Exercises PDEs I: Set 6 (November 21, 2025)

Elliptic PDEs

1. Let Ω be an open Lipschitz domain in $\mathbb{R}^d$. Assume that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where $\Omega_1 \cap \Omega_2 = \emptyset$, Ω_i are open, $i = 1, 2$ and let $|\Gamma|_{d-1} > 0$, finite, be the common part of the of the boundaries of Ω_1 and Ω_2 (i.e., $\overline{\Omega_1} \cap \overline{\Omega_2} = \overline{\Gamma}$). Under standard assumptions on a_{ij}, f consider the weak solution to

$$-\frac{\partial}{\partial x_i}\Big(\sum_{i,j=1} a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) = f \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Discuss, under suitable conditions on u, which weak problem satisfies the function u separately on Ω_1 and Ω_2. This problem can be also viewed as follows.

Assume additionally that $u \in W^{2,2}(\Omega)$. What kind of problems satisfies u in Ω_1 and in Ω_2. Or another point of view, assume that $u \in W^{1,2}(\Omega_i)$ satisfies in the weak setting

$$-\frac{\partial}{\partial x_i}\Big(\sum_{i,j=1} a_{ij}(x)\frac{\partial u_i}{\partial x_j}\Big) = f \qquad \text{in } \Omega_i$$
$$u_i = u_i^0 \qquad \text{on } \partial\Omega_i, \quad i = 1, 2.$$

What are the necessary conditions on the common part of the boundary such that

$$u = u 1_{\Omega_1} + u_2 1_{\Omega_2}$$

is a weak solution to the original problem on Ω?

2. Find the real spectrum of the second derivative ("Laplace operator") in one space dimension with Dirichlet boundary conditions. More precisely, find all $\lambda \in \mathbb{R}$ such that there exists a nontrivial solution to problem

$$-u'' = \lambda u \qquad \text{in } (0, 1)$$
$$u(0) = u(1) = 0.$$

3. Find the real spectrum of the Laplace operator in two space dimensions with Dirichlet boundary conditions on a square. More precisely, find all $\lambda \in \mathbb{R}$ such that there exists a nontrivial solution to problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } (0,1)^2 \\ u &= 0 && \text{on } \partial(0,1)^2. \end{aligned}$$

Try to generalize the problem to higher space dimensions or also for other boundary conditions (at least for one space dimension).

4. Consider the classical formulation of the problem

$$\begin{aligned} -\Delta u + b_0 u &= f && \text{in } \Omega = B_1(0) \subset \mathbb{R}^d \\ \frac{\partial u}{\partial \nu} &= g && \text{on } \partial\Omega. \end{aligned}$$

Formulate the problem weakly.

For particular choice

$$f(x) = |x|^\alpha, \qquad g(x) = |x - (1,0,\dots,0)|^\beta \qquad b_0 = \text{const}$$

find conditions on α, β and b_0 so that there exists a unique weak solution to the problem above. Or to simplify, replace the ball by the upper halfball

$$B_1^+(0) := \left\{ x \in \mathbb{R}^d \,|\, |x| < 1,\, x_d > 0 \right\}$$

and the function $g(x) = |x|^\beta$.