Exercises PDEs I: Set 7 (November 28, 2025)

Elliptic PDEs

1. Find the real spectrum of the Laplace operator in two space dimensions with Dirichlet boundary conditions on a square. More precisely, find all $\lambda \in \mathbb{R}$ such that there exists a nontrivial solution to problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } (0,1)^2 \\ u &= 0 && \text{on } \partial(0,1)^2. \end{aligned}$$

   Try to generalize the problem to higher space dimensions or also for other boundary conditions (at least for one space dimension).

2. Consider the classical formulation of the problem

$$\begin{aligned} -\Delta u + b_0 u &= f && \text{in } \Omega = B_1(0) \subset \mathbb{R}^d \\ \frac{\partial u}{\partial \nu} &= g && \text{on } \partial\Omega. \end{aligned}$$

   Formulate the problem weakly.

   For particular choice

$$f(x) = |x|^\alpha, \qquad g(x) = |x - (1,0,\dots,0)|^\beta \qquad b_0 = \text{ const}$$

   find conditions on α, β and b_0 so that there exists a unique weak solution to the problem above. Or to simplify, replace the ball by the upper halfball

$$B_1^+(0) := \left\{ x \in \mathbb{R}^d \,\middle|\, |x| < 1,\, x_d > 0 \right\}$$

   and the function $g(x) = |x|^\beta$.

3. Consider the Neumann boundary problem in $C^+ = (-1,1)^{d-1} \times (0,1)$, i.e. we look for $u \in W^{1,2}(C^+)$ such that

$$\int_{C^+} \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + bu\varphi \Big) \, \mathrm{d}x = \int_{C^+} f\varphi \, \mathrm{d}x + \int_{(-1,1)^{d-1}} g\varphi \, \mathrm{d}x'$$

   for all $\varphi \in W^{1,2}(C^+)$. The operator is elliptic with our standard assumptions on the data, $b \geq 0$, but not identically zero. We further assume that there exists unique solution with $\operatorname{supp} u \subset (-1,1)^{d-1} \times [0,1)$, exactly as in the regularity proof for the Dirichlet boundary conditions. Moreover

$$\|u\|_{W^{1,2}(C^+)} \le C(\|f\|_{L^2(C^+)} + \|g\|_{L^2((-1,1)^{d-1})}).$$

Show that if $\{a_{ij}\}_{i,j=1}^d \in W^{1,\infty}(C^+)$, $b \in L^\infty(C^+)$, $f \in L^2(C^+)$ and $g \in W^{1,2}((-1,1)^{d-1})$, then the solution above belongs to $W^{2,2}(C^+)$ and

$$\|u\|_{W^{2,2}(C^+)} \le C\big(\|f\|_{L^2(C^+)} + \|g\|_{W^{1,2}((-1,1)^{d-1})}\big).$$