Exercises PDEs I: Set 1 (October 10, 2025)

Introduction and function spaces

1. Deduce the Euler–Lagrange equations for the function

$$L(x, u, \nabla u) = \frac{1}{p}|u|^p + \frac{1}{p}|\nabla u|^p - fu,$$

where $1 < p < \infty$. More precisely, consider the minimization problem: Find $u \in W_0^{1,p}(\Omega)$ such that

$$\Phi[u] = \min_{v \in W_0^{1,p}(\Omega)} \int_\Omega L(x, v(x), \nabla v(x))\, \mathrm{d}x.$$

2. Prove that

$$(u, v) = \int_\Omega (uv + \nabla u \cdot \nabla v)\, \mathrm{d}x$$

is on

$$V = \{w \in C^1(\overline{\Omega}); w = 0 \text{ na } \partial\Omega\}$$

a scalar product, but the space V is not complete with respect to the norm, induced by the scalar product.

3. Prove that for $\Omega \subset \mathbb{R}^d$, open, bounded, connected, it holds:

$$\{u \in C(\Omega); u \text{ bounded and uniformly continuous } \Omega\} =$$
$$\{u \in C(\Omega); \exists \text{ uniquely defined continuous extension of } u \text{ to } \overline{\Omega}\}$$

4. Prove the generalized Hölder inequality:
If $f_i \in L^{p_i}(\Omega)$, $1 \leq p_i \leq \infty$, $i = 1, \ldots, n$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, then $\prod_{i=1}^n f_i \in L^1(\Omega)$ and

$$\|\prod_{i=1}^n f_i\|_1 \leq \prod_{i=1}^n \|f_i\|_{p_i}.$$

5. Let $|\Omega| < \infty$, $1 \leq p_1 \leq p_2 \leq \infty$. If $f \in L^{p_2}(\Omega)$, then $f \in L^{p_1}(\Omega)$ and it holds:

$$\|f\|_{p_1} \leq |\Omega|^{\frac{p_2 - p_1}{p_1 p_2}} \|f\|_{p_2}.$$

(If $p_2 = \infty$, then $\|f\|_{p_1} \leq |\Omega|^{\frac{1}{p_1}} \|f\|_\infty$.)

6. Let $\Omega \subset \mathbb{R}^d$, $1 \le p_1 < p_2 \le \infty$. If $f \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, then $f \in L^r(\Omega)$ $\forall r \in [p_1, p_2]$ and it holds

$$\|f\|_r \le \|f\|_{p_1}^{\alpha} \|f\|_{p_2}^{1-\alpha}, \qquad \frac{1}{r} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}.$$

7. Show that for any $1 \le p < \infty$ there exists $f \in L^p(\mathbb{R}^d)$ such that

$$\lim_{R \to \infty} \operatorname*{ess\,sup}_{|x| > R} |f(x)| = \infty.$$

8. Show that Hölder continuous functions

$$C^{0,\lambda}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega});\ \sup_{x,y \in \Omega; x \neq y} \frac{|u(x) - u(y)|}{|x-y|^{\lambda}} < \infty \right\}$$

form the Banach space with respect to the norm

$$\|u\|_{C^{0,\lambda}(\overline{\Omega})} = \|u\|_{C(\overline{\Omega})} + \sup_{x,y \in \Omega; x \neq y} \frac{|u(x) - u(y)|}{|x-y|^{\lambda}},$$

$\lambda \in (0, 1]$, which is not separable.

9. Prove:
Let Ω be an open bounded set. Then for any $\lambda \in (0, 1]$ it holds

$$\mathcal{C}^{0,\lambda}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}(\overline{\Omega}).$$

(10b) Let Ω be an open bounded set. Then for any $\alpha, \beta \in [0, 1]$ such that $0 \le \alpha < \beta \le 1$ it holds

$$\mathcal{C}^{0,\beta}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega}).$$

10. Let $\Omega \subset \mathbb{R}^d$ have a finite $d-$dimensional measure. Prove the following claims:
(i) If $f \in L^{\infty}(\Omega)$, then $f \in L^p(\Omega)$ $\forall 1 \le p < \infty$ and, moreover, $\lim_{p \to \infty} \|f\|_p = \|f\|_{\infty}$. Does the same hold for $|\Omega| = \infty$? If not, find an additional condition on f, so that the claim holds true.
(ii) If $f \in \cap_{p_k} L^{p_k}(\Omega)$ for a certain subsequence $p_k \to \infty$ and, moreover, $\sup_{p_k} \|f\|_{p_k} = C < \infty$, then $f \in L^{\infty}(\Omega)$ and $\|f\|_{\infty} \le C$. Is the assumption that Ω has a finite measure necessary?

11. For arbitrary $p \in [1, \infty]$ find a function which belongs to $L^p(\mathbb{R}^d)$, but does not belong to $L^q(R^d)$ for any $q \neq p$.

12. Show that there exists a function which belongs to any $L^p(B_1(0))$, $1 \leq p < \infty$, but does not belong to $L^\infty(B_1(0))$.

13. Show that the space $\widetilde{W}^{k,\infty}(\Omega)$ is isometrically isomorphic to $C^k(\overline{\Omega})$ and $\widetilde{W}_0^{k,\infty}(\Omega)$ to the space $\{v \in C^k(\overline{\Omega}); D^\alpha v = 0 \text{ on } \partial\Omega, \forall |\alpha| \leq k\}$.