(1) Rozviňte funkci $\cosh(x)|_{[-\pi,\pi]}$, periodicky rozšířenou na $\mathbb{R}$, do trigonometrické Fourierovy řady na $(-\pi,\pi)$. Určete, na jaké množině konverguje příslušná Fř. stejnoměrně a k jaké funkci a zda totéž platí i pro formální derivaci řady člen po členu. Odůvodněte!

10b

Řešení

$\cosh x$ je sudá na $(-\pi,\pi) \Rightarrow b_k = 0 \quad \forall k \in \mathbb{N}$ 1b

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} \cosh x \, dx = \frac{2}{\pi} \cdot \sinh \pi$$ 1b

$$a_k = \frac{2}{2\pi}\int_{-\pi}^{\pi} \cosh x \cos(kx)\, dx = \frac{2}{\pi}\frac{1}{2}\int_0^{\pi} (e^x + e^{-x}) \cos(kx)\, dx$$

$$= \frac{1}{\pi} \operatorname{Re} \int_0^{\pi} (e^{x+ikx} + e^{-x+ikx})\, dx = \frac{1}{\pi} \operatorname{Re}\left[\frac{1}{1+ik} e^{x(1+ik)} + \frac{1}{-1+ik} e^{x(-1+ik)}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \operatorname{Re}\left(\frac{e^{\pi}e^{ik\pi}-1}{1+ik} + \frac{e^{-\pi}e^{ik\pi}-1}{-1+ik}\right)$$

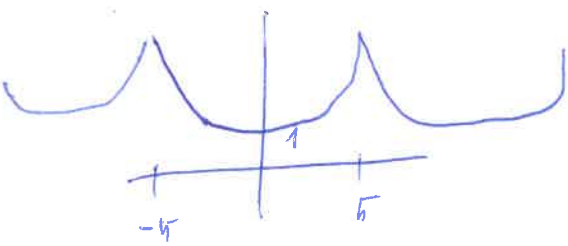
$$= \frac{1}{\pi} \operatorname{Re}\left(\frac{1}{1+k^2}(1-ik)[e^{\pi}(\cos k\pi + i\underbrace{\sin k\pi}_{0}) - 1] + \frac{1}{1+k^2}(-1-ik)(e^{-\pi}(\cos k\pi + i\underbrace{\sin k\pi}_{0})-1)\right)$$

$$= \frac{1}{\pi}\frac{1}{1+k^2}(e^{\pi}(-1)^k - 1) - \frac{1}{\pi}\frac{1}{1+k^2}(e^{-\pi}(-1)^k - 1)$$ 3b

$$= \frac{1}{\pi}\frac{1}{1+k^2}(-1)^k(e^{\pi} - e^{-\pi})$$

$$f(x) = \frac{1}{\pi}\sinh\pi + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{1+k^2}(-1)^k(e^{\pi}-e^{-\pi})\cos(kx)$$ 1b

(po částech ~~spojitá~~) diferencovatelná, hladká na $\mathbb{R}$ 1b



řada $\overset{1b}{\underset{\mathbb{R}}{\rightrightarrows}} \cosh(x)$ (2π-periodické rozšíření z $[-\pi,\pi]$)

derivovaná řada $\underset{\text{lok}}{\overset{(-\pi,\pi)}{\rightrightarrows}} \sinh(x)$ 1b (pouze 2π-per. rozšíření) 1b

(lze derivovat člen po členu – pomocí derivace stejnoměrné a limita míří k derivaci součtu)

2

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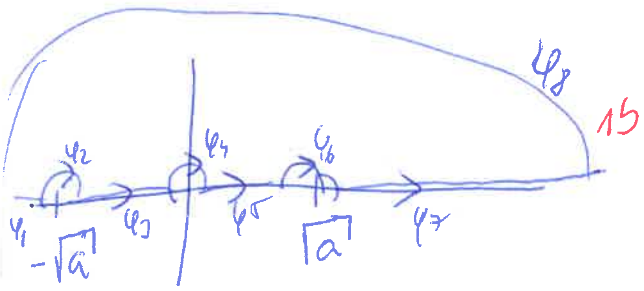
V závislosti na parametru $a \in \mathbb{R}$ spočtěte
$$\int_0^\infty \frac{\ln x}{x^2-a}.$$
V jakém smyslu integrál počítáte?

Řešení

a) $a=0$ .. integrál nekonverguje (jde $-\infty$) .. kvůli chování kolem $x=0$ 1b

b) $a \in \mathbb{R}$, $a>0$ (jako hlavní hodnota) 1b



1b

$$\int_{\varphi_1+\dots+\varphi_8} F(z)\,dz = 0 \quad 1b$$

$$F(z) = \frac{\log z}{z^2-a} \quad 1b$$

$$\int_{\varphi_8} F(z)\,dz \xrightarrow[R\to\infty]{} 0 \quad \left(\sim \frac{1}{R^2}\cdot R\cdot \ln R \xrightarrow[R\to\infty]{} 0\right) \quad 1b$$

$$\left(\int_{\varphi_1}+\int_{\varphi_3}+\int_{\varphi_5}+\int_{\varphi_7}\right)\frac{\log z}{z^2-a}\,dz \xrightarrow[\substack{\varepsilon\to 0+\\ R\to\infty}]{} \text{v.p. } 2\int_0^\infty \frac{\ln x}{x^2-a}\,dx + i\pi\,\text{v.p.}\int_0^\infty \frac{1}{x^2-a}\,dx \quad 2b$$

$$\int_{\varphi_4} \frac{\log z}{z^2-a}\,dz \xrightarrow[\varepsilon\to 0]{} 0 \quad 1b \quad \left(\varepsilon \underset{\to 0}{\ln \varepsilon}\right)$$

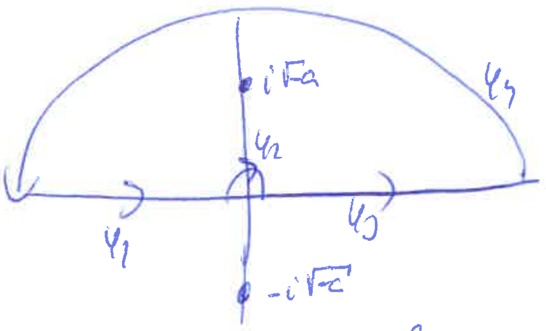
$$\lim_{\varepsilon\to 0+}\int_{\varphi_2}\frac{\log(z)}{z^2-a}\,dz = -\pi i \operatorname{Res}_{-\sqrt{a}}\frac{\log z}{z^2-a} = -\pi i\,\frac{\ln(\sqrt{a})+i\pi}{-2\sqrt{a}} = \frac{-\pi^2}{2\sqrt{a}} + i\pi\frac{\ln\sqrt{a}}{+2\sqrt{a}} \quad 2b$$

$$\lim_{\varepsilon\to 0^+}\int_{\varphi_6}\frac{\log z}{z^2-a} = -\pi i \operatorname{Res}_{\sqrt{a}}\frac{\log z}{z^2-a} = -\pi i\,\frac{\ln\sqrt{a}}{2\cdot\sqrt{a}} \quad 1b$$

Celkem v.p. $\boxed{\int_0^\infty \frac{\ln x}{x^2-a}\,dx = +\frac{\pi^2}{4\sqrt{a}}}$ 1b

c) $a < 0$, $a \in \mathbb{R}$ (Zde o Nevlastní int.) 1b

$x^2 = +a$



1b

1b

$$\int_{\varphi_1 \oplus \dots \oplus \varphi_4} F(z)\,dz = 2\pi i \operatorname{Res}_{i\sqrt{-a}} F(z)$$

$$\int_{\varphi_4} \frac{\ln z}{z^2-c^2}\,dz \xrightarrow[R\to\infty]{} 0$$ 1b

$$\int_{\varphi_2} \frac{\ln z}{z^2-c^2}\,dz \xrightarrow[\varepsilon\to 0]{} 0$$ 1b

$$\int_{\varphi_1} \frac{\ln z}{z^2-a}\,dz \xrightarrow[\substack{R\to\infty\\ \varepsilon\to 0+}]{} \int_0^\infty \frac{\ln x}{x^2-a}\,dx + i\pi \int_0^\infty \frac{1}{x^2-a}\,dx$$ 1b

$$\int_{\varphi_3} \frac{\ln z}{z^2-c^2}\,dz \xrightarrow[\substack{R\to\infty\\ \varepsilon\to 0+}]{} \int_0^\infty \frac{\ln x}{x^2-c^2}\,dx$$ 1b

Ale
$$2\int_0^\infty \frac{\ln x}{x^2-a}\,dx + i\pi \int_0^\infty \frac{1}{x^2-a}\,dx = 2\pi i \operatorname{Res}_{i\sqrt{|a|}} \frac{\log z}{z^2-a} = 2\pi i\, \frac{\frac{1}{2}\ln|a| + i\frac{\pi}{2}}{2i\sqrt{|a|}}$$ 2b

Tedy
$$\boxed{\int_0^\infty \frac{\ln x}{x^2-a}\,dx = \frac{1}{4}\pi\, \frac{\ln|a|}{\sqrt{|a|}}}$$ 1b

(3) Spočtěte

(15b) $\mathcal{F}\left(\frac{\sin x}{x}\right)(s)$. Vyjdou nějaké kři. potíže?

Řešení

Zřejmě $\frac{\sin x}{x} \in L^2(\mathbb{R}) \Rightarrow \mathcal{F}\left(\frac{\sin x}{x}\right) = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x} e^{-i2\pi x s}\,dx.$ 1b 1b

Protože $\frac{\sin x}{x}$ je sudá funkce, 1b máme

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin x}{x}e^{-i2\pi x s}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{\sin x\cos 2\pi x s}{x}\,dx =$$

1b

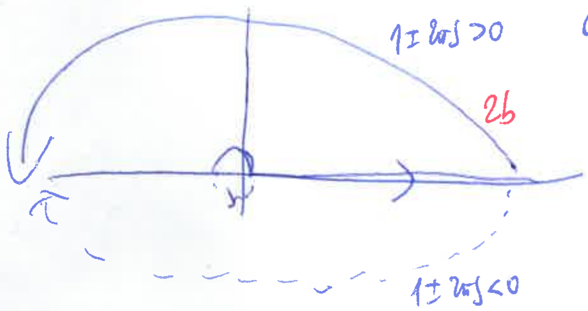
$$\sin x \cdot \cos 2\pi x s = \frac{1}{2}\left(\sin x(1+2\pi s) + \sin x(1-2\pi s)\right)$$

1b

$$= \lim_{R\to\infty}\frac{1}{2}\int_{-R}^{R}\left[\frac{\sin x(1+2\pi s)}{x} + \frac{\sin x(1-2\pi s)}{x}\right]dx$$

1b

$$= \lim_{R\to\infty} \text{p.v.}\ \frac{1}{2i}\int_{-R}^{R}\left[\frac{e^{ix(1+2\pi s)}}{x} + \frac{e^{ix(1-2\pi s)}}{x}\right]dx$$



a) $s \in \left(-\frac{1}{2\pi}, \frac{1}{2\pi}\right)$

oba kladné 1b

$$I = \frac{1}{2}2\pi i\,\mathrm{Res}_0\left(\frac{e^{ix(1+2\pi s)}}{x} + \frac{e^{ix(1-2\pi s)}}{x}\right)$$

$= \pi$ 1b

b) $s \in \left(-\infty, -\frac{1}{2\pi}\right)$ 1b

$$I = \frac{1}{2\pi i}\mathrm{Res}_0\left(-\frac{e^{ix(1+2\pi s)}}{x} + \frac{e^{ix(1-2\pi s)}}{x}\right)$$

$= 0$ 1b

$\mathcal{F}\left(\frac{\sin x}{x}\right) = \pi\chi_{\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]}$, 1b

Pochopitelně ve smyslu $L^2(\mathbb{R})$.

c) $s \in \left(\frac{1}{2\pi}, \infty\right)$ 1b

$$I = \frac{1}{2}\pi i\,\mathrm{Res}_0\left(\frac{e^{ix(1+2\pi s)}}{x} - \frac{e^{ix(1-2\pi s)}}{x}\right) = 0$$

1b

5

(i) Ukažte, že definice

13b

$$H_{x^{-2}} := \lim_{\lambda\to -2} H_{|x|^\lambda}$$

je dobře definována.

(ii) Ukažte, že

$$H_{x^{-2}} = H_{f.p.\frac{1}{x^2}},$$

kde $\langle H_{f.p.\frac{1}{x^2}}, \varphi \rangle = p.v. \int_{\mathbb{R}} \frac{\varphi(x)-\varphi(0)}{x^2}\, dx.$

Řešení:

Mám pro $\lambda \in (-3,-1)\setminus\{-2\}$ 1b

$$\langle H_{|x|^\lambda}, \varphi\rangle \overset{1b}{=} \left\langle \frac{D^2 H_{|x|^{\lambda+2}}}{(\lambda+2)(\lambda+1)}, \varphi \right\rangle = \frac{1}{(\lambda+1)(\lambda+2)} \langle H_{|x|^{\lambda+2}}, \varphi''\rangle$$

$$= \lim_{\varepsilon\to 0+}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{1}{(\lambda+2)(\lambda+1)} |x|^{\lambda+2}\varphi''\, dx \quad 1b$$

$$= -\lim_{\varepsilon\to 0+}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{1}{\lambda+1} |x|^{\lambda+1}\operatorname{sign} x\, \varphi'(x)\, dx \overset{1b}{+} \frac{1}{(\lambda+2)(\lambda+1)} \underbrace{\left(\varphi'(-\varepsilon)|\varepsilon|^{\lambda+2} - \varphi'(\varepsilon)|\varepsilon|^{\lambda+2}\right)}_{\sim |\varepsilon|^{\lambda+1} \xrightarrow[\varepsilon\to 0+]{} 0} \quad 1b$$

$= 0$

$$= \lim_{\varepsilon\to 0+}\Big[\frac{1}{\lambda+1}\Big(-\left[x^{\lambda+1}(\varphi(x)-\varphi(0))\right]_{\varepsilon}^{\infty} + \left[(-x)^{\lambda+1}(\varphi(x)-\varphi(0))\right]_{-\infty}^{-\varepsilon}\Big) + \int_{\varepsilon}^{\infty} x^\lambda(\varphi(x)-\varphi(0))\, dx + \int_{-\infty}^{-\varepsilon} (-x)^\lambda (\varphi(x)-\varphi(0))\, dx\Big]; \quad 1b$$

$$= \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{\infty} x^\lambda(\varphi(x)-\varphi(0))\, dx + \int_{-\infty}^{-\varepsilon}(-x)^\lambda(\varphi(x)-\varphi(0))\, dx; \quad 1b$$

Nyní lze provést $\lim_{\lambda\to -2}$, neboť je stejnoměrná vůči $\lim_{\varepsilon\to 0+}$ a tudíž lze prohodit obě limity, dostaneme

$$\lim_{\lambda\to-2} \langle H_{|x|^\lambda}, \varphi\rangle$$

$$= \lim_{\lambda\to-2}\Big(\lim_{\varepsilon\to0^+}\int_\varepsilon^1 x^\lambda(\varphi(x)-\varphi(0)-x\varphi'(0))\,dx + \int_{-1}^{-\varepsilon}(-x)^\lambda(\varphi(x)-\varphi(0)-x\varphi'(0))\,dx$$

1b

$$+\int_1^\infty x^\lambda(\varphi(x)-\varphi(0))\,dx + \int_{-\infty}^{-1}(-x)^\lambda(\varphi(x)-\varphi(0))\,dx\Big)$$

1b

$$= \lim_{\varepsilon\to0^+}\int_\varepsilon^1 x^{-2}(\varphi(x)-\varphi(0)-x\varphi'(0))\,dx + \int_{-1}^{-\varepsilon}(-x)^{-2}(\varphi(x)-\varphi(0)-x\varphi'(0))\,dx$$

1b

$$+\int_1^\infty x^{-2}(\varphi(x)-\varphi(0))\,dx + \int_{-\infty}^{-1}(-x)^2(\varphi(x)-\varphi(0))\,dx$$

1b

$$= \lim_{\varepsilon\to0^+}\Big(\int_\varepsilon^\infty \frac{\varphi(x)-\varphi(0)}{x^2}\,dx + \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)-\varphi(0)}{x^2}\,dx\Big)$$

1b

$$= \text{p.v.}\int_{-\infty}^\infty \frac{\varphi(x)-\varphi(0)}{x^2}\,dx =: \langle T_{\text{f.p.}\frac{1}{x^2}}, \varphi\rangle.$$