Exercises PDEs I: Set 8 (December 19, 2025)

Parabolic PDEs

1. Consider the inhomogeneous Dirichlet boundary value problem for parabolic problem, i.e.,

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + bu = f \qquad && \text{in } (0,T)\times\Omega \\
&u(t,x) = u_0(t,x) && \text{on } (0,T)\times\partial\Omega \\
&u(0,x) = g(x) && \text{in } \Omega.
\end{aligned}$$

Assume that there exists U_0 defined in $(0,T)\times\Omega$ which attains the given boundary value u_0. Assume that the operator is parabolic.
(i) Formulate the problem weakly.
(ii) Formulate assumptions on U_0 so that there exists unique weak solution to this problem and explain the main steps of the proof.

2. Consider the homogeneous Dirichlet boundary value problem for parabolic problem, i.e.,

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + bu = f \qquad && \text{in } (0,T)\times\Omega \\
&u(t,x) = 0 && \text{on } (0,T)\times\partial\Omega \\
&u(0,x) = g(x) && \text{in } \Omega.
\end{aligned}$$

The matrix $\mathbb{A}$ is positive definite in the usual sense and essentially bounded in $(0,T)\times\Omega$. Assume that the functions $f \in L^{t_f}(0,T;L^{s_f}(\Omega))$, $b \in L^{t_b}(0,T;L^{s_b}(\Omega))$ and $c_i \in L^{t_c}(0,T;L^{s_c}(\Omega))$, $i = 1,2,\ldots,d$.
(i) Formulate possibly minimal assumptions on the exponents so that there still exists a weak solution to this problem.
(ii) Do the same (and add possibly further assumptions on the derivatives of the functions) so that the weak solution fulfills the regularity as in the theorem (see Theorem 5.1.13 in the Lecture Notes).