Exercises PDEs I: Set 9 (January 9, 2026)

Parabolic and hyperbolic PDEs

1. Consider the homogeneous Dirichlet boundary value problem for parabolic problem, i.e.,

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + a(x_1)\frac{\partial^2 u}{\partial x_1 \partial x_2} + \sum_{i=1}^{2} c_i \frac{\partial u}{\partial x_i} + bu = f && \text{in } (0,T)\times C\\
&u(t,x) = 0 && \text{on } (0,T)\times \partial C\\
&u(0,x) = g(x) && \text{in } C,
\end{aligned}
$$

where $C = (0,1)\times(0,1) \subset \mathbb{R}^2$. Assume that $a \in L^\infty((0,1))$, b and c_1, $c_2 \in L^\infty(C)$, $f \in L^2((0,T)\times C)$.
(i) Formulate the problem weakly.
(ii) Formulate assumptions on a so that there exists unique weak solution to this problem and explain the main steps of the proof.

2. Consider the homogeneous Dirichlet boundary value problem for the hyperbolic problem, i.e.,

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + bu = f && \text{in } (0,T)\times \Omega\\
&u(t,x) = 0 && \text{on } (0,T)\times \partial\Omega\\
&u(0,x) = g(x) && \text{in } \Omega\\
&\frac{\partial u(0,x)}{\partial t} = h(x) && \text{in } \Omega,
\end{aligned}
$$

$\Omega \subset \mathbb{R}^d$ bounded, open. The matrix $\mathbb{A}$ is positive definite in the usual sense and belongs to $W^{1,\infty}((0,T)\times\Omega)$, $f \in L^2((0,T)\times C)$. Formulate the problem weakly. Review the proof of existence of a unique weak solution and try to formulate conditions (weaker than in the lecture) on the coefficients of the hyperbolic operator so that there still exists (unique) weak solution to this problem.

3. Consider the inhomogeneous Dirichlet boundary value problem for the hyperbolic problem, i.e.,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + bu = f && \text{in } (0,T)\times\Omega \\
&u(t,x) = u_0 && \text{on } (0,T)\times\partial\Omega \\
&u(0,x) = g(x) && \text{in } \Omega \\
&\frac{\partial u(0,x)}{\partial t} = h(x) && \text{in } \Omega,
\end{aligned}$$

$\Omega \subset \mathbb{R}^d$ bounded, open. Assume that there exists U_0 defined in $(0,T)\times\Omega$ such that $U_0 = u_0$ on $(0,T)\times\partial\Omega$. The matrix $\mathbb{A}$ is positive definite in the usual sense and belongs to $W^{1,\infty}((0,T)\times\Omega)$. Formulate the problem weakly. Review the proof of existence of a unique weak solution and try to formulate conditions on U_0 (for the coefficients and f use the same conditions as in the lecture) so that there exists (unique) weak solution to this problem.

4. Consider the homogeneous Dirichlet boundary value problem for the hyperbolic problem, i.e.,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} + x_1 a(x_2)\frac{\partial^2 u}{\partial x_1 \partial x_2} \\
&+ x_3 b(x_1)\frac{\partial^2 u}{\partial x_1 \partial x_3} + x_2 c(x_3)\frac{\partial^2 u}{\partial x_2 \partial x_3} = f && \text{in } (0,T)\times C \\
&u(t,x) = 0 && \text{on } (0,T)\times\partial C \\
&u(0,x) = g(x) && \text{in } C \\
&\frac{\partial u(0,x)}{\partial t} = h(x) && \text{in } C,
\end{aligned}$$

where $C = (0,1)^3 \subset \mathbb{R}^3$. Assume that a, b, $c \in W^{1,\infty}((0,1))$, $f \in L^2((0,T)\times C)$.
(i) Formulate the problem weakly.
(ii) Formulate assumptions on a, b and c so that there exists unique weak solution to this problem and explain the main steps of the proof.