Continuity equation and vacuum regions in compressible flows

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joint work with A. Novotný (Toulon)

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Introduction

Consider compressible Navier–Stokes equations:

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \operatorname{div}(\mathbb{S}(\nabla \mathbf{u})) + \nabla p(\varrho) &= \varrho \mathbf{f}
\end{aligned}
$$

Open problem: regularity of solutions for regular data
Known: it is connected with the presence of vacuum regions
Questions: if the vacuum zone develops, is it possible that it appears instantaneously on a "large" set?
What are the conditions to exclude the presence of vacuum regions without assuming small data or short time interval?

Aim of this talk is to discuss these problems from the point of view of properties of continuity equation

Based on the paper:

A. Novotný, M. Pokorný: *Continuity equation and vacuum regions in compressible flows.* Journal of Evolution Equations, online first. https://doi.org/10.1007/s00028-021-00704-3.

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Outline of the talk

- Solutions to continuity (transport) equation
- Main results
- Applications to the compressible Navier–Stokes equations
- Auxiliary results for continuity (transport) equation
- Proof of the main theorems

Continuity (transport) equation I

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0 \qquad \text{in } I \times \Omega, \\ \varrho(0, \cdot) &= \varrho_0 \qquad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset R^d$, $d \geq 2$ is an open set and $I = (0, T)$, $0 < T < \infty$.
Assume: $\mathbf{u} \in L^1(I \times \Omega; R^d)$, $\operatorname{div} \mathbf{u} \in L^1(I \times \Omega)$
Then the function $\varrho \in L^1(I \times \Omega)$ with $\varrho \mathbf{u} \in L^1(I \times \Omega; R^d)$ is

Distributional solution to the continuity equation, if

$$\int_0^T \int_\Omega (\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla \varphi) \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{2}$$

holds for arbitrary $\varphi \in C_c^\infty(I \times \Omega)$.

Weak solution to the continuity equation, if (2) holds for any $\varphi \in C_c^\infty(I \times \overline{\Omega})$.

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Continuity equation II

Time integrated distributional solution to the continuity equation, if $\varrho \in C_{weak}(\bar{I}; L^1(\Omega))$ and

$$\int_\Omega (\varrho\varphi)(\tau,\cdot)\mathrm{d}x - \int_\Omega (\varrho\varphi)(0,\cdot)\mathrm{d}x = \int_0^\tau \int_\Omega (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0 \quad (3)$$

holds for arbitrary $\varphi \in C_c^\infty(\bar{I}\times\Omega)$ and any $\tau \in \bar{I}$.

Time integrated weak solution to the continuity equation, if $\varrho \in C_{weak}(\bar{I}; L^1(\Omega))$ and (3) holds for any $\varphi \in C^\infty(\bar{I}\times\bar{\Omega})$ and any $\tau \in \bar{I}$.

Renormalized distributional solution to the continuity equation, if in addition to (2)

$$\int_0^T \int_\Omega \Big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla\varphi - \big(b'(\varrho)\varrho - b(\varrho)\big)\,\mathrm{div}\,\mathbf{u}\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \quad (4)$$

holds for arbitrary $\varphi \in C_c^\infty(I\times\Omega)$ and all $b \in C^1([0,\infty))$, $b' \in C_c([0,\infty))$.

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Renormalized weak solution to the continuity equation, if it is a weak solution and (4) holds for any $\varphi \in C_c^\infty(I \times \overline{\Omega})$ and any b specified above.

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$$\int_\Omega (b(\varrho)\varphi)(\tau,\cdot)\mathrm{d}x - \int_\Omega (b(\varrho)\varphi)(0,\cdot)\mathrm{d}x$$
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Continuity (transport) equation IV

$$\partial_t s + \mathbf{u} \cdot \nabla s = 0 \qquad \text{in } I \times \Omega,$$
$$s(0, \cdot) = s_0 \qquad \text{in } \Omega \tag{6}$$

Let $\mathbf{u} \in L^1(I \times \Omega; R^d)$, $\operatorname{div} \mathbf{u} \in L^1(I \times \Omega)$. Then for $s \in L^1(I \times \Omega)$, $s\mathbf{u}$ and $s \operatorname{div} \mathbf{u} \in L^1(I \times \Omega)$ we define by analogy the Distributional, Weak, Time integrated distributional, etc. solutions to the transport equation. Recall that the distributional formulation means

$$\int_0^T \int_\Omega (s\partial_t \varphi + s\mathbf{u} \cdot \nabla \varphi + s \operatorname{div} \mathbf{u} \varphi) \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{7}$$

for all $\varphi \in C_c^\infty(I \times \Omega)$ and the renormalized distributional formulation is

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Main results I

Theorem (1)

Let $\Omega \subset R^d$ *be a bounded domain. Let* $1 \le q, p \le \infty$ *and* $\mathbf{u} \in L^p(0, T; W^{1,q}(\Omega; R^d))$. *Let*

$$0 \le \varrho \in C_{weak}(\bar{I}; L^\gamma(\Omega)),\ \gamma > 1 \tag{9}$$

be a renormalized time integrated weak solution to the continuity equation (1) with transporting velocity $\mathbf{u}$.

Then the map $t \mapsto s_\varrho(t,\cdot) := 1_{\{x\in\Omega|\varrho(t,x)=0\}}(\cdot)$ *belongs to* $C([0,T]; L^r(\Omega))$ *with any* $1 \le r < \infty$ *and it is a time integrated renormalized weak solution of the transport equation (6) with transporting velocity* $\mathbf{u}$. *In particular,*

$$|\{x \in \Omega | \varrho(t,x) = 0\}|_d \in C([0,T]). \tag{10}$$

In the above $|A|_d$ *denotes the d-dimensional Lebesgue measure of the set A.*

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Main results II

Theorem (2)

Let $\Omega \subset R^d$ be a bounded Lipschitz domain. Let

$$1 \le q, p, \alpha, \beta \le \infty, \ (q, \beta) \neq (1, \infty), \ \frac{1}{\beta} + \frac{1}{q} \le 1, \ \frac{1}{\alpha} + \frac{1}{p} \le 1. \tag{11}$$

Let ϱ from class (9) be a renormalized time integrated weak solution to the continuity equation (1) with transporting velocity $\mathbf{u} \in L^p(0, T; W_0^{1,q}(\Omega; R^d))$. Let

$$0 \le R \in L^\infty(0, T; L^{\widetilde{\gamma}}(\Omega)) \cap L^\alpha(0, T; L^\beta(\Omega)), \ \widetilde{\gamma} > 1 \tag{12}$$

be a distributional solution to the continuity equation (1) with the same transporting velocity $\mathbf{u}$.

Then

1/ Function R belongs to

$$R \in C_{weak}([0, T]; L^{\widetilde{\gamma}}(\Omega)) \cap C([0, T]; L^r(\Omega)), \ 1 \le r < \widetilde{\gamma} \tag{13}$$

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Main results III

Theorem (2 cont.)

2/ The map $t \mapsto (s_\varrho R)(t)$ *belongs to* $C([0,T];L^r(\Omega))$ *with any* $1 \le r < \widetilde{\gamma}$ *and it is a renormalized time integrated weak solution of the continuity equation (1) (with the same transporting velocity). In particular,*

$$\int_\Omega (s_\varrho R)(t,\cdot)\mathrm{d}x = \int_\Omega (s_\varrho R)(0,\cdot)\mathrm{d}x \tag{14}$$

for all $t \in [0,T]$.

3/ If further $\varrho(0,\cdot) > 0$ *a.e. in* Ω*, then, up to sets of* d*-dimensional Lebesgue measure zero, for all* $t \in (0,T]$

$$\{x \in \Omega | \varrho(t,x) = 0\} \subset \{x \in \Omega | R(t,x) = 0\}.$$

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Comments and corollaries I

First theorem:

The result means that it is not possible that the vacuum region appears at one instant on a large portion of the domain.

Second theorem:

The result indicates that either there are not too many solutions of the continuity equation with the given velocity field, or there is no vacuum created, if there was no vacuum region at the initial time.

More precisely, we have the following corollaries:

Corollary (4)

Let q, p, α, β verify conditions (11) and $\widetilde{\gamma}, \gamma > 1$. Let Ω, ϱ, $\mathbf{u}$ verify assumptions of Theorem 2, where $\varrho(0,x) > 0$. (In particular, ϱ is a renormalized time integrated weak solution of the continuity equation (1) with transporting velocity $\mathbf{u}$.)

Let $\tau \in (0,T)$. Suppose that continuity equation (1) with transporting velocity $\mathbf{u}$ admits at least one distributional solution R belonging to class (12) which does not admit in Ω a vacuum at time τ, i.e. $R(\tau) > 0$ a.e. in Ω. Then ϱ does not admit a vacuum at time τ, i.e.

$$|\{x \in \Omega | \varrho(\tau, x) = 0\}|_d = 0.$$

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Let $\tau \in (0,T)$. Suppose that continuity equation (1) with transporting velocity $\mathbf{u}$ admits at least one distributional solution R belonging to class (12) which does not admit in Ω a vacuum at time τ, i.e. $R(\tau) > 0$ a.e. in Ω. Then ϱ does not admit a vacuum at time τ, i.e.

$$|\{x \in \Omega | \varrho(\tau,x) = 0\}|_d = 0.$$

Comments and corollaries I

First theorem:

The result means that it is not possible that the vacuum region appears at one instant on a large portion of the domain.

Second theorem:

The result indicates that either there are not too many solutions of the continuity equation with the given velocity field, or there is no vacuum created, if there was no vacuum region at the initial time.

More precisely, we have the following corollaries:

Corollary (4)

Let q, p, α, β verify conditions (11) and $\widetilde{\gamma}, \gamma > 1$. Let Ω, ϱ, $\mathbf{u}$ verify assumptions of Theorem 2, where $\varrho(0,x) > 0$. (In particular, ϱ is a renormalized time integrated weak solution of the continuity equation (1) with transporting velocity $\mathbf{u}$.)
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Corollary (5)

Let (q, p, α, β) *and* $(q, p, \tilde{\alpha}, \tilde{\beta})$ *verify conditions (11) and* $\widetilde{\gamma}, \gamma > 1$. *Let* Ω, $\mathbf{u}$ *be as in Theorem 2. Suppose that* ϱ *belongs to class*

$$0 \le \varrho \in L^{\infty}(I; L^{\gamma}(\Omega)) \cap L^{\alpha}(I; L^{\beta}(\Omega)) \tag{15}$$

while R *belongs to class* $(15)_{\tilde{\alpha},\tilde{\beta},\tilde{\gamma}}$, *and that each of* ϱ *and* R *represents a distributional solution to the continuity equation (1) with the transporting velocity* $\mathbf{u}$. *Then,* $\varrho \in C_{weak}([0,T]; L^{\gamma}(\Omega))$, $R \in C_{weak}([0,T]; L^{\tilde{\gamma}}(\Omega))$, *and they are both renormalized time integrated solutions of the continuity equation (1). Moreover, if* $\varrho(0,\cdot) > 0$ *and* $R(0,\cdot) > 0$ *a.e. in* Ω *then up to sets of d-dimensional Lebesgue measure zero, for all* $t \in (0,T]$

$$\{x|\varrho(t,x) = 0\} = \{x|R(t,x) = 0\}.$$

Comments and corollaries II

Corollary (5)

Let (q, p, α, β) *and* $(q, p, \tilde{\alpha}, \tilde{\beta})$ *verify conditions (11) and* $\widetilde{\gamma}, \gamma > 1$. *Let* Ω, $\mathbf{u}$ *be as in Theorem 2. Suppose that* ϱ *belongs to class*

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Corollary (6)

Let $q, \alpha, \beta, \gamma, \widetilde{\gamma}$ *verify assumptions of Corollary 4 with* $p = \infty$*.*
Let Ω, ϱ, $\mathbf{u}$ *verify assumptions of Corollary 4. (In particular,* $0 \leq \varrho$ *is a renormalized time integrated weak solution of the continuity equation (1) with transporting velocity* $\mathbf{u}$ *and* $\varrho(0,x) > 0$*.) We assume that* $\mathbf{u}$ *is time independent, i.e.* $\mathbf{u} = \mathbf{u}(x)$, $\mathbf{u} \in W_0^{1,q}(\Omega; R^d)$*.*
Suppose that continuity equation (1) with transporting velocity $\mathbf{u}$ *admits at least one (local in time) distributional solution* R *on* $(0, T') \times \Omega$ *with some* $T' > 0$ *belonging to class* $(12)_{T=T'}$ *which does not admit in* Ω *a vacuum at time* $\tau \in (0, T')$*, i.e. there exists* $\tau \in (0, T')$ *such that* $R(\tau) > 0$ *a.e. in* Ω*.*
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$$\forall t \in [0, T], \ |\{x \in \Omega | \varrho(t,x) = 0\}|_d = 0.$$

Comments and corollaries III

Corollary (6)

Let $q, \alpha, \beta, \gamma, \widetilde{\gamma}$ *verify assumptions of Corollary* 4 *with* $p = \infty$.
Let Ω, ϱ, $\mathbf{u}$ *verify assumptions of Corollary* 4. *(In particular,* $0 \le \varrho$ *is a renormalized time integrated weak solution of the continuity equation (1) with transporting velocity* $\mathbf{u}$ *and* $\varrho(0, x) > 0$.*) We assume that* $\mathbf{u}$ *is time independent, i.e.* $\mathbf{u} = \mathbf{u}(x)$, $\mathbf{u} \in W_0^{1,q}(\Omega; R^d)$.
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Application to the compressible NS equations I

Compressible Navier–Stokes equations in barotropic regime:

$$\begin{aligned} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) &= 0 \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho) &= \mathrm{div}\, \mathbb{S}(\nabla \mathbf{u}) + \varrho \mathbf{f} \end{aligned} \tag{16}$$

which we consider in $(0, T) \times \Omega$, together with the initial conditions in Ω

$$\varrho(0, \cdot) = \varrho_0, \qquad (\varrho \mathbf{u})(0, \cdot) = \mathbf{m}_0 \tag{17}$$

and so called no-slip boundary condition on $(0, T) \times \partial\Omega$

$$\mathbf{u}(t, x) = \mathbf{0}. \tag{18}$$

The homogeneous boundary condition (18) can be replaced by Navier (slip) boundary conditions or by periodic boundary conditions if Ω is a periodic cell.

$$\mathbb{S}(\nabla \mathbf{u}) = \mu \Big(\nabla \mathbf{u} + \nabla \mathbf{u}^t - \frac{2}{d} \mathrm{div}\, \mathbf{u} \mathbb{I} \Big) + \lambda \mathrm{div}\, \mathbf{u} \mathbb{I}. \tag{19}$$

The viscosity coefficients are assumed to be constant: $\mu > 0$ and $\lambda \geq 0$. Function $\varrho \mapsto p(\varrho)$ denotes the pressure. One supposes that

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Application to the compressible NS equations II

Definition

Let $\varrho_0 \in L^\gamma(\Omega)$, $0 \leq \varrho_0 \in L^\gamma(\Omega)$ a.e. in Ω, $\gamma > 1$, $r > 1$, $(\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0 \in L^1(\Omega; R^d)$ and $\mathbf{f} \in L^\infty((0,T)\times\Omega; R^d)$. A couple $(\varrho, \mathbf{u})$ is a renormalized weak solution to the initial boundary value problem (16–18) iff:

1. The couple $(\varrho, \mathbf{u})$ belongs to functional spaces

$$0 \leq \varrho \in C_{weak}(\bar{I}; L^\gamma(\Omega)),\ \mathbf{u} \in L^2(I; W_0^{1,2}(\Omega; R^d)),\ p(\varrho) \in L^1(Q),$$

$$\varrho\mathbf{u} \in C_{weak}(\bar{I}; L^r(\Omega; R^d)),\ \varrho(\mathbf{u}\otimes\mathbf{u}), \mathbb{S}(\nabla\mathbf{u}) \in L^1((0,T)\times\Omega; R^{d\times d}).$$

2. ϱ is a time integrated renormalized weak solution to the continuity equation $(16)_1$ with transporting velocity $\mathbf{u}$.
3. The couple $(\varrho, \mathbf{u})$ verifies the momentum equation $(16)_2$ in the following sense:

$$\int_0^T \int_\Omega \Big(-\varrho\mathbf{u}\cdot\partial_t\varphi - \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla\varphi + \mathbb{S}(\nabla\mathbf{u}) : \nabla\varphi - \varrho^\gamma \operatorname{div}\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_\Omega \mathbf{m}_0\cdot\varphi(0,\cdot)\mathrm{d}x = \int_0^T \int_\Omega \varrho\mathbf{f}\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{20}$$

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Application to the compressible NS equations III

Corollary (8)

Let $\gamma > 1$ *verify* $\gamma \geq \frac{2d}{d+2}$. *Then the claims of Theorems 1 and 2 (and Corollaries 4, 5 and 6) hold for any renormalized weak solution to the compressible Navier–Stokes equations specified in Definition 7 (with* $p = 2$ *in Theorem 1).*

Note that existence of such solutions is known, e.g., for $p(\varrho) = \varrho^\gamma$ and $\gamma > 1$ if $d = 2$, $\gamma > \frac{3}{2}$ if $d = 3$.

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Preliminaries I

Proposition (9)

Let $\Omega \subset R^d$, $d \geq 2$ *be a bounded domain with Lipschitz boundary. Let* $\mathbf{u} \in L^p(I; W^{1,q}(\Omega; R^d))$, $1 \leq p, q \leq \infty$. *Suppose that*

$$0 \leq \varrho \in L^\infty(I; L^\gamma(\Omega)), \; \gamma > 1. \tag{21}$$

Then the following statements are true:

1. *If* ϱ *is a* renormalized distributional solution *of the continuity equation with transporting velocity* $\mathbf{u}$, *then function* ϱ *and functions* $b(\varrho)$ *with any* b *from the definition of the renormalized solution belong to* $C_{weak}(\bar{I}; L^\gamma(\Omega)) \cap C(\bar{I}; L^r(\Omega))$ *for any* $1 \leq r < \gamma$ *and* ϱ *is a* renormalized time integrated distributional solution *of the continuity equation with transporting velocity* $\mathbf{u}$.
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A similar statement holds for solutions to the transport equation.

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Proposition (10)

Let $\Omega \subset R^d$, $d \geq 2$ *be a bounded domain with Lipschitz boundary. Further, let* $\mathbf{u} \in L^p(I; W^{1,q}(\Omega; R^d))$, $0 \leq \varrho \in L^\alpha(I; L^\beta(\Omega))$, *where* p, q, α, β *satisfy*

$$1 \leq q, p, \alpha, \beta \leq \infty, \ (q, \beta) \neq (1, \infty), \ \frac{1}{\beta} + \frac{1}{q} \leq 1, \ \frac{1}{\alpha} + \frac{1}{p} \leq 1.$$

1/ Assume that ϱ *is a* distributional solution *of the continuity equation with transporting velocity* $\mathbf{u}$. *Then the following statements are true:*

1.1 ϱ *is a* renormalized distributional solution.

1.2 *If moreover*

$$\mathbf{u} \in L^p(I; W_0^{1,q}(\Omega; R^d)), \tag{22}$$

then ϱ *is a* renormalized weak solution *of the continuity equation.*

Preliminaries II

Proposition (10)

Let $\Omega \subset R^d$, $d \geq 2$ *be a bounded domain with Lipschitz boundary. Further, let* $\mathbf{u} \in L^p(I; W^{1,q}(\Omega; R^d))$, $0 \leq \varrho \in L^\alpha(I; L^\beta(\Omega))$, *where* p, q, α, β *satisfy*

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Preliminaries III

Proposition (10 cont.)

2/ Assume that ϱ belongs to class

$$\varrho \in C_{weak}(\bar{I}; L^{\gamma}(\Omega)) \text{ with some } \gamma > 1 \tag{23}$$

and is a time integrated distributional solution *of the continuity equation with transporting velocity* $\mathbf{u}$*. Then the following statements are true:*

2.1 *Functions ϱ and $b(\varrho)$ belong to $C_{weak}(\bar{I}; L^{\gamma}(\Omega)) \cap C(\bar{I}; L^{r}(\Omega))$ for any $1 \leq r < \gamma$. Moreover, ϱ is a renormalized time integrated distributional solution.*

2.2 *If moreover $\mathbf{u}$ has zero traces, then ϱ is a* renormalized time integrated weak solution *of the continuity equation.*

A similar statement holds for solutions to the transport equation.

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A similar statement holds for solutions to the transport equation.

Proposition (12)

Let Ω be a bounded domain with Lipschitz boundary. Suppose that

$$1 \leq q, p, \alpha_\varrho, \beta_\varrho, \alpha_s, \beta_s \leq \infty, \ (q, \beta_\varrho) \neq (1, \infty), \ (q, \beta_s) \neq (1, \infty),$$

$$\frac{1}{\alpha_\varrho} + \frac{1}{\alpha_s} + \frac{1}{p} \leq 1, \ \frac{1}{r_\varrho} + \frac{1}{r_s} + \frac{1}{q} \leq 1,$$

where

$$r_\varrho \left\{ \begin{array}{c} \in [1, \infty) \text{ if } q > 1 \text{ and } \beta_\varrho = \infty \\ = \beta_\varrho \text{ otherwise} \end{array} \right\}, \ r_s \left\{ \begin{array}{c} \in [1, \infty) \text{ if } q > 1 \text{ and } \beta_s = \infty \\ = \beta_s \text{ otherwise} \end{array} \right\}.$$

Let

$$\varrho \in L^{\alpha_\varrho}(I; L^{\beta_\varrho}(\Omega)), \ s \in L^{\alpha_s}(I; L^{\beta_s}(\Omega)), \ \mathbf{u} \in L^p(I; W^{1,q}(\Omega; R^d)).$$

Then there holds:

Preliminaries V

Proposition (12 cont.)

1. *Assume additionally that*

$$\frac{1}{t_\varrho} + \frac{1}{t_s} + \frac{1}{p} \le 1,$$

where

$$t_\varrho \left\{ \begin{array}{c} \in [1,\infty) \text{ if } p>1 \text{ and } \alpha_\varrho = \infty \\ = \alpha_\varrho \text{ otherwise} \end{array} \right\},\ t_s \left\{ \begin{array}{c} \in [1,\infty) \text{ if } p>1 \text{ and } \alpha_s = \infty \\ = \alpha_s \text{ otherwise} \end{array} \right\}.$$

If ϱ is a distributional (resp. weak) solution of the continuity equation and s a distributional (resp. weak) solution of the transport equation with transporting velocity $\mathbf{u}$, then ϱs is a renormalized distributional (resp. weak) solution of the continuity equation with the same transporting velocity $\mathbf{u}$.

2. *If $\varrho \in C_{weak}(\bar{I}; L^{\gamma_\varrho}(\Omega))$ is a time integrated distributional (resp. weak) solution of the continuity equation and $s \in C_{weak}(\bar{I}; L^{\gamma_s}(\Omega))$ a time integrated distributional (resp. weak) solution of the transport equation with transporting velocity $\mathbf{u}$ (where $1 < \gamma_\varrho, \gamma_s \le \infty$, $\frac{1}{\gamma_\varrho} + \frac{1}{\gamma_s} := \frac{1}{\gamma} < 1$), then $\varrho s \in C(\bar{I}; L^r(\Omega))$, $1 \le r < \gamma$ is a renormalized distributional (resp. weak) solution of the continuity equation with the same transporting velocity $\mathbf{u}$.*

Preliminaries V

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Preliminaries V

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Proof of Theorem 1

- $(\varrho, \mathbf{u})$ is renormalized solution, use $b_\delta(\varrho) := \frac{\delta}{\delta+\varrho}$ with $\delta > 0$ in the renormalized formulation

- $$\partial_t\Big(\frac{\delta}{\delta+\varrho}\Big) + \operatorname{div}\Big(\frac{\delta}{\delta+\varrho}\mathbf{u}\Big) - \frac{\delta}{\delta+\varrho}\operatorname{div}\mathbf{u} + \frac{\delta\varrho}{(\delta+\varrho)^2}\operatorname{div}\mathbf{u} = 0.$$

- $\varrho \in C_{weak}([0,T]; L^\gamma(\Omega)) \cap C([0,T]; L^r(\Omega))$, $1 \le r < \gamma$, thus $\frac{\delta}{\delta+\varrho} \in C([0,T]; L^r(\Omega))$, $1 \le r < \infty$.

- weak time integrated formulation of the renormalized equation

$$\int_\Omega \frac{\delta}{\delta+\varrho(t,\cdot)}\varphi(t,\cdot)\mathrm{d}x - \int_\Omega \frac{\delta}{\delta+\varrho(0,\cdot)}\varphi(0,\cdot)\mathrm{d}x$$
$$-\int_0^T\int_\Omega \Big(\frac{\delta}{\delta+\varrho}\partial_t\varphi + \frac{\delta}{\delta+\varrho}\mathbf{u}\cdot\nabla\varphi + \Big(\frac{\delta}{\delta+\varrho} - \frac{\delta\varrho}{(\delta+\varrho)^2}\Big)\operatorname{div}\mathbf{u}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0 \quad (24)$$

for all $\varphi \in C^\infty([0,T]\times\overline{\Omega})$

Proof of main theorems I

Proof of Theorem 1

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- $\varrho \in C_{weak}([0,T]; L^\gamma(\Omega)) \cap C([0,T]; L^r(\Omega))$, $1 \le r < \gamma$, thus $\frac{\delta}{\delta+\varrho} \in C([0,T]; L^r(\Omega))$, $1 \le r < \infty$.
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Proof of main theorems II

- weak formulation with constant test function

$$\int_\Omega \frac{\delta}{\delta + \varrho(\tau,\cdot)}\mathrm{d}x - \int_\Omega \frac{\delta}{\delta + \varrho(0,\cdot)}\mathrm{d}x = \int_0^\tau \int_\Omega \left(\frac{\delta}{\delta+\varrho} - \frac{\delta\varrho}{(\delta+\varrho)^2} \right) \operatorname{div}\mathbf{u}\,\mathrm{d}x\ \mathrm{d}t \tag{25}$$

for all $\tau \in (0, T]$

- let $\delta \to 0+$ both in (24) and (25) to get (recall $\frac{\delta}{\delta+\varrho(t,x)} = 1$ provided $\varrho(t,x) = 0$)

$$\int_0^T \int_\Omega (s_\varrho \partial_t \varphi + s_\varrho \mathbf{u}\cdot\nabla\varphi + s_\varrho \operatorname{div}\mathbf{u}\varphi)\,\mathrm{d}x\ \mathrm{d}t = 0 \tag{26}$$

for all $\varphi \in C_c^\infty((0,T)\times\Omega)$, and

$$\int_\Omega s_\varrho(\tau,\cdot)\,\mathrm{d}x - \int_\Omega s_\varrho(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_\Omega s_\varrho \operatorname{div}\mathbf{u}\,\mathrm{d}x\ \mathrm{d}t, \tag{27}$$

where s_ϱ denotes the characteristic function of the set, where $\varrho = 0$

- $(s_\varrho, \mathbf{u})$ fulfills the transport equation; recall $\int_\Omega s_\varrho(\tau,\cdot)\mathrm{d}x = |\{x\in\Omega; \varrho(\tau,x) = 0\}|_d$.

Proof of main theorems II

- weak formulation with constant test function

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Proof of main theorems III

- subtract equations (27) for $\tau := \tau_1$ and $\tau := \tau_2$ to get

$$\Big| \int_{\tau_1}^{\tau_2} s_\varrho \operatorname{div} \mathbf{u} \, \mathrm{d}x \; \mathrm{d}t \Big| \to 0 \qquad \text{for } \tau_1 \to \tau_2.$$

-

$$|\{x \in \Omega; \varrho(\tau, x) = 0\}|_d \in C([0, T]).$$

- furthermore $s_\varrho \in C_{weak}([0, T]; L^r(\Omega))$ for any $1 \leq r < \infty$, and

$$s_\varrho \in C([0, T]; L^r(\Omega)), \quad \text{for any} \quad 1 \leq r < \infty.$$

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- $$|\{x \in \Omega; \varrho(\tau, x) = 0\}|_d \in C([0, T]).$$

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$$s_\varrho \in C([0, T]; L^r(\Omega)), \quad \text{for any} \quad 1 \le r < \infty.$$

Proof of main theorems IV

Proof of Theorem 2

• assumptions of the first theorem are fulfilled, thus $s_\varrho = \chi_{\{x\in\Omega;\varrho(t,x)=0\}}$ solves (in the time integrated renormalized weak sense) the transport equation

$$\partial_t s_\varrho + \mathbf{u}\cdot\nabla s_\varrho = 0$$

and $s_\varrho \in C([0,T];L^r(\Omega)) \cap L^\infty((0,T)\times\Omega)$, $1\le r<\infty$, arbitrary

• R is also a renormalized up to the boundary solution to the continuity equation and $R\in C([0,T];L^r(\Omega))$, $1\le r<\widetilde{\gamma}$. Hence $s_\varrho R\in C([0,T];L^r(\Omega))$ for $1\le r<\widetilde{\gamma}$. By Propositions 12 and 10 $s_\varrho R$ fulfills the continuity equation

$$\partial_t(s_\varrho R) + \operatorname{div}(s_\varrho R\mathbf{u}) = 0 \tag{28}$$

in the time integrated renormalized weak sense.

• then

$$\int_\Omega R(t,\cdot)\chi_{\{x\in\Omega;\varrho(t,x)=0\}}\mathrm{d}x = \int_\Omega R(0,\cdot)\chi_{\{x\in\Omega;\varrho(0,x)=0\}}\mathrm{d}x$$

The last claim is evident.

Proof of Theorem 2

• assumptions of the first theorem are fulfilled, thus $s_\varrho = \chi_{\{x\in\Omega;\varrho(t,x)=0\}}$ solves (in the time integrated renormalized weak sense) the transport equation

$$\partial_t s_\varrho + \mathbf{u}\cdot\nabla s_\varrho = 0$$

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THANK YOU FOR

YOUR ATTENTION