~~T_k~~ $=0 \Rightarrow T_k(0)=0$

$\Rightarrow C_2 = 0$

$T_k'(0)=0 \Rightarrow \quad C_1\, k\pi + \frac{\alpha}{(k\pi)^2-\alpha^2} = 0$

$C_1 = -\frac{\alpha}{k\pi((k\pi)^2-\alpha^2)}$

$u(t,x) = \left(-\frac{\alpha}{k\pi((k\pi)^2-\alpha^2)} \sin(k\pi t) + \frac{1}{(k\pi)^2-\alpha^2}\sin(\alpha t)\right)\sin(k\pi x) \quad t\geq 0 \quad x\in(0,1)$

ali $\alpha \to k\pi$ ~~sestava~~ nuo jasno

b) $\alpha = k\pi$

$u_p(t) = t(A_1 \cos(k\pi t) + A_2 \sin(k\pi t))$

$2(-A_1(k\pi)\sin(k\pi t) + A_2 \cos(k\pi t)) = \sin(k\pi t) \qquad A_2 = 0$

$A_1 = -\frac{1}{2k\pi}$

$T_p(t) = -t\frac{1}{2k\pi}\cos(k\pi t)$

~~$u(t,x)$~~ $T(t) = C_1 \sin(k\pi t) + C_2\cos(k\pi t) - t\frac{1}{2k\pi}\cos(k\pi t)$

$T(0)=0 \Rightarrow \quad C_2 = 0$

$T'(0)=0 \Rightarrow \quad C_1 \cdot k\pi - \frac{1}{2\pi k} = 0 \qquad C_1 = \frac{1}{2(k\pi)^2}$

$T(t) = \frac{1}{2(k\pi)^2}\sin(k\pi t) - \frac{t}{2k\pi}\cos(k\pi t)$

Celk $\underline{u(t,x) = \left(\frac{1}{2(k\pi)^2}\sin(k\pi t) - \frac{t}{2k\pi}\cos(k\pi t)\right)\sin(k\pi x)}$ rešenje je neomeđeno!

<u>Primjer</u>

$\partial_{tt}u - \partial_{xx}u = 0 \qquad$ na $(0,\infty)\times(0,1)$

$u(0,x) = 0 \qquad$ na $(0,1)$

$\partial_t u(0,x) = (x-1)^2 \cdot \sin^2 x \qquad$ na $(0,1)$

$\partial_x u(t,0) = \partial_x u(t,1) = 0$

<u>Rešenje</u>:

$X'' + \lambda X = 0 \qquad \Rightarrow \qquad X_k(x) = \cos(k\pi x)$

$X'(0) = X'(1) = 0$

$T_k''(t) + (k\pi)^2 T_k(t) = 0$

$T_k(0) = 0 \quad T_k'(0) = B_k$

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$T_k(t) = C_1 \sin(k\pi t) + C_2 \cos(k\pi t)$ $\quad T_k(0) = 0 \Rightarrow C_2 = 0$

$T_k'(0) = C_1(k\pi) \cdot 1 \Rightarrow \quad C_1 = \frac{1}{k\pi} \cdot B_k \quad k = 1, 2, 3 \ldots$

$T_0(t) = At + B$

$T_0(0) = 0 \Rightarrow B = 0$

$T_0'(0) = B_0 \Rightarrow \quad T_0(t) = B_0 t$

$$T_k(t) = \sum_{k=1}^{\infty} \frac{B_k}{k\pi} \sin(k\pi t) \cos(k\pi x) + B_0 t$$

pitomá $\quad$ sudé 2 ...

$$(x-1)^2 \sin^2 x \Big/_{...} = \sum_{k=1}^{\infty} B_k \cos(k\pi x) + B_0$$

$$\frac{B_0}{2} = 2\int_0^1 (x-1)^2 \sin^2 x \, dx = \int_0^1 (x-1)^2 \left(1 - \cos 2x\right) dx =$$

$$= \frac{1}{3}\left[(x-1)^3\right]_0^1 - \Big[\underbrace{\frac{\sin 2x}{2} \cdot (x-1)^2}_{=0}\Big]_0^1 + \int_0^1 2(x-1) \cdot \frac{\sin(2x)}{2} \, dx$$

$$= \frac{1}{3} + \int_0^1 (x-1) \sin(2x) \, dx = \frac{1}{3} + \left[(x-1) \cdot -\frac{\cos 2x}{2}\right]_0^1 + \frac{1}{3}\int_0^1 \frac{\cos 2x}{2} \, dx$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{6}\left[\frac{\sin 2x}{2}\right]_0^1 = -\frac{1}{6} + \frac{1}{12} \sin 2$$

$$B_k = 2\int_0^1 (x-1)^2 \, \frac{1 - \cos 2x}{2} \cdot \cos(k\pi x) \, dx$$

$$= I_1 + I_2$$

$$I_1 = \int_0^1 (x-1)^2 \cos(k\pi x) \, dx = \Big[\underbrace{(x-1)^2 \frac{\sin(k\pi x)}{k\pi}}_{=0}\Big]_0^1 - 2\int_0^1 (x-1) \frac{\sin k\pi x}{k\pi} \, dx$$

$$= -\frac{2}{k\pi}\left[(x-1) \cdot \frac{-\cos(k\pi x)}{k\pi}\right]_0^1 - \frac{2}{(k\pi)^2}\int_0^1 \cos(k\pi x) \, dx = + \frac{2}{(k\pi)^2}$$

$$-I_2 = \frac{1}{2}\int_0^1 \left[\cos((k\pi+2)x) + \cos((k\pi-2)x)\right](x-1)^2 \, dx = \frac{1}{2}\Big[\underbrace{(x-1)^2 \left(\frac{\sin(k\pi+2)x}{(k\pi+2)} + \frac{\sin(k\pi-2)x}{(k\pi-2)}\right)}_{=0}\Big]_0^1$$

$$- \frac{2}{2}\int_0^1 (x-1)\left[\frac{\sin(k\pi+2)x}{(k\pi+2)} + \frac{\sin(k\pi-2)x}{(k\pi-2)}\right] dx =$$

$$= + \left[(x-1)\left(\frac{\cos(k\pi+2)x}{(k\pi+2)^2} + \frac{\cos((k\pi-2)x)}{(k\pi-2)^2}\right)\right]_0^1 - \int_0^1 \left(\frac{\cos(k\pi+2)x}{(k\pi+2)^2} + \frac{\cos(k\pi-2)}{(k\pi-2)^2}\right) dx$$

$$= -\left(\frac{1}{(k\pi+2)^2} + \frac{1}{(k\pi-2)^2}\right) + \left[\frac{\sin(k\pi+2)x}{(k\pi+2)^3} + \frac{\cos(k\pi-2)x}{(k\pi-2)^3}\right]_0^1 = -\left(\frac{1}{(k\pi+2)^2} + \frac{1}{(k\pi-2)^2}\right) + \frac{(-1)^k \sin 2x}{(k\pi+2)^3} - \frac{(-1)^k \sin 2x}{(k\pi+2)^3}$$

...

Příklad

Pomocí Fourierovy metody nalezněte řešení úlohy

$$\partial_{tt} u - \Delta u = 0 \quad \text{na } (0,\infty)\times(0,\pi)^2$$
$$u(t,x) = 0 \quad \text{na } (0,\infty)\times \partial\big((0,\pi)^2\big)$$
$$u(0,x) = u_0(x)$$
$$\partial_t u(0,x) = u_1(x)$$

Speciálně pak pro $u_0(x,y) = \sin^3 x \sin^3 y$,
$u_1(x,y) = 0$.

Řešení:

Protože jsou konečné kombinace typu $\sum_{k=1}^{m} v_i(x)\, w_j(y)$ husté v $C^1([0,\pi]^2)$, je vidět, že funkce
$$\{\sin(kx)\sin(ly)\}_{k,l=1}^{\infty}$$
tvoří úplný systém v $L^2((0,\pi)^2)$.

Proto bude stačit hledat řešení pomocí separace proměnných.

Řešíme

$$\frac{\ddot T(t)}{T(t)} = \frac{\Delta R(x,y)}{R(x,y)} = -\lambda \qquad \lambda \le 0 \ \ldots \text{ nelze pak}$$

$$\int \Delta R \cdot R \, dx + \lambda \int R^2\, dx = 0$$
$$-\int |\nabla R|^2\, dx + \lambda \int R^2\, dx = 0$$

$\lambda < 0$ pak $R \equiv 0$
$\lambda = 0$ ∇R (R = konst, ale 0 na $\partial\Omega$)

$$\frac{\Delta R}{R} = -\lambda \qquad \lambda > 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda \qquad \frac{X''}{X} = -\lambda \quad X(0) = X(\pi) = 0$$

$$X_k = \{\sin kx\}_{k=1}^{\infty}$$

$$\frac{Y''}{Y} = -l \qquad Y = \{\sin ly\}_{l=1}^{\infty}$$

$$\frac{\ddot T}{T} = -(k^2 + l^2)$$
$$\ddot T(t) + (k^2 + l^2)\, T(t) = 0$$
$$T(0) = A_{kl}$$
$$\dot T(0) = B_{kl} \quad \Longrightarrow$$

$$\boxed{T_{kl}(t) = A_{kl}\cos\left(\sqrt{k^2+l^2}\, t\right) + \frac{B_{kl}}{\sqrt{k^2+l^2}}\sin\left(\sqrt{k^2+l^2}\, t\right)}$$

Speciálně pro $u_1(x) = 0$ je $B_{kl} = 0 \quad \forall k,l \in \mathbb{N}$

$$u_0(x) = \sin^3 x \cdot \sin^3 y = \left(\tfrac34 \sin x - \tfrac14 \sin 3x\right)\left(\tfrac34 \sin y - \tfrac14 \sin 3y\right)$$

$$\sin^3 x = \sin x\,\cancel{(1-\cos^2 x)}\ \sin^2 x = \tfrac12 \sin x\,(1 - \cos 2x) = \tfrac12 \sin x - \tfrac12\tfrac12\left[\sin(x+2x) + \sin(x-2x)\right]$$
$$= \tfrac34 \sin x - \tfrac14 \sin 3x$$

$$\boxed{u(t,x) = \tfrac{9}{16}\sin x \sin y \cos(\sqrt{2}\, t) - \tfrac{3}{16}\sin x \sin 3y \cos(\sqrt{10}\, t) - \tfrac{3}{16}\sin 3x \sin y \cos(\sqrt{10}\, t) + \tfrac{1}{16}\sin 3x \sin 3y \cos(\sqrt{18}\, t)}$$

Hledejme řešení

$$\partial_{tt} u - \Delta u = 0 \quad \text{na} \quad (0,\infty) \times B_1(0)$$

$$u(0,x) = 0$$

$$\partial_t u(0,x) = (r-1) \sin \tfrac{\pi}{2} r$$

$$\frac{\partial u}{\partial n}(t, x) + u(|x|) = 0 \qquad |x| = 1$$

Řešení :

Hledáme řešení ve tvaru $u(t,x) = v(t,r)$ $\quad r = |x|$

$$\partial_{tt} v - \partial_{rr} v - \frac{2}{r} \partial_r v = 0 \quad \text{na } (0,\infty) \times (0,1) \qquad / r \qquad w(t,r) = r v(t,r)$$

$$v(0,r) = 0$$

$$\partial_t v(0,r) = (r-1) \sin\left(\frac{\pi}{2} r\right)$$

$$\frac{\partial v}{\partial r} + v = 0 \qquad r = 1$$

$$\partial_{tt} w - \partial_{rr} w = 0 \quad \text{na } (0,\infty) \times (0,1)$$

$$w(0,r) = 0$$

$$\partial_t w(0,r) = r(r-1) \sin\left(\frac{\pi}{2} r\right)$$

$$\frac{\partial}{\partial r}(w) = 0 \qquad r = 1$$

Doplním $w(t,0) = 0$

a máme úlohu, kterou umíme řešit

$$\frac{R''}{R} = \frac{T''}{T} = -\lambda$$

$$R'' + \lambda R = 0 \qquad R(0) = 0 \qquad R'(1) = 0$$

$$\boxed{R(r) = A_k \sin\left(\left(\frac{2k-1}{2}\right)\pi \cdot r\right)}$$

$$T'' + \left(\frac{(2k-1)\pi}{2}\right)^2 T = 0$$

$$T(0) = 0,$$

$$T'(0) = B_k$$

$$T_k(t) = \frac{2B_k}{(2k-1)\pi} \cdot \sin\left(\frac{2k-1}{2} \pi t\right)$$

$$\boxed{u(t,r) = \sum_{k=1}^{\infty} \frac{2B_k}{(2k-1)\pi} \sin\left(\frac{2k-1}{2} \pi t\right) \sin\left(\frac{2k-1}{2} \pi r\right)}$$

$$A_k = 2\int_0^1 (r-1)\cdot r \,\sin\left(\frac{\pi}{2}r\right) \sin\left(\frac{2k-1}{2}\pi\right) dr$$

a) $k=1$

$$2\int_0^1 r(r-1)\cdot \sin^2\left(\frac{\pi}{2}r\right)dr = \int_0^1 r(r-1)\,(1-\cos\pi r)\,dr$$

$$= \frac{1}{3}-\frac{1}{2} - \int_0^1 \underset{v}{r(r-1)}\,\underbrace{\cos(\pi r)}_{u'}dr = -\frac{1}{6} - \underbrace{\left[r(r-1)\frac{\sin\pi r}{\pi}\right]_0^1}_{=0}$$

$$+\frac{1}{\pi}\int_0^1 \underset{v}{(2r-1)}\,\underset{u'}{\sin(\pi r)}\,dr = -\frac{1}{6}+\frac{1}{\pi}\left[(2r-1)\left(-\frac{\cos(\pi r)}{\pi}\right)\right]_0^1 + \frac{1}{\pi^2}\underbrace{\int_0^1 2\cos(\pi r)dr}_{=0}$$

$$= -\frac{1}{6}+\frac{1}{\pi^2}(1-1) = \underline{-\frac{1}{6}}$$

b) $k>1$ $\quad 2\int_0^1 r(r-1)\,\sin\left(\frac{\pi}{2}r\right)\sin\left(\frac{2k-1}{2}\pi r\right)dr =$

$$= -2\int_0^1 r(r-1)\,\frac{1}{2}\Big(\cos(k\pi r)-\cos((k-1)\pi r)\Big)dr$$

$$= -\left[r(r-1)\left(\frac{\sin k\pi r}{k\pi}-\frac{\sin((k-1)\pi r)}{(k-1)\pi}\right)\right]_0^1 + \int_0^1 (2r-1)\left(\frac{1}{k\pi}\sin(k\pi r)-\frac{1}{(k-1)\pi}\sin(k\pi r)\right)dr$$

$$= \left[(2r-1)\left(-\frac{\cos(k\pi r)}{(k\pi)^2}+\frac{\cos((k-1)\pi r)}{((k-1)\pi)^2}\right)\right]_0^1 - \underbrace{\int_0^1 2\left(\frac{\cos(k-1)\pi r}{(k-1)^2\pi^2}-\frac{\cos k\pi r}{(k\pi)^2}\right)dr}_{=0}$$

$$= \underline{\left(\frac{1}{(k\pi)^2}\left((-1)^{k+1}-1\right)+\frac{1}{((k-1)\pi)^2}\left(1-(-1)^k\right)\right)}$$

<u>Příklad</u> – ne zadání ?

<u>Řešení</u>

$\partial_{tt}u - \Delta u = 0 \quad n\in\mathbb{N} \qquad$ na $(0,\infty)\times\mathbb{R}^2$

$u(0,x) = |x|^m \qquad$ na $\mathbb{R}^2$

$\partial_t u(0,x) = 0 \qquad$ na $\mathbb{R}^2 \qquad$ Nalezn. $u(t,0)$

Řešení.

$$u(t,x) = \frac{c}{2}\cdot\frac{1}{\pi(ct)^2}\int_{B_{ct}}\frac{t|y|^m + t\cdot m|y|^{m-2}\,(y\cdot(y-x))}{\sqrt{c^2t^2-|y-x|^2}}dy$$

$$= \frac{1}{2\pi ct}\int_{B_{ct}}\frac{|y|^m(1+m)}{\sqrt{(ct)^2-|y|^2}}dy = \frac{1}{2\pi ct}\int_0^{ct}\frac{|y|^m(m+1)}{\sqrt{(ct)^2-|y|^2}}d|y|\cdot ct\cdot 2\pi$$

$$= \int_0^{ct}\frac{r^m(m+1)}{\sqrt{(ct)^2-r^2}}\,r\,dr = [r=ct\varrho] = (ct)^m(m+1)\int_0^1\frac{\varrho^{m+1}}{\sqrt{1-\varrho^2}}d\varrho$$

$$= [\varrho^2=\tau] = (ct)^m\frac{(m+1)}{2}\int_0^1 \tau^{\frac{m}{2}}(1-\tau)^{-\frac{1}{2}}d\tau = \underbrace{(ct)^m\frac{(m+1)}{2}B\left(\frac{m}{2}+1,\frac{1}{2}\right)}_{\text{Beta-funkce}}$$

(lze přepsat na Γ-funkci) $\quad \frac{\Gamma(\frac{m}{2}+1)\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{m+3}{2})}$