Omega |\mathbf{w}|\,\mathrm{d}x \le K(\mathrm{diam}\ \Omega)\int_\Omega|\nabla\mathbf{w}|\,\mathrm{d}x,$$

where the constant K is independent of Ω. Thus

$$\begin{aligned}\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x &\le K\rho\Big(\int_{B_\rho}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac12} + \Big(\frac{r}{\rho}\Big)^3\int_{B_\rho}|\mathbf{u}|^2\,\mathrm{d}x\\ &\le K\rho^{\frac32}A^{\frac12}(\rho)\Big(\int_{B_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac12} + \Big(\frac{r}{\rho}\Big)^3\rho A(\rho).\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_{B_r}|\mathbf{u}|^3\,\mathrm{d}x &\le K\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34}\Big(\int_{B_r}|\mathbf{u}|^6\,\mathrm{d}x\Big)^{\frac{3}{12}}\\ &\le K\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34}\Big(\int_{B_r}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34} + K(r)\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac32},\end{aligned}$$

where for the ball $B_r\subset\mathbb{R}^3$ the constant $K(r)$ can be computed by the scaling argument:

$$\int_{B_1}|\mathbf{w}|^3\,\mathrm{d}x \le K\Big(\int_{B_1}|\mathbf{w}|^2\,\mathrm{d}x\Big)^{\frac34}\Big(\int_{B_1}|\nabla_x\mathbf{w}|^2\,\mathrm{d}x\Big)^{\frac34} + K(1)\Big(\int_{B_1}|\mathbf{w}|^2\,\mathrm{d}x\Big)^{\frac32},$$

$\mathbf{w}(x) = \mathbf{u}(rx)$, $y = rx$, $\nabla_x = r\nabla_y$. Thus

$$\begin{aligned}\frac{1}{r^3}\int_{B_r}|\mathbf{u}|^3\,\mathrm{d}y \le K\Big(\frac{1}{r^3}\Big)^{\frac34}\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}y\Big)^{\frac34}\Big(\int_{B_r}|\nabla_y\mathbf{u}|^2\,\mathrm{d}y\Big)^{\frac34}\Big(\frac{1}{r^3}\Big)^{\frac34}(r^2)^{\frac34}\\ + K(1)\Big(\frac{1}{r^3}\Big)^{\frac32}\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}y\Big)^{\frac32}.\end{aligned}$$

It yields $K(r) = \frac{1}{r^{\frac32}}$. Therefore

$$\begin{aligned}\int_{B_r}|\mathbf{u}|^3\,\mathrm{d}x &\le K\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34}\Big(\int_{B_r}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34} + \frac{K}{r^{\frac32}}\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac32}\\ &\le K\Big(\int_{B_\rho}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34}\Big(\int_{B_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34} + \frac{K}{r^{\frac32}}\Big(\int_{B_r}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac32}\\ &\le K\rho^{\frac34}A^{\frac34}(\rho)\Big(\int_{B_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34}\\ &\quad + \frac{K}{r^{\frac32}}\Big[\Big(\rho^{\frac32}A^{\frac12}(\rho)\Big(\int_{B_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac12}\Big)^{\frac32} + \Big(\Big(\frac{r}{\rho}\Big)^3\rho A(\rho)\Big)^{\frac32}\Big]\\ &\le K\Big[\rho^{\frac34}A^{\frac34}(\rho)\Big(\int_{B_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34} + \frac{\rho^{\frac94}}{r^{\frac32}}A^{\frac34}(\rho)\Big(\int_{B_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac34} + \frac{r^3}{\rho^3}A^{\frac32}(\rho)\Big].\end{aligned}$$

We now integrate over the time variable $\int_{-r^2}^0 \cdot \,\mathrm{d}t$:

$$\int_{Q_r} |\mathbf{u}|^3 \,\mathrm{d}x\,\mathrm{d}t \leq K\Big[\int_{-r^2}^0 \Big(\int_{B_\rho} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\Big)^{\frac{3}{4}} \mathrm{d}t \Big(\rho^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} A^{\frac{3}{4}}(\rho)\Big) + \frac{r^5}{\rho^3} A^{\frac{3}{2}}(\rho)\Big]$$
$$\leq K r^{\frac{1}{2}} \Big(\int_{Q_\rho} |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{3}{4}} \Big(\rho^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} A^{\frac{3}{4}}(\rho)\Big) + K \frac{r^5}{\rho^3} A^{\frac{3}{2}}(\rho).$$

This implies

$$C(r) \leq K \frac{r^3}{\rho^3} A^{\frac{3}{2}}(\rho) + K A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \Big[\Big(\frac{\rho}{r}\Big)^{\frac{3}{2}} + \Big(\frac{\rho}{r}\Big)^3\Big].$$

As $\rho \geq r$, we have $\frac{\rho^{\frac{3}{2}}}{r^{\frac{3}{2}}} \leq \frac{\rho^3}{r^3}$. It gives

$$C(r) \leq K\Big[\Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho)\Big].$$

The proof is complete. □

Lemma 2.24. *It holds for* $0 < r \leq \rho$

$$D(r) \leq K\Big[\Big(\frac{\rho}{r}\Big)^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) + \frac{r}{\rho} D(\rho)\Big]. \tag{2.16}$$

Proof. We know that the pressure can be written as $p = p_1 + p_2$, where

$$\begin{aligned} \Delta p_2 &= 0 \quad \text{in } B_\rho \\ \Delta p_1 &= -\operatorname{div}\operatorname{div}\Big(\mathbf{u}\otimes\mathbf{u} - \overline{(\mathbf{u}\otimes\mathbf{u})_\rho}\Big) \quad \text{in } B_\rho, \\ p_1 &= 0 \quad \text{on } \partial B_\rho. \end{aligned}$$

Due to the Calderón–Zygmund theory

$$\int_{B_1} |p_1|^{\frac{3}{2}} \,\mathrm{d}x \leq K \int_{B_1} |\mathbf{u}\otimes\mathbf{u} - \overline{(\mathbf{u}\otimes\mathbf{u})_1}|^{\frac{3}{2}} \,\mathrm{d}x \leq K\Big(\int_{B_1} |\nabla(\mathbf{u}\otimes\mathbf{u})| \,\mathrm{d}x\Big)^{\frac{3}{2}}.$$

We have using the Poincaré inequality and the scaling argument

$$\int_{B_\rho} |p_1|^{\frac{3}{2}} \,\mathrm{d}x \leq K \|\nabla(\mathbf{u}\otimes\mathbf{u})\|^{\frac{3}{2}}_{L^1(B_\rho)} \leq K\Big(\int_{B_\varrho} |\nabla\mathbf{u}||\mathbf{u}| \,\mathrm{d}x\Big)^{\frac{3}{2}}$$
$$\leq K \rho^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) \Big(\int_{B_\rho} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\Big)^{\frac{3}{4}}.$$

Therefore

$$\int_{Q_\rho} |p_1|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t \leq K \rho^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) \rho^{\frac{1}{2}} \Big(\frac{1}{\rho}\int_{Q_\rho} |\nabla\mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{3}{4}} \rho^{\frac{3}{4}} = K \rho^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho).$$

Hence

$$\int_{Q_\rho} |p_2|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t \leq K\Big(\int_{Q_\rho} |p|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t + \int_{Q_\rho} |p_1|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t\Big) \leq K \varrho^2 \Big[D(\rho) + A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho)\Big].$$

As p_2 is harmonic in B_ρ, we have

$$\frac{1}{r^3} \int_{B_r} |p_2|^{\frac{3}{2}} \,\mathrm{d}x \leq \frac{K}{\rho^3} \int_{B_\rho} |p_2|^{\frac{3}{2}} \,\mathrm{d}x,$$

where K is independent of ρ and r. This follows from the mean value theorem for harmonic functions. If $\frac{1}{2}\rho \leq r \leq \rho$, the claim is obvious. Therefore it is enough to assume $r < \frac{1}{2}\rho$. In this case, for arbitrary $x \in B_r$ and $s < \frac{1}{2}\rho$ we have

$$p_2(x) = \frac{1}{4\pi s^2} \int_{\partial B_s(x)} p_2(y) \,\mathrm{d}S_y,$$

hence

$$s^2|p_2(x)|^{\frac{3}{2}} \leq C \int_{\partial B_s(x)} |p_2(y)|^{\frac{3}{2}}\, \mathrm{d}S_y.$$

Integrating this inequality over $s \in (0, \frac{1}{2}\rho)$ yields

$$|p(x)|^{\frac{3}{2}} \leq \frac{C}{\rho^3} \int_{B_{\frac{1}{2}\rho}(x)} |p_2|^{\frac{3}{2}}\, \mathrm{d}y \leq \frac{C}{\rho^3} \int_{B_\rho} |p_2|^{\frac{3}{2}}\, \mathrm{d}y.$$

Integrating this inequality over B_r we get the claim. Thus for $r \leq \rho$

$$\frac{1}{r^3} \int_{Q_r} |p_2|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t \leq \frac{K}{\rho^3} \int_{Q_\rho} |p_2|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t.$$

Finally

$$\begin{aligned} D(r) = \frac{1}{r^2} \int_{Q_r} |p|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t &\leq \frac{K}{r^2}\Big(\int_{Q_r} |p_1|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t + \int_{Q_r} |p_2|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t\Big) \\ &\leq K\Big(\frac{1}{r^2} \int_{Q_r} |p_1|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t + \frac{r}{\rho}\frac{1}{\rho^2} \int_{Q_\rho} |p_2|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t\Big) \\ &\leq K\Big[\Big(\frac{\rho}{r}\Big)^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) + \frac{r}{\rho} D(\rho) + \frac{r}{\rho} A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho)\Big], \\ &\leq K\Big[\Big(\frac{\rho}{r}\Big)^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) + \frac{r}{\rho} D(\rho)\Big], \end{aligned}$$

as $0 < r \leq \rho$. □

We can now start with the proof of Theorem 2.14.

Proof. (Theorem 2.14) Without loss of generality we assume $z_0 = (0,0)$ and we replace Q^* by Q. We take $r \leq \frac{\rho}{2}$ and the generalized energy inequality (2.1) with

$$\Phi \begin{cases} = 1 & \mathrm{na}\, Q_r, \\ = 0 & \mathrm{na}\, (\mathbb{R}^- \times \mathbb{R}^3)\backslash Q_\rho, \\ \in C^\infty & \mathrm{na}\, Q_\rho \backslash Q_r, \end{cases}$$

where $0 \leq \Phi \leq 1$ and $\nabla^k \Phi \leq \frac{K}{\rho^k}$, $k = 1, 2$, $\frac{\partial \Phi}{\partial t} \leq \frac{K}{\rho^2}$. Then we have

(2.17)
$$\begin{aligned} &\sup_{-r^2 \leq t \leq 0} \frac{1}{r} \int_{B_r} |\mathbf{u}|^2(t, \cdot)\, \mathrm{d}x + \frac{1}{r} \int_{Q_r} |\nabla \mathbf{u}|^2\, \mathrm{d}x\, \mathrm{d}t \leq \frac{K}{r} \int_{Q_\rho} |\mathbf{u}|^2 \Big(\frac{\partial \Phi}{\partial t} + \Delta \Phi\Big)\, \mathrm{d}x\, \mathrm{d}t \\ &+ \frac{K}{r} \int_{Q_\rho} \Big(|\mathbf{u}|^2 - \overline{|\mathbf{u}|^2_\rho}\Big) \mathbf{u} \cdot \nabla \Phi\, \mathrm{d}x\, \mathrm{d}t + \frac{K}{r} \int_{Q_\rho} 2p\, \mathbf{u} \cdot \nabla \Phi\, \mathrm{d}x\, \mathrm{d}t = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

On the left-hand side we have $A(r) + B(r)$, we estimate the terms on the right-hand side

$$|\mathcal{I}_1| \leq \frac{K}{r}\frac{1}{\rho^2}\Big(\int_{Q_\rho} |\mathbf{u}|^3\, \mathrm{d}x\, \mathrm{d}t\Big)^{\frac{2}{3}} \rho^{\frac{5}{3}} \leq K \frac{\rho}{r} C^{\frac{2}{3}}(\rho). \tag{2.18}$$

It holds for the second term (without loss of generality we assume $\rho \le 1$)

$$
\begin{aligned}
|\mathcal{I}_2| &\le \frac{K}{r\rho}\Big(\int_{Q_\rho} |\mathbf{u}|^3 \,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac13}\Big(\int_{Q_\rho} \big||\mathbf{u}|^2 - \overline{|\mathbf{u}|^2_\rho}\big|^{\frac32}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac23} \\
&\le \frac{K}{r\rho^{\frac13}} C^{\frac13}(\rho)\int_{Q_\rho} |\nabla|\mathbf{u}|^2|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{K}{r\rho^{\frac13}} C^{\frac13}(\rho)\sup_{-\rho^2\le t\le 0}\Big(\int_{B_\rho}|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac12}\Big(\int_{Q_\rho}|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac12}\Big(\int_{-\rho^2}^0 1\,\mathrm{d}t\Big)^{\frac12} \\
&\le \frac{K\rho^2}{r\rho^{\frac13}} C^{\frac13}(\rho)A^{\frac12}(\rho)B^{\frac12}(\rho) \le K\Big(\frac{\rho}{r}\Big)A^{\frac12}(\rho)B^{\frac12}(\rho)C^{\frac13}(\rho).
\end{aligned} \tag{2.19}
$$

We estimate the last term

$$
|\mathcal{I}_3| \le \frac{K}{r\rho}\Big(\int_{Q_\rho}|\mathbf{u}|^3\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac13}\Big(\int_{Q_\rho}|p|^{\frac32}\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac23} \le K\Big(\frac{\rho}{r}\Big)C^{\frac13}(\rho)D^{\frac23}(\rho). \tag{2.20}
$$

Altogether we have

$$
A^{\frac32}(r) \le K\Big[\Big(\frac{\rho}{r}\Big)^{\frac32}C(\rho) + \Big(\frac{\rho}{r}\Big)^{\frac32}A^{\frac34}(\rho)B^{\frac34}(\rho)C^{\frac12}(\rho) + \Big(\frac{\rho}{r}\Big)^{\frac32}C^{\frac12}(\rho)D(\rho)\Big]. \tag{2.21}
$$

Lemma 2.24 yields

$$
D^2(r) \le K\Big[\Big(\frac{\rho}{r}\Big)^4 A^{\frac32}(\rho)B^{\frac32}(\rho) + \Big(\frac{r}{\rho}\Big)^2 D^2(\rho)\Big]. \tag{2.22}
$$

Therefore

$$
\begin{aligned}
A^{\frac32}(r) + D^2(r) \le K\Big\{&\Big(\frac{\rho}{r}\Big)^{\frac32}C(\rho) + \Big(\frac{\rho}{r}\Big)^{\frac32}A^{\frac32}(\rho)B^{\frac32}(\rho) \\
&+\Big(\frac{\rho}{r}\Big)^{\frac32}D^2(\rho) + \Big(\frac{\rho}{r}\Big)^4 A^{\frac32}(\rho)B^{\frac32}(\rho) + \Big(\frac{r}{\rho}\Big)^2 D^2(\rho)\Big\}.
\end{aligned} \tag{2.23}
$$

We now apply (2.23) with $r := r/2$, $\rho := r$ and get

$$
A^{\frac32}\Big(\frac{r}{2}\Big) + D^2\Big(\frac{r}{2}\Big) \le K\Big(C(r) + A^{\frac32}(r)B^{\frac32}(r) + D^2(r)\Big). \tag{2.24}
$$

We further use Lemma 2.23, (2.21), (2.22) and we have

$$
\begin{aligned}
A^{\frac32}\Big(\frac{r}{2}\Big) + D^2\Big(\frac{r}{2}\Big) \le K\Big\{&\Big(\frac{r}{\rho}\Big)^3 A^{\frac32}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac34}(\rho)B^{\frac34}(\rho) \\
&+B^{\frac32}(r)\Big[\Big(\frac{\rho}{r}\Big)^{\frac32}C(\rho) + \Big(\frac{\rho}{r}\Big)^{\frac32}A^{\frac32}(\rho)B^{\frac32}(\rho) + \Big(\frac{\rho}{r}\Big)^{\frac32}D^2(\rho)\Big] \\
&+\Big(\frac{\rho}{r}\Big)^4 A^{\frac32}(\rho)B^{\frac32}(\rho) + \Big(\frac{r}{\rho}\Big)^2 D^2(\rho)\Big\}.
\end{aligned} \tag{2.25}
$$

The term $C(\rho)$ can be again estimated by Lemma 2.23 for $r = \rho$. Altogether we have

$$
\begin{aligned}
A^{\frac32}\Big(\frac{r}{2}\Big) + D^2\Big(\frac{r}{2}\Big) \le K\Big\{&\Big(\frac{r}{\rho}\Big)^3 A^{\frac32}(\rho) + \Big(\frac{r}{\rho}\Big)^2 D^2(\rho) \\
&+B^{\frac32}(r)\Big[\Big(\frac{\rho}{r}\Big)^{\alpha}A^{\frac32}(\rho) + \Big(\frac{\rho}{r}\Big)^{\beta}A^{\frac32}(\rho)B^{\frac32}(\rho) + \Big(\frac{\rho}{r}\Big)^{\gamma}D^2(\rho) + \Big(\frac{\rho}{r}\Big)^{\delta}B^{\frac32}(\rho)\Big] \\
&+\Big(\frac{\rho}{r}\Big)^{\varepsilon}A^{\frac32}(\rho)B^{\frac32}(\rho)\Big\}
\end{aligned} \tag{2.26}
$$

for certain α, β, γ, δ, $\varepsilon > 0$. We choose $r = \theta\rho$, $0 < \theta < 1$ with θ sufficiently small and use the assumption from Theorem 2.14 for ϵ^* sufficiently small. We obtain

$$
A^{\frac32}\Big(\frac12\theta\rho\Big) + D^2\Big(\frac12\theta\rho\Big) \le \frac12\big[A^{\frac32}(\rho) + D^2(\rho)\big] + \widetilde{\epsilon}_1^* \tag{2.27}
$$

for a fixed $\theta \in (0,1)$, $\widetilde{\epsilon}_1^* \ll 1$ and arbitrary $\rho > 0$. We denote $\theta_1 = \frac{\theta}{2}$. Then, iterating (2.27) we get

$$A^{\frac{3}{2}}\big(\theta_1^{k+1}\rho\big) + D^2\big(\theta_1^{k+1}\rho\big) \leq \frac{1}{2^{k+1}}\big[A^{\frac{3}{2}}(\rho) + D^2(\rho)\big] + \epsilon_1^* \quad \forall k \in \mathbb{N}.$$

Lemma 2.23 yields

$$C(\theta_1^{k+1}\rho) \leq K\Big(A^{\frac{3}{2}}\big(\theta_1^{k+1}\rho\big) + B^{\frac{3}{2}}\big(\theta_1^{k+1}\rho\big)\Big) \leq \frac{K}{2^{k+1}}\big[A^{\frac{3}{2}}(\rho) + D^2(\rho)\big] + \epsilon_2^*.$$

Hence it holds for a certain $\theta_1 \in (0,1)$ and arbitrary $\rho > 0$

$$C\big(\theta_1^{k+1}\rho\big) + D\big(\theta_1^{k+1}\rho\big) \leq \epsilon_0,$$

where ϵ_0 is the number from Theorem 2.19.

Now it is enough to recall that if $(\mathbf{u}, p)$ solves the Navier–Stokes equations, then

$$\mathbf{u}_\lambda(t,x) = \lambda \mathbf{u}(\lambda^2 t, \lambda x),$$
$$p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)$$

solves the same system, while

$$C_{\mathbf{u}}(\rho) = C_{\mathbf{u}_\rho}(1),$$
$$D_p(\rho) = D_{p_\rho}(1).$$

This is a direct consequence of the change of variables

$$\int_{Q_1} |\mathbf{u}_\rho(t,x)|^3 \,\mathrm{d}x\,\mathrm{d}t = \frac{\rho^3}{\rho^5}\int_{Q_1} |\mathbf{u}(\rho^2 t, \rho x)|^3 \rho^5 \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\rho^2}\int_{Q_\rho} |\mathbf{u}(\tau,y)|^3 \,\mathrm{d}y\,\mathrm{d}\tau$$

$$\int_{Q_1} |p_\rho(t,x)|^{\frac{3}{2}} \,\mathrm{d}x\,\mathrm{d}t = \frac{\rho^3}{\rho^5}\int_{Q_1} |p(\rho^2 t, \rho x)|^{\frac{3}{2}} \rho^5 \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{\rho^2}\int_{Q_\rho} |p(\tau,y)|^{\frac{3}{2}} \,\mathrm{d}y\,\mathrm{d}\tau.$$

Hence the assumptions of Theorem 2.19 are fulfilled. The proof of Theorem 2.14 is finished. □

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