Exercises PDEs I: Set 4 (October 31, 2025)

Introduction and function spaces. Elliptic PDEs

1. Let $f \in C^{0,1}(\mathbb{R})$ (with $\|f\|_{C^{0,1}(\mathbb{R})} < \infty$). Show that there exists a sequence $f_n \in C^1(\mathbb{R})$ such that:
$$\begin{aligned} f_n &\rightrightarrows f && \text{in } \mathbb{R} \\ f_n' &\to f' && \text{a.e. in } \mathbb{R}. \end{aligned}$$

2. Show the interpolation inequality: Let $\alpha \in [0,1)$, $r,p,q \in [1,\infty]$ such that
$$\frac{1}{r} = \alpha\Big(\frac{1}{p} - \frac{1}{d}\Big) + (1-\alpha)\frac{1}{q}.$$
Then there exists $C > 0$ such that for any $u \in C_0^\infty(\mathbb{R}^d)$ there holds
$$\|u\|_{L^r(\mathbb{R}^d)} \le C\|\nabla u\|^{\alpha}_{L^p(\mathbb{R}^d)}\|u\|^{1-\alpha}_{L^q(\mathbb{R}^d)}.$$
Moreover, if $p < d$, we may take also $\alpha = 1$.

3. Consider the differential operator
$$Lu = -xu_{xx} - yu_{yy} + 2(u_x + u_y + u_{xy}) - 6u.$$
Rewrite it into the standard form
$$Lu = -\mathrm{div}\,(\mathbb{A}\nabla u) + \mathbf{c}\cdot\nabla u + bu$$
and verify, where in $\mathbb{R}^2$ the operator fulfils the ellipticity condition.

4. Consider the space
$$\dot{D}^{1,2}(\mathbb{R}^2) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2); \nabla u \in L^2(\mathbb{R}^2;\mathbb{R}^2)\}.$$
Show that $\dot{D}^{1,2}(\mathbb{R}^2)$ can be identified with classes of functions with square integrable gradient over $\mathbb{R}^2$ such that two functions belong to the same class if they differ by a constant a.e. in $\mathbb{R}^2$. Using this identification show that $\dot{D}^{1,2}(\mathbb{R}^2)$ is with respect to the scalar product $(u,v) = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x$ a Hilbert space. Finally, assuming that $f = \mathrm{div}\,\mathbf{g}$, $\mathbf{g} \in L^2(\mathbb{R}^2;\mathbb{R}^2)$, show that there exists unique weak solution in $\dot{D}^{1,2}(\mathbb{R}^2)$ to
$$-\Delta u = f \qquad \text{in } \mathbb{R}^2.$$

5. Consider the problem

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(d_i u) + bu = f$$

in $\Omega \subset \mathbb{R}^d$ bounded open set with the boundary condition

$$u = 0$$

on $\partial\Omega$. Formulate the problem weakly and specify the minimal integrability conditions on the functions a_{ij}, c_i, d_i, b and f so that all integrals are finite. Then specify further restrictions under which there exists unique weak solution to this problem (use only Riesz representation Theorem).

6. Consider the problem

$$\Delta^2 u + \alpha\Delta u + \beta u = f$$

in $\Omega \subset \mathbb{R}^d$ a bounded open set, where $\Delta^2 u = \Delta(\Delta u)$, α, β are real constants and $f \in L^2(\Omega)$. Consider the following boundary conditions

$$u = \frac{\partial u}{\partial \boldsymbol{\nu}} = 0$$

on $\partial\Omega$, where $\boldsymbol{\nu}$ is the (unit) exterior normal vector to $\partial\Omega$.

Formulate the problem weakly and discuss conditions under which there exists a unique weak solution to the problem above.

7. Consider the system of equations in $\Omega \subset \mathbb{R}^d$

$$-\sum_{\beta=1}^{N}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j}\Big) + \sum_{\beta=1}^{N}\sum_{i=1}^{d} c_i^{\alpha\beta}\frac{\partial u^\beta}{\partial x_i} + \sum_{\beta=1}^{N}\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(d_i^{\alpha\beta}u^\beta\Big) + \sum_{\beta=1}^{N} b^{\alpha\beta}u^\beta = f^\alpha, \qquad \alpha = 1, 2, \ldots, N$$

with the boundary conditions

$$
\begin{aligned}
u^\alpha &= u_0^\alpha \qquad && \text{on } \Gamma_1, \\
\sum_{\beta=1}^{N}\sum_{i=1}^{d}\Big(\sum_{j=1}^{d} a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j} - d_i^{\alpha\beta}u^\beta\Big)\nu_i &= g^\alpha \qquad && \text{on } \Gamma_2, \\
\sum_{\beta=1}^{N}\sum_{i=1}^{d}\Big(\sum_{j=1}^{d} a_{ij}^{\alpha\beta}\frac{\partial u^\beta}{\partial x_j} - d_i^{\alpha\beta}u^\beta\Big)\nu_i + \sum_{\beta+1}^{N}\sigma^{\alpha\beta}u^\beta &= g^\alpha \qquad && \text{on } \Gamma_3,
\end{aligned}
$$

$\alpha = 1, 2, \ldots, N$. Formulate the problem weakly and for $c_i^{\alpha\beta} = d_i^{\alpha\beta} = 0$ formulate conditions on the other functions such that (using Riesz representation Theorem) there exists unique weak solution to this problem.