Homework PDEs I: Set 2 (Deadline December 12, 2025, 16 30)
Return either personally or
send a reasonably readable form to `pokorny@karlin.mff.cuni.cz`

Elliptic PDEs

1. (8 pts) Consider the following problem

$$\begin{aligned} -u'' + u &= F \qquad \text{in } (0,1) \\ u(0) = u(1) &= 0, \end{aligned}$$

where the functional F is defined for any $\varphi \in C_0^\infty((0,1))$ as

$$\langle F, \varphi \rangle = \int_0^1 f\varphi \, \mathrm{d}x + \langle \delta_a, \varphi \rangle = \int_0^1 f\varphi \, \mathrm{d}x + \varphi(a)$$

for some $a \in (0,1)$ (δ_a is the Dirac distribution sitting at the point a), $f \in L^1((0,1))$. Formulate the problem weakly and show existence and uniqueness of the weak solution.

2. (12 pts) Consider the following elliptic problem ($d \geq 2$, Ω bounded open)

$$\begin{aligned} -\Delta u + bu + \operatorname{div}(\mathbf{d}u) &= f \qquad \text{in } \Omega \subset \mathbb{R}^d \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Assume that $b \in L^p(\Omega)$, $b \geq 0$ a.e. in Ω, $\mathbf{d} \in L^q(\Omega; \mathbb{R}^d)$, $f \in L^r(\Omega)$ $1 \leq p, q, r \leq \infty$ and $\operatorname{div} \mathbf{d} \geq 0$ in the sense of distributions.

Formulate the problem weakly and find minimal assumptions on p, q and r so that there exists a unique weak solution to this problem.

3. (15 pts) Consider the same problem as in the previous problem. Formulate the conditions under which you can get that $u \in W^{2,2}(\Omega)$ and sketch the proof. In fact, all the conditions appear if you consider the case $\Omega = C^+$ with the corresponding assumption on the support of u.

4. (10 pts) Find the real spectrum for the operator

$$-u'' + u'$$

in $(0,1)$ with homogeneous Dirichlet boundary conditions, i.e., find for which $\lambda \in \mathbb{R}$ there exists a nontrivial solution to

$$\begin{aligned} -u'' + u' &= \lambda u \qquad \text{in } (0,1) \\ u(0) = u(1) &= 0. \end{aligned}$$