Exercises PDEs I: Set 3 (October 24, 2025)

Introduction and function spaces

1. Show that the space $\widetilde{W}^{k,\infty}(\Omega)$ is isometrically isomorphic to $C^k(\overline{\Omega})$ and $\widetilde{W}_0^{k,\infty}(\Omega)$ to the space $\{v \in C^k(\overline{\Omega}); D^\alpha v = 0 \text{ on } \partial\Omega, \forall |\alpha| \leq k\}$.

2. Prove the following
Let $u \in L^1_{\mathrm{loc}}(\Omega)$, Ω open, be such that
$$\int_\Omega u\varphi \,\mathrm{d}x = 0$$
for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Then $u = 0$ a.e. in Ω.

3. Let Ω_1, $\Omega_2 \subset \mathbb{R}^d$ be open, $p \in [1, \infty]$ and $u \in W^{1,p}(\Omega_1) \cap W^{1,p}(\Omega_2)$. Show that $u \in W^{1,p}(\Omega_1 \cap \Omega_2)$.

4. Let $f \in C^{0,1}(\mathbb{R})$ (with $\|f\|_{C^{0,1}(\mathbb{R})} < \infty$). Show that there exists a sequence $f_n \in C^1(\mathbb{R})$ such that:
$$\begin{aligned} f_n &\rightrightarrows f && \text{in } \mathbb{R} \\ f_n' &\to f && \text{a.e. in } \mathbb{R}. \end{aligned}$$

5. Let $u \in W^{2,2}(\mathbb{R}^d)$. Then
$$\|\nabla u\|_{L^2(\mathbb{R}^d)} \leq \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)} \|\Delta u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)}.$$

6. Show the interpolation inequality: Let $\alpha \in [0, 1)$, $r, p, q \in [1, \infty]$ such that
$$\frac{1}{r} = \alpha\Big(\frac{1}{p} - \frac{1}{d}\Big) + (1 - \alpha)\frac{1}{q}.$$
Then there exists $C > 0$ such that for any $u \in C_0^\infty(\mathbb{R}^d)$ there holds
$$\|u\|_{L^r(\mathbb{R}^d)} \leq C\|\nabla u\|^\alpha_{L^p(\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^q(\mathbb{R}^d)}.$$
Moreover, if $p < d$, we may take also $\alpha = 1$.