(1) 10b

Rozviňte do trigonometrické řady na $(-\pi,\pi)$ funkci $f(x)= x|x|$. Vyšetřete bodovou (stejnoměrnou) konvergenci na $[-\pi,\pi]$ (včetně krajních bodů) a lokálně stejnoměrnou konvergenci na $(-\pi,\pi)$.

Řešení

$x|x|$ je lichá fce $\Rightarrow a_k = 0 \quad \forall k \in \mathbb{N}_0$ 1b

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x|x| \sin(kx)\,dx \overset{1b}{=} \frac{2}{\pi}\int_0^{\pi} x^2 \sin(kx)\,dx = \frac{2}{\pi}\left[-x^2 \frac{\cos(kx)}{k}\right]_0^{\pi}$$

$$+ \frac{2}{\pi}\int_0^{\pi} 2x \cdot \frac{\cos(kx)}{k}\,dx = -\frac{2\pi}{k}(-1)^k + \frac{4}{k\pi}\underbrace{\left[x\frac{\sin(kx)}{k}\right]_0^{\pi}}_{=0}$$

$$- \frac{4}{k^2\pi}\int_0^{\pi} \sin(kx)\,dx = -\frac{2\pi}{k}(-1)^k + \frac{4}{k^3\pi}\left[\cos(kx)\right]_0^{\pi}$$

$$= -\frac{2\pi}{k}(-1)^k + \frac{4}{k^3\pi}\left((-1)^k - 1\right)$$

3b

1b

$$\boxed{x|x| = -\sum_{k=1}^{\infty}\left(\frac{2\pi}{k}(-1)^k + \left(1-(-1)^k\right)\frac{4}{k^3\pi}\right)\sin(kx)} \quad \text{na } (-\pi,\pi)$$

- Bodově konv. na $(-\pi,\pi)$ … je fce $x|x|$ 1b
- Bodová kon. pro $\pi/-\pi$ je 0 1b
- Stejnoměrná kon. na $[-\pi,\pi]$ … ne 1b
- Lokálně stejn.: Ano k $x|x|$ na $(-\pi,\pi)$ 1b — na $[-\pi+\delta, \pi-\delta]$ je $x|x|$ spojitě diferencovatelná ~~po částech~~ ~~trig~~ $\forall \delta>0$.

2 Pro které hodnoty $a \in \mathbb{R}$ konverguje Lebesgueův integrál

153 $$\int_0^\infty \frac{x^a}{(x^2+1)(x^2+4)}\,dx\ ?$$

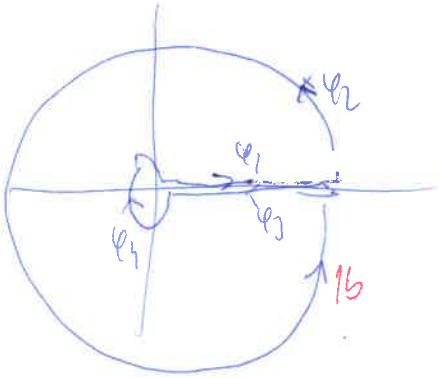
Pro tyto hodnoty ho spočítejte.

Pozn.: Nezapomeňte na jiné obecné hodnoty a!

Řešení

Vzhledem k podmínkám: v 0 $a>-1$ 1b, v ∞ $a<3$ 1b $\Rightarrow$ $a\in(-1,3)$ integrál konverguje.

Integrační cesta (po zjednodušení)



φ_1: t, $t\in(\varepsilon,R)$
φ_2: Re^{it}, $t\in(0,2\pi)$
φ_3: t, $t\in(R,\varepsilon)$
φ_4: εe^{it}, $t\in(2\pi,0)$ 2b

$$F(z)=\frac{z^a}{(z^2+1)(z^2+4)}$$ 1b

$\arg z\in(0,2\pi)$

$$\Rightarrow \int_{\varphi_1} F(z)\,dz \sim \int_\varepsilon^R \frac{x^a}{(x^2+1)(x^2+4)}dx \to \int_0^\infty \frac{x^a}{(x^2+1)(x^2+4)}dx \quad 0{,}5b$$

$$\int_{\varphi_2} F(z)\,dz \sim CR^{a-4}R \xrightarrow[R\to\infty]{} 0 \quad a<3! \quad 0{,}5b$$

$$\int_{\varphi_3} F(z)\,dz \sim -\int_\varepsilon^R \frac{e^{ia2\pi}x^a}{(x^2+1)(x^2+4)}dx \to -e^{ia2\pi}\int_0^\infty \frac{x^a}{(x^2+1)(x^2+4)}dx \quad 0{,}5b$$

$z^a = e^{ia2\pi}|z|^a$

$$\int_{\varphi_4} F(z)\,dz \sim C\varepsilon^{a+1} \xrightarrow[\varepsilon\to0^+]{} 0 \quad a>-1 \quad 0{,}5b$$

Celkem: $$\left(1-e^{ia2\pi}\right)\int_0^\infty \frac{x^a}{(x^2+1)(x^2+4)}dx = 2\pi i\left(\operatorname{Res}_i + \operatorname{Res}_{-i} + \operatorname{Res}_{2i} + \operatorname{Res}_{-2i}\right) F(z)$$

1b

$$\operatorname{Res}_{i} \frac{z^a}{(z^2+1)(z^2+4)} = \frac{i^a}{2i\cdot 3} = \frac{e^{ia\frac{\pi}{2}}}{6i}$$

$$\operatorname{Res}_{-i} \frac{z^a}{(z^2+1)(z^2+4)} = \frac{(-i)^a}{(-2i)\cdot 3} = \frac{e^{i\frac{3}{2}\pi a}}{-6i}$$

$$\operatorname{Res}_{2i} \frac{z^a}{(z^2+1)(z^2+4)} = \frac{2^a\cdot i^a}{-3\cdot 4i} = -2^a\,\frac{e^{ia\frac{\pi}{2}}}{12i}$$

$$\operatorname{Res}_{-2i} \frac{z^a}{(z^2+1)(z^2+4)} = \frac{2^a\cdot(-i)^a}{-3\cdot(-4i)} = \frac{2^a\cdot e^{i\frac{3}{2}\pi a}}{12i}$$

2b

$$(1-e^{ia2\pi})I = \pi\left[\frac{e^{ia\frac{\pi}{2}}}{3} - \frac{e^{i\frac{3}{2}\pi a}}{3} + 2^a\left(\frac{e^{i\frac{3}{2}\pi a}}{6} - \frac{e^{ia\frac{\pi}{2}}}{6}\right)\right]$$ 1b

$$\Rightarrow I = \frac{\pi}{3}\,\frac{e^{ia\frac{\pi}{2}}\cancel{(1-e^{i\pi a})}}{\cancel{(1-e^{ia\pi})}(1+e^{ia\pi})} + \frac{2^a\pi}{6}\,\frac{e^{ia\frac{\pi}{2}}\cancel{(1-e^{i\pi a})}}{\cancel{(1-e^{ia\pi})}(1+e^{ia\pi})}$$

$$= \boxed{\frac{\pi}{6\cos\frac{a\pi}{2}}\left(1-\frac{2^a}{2}\right)}$$ 1b

Je dobro definirana za $a \in (-1,3)\setminus\{1\}$

Ker lahko pomoč Leb.-ovega izreka, integral je zvezen za v a (majoranta 1b

dobimo C, v ∞ $\frac{x^2}{(x^2+1)(x^2+4)}$)), ker

$$I(1) = \lim_{a\to 1} \frac{\pi}{6\cos\frac{a\pi}{2}}\left(1-2^{a-1}\right) = \frac{\pi}{6}\,\frac{(+\ln 2)}{+\frac{\pi}{2}\sin\frac{\pi}{2}} = \boxed{\frac{\ln 2}{3}}$$ 1b

$$I(a) = \frac{\pi}{6\cos\frac{a\pi}{2}}\left(1-2^{a-1}\right) \qquad a\in(-1,3)\setminus\{1\}$$

$$= \frac{\ln 2}{3} \qquad a=1.$$

(1) ~~Spočítat~~ Výpočet zobecněné pro $\frac{x^2+1}{x^4+1}$? (zde z $\mathbb{R}$ do $\mathbb{R}$) Spočítal by

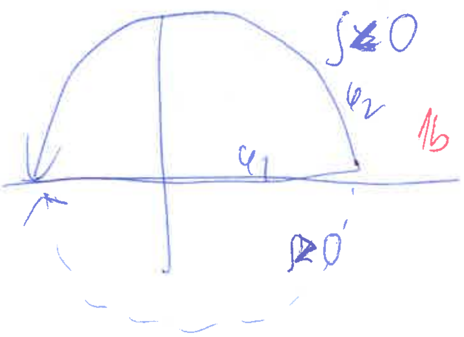
(15b) F.T. a vysvětlit, v jakém smyslu je použita.

Řešení:

$\frac{x^2+1}{x^4+1} \notin L^1(\mathbb{R})$, ale $\in L^2(\mathbb{R})$ Tedy Riemann, 1b $\frac{x^2+1}{x^4+1} \in L^p(\mathbb{R})\ \forall p>1$. 1b

Tedy $\mathcal{F}\left(\frac{x^2+1}{x^4+1}\right)(\xi) = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2+1}{x^4+1} e^{-2\pi i x\xi}\, dx$. 1b

Výpočet lze provést pomocí integrace cesty



a) $\xi<0$

Jordan $\int_{\varphi_2} \frac{z^2+1}{z^4+1} e^{-2\pi i z\xi}\, dz \xrightarrow[R\to\infty]{} 0$ 1b

$\int_{-R}^{R} \frac{z^2+1}{z^4+1} e^{-2\pi i z\xi} \xrightarrow[R\to\infty]{} 2\pi i \left(\mathrm{Res}_{e^{i\frac{\pi}{4}}} + \mathrm{Res}_{e^{i\frac{3\pi}{4}}}\right) \frac{z^2+1}{z^4+1} e^{-2\pi i z\xi}$ 1b

$\mathcal{F}\left(\frac{x^2+1}{x^4+1}\right)(\xi) = 2\pi i \left(\frac{1}{4} + \frac{1}{4z^3}\right) e^{-2\pi i z\xi}\Big|_{z=e^{i\frac{\pi}{4}}}$

$+ 2\pi i \left(\frac{1}{4} + \frac{1}{4z^3}\right) e^{-2\pi i z\xi}\Big|_{z=e^{i\frac{3\pi}{4}}}$ ~~1b~~ 2b

$= \frac{\pi i}{2}\left(1 + e^{-i\frac{3}{4}\pi}\right) e^{-2\pi i\xi\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)}$

$+ \frac{\pi i}{2}\left(1 + e^{-i\frac{\pi}{4}}\right) e^{-2\pi i\xi\left(-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)}$ 1b

$= \frac{\pi i}{2}\left(1 - \frac{\sqrt2}{2} - i\frac{\sqrt2}{2}\right) e^{+\pi\sqrt2\xi}\left(\cos(\sqrt2\pi\xi) - i\sin(\sqrt2\pi\xi)\right)$ 1b $\xi<0$,

$+ \frac{\pi i}{2}\left(1 + \frac{\sqrt2}{2} - i\frac{\sqrt2}{2}\right) e^{\pi\sqrt2\xi}\left(\cos(\sqrt2\pi\xi) + i\sin(\sqrt2\pi\xi)\right)$

$= \pi i\left(1 - i\frac{\sqrt2}{2}\right) e^{\pi\sqrt2\xi}\left(\cos(\sqrt2\pi\xi)\right)$ ~~[illegible]~~ $+ \frac{\pi i}{2}\frac{\sqrt2}{2} e^{\pi\sqrt2\xi}\, i\sin(\sqrt2\pi\xi) - \frac{\pi\sqrt2}{2} e^{\pi\sqrt2\xi}\sin(\sqrt2\pi\xi)$ 1b

Pro $\zeta>0$

Opět $\displaystyle\int_{\varphi_2} \frac{z^3+1}{z^4+1} e^{-2\pi i z\zeta}\,d\zeta \to 0$ (Jordan) 1b

1+4

1b

$$\int_{-R}^{R} \frac{z^3+1}{z^4+1} e^{-2\pi i z\zeta}\,dz \to -2\pi i\left(\operatorname{Res}_{e^{i\frac{5}{4}\pi}} + \operatorname{Res}_{e^{i\frac{7}{4}\pi}}\right)\left(\frac{z^3+1}{z^4+1} e^{-2\pi i z\zeta}\right)$$

$$= -2\pi i\left(\frac{1}{4}+\frac{1}{4z^3}\right) e^{-i2\pi i z\zeta}\Big|_{z=e^{i\frac{5}{4}\pi}} - 2\pi i\left(\frac{1}{4}+\frac{1}{4z^3}\right) e^{-i2\pi i z\zeta}\Big|_{z=e^{i\frac{7}{4}\pi}}$$

$-\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}$ ~~1b~~ 2b $\quad \frac{\sqrt2}{2}-i\frac{\sqrt2}{2}$

$\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}$

$$= -\frac{\pi i}{2}\left(1+e^{i\frac{\pi}{4}}\right) e^{-\pi\sqrt2\zeta}\left(\cos(\pi\sqrt2\zeta)+i\sin(\pi\sqrt2\zeta)\right)$$

$$-\frac{\pi i}{2}\left(1+e^{i\frac{3}{4}\pi}\right) e^{-\pi\sqrt2\zeta}\left(\cos(\pi\sqrt2\zeta) - i\sin(\pi\sqrt2\zeta)\right)$$ ~~1b~~

$-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}$

$(+\pi)\cdot\frac{\sqrt2}{2} e^{-\pi\sqrt2\zeta}\sin(\pi\sqrt2\zeta)$

$$= \underbrace{-\pi i\left(1+i\frac{\sqrt2}{2}\right) e^{-\pi\sqrt2\zeta}\cos(\pi\sqrt2\zeta)}_{1b} \qquad \zeta>0$$

4) Zadana je distribucija

10b (u rezultatu nam mora biti samo konačna lin. kombinacija Diracovih i njihovih derivacija u nuli i točki b)

$$G = e^{-(x+1)^2}\left[(T_{|x|} * \delta_0''') + \delta_b'' * T_{|x|} \right].$$

Rješenje

a) $T_{|x|} * \delta_0''' = T_{\operatorname{sign} x} * \delta_0'' = (2\delta_0) * \delta_0' = 2\delta_0'.$ 1b 1b

Prema

$$\langle e^{-(x+1)^2} T_{|x|} * \delta_0''', \varphi \rangle = \langle T_{|x|} * \delta_0''', e^{-(x+1)^2}\varphi \rangle =$$ 1b

$$= \langle 2\delta_0', e^{-(x+1)^2}\varphi \rangle = -2\left(\varphi'(0)\cdot e^{-1} + \varphi(0)\cdot 2e^{-1}\right)$$ 1b

$$= 2e^{-1}\left(\langle \delta_0', \varphi \rangle + 2\langle \delta_0, \varphi \rangle \right)$$ 1b

b) $\delta_b'' * T_{|x|} = \delta_b' * T_{\operatorname{sign} x} = \delta_b * 2\delta_0 = 2\delta_b$ 1b 1b

$$\langle e^{-(x+1)^2} \delta_b'' * T_{|x|}, \varphi \rangle = \langle 2\delta_b, e^{-(x+1)^2}\varphi \rangle = 2e^{-(b+1)^2}\varphi(b)$$ 1b

$$= 2e^{-(b+1)^2}\langle \delta_b, \varphi \rangle.$$ 1b

Dakle

$$\underbrace{G = 2e^{-1}(\delta_0' + 2\delta_0) + 2e^{-(b+1)^2}\delta_b}_{1b}$$