Homework PDEs I: Set 1 (Deadline November 14, 2025, 15 30)
Return either personally at the exercises or
send a reasonably readable form to `pokorny@karlin.mff.cuni.cz`

Introduction and function spaces. Elliptic PDEs

1. (10 pts) Determine the exponents α_1, α_2, β_1 and β_2 such that the function

$$f(x) = |x|^{\alpha_1} \ln^{\beta_1}(|x|) 1_{x \in B_{\frac{1}{2}}(0)} + |x|^{\alpha_2} \ln^{\beta_2}(|x|) 1_{x \in \mathbb{R}^d \setminus B_2(0)}$$

belongs to $L^p(\mathbb{R}^d)$ for a certain $p \in (1, \infty)$ but it does not belong to any $L^q(\mathbb{R}^d)$, $q \in (1, \infty)$, $q \neq p$. The function 1_Ω is the characteristic function of the set Ω.

2. (5 pts) Show that for $p \geq 2$ there exists a constant $C = C(d, p)$ such that for any $u \in W_0^{2,p}(\Omega)$, $\Omega \subset \mathbb{R}^d$ bounded, it holds

$$\| |\nabla u| \|_{L^p(\Omega)} \leq C \| |\nabla^2 u| \|_{L^p(\Omega)}^{\frac{1}{2}} \|u\|_{L^p(\Omega)}^{\frac{1}{2}}.$$

Note that $|\nabla u|^p = \big(\sum_{i=1}^d \big| \frac{\partial u}{\partial x_i} \big|^2 \big)^{\frac{p}{2}}$, similarly for the second gradient.

3. (10 pts) For $K = (0,1)^d$ show that there exists $C = C(p, d)$, $p < d$ such that for any $u \in C^1(\overline{K})$, $\operatorname{supp} u \subset (0,1)^{d-1} \times [0, 1)$ it holds

$$\|u\|_{L^{\frac{p(d-1)}{d-p}}((0,1)^{d-1} \times \{0\})} \leq C \|u\|_{W^{1,p}(K)}.$$

Hint: Consider the integral

$$\int_0^1 \frac{\partial}{\partial x_d} \Big(\int_{(0,1)^{d-1}} |u|^{\frac{p(d-1)}{d-p}} \mathrm{d}x_1 \ldots \mathrm{d}x_{d-1} \Big) \mathrm{d}x_d.$$

4. (10 pts) Consider the problem

$$-u'' + bu = f$$

in $(0,1) \subset \mathbb{R}$ with the boundary conditions $u(0) = 0$, $u'(1) = 1$, where $f \in L^p((0,1))$ and $b \in L^q((0,1))$. Formulate the problem weakly and show under which conditions on p, q and further properties of the function b there exists unique weak solution to this problem.

5. (10 pts) Consider the problem

$$-\Delta u + bu = f$$

in $B_1(0) \subset \mathbb{R}^d$, $d \geq 2$ with the boundary condition $u = 0$ on $\partial B_1(0)$ with

$$\begin{aligned} b(x) &= |x|^{-\alpha} \\ f(x) &= |x|^{-\beta} \sin(1 + |x|). \end{aligned}$$

Formulate the problem weakly and specify the maximal intervals for α and β for which there exists a unique weak solution to the problem above.