Exercises PDEs I: Set 2 (October 17, 2025)

Introduction and function spaces

1. Prove:
   Let Ω be an open bounded set. Then for any $\lambda \in (0,1]$ it holds
   $$\mathcal{C}^{0,\lambda}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}(\overline{\Omega}).$$
   (10b) Let Ω be an open bounded set. Then for any $\alpha, \beta \in [0,1]$ such that $0 \leq \alpha < \beta \leq 1$ it holds
   $$\mathcal{C}^{0,\beta}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}^{0,\alpha}(\overline{\Omega}).$$
2. Let $\Omega \subset \mathbb{R}^d$ have a finite $d-$dimensional measure. Prove the following claims:
   (i) If $f \in L^\infty(\Omega)$, then $f \in L^p(\Omega)$ $\forall 1 \leq p < \infty$ and, moreover, $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. Does the same hold for $|\Omega| = \infty$? If not, find an additional condition on f, so that the claim holds true.
   (ii) If $f \in \cap_{p_k} L^{p_k}(\Omega)$ for a certain subsequence $p_k \to \infty$ and, moreover, $\sup_{p_k} \|f\|_{p_k} = C < \infty$, then $f \in L^\infty(\Omega)$ and $\|f\|_\infty \leq C$. Is the assumption that Ω has a finite measure necessary?
3. For arbitrary $p \in [1, \infty]$ find a function which belongs to $L^p(\mathbb{R}^d)$, but does not belong to $L^q(R^d)$ for any $q \neq p$.
4. Show that there exists a function which belongs to any $L^p(B_1(0))$, $1 \leq p < \infty$, but does not belong to $L^\infty(B_1(0))$.
5. Show that the space $\widetilde{W}^{k,\infty}(\Omega)$ is isometrically isomorphic to $C^k(\overline{\Omega})$ and $\widetilde{W}_0^{k,\infty}(\Omega)$ to the space $\{v \in C^k(\overline{\Omega}); D^\alpha v = 0 \text{ on } \partial\Omega, \forall |\alpha| \leq k\}$.
6. Prove the following
   Let $u \in L^1_{\text{loc}}(\Omega)$, Ω open, be such that
   $$\int_\Omega u\varphi \,\mathrm{d}x = 0$$
   for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Then $u = 0$ a.e. in Ω.

7. Let $\Omega_1,\ \Omega_2 \subset \mathbb{R}^d$ be open, $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega_1) \cap W^{1,p}(\Omega_2)$. Show that $u \in W^{1,p}(\Omega_1 \cap \Omega_2)$.

8. Let $f \in C^{0,1}(\mathbb{R})$ (with $\|f\|_{C^{0,1}(\mathbb{R})} < \infty$). Show that there exists a sequence $f_n \in C^1(\mathbb{R})$ such that:

$$\begin{aligned} f_n &\rightrightarrows f && \text{in } \mathbb{R} \\ f_n' &\to f && \text{a.e. in } \mathbb{R}. \end{aligned}$$