Homogenization of compressible Navier–Stokes–Fourier system in domains with tiny holes

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Joint work with Yong Lu (Nanjing) and Emil Skříšovský (Praha)

Prague Sum Workshop, September 25, 2021

Dedicated to the memory on Antonín Novotný

Olomouc 1999

Dedicated to the memory on Antonín Novotný

Kácov 2015

The domain

Let $\varepsilon > 0$ be a small number, it measures the mutual distance between the holes

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{n=1}^{N(\varepsilon)} \overline{T}_{n,\varepsilon}, \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded C^2-domain and $\{T_{n,\varepsilon}\}_{n=1}^{N(\varepsilon)}$ are C^2-domains of the diameter comparable to ε^α for some $\alpha \geq 1$ such that there exist δ_0, δ_1 and δ_2 positive for which

$$T_{n,\varepsilon} = x_{\varepsilon,n} + \varepsilon^\alpha T^0_{n,1} \subset B_{\delta_0 \varepsilon^\alpha}(x_{n,\varepsilon}) \subset B_{2\delta_0 \varepsilon^\alpha}(x_{n,\varepsilon}) \subset B_{\delta_1 \varepsilon}(x_{n,\varepsilon}) \subset B_{\delta_2 \varepsilon}(x_{n,\varepsilon}) \subset \Omega. \tag{2}$$

The balls $B_{\delta_2 \varepsilon}(x_{n,\varepsilon})$ centred at $x_{\varepsilon,n}$ with diameter $\delta_2 \varepsilon$ are pairwise disjoint and we assume that the domains $\{T^0_{n,1}\}_{n=1}^{N(\varepsilon)}$ are uniformly C^2-domains.

We want to study homogenization for tiny holes in case of the steady (joint work with Y. Lu) and evolutionary (joint work with E. Skříšovský) compressible Navier–Stokes–Fourier system.

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- $1 \leq \alpha < 3$ Limit system is the Darcy system
- $\alpha = 3$ Limit system is the Brinkman system
- $\alpha > 3$ Limit system is the original one

Similar results for evolutionary case were proved in

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In both cases, one gets porous medium equation with Darcy law.

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Homogenization III

Similarly, for large γ in the evolutionary case

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Outline of the talk

The steady case

- The problem in Ω_ε
- Main result
- Proof of the main result
    - Estimates independent of ε for the functions defined in Ω_ε
    - Extension of functions to Ω
    - Estimates of extended functions in Ω
    - Limit passage
    - Properties of temperature, entropy inequality

The evolutionary case (main result)

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The evolutionary case (main result)

Original steady system

We consider in Ω_ε given by (1) and (2):

$$\operatorname{div}(\varrho \mathbf{u}) = 0, \tag{3}$$

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta) - \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \varrho \mathbf{f}, \tag{4}$$

$$\operatorname{div} \left(\varrho E \mathbf{u} + p \mathbf{u} - \mathbb{S}(\vartheta, \nabla \mathbf{u}) \mathbf{u} + \mathbf{q} \right) = \varrho \mathbf{f} \cdot \mathbf{u}. \tag{5}$$

We complete the system by the boundary conditions on $\partial\Omega_\varepsilon$

$$\mathbf{u} = \mathbf{0}, \tag{6}$$

$$\mathbf{q} \cdot \mathbf{n} + L(\vartheta - \vartheta_0) = 0 \tag{7}$$

and by prescribing the total mass

$$\int_{\Omega_\varepsilon} \varrho \, \mathrm{d}x = M_\varepsilon > 0. \tag{8}$$

The unknown quantities are the density $\varrho \colon \Omega_\varepsilon \to \mathbb{R}_{\geq 0}$, the velocity $\mathbf{u} \colon \Omega_\varepsilon \to \mathbb{R}^3$ and the temperature $\vartheta \colon \Omega_\varepsilon \to \mathbb{R}_+$.

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Constitutive relations I

Pressure:

$$p(\varrho, \vartheta) = \varrho^\gamma + \varrho\vartheta, \qquad \gamma > 2. \tag{9}$$

Stress tensor:

$$\mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\Big(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u}\, \mathbb{I}\Big) + \nu(\vartheta) \operatorname{div} \mathbf{u}\, \mathbb{I}, \tag{10}$$

where the viscosity coefficients are continuous functions of the temperature on $\mathbb{R}_+$, the shear viscosity $\mu(\cdot)$ is moreover globally Lipschitz continuous, and

$$C_1(1+\vartheta) \le \mu(\vartheta) \le C_2(1+\vartheta), \qquad 0 \le \nu(\vartheta) \le C_2(1+\vartheta). \tag{11}$$

Heat flux:

$$\mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{12}$$

where the heat conductivity

$$C_3(1+\vartheta^m) \le \kappa(\vartheta) \le C_4(1+\vartheta^m), \qquad m > 2. \tag{13}$$

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Constitutive relations II

Total energy:

$$E = e + \frac{1}{2}|\mathbf{u}|^2,$$

and the specific internal energy e fulfils the Gibbs relation

$$\frac{1}{\vartheta}\left(De + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right)\right) = Ds(\varrho,\vartheta), \tag{14}$$

$$e(\varrho,\vartheta) = c_v\vartheta + \frac{\varrho^{\gamma-1}}{\gamma-1}. \tag{15}$$

The entropy fulfils formally

$$\operatorname{div}\left(\varrho s\mathbf{u} + \frac{\mathbf{q}}{\vartheta}\right) = \sigma = \frac{\mathbb{S} : \nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}.$$

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Weak solution I

Continuity equation:

$$\int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \tag{16}$$

for all $\psi \in C_c^1(\mathbb{R}^3)$, where ϱ and $\mathbf{u}$ are extended by zero outside of Ω_ε.

Renormalized continuity equation:

$$\int_{\mathbb{R}^3} \Big(b(\varrho)\mathbf{u} \cdot \nabla \psi + (b(\varrho) - \varrho b'(\varrho)) \operatorname{div} \mathbf{u} \psi \Big) \, \mathrm{d}x = 0 \tag{17}$$

for all $\psi \in C_c^1(\mathbb{R}^3)$ and all $b \in C^1([0,\infty))$ such that $b' \in C_0([0,\infty))$, and both ϱ and $\mathbf{u}$ are extended by zero outside of Ω_ε.

Momentum equation:

$$\int_{\Omega_\varepsilon} \Big(-\varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} - p(\varrho, \vartheta) \operatorname{div} \boldsymbol{\varphi} + \mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \boldsymbol{\varphi} \Big) \mathrm{d}x = \int_{\Omega_\varepsilon} \varrho \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \tag{18}$$

for all $\boldsymbol{\varphi} \in C_c^1(\Omega_\varepsilon; \mathbb{R}^3)$.

Weak solution I

Continuity equation:

$$\int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0 \tag{16}$$

for all $\psi \in C_c^1(\mathbb{R}^3)$, where ϱ and $\mathbf{u}$ are extended by zero outside of Ω_ε.

Renormalized continuity equation:

$$\int_{\mathbb{R}^3} \Big(b(\varrho)\mathbf{u} \cdot \nabla \psi + (b(\varrho) - \varrho b'(\varrho)) \operatorname{div} \mathbf{u} \psi \Big) \, \mathrm{d}x = 0 \tag{17}$$

for all $\psi \in C_c^1(\mathbb{R}^3)$ and all $b \in C^1([0,\infty))$ such that $b' \in C_0([0,\infty))$, and both ϱ and $\mathbf{u}$ are extended by zero outside of Ω_ε.

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Weak solution II

Total energy balance:

$$-\int_{\Omega_\varepsilon}\Big(\varrho E\mathbf{u}+p(\varrho,\vartheta)\mathbf{u}-\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u}+\mathbf{q}\Big)\cdot\nabla\psi\,\mathrm{d}x+\int_{\partial\Omega_\varepsilon}L(\vartheta-\vartheta_0)\psi\,\mathrm{d}S=\int_{\Omega_\varepsilon}\varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x \tag{19}$$

for all $\psi\in C^1(\overline{\Omega_\varepsilon})$.

Entropy inequality:

$$\begin{aligned}&\int_{\Omega_\varepsilon}\Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u})}{\vartheta}-\frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2}\Big)\psi\,\mathrm{d}x+\int_{\partial\Omega_\varepsilon}\frac{L\vartheta_0}{\vartheta}\psi\,\mathrm{d}S\\ &\leq L\int_{\partial\Omega_\varepsilon}\psi\,\mathrm{d}S+\int_{\Omega_\varepsilon}\Big(-\frac{\mathbf{q}\cdot\nabla\psi}{\vartheta}-\varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\psi\Big)\,\mathrm{d}x\end{aligned} \tag{20}$$

for all $\psi\in C^1(\overline{\Omega_\varepsilon})$, non-negative.

Definition

We say that the triple $(\varrho,\,\mathbf{u},\,\vartheta)$, $\varrho\geq 0$ and $\vartheta>0$ a.e. in Ω_ε, is a renormalized weak entropy solution to our problem (3)–(15), if $\varrho\in L^\gamma(\Omega_\varepsilon)$, $\int_{\Omega_\varepsilon}\varrho\,\mathrm{d}x=M_\varepsilon$, $\mathbf{u}\in W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^3)$, $\vartheta^{\frac{m}{2}}$ and $\log\vartheta\in W^{1,2}(\Omega_\varepsilon)$ such that $\varrho|\mathbf{u}|^3$, $|\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u}|$ and $p(\varrho,\mathbf{u})|\mathbf{u}|\in L^1(\Omega_\varepsilon)$ and the relations (16), (17), (18), (19) and (20) are fulfilled with test functions specified above.

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Existence result in perforated domains

Theorem

Let $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $\vartheta_0 \in L^1(\partial\Omega_\varepsilon)$, $\vartheta_0 \geq T_0 > 0$ *a.e. on* $\partial\Omega_\varepsilon$, $L > 0$, $M_\varepsilon > 0$. *Let* $\gamma > \frac{5}{3}$ *and* $m > 1$. *Then there exists a renormalized weak entropy solution* $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon)$ *to our problem (3)–(15) in the sense of Definition 1.*

The proof can be found in

A. Novotný, M. P. Steady compressible Navier–Stokes–Fourier system for monoatomic gas and its generalizations. J. Differential Equations **251** (2011), 270–315.

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Main result

Theorem

Let $\mathbf{f} \in L^\infty(\Omega;\mathbb{R}^3)$, $M_\varepsilon > 0$ *with* $\sup_\varepsilon M_\varepsilon = M_1 < \infty$, $\inf_\varepsilon M_\varepsilon = M_0 > 0$, $L > 0$ *and let* $\vartheta_0 \geq T_0 > 0$ *in* Ω *be defined so that it has finite* L^q*-norm over arbitrary smooth two-dimensional surface with finite surface area contained in* Ω *for some* $q > 1$*. Let* $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon)$ *denote the corresponding renormalized weak entropy solution to (3)–(15) for fixed* $\varepsilon > 0$*, extended suitably to the whole* Ω*, for which in particular the extensions preserve their values in* Ω_ε*. Let* $\alpha > 3$, $m > 2$ *and* $\gamma > 2$ *fulfil* $\alpha > \max\{\frac{2\gamma-3}{\gamma-2}, \frac{3m-2}{m-2}\}$*. Then, for* $\varepsilon \in (0,1]$ *the solutions are uniformly bounded*

$$\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(\Omega)} + \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega)} + \|\vartheta_\varepsilon\|_{W^{1,2}\cap L^{3m}(\Omega)} \leq C, \qquad (21)$$

where $\Theta := \min\left\{2\gamma - 3, \gamma\frac{3m-2}{3m+2}\right\}$ *and* C *is independent of* ε*. Moreover, the corresponding weak limit of the sequence for* $\varepsilon \to 0^+$ *is a renormalized weak solution to problem (3)–(15) in* Ω*, i.e., it fulfils the continuity equation in the weak and renormalized sense, the mass balance, the entropy inequality and the total energy balance in the weak sense in* Ω*, and* $\varrho \geq 0$ *and* $\vartheta > 0$ *a.e. in* Ω*.*

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Estimates independent of ε in Ω_ε

From the entropy inequality and the total energy balances (in both cases, with the test function identically equal to one) we get

$$\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon)} + \|\nabla\vartheta_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla\log\vartheta_\varepsilon\|_{L^2(\Omega_\varepsilon)}$$
$$+ \|\nabla|\vartheta_\varepsilon|^{\frac{m}{2}}\|_{L^2(\Omega_\varepsilon)} + \left\|\frac{1}{\vartheta_\varepsilon}\right\|_{L^1(\partial\Omega_\varepsilon)} \le C,$$
$$\|\vartheta_\varepsilon\|_{L^1(\partial\Omega_\varepsilon)} \le C\big(1 + \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)}\big).$$

We further need:

- to find a bound for the sequence of densities
- to check that the $L^{3m}(\Omega_\varepsilon)$-norm of the sequence of temperatures is bounded independently of ε
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Extensions of the functions I

We extend the density and the velocity by zero. After this extension we still have

$$\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega)} = \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon)} \le C, \quad \|\varrho_\varepsilon\|_{L^r(\Omega)} = \|\varrho_\varepsilon\|_{L^r(\Omega_\varepsilon)}, \ 1 \le r \le \infty.$$

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Extensions of the functions II

For the sequence of temperatures we use a slightly stronger version of the extension from

D. Cioranescu, J. S. J. Paulin. Homogenization in open sets with holes. J. Math. Anal. Appl. **71** (1979), 590–607.

Lemma

Let Ω_ε be given by (1) *and* (2). *There exists an extension operator E_ε: $W^{1,2}(\Omega_\varepsilon) \to W^{1,2}(\Omega)$ such that for each $\varphi \in W^{1,2}(\Omega_\varepsilon)$,*

$$E_\varepsilon\varphi(x) = \varphi(x), \quad x \in \Omega_\varepsilon,$$
$$\|\nabla E_\varepsilon \varphi\|_{L^2(T_{n,\varepsilon})} \leq C\|\nabla\varphi\|_{L^2(B_{2\delta_0\varepsilon^\alpha}(x_{n,\varepsilon})\setminus T_{n,\varepsilon})}$$

and hence $\|\nabla E_\varepsilon\varphi\|_{L^2(\Omega)} \leq C\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}$. Moreover, for all $1 \leq q \leq \infty$,

$$\|E_\varepsilon\varphi\|_{L^q(T_{n,\varepsilon})} \leq C\|\varphi\|_{L^q(B_{2\delta_0\varepsilon^\alpha}(x_{n,\varepsilon})\setminus T_{n,\varepsilon})}.$$

The constant C is independent of ε and n. Furthermore, there is an extension operator $\tilde{E}_\varepsilon : W^{1,2}_{\geq 0}(\Omega_\varepsilon) \to W^{1,2}_{\geq 0}(\Omega)$ such that the above properties are also satisfied. Here $W^{1,2}_{\geq 0}(\Omega_\varepsilon)$ denotes the set of nonnegative functions in $W^{1,2}(\Omega_\varepsilon)$.

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Extensions of the functions III

The extension fulfils the estimate

$$\|\tilde{E}_\varepsilon \vartheta_\varepsilon\|_{W^{1,2}(\Omega)} + \|\tilde{E}_\varepsilon \vartheta_\varepsilon\|_{L^{3m}(\Omega)} \leq C(1 + \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega_\varepsilon)}),$$

where C is independent of ε.

Estimates independent of ε in Ω I

We need to estimate the density. To this aim, we use the following version of the Bogovskii operator from

L. Diening, E. Feireisl, Y. Lu. The inverse of the divergence operator on perforated domains with applications to homogenization problems for the compressible Navier–Stokes system. ESAIM Control Optim. Calc. Var. **23** (2017), 851–868.

Theorem

Let a family of domains Ω_ε be defined by (1) *and* (2)*. Then there exists a family of linear operators*

$$\mathcal{B}_\varepsilon : L^q_0(\Omega_\varepsilon) \to W^{1,q}_0(\Omega_\varepsilon; \mathbb{R}^3), \quad 1 < q < \infty,$$

such that for arbitrary $f \in L^q_0(\Omega_\varepsilon)$ it holds

$$\operatorname{div} \mathcal{B}_\varepsilon(f) = f \qquad \text{a.e. in } \Omega_\varepsilon,$$

$$\|\mathcal{B}_\varepsilon(f)\|_{W^{1,q}_0(\Omega_\varepsilon)} \le C\left(1 + \varepsilon^{\frac{(3-q)\alpha-3}{q}}\right)\|f\|_{L^q(\Omega_\varepsilon)},$$

where the constant C is independent of ε. Here $L^q_0(\Omega_\varepsilon)$ denote the set of $L^q(\Omega_\varepsilon)$ functions which have zero mean value.

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such that for arbitrary $f \in L_0^q(\Omega_\varepsilon)$ it holds

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We get

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Let $\gamma > 2$, $m > 2$ *and* $\alpha > \max\left\{\frac{2\gamma-3}{\gamma-2}, \frac{3m-2}{m-2}\right\}$. *Then the sequence* $\{\varrho_\varepsilon\}$ *is bounded in* $L^{\gamma+\Theta}(\Omega_\varepsilon)$, *where*

$$\Theta = \min\left\{2\gamma - 3, \gamma\frac{3m-2}{3m+2}\right\}.$$

The proof is based on the use of the test function

$$\varphi := \mathcal{B}_\varepsilon\left(\varrho_\varepsilon^\Theta - \langle\varrho_\varepsilon^\Theta\rangle\right), \quad \langle\varrho_\varepsilon^\Theta\rangle := \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon^\Theta \mathrm{d}x.$$

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We get

Lemma

Let $\gamma > 2$, $m > 2$ *and* $\alpha > \max\left\{\frac{2\gamma-3}{\gamma-2}, \frac{3m-2}{m-2}\right\}$. *Then the sequence* $\{\varrho_\varepsilon\}$ *is bounded in* $L^{\gamma+\Theta}(\Omega_\varepsilon)$, *where*

$$\Theta = \min\left\{2\gamma - 3, \gamma\frac{3m-2}{3m+2}\right\}.$$

The proof is based on the use of the test function

$$\varphi := \mathcal{B}_\varepsilon\left(\varrho_\varepsilon^\Theta - \langle\varrho_\varepsilon^\Theta\rangle\right), \quad \langle\varrho_\varepsilon^\Theta\rangle := \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} \varrho_\varepsilon^\Theta \mathrm{d}x.$$

Estimates independent of ε in Ω III

To summarize, we get

$$\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(\Omega_\varepsilon)} + \|\vartheta_\varepsilon\|_{W^{1,2}(\Omega_\varepsilon)}$$
$$+ \|\nabla \log \vartheta_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\vartheta_\varepsilon^{\frac{m}{2}}\|_{W^{1,2}(\Omega_\varepsilon)} + \|\vartheta_\varepsilon\|_{L^{3m}(\Omega_\varepsilon)} \leq C,$$
$$\|\vartheta_\varepsilon\|_{L^1(\partial\Omega_\varepsilon)} + \|\vartheta_\varepsilon^{-1}\|_{L^1(\partial\Omega_\varepsilon)} \leq C,$$

where Θ is as in the Lemma above.

For simplicity, we denote the extensions of our functions to Ω as $(\mathbf{u}_\varepsilon, \varrho_\varepsilon, \vartheta_\varepsilon)$. Then we have the following uniform estimates in Ω

$$\|\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega)} \leq C, \quad \|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(\Omega)} \leq C, \quad \|\vartheta_\varepsilon\|_{W^{1,2}(\Omega)} + \|\vartheta_\varepsilon\|_{L^{3m}(\Omega)} \leq C. \quad (22)$$

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Trace estimates for the temperature

For the traces of the temperature, we have

Lemma

Under the assumptions stated in the main Theorem there holds

$$\|\vartheta_\varepsilon\|^{2m}_{L^{2m}(\partial T_{n,\varepsilon})} \le C\Big(\|\nabla|\vartheta_\varepsilon|^{\frac{m}{2}}\|^2_{L^2(B_{2\delta_0\varepsilon^\alpha}(x_{n,\varepsilon})\setminus T_{n,\varepsilon})} + \|\vartheta_\varepsilon\|^{3m}_{L^{3m}(B_{2\delta_0\varepsilon^\alpha}(x_{n,\varepsilon})\setminus T_{n,\varepsilon})} + \|\vartheta_\varepsilon\|^{2m}_{L^{3m}(B_{2\delta_0\varepsilon^\alpha}(x_{n,\varepsilon})\setminus T_{n,\varepsilon})}\Big),$$

where the constant C is independent of ε and n.

The previous lemma implies

Corollary

Under the assumptions stated in the main Theorem there holds

$$\|\vartheta_\varepsilon\|_{L^{2m}(\cup_{n=1}^{N(\varepsilon)}\partial T_{n,\varepsilon})} \le C\varepsilon^{-\frac{1}{2m}}.$$

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Limit passage in the total energy balance I

We have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } W_0^{1,2}(\Omega;\mathbb{R}^3), \quad \mathbf{u}_\varepsilon \to \mathbf{u} \text{ strongly in } L^r(\Omega;\mathbb{R}^3), \text{ for all } 1 \le r < 6,$$
$$\varrho_\varepsilon \to \varrho \text{ weakly in } L^{\gamma+\Theta}(\Omega),$$
$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } W^{1,2}(\Omega), \quad \vartheta_\varepsilon \to \vartheta \text{ strongly in } L^r(\Omega), \text{ for all } 1 \le r < 3m.$$

As $\mathbf{u}_\varepsilon = \mathbf{0}$ and $\varrho_\varepsilon = 0$ on $\Omega \setminus \Omega_\varepsilon$, we can rewrite the weak formulation of the total energy balance as follows:

$$\begin{aligned}
&-\int_\Omega \Big(\varrho_\varepsilon\big(e(\varrho_\varepsilon,\vartheta_\varepsilon)+\frac{1}{2}|\mathbf{u}_\varepsilon|^2\big)\mathbf{u}_\varepsilon + p(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon - \mathbb{S}(\vartheta_\varepsilon,\nabla\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon - \kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon\Big)\cdot\nabla\psi\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} L(\vartheta_\varepsilon-\vartheta_0)\psi\,\mathrm{d}S - \int_\Omega \varrho_\varepsilon\mathbf{f}\cdot\mathbf{u}_\varepsilon\psi\,\mathrm{d}x \\
&= \int_{\Omega\setminus\Omega_\varepsilon} \kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon\cdot\nabla\psi\,\mathrm{d}x - \int_{\cup_{n=1}^{N(\varepsilon)}\partial T_{n,\varepsilon}} L(\vartheta_\varepsilon-\vartheta_0)\psi\,\mathrm{d}S \\
&=: I_1 + I_2.
\end{aligned}$$

Using the estimates above we can show that

$$|I_1| + |I_2| \to 0 \qquad \text{as } \varepsilon \to 0.$$

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&- \int_\Omega \Big(\varrho_\varepsilon \big(e(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{2}|\mathbf{u}_\varepsilon|^2\big)\mathbf{u}_\varepsilon + p(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon - \mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon - \kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon \Big) \cdot \nabla\psi \,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} L(\vartheta_\varepsilon - \vartheta_0)\psi \,\mathrm{d}S - \int_\Omega \varrho_\varepsilon \mathbf{f} \cdot \mathbf{u}_\varepsilon \psi \,\mathrm{d}x \\
&= \int_{\Omega\setminus\Omega_\varepsilon} \kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon \cdot \nabla\psi \,\mathrm{d}x - \int_{\cup_{n=1}^{N(\varepsilon)} \partial T_{n,\varepsilon}} L(\vartheta_\varepsilon - \vartheta_0)\psi \,\mathrm{d}S \\
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Using the estimates above we can show that

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Limit passage in the total energy balance II

Therefore we get

$$-\int_{\Omega}\left(\left(\overline{\varrho e(\varrho,\vartheta)}+\varrho\frac{1}{2}|\mathbf{u}|^2\right)\mathbf{u}+\overline{p(\varrho,\vartheta)}\mathbf{u}-\mathbb{S}(\vartheta,\nabla\mathbf{u})\mathbf{u}-\kappa(\vartheta)\nabla\vartheta\right)\cdot\nabla\psi\,\mathrm{d}x$$
$$+\int_{\partial\Omega}L(\vartheta-\vartheta_0)\psi\,\mathrm{d}S=\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}\psi\,\mathrm{d}x.$$

We need to show the strong convergence of the density.

Using the properties of the extension and the control of $\|E_\varepsilon(\log\vartheta_\varepsilon)\|_{L^2(\Omega)}$ we can verify that the limit temperature is positive a.e. in Ω.

Limit passage in the total energy balance II

Therefore we get

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&-\int_\Omega \left(\left(\overline{\varrho e(\varrho, \vartheta)} + \varrho \frac{1}{2}|\mathbf{u}|^2 \right)\mathbf{u} + \overline{p(\varrho, \vartheta)}\mathbf{u} - \mathbb{S}(\vartheta, \nabla \mathbf{u})\mathbf{u} - \kappa(\vartheta)\nabla\vartheta \right) \cdot \nabla\psi \,\mathrm{d}x \\
&+ \int_{\partial\Omega} L(\vartheta - \vartheta_0)\psi \,\mathrm{d}S = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\psi \,\mathrm{d}x.
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Limit passage in the entropy inequality I

We want to keep the entropy inequality also for the limit problem. This was not shown in the paper with Y. Lu, however, it was clarified in the paper for the evolutionary problem. Recall that we have

$$\int_{\Omega_\varepsilon} \Big(\frac{\mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon)}{\vartheta_\varepsilon} - \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla\vartheta_\varepsilon)\cdot\nabla\vartheta_\varepsilon}{\vartheta_\varepsilon^2}\Big)\psi \,\mathrm{d}x + \int_{\partial\Omega_\varepsilon} \frac{L\vartheta_0}{\vartheta_\varepsilon}\psi\,\mathrm{d}S \\ \leq L\int_{\partial\Omega_\varepsilon}\psi\,\mathrm{d}S + \int_{\Omega_\varepsilon}\Big(-\frac{\mathbf{q}(\vartheta_\varepsilon,\nabla\vartheta_\varepsilon)\cdot\nabla\psi}{\vartheta_\varepsilon} - \varrho s(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon\cdot\nabla\psi\Big)\,\mathrm{d}x \tag{23}$$

for all $\psi \in C^1(\overline{\Omega_\varepsilon})$, non-negative. To avoid technical complications, we assume that $\kappa(z) = (1+z^m)$. We extend the velocity and the density by zero to Ω. For the temperature we use both the extension $\tilde\vartheta_\varepsilon := \tilde E_\varepsilon \vartheta_\varepsilon$ and $E_\varepsilon \log\vartheta_\varepsilon$. We have

$$\int_{\Omega}\Big(\frac{\mathbb{S}(\tilde\vartheta_\varepsilon,\nabla\mathbf{u}_\varepsilon)}{\tilde\vartheta_\varepsilon} + |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 + \tilde\vartheta_\varepsilon^{m-2}|\nabla\tilde\vartheta_\varepsilon|^2\Big)\psi\,\mathrm{d}x + \int_{\partial\Omega}\frac{L\vartheta_0}{\vartheta_\varepsilon}\psi\,\mathrm{d}S \\ \leq \int_\Omega\Big(\big(-\nabla E_\varepsilon\log(\vartheta_\varepsilon) - \tilde\vartheta_\varepsilon^{m-1}\nabla\tilde\vartheta_\varepsilon\big)\cdot\nabla\psi - \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon\cdot\nabla\psi\Big)\,\mathrm{d}x \\ + L\int_{\partial\Omega}\psi\,\mathrm{d}S + R_\varepsilon, \tag{24}$$

where $R_\varepsilon \to 0$ for $\varepsilon \to 0$. Note also that $\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon$ should be understood in the sense that it is zero outside of Ω_ε.

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Limit passage in the entropy inequality II

The only complicated terms are

$$\int_{\Omega} \left(|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 + \tilde{\vartheta}_\varepsilon^{m-2}|\nabla \tilde{\vartheta}_\varepsilon|^2\right)\psi \, \mathrm{d}x,$$

in the other terms we may employ the strong convergence of velocity and temperature, the weak convergence of the density and the weak lower semicontinuity of convex functionals as well as the fact that due to a.e. positivity of the temperature and the strong convergence we have $\lim_{\varepsilon\to 0} E_\varepsilon \log(\vartheta_\varepsilon) = \log \vartheta$ a.e. We consider the first term, the other can be treated similarly. We fix $\delta > 0$ and compute

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 \, \mathrm{d}x \geq \\
&\geq -\int_{\Omega\setminus\Omega_\varepsilon} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta} \chi_{\{|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|>1\}} \, \mathrm{d}x + \int_{\Omega} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta} \, \mathrm{d}x \\
&+ \int_{\Omega_\varepsilon} \left(|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 - |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta}\right) \chi_{\{|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|\leq 1\}} \, \mathrm{d}x =: \sum_{i=1}^{3} I_i.
\end{aligned} \tag{25}$$

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Limit passage in the entropy inequality III

For I_1 we get

$$|I_1| = \int_{\Omega\setminus\Omega_\varepsilon} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta} \chi_{\{|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|>1\}} \,\mathrm{d}x$$

$$\leq C\|\nabla E_\varepsilon \log(\vartheta_\varepsilon)\|^{2-\delta}_{L^2(\Omega)} |\Omega \setminus \Omega_\varepsilon|^{\frac{2}{\delta}}.$$

Thus for fixed $\delta > 0$ the term converges to zero for $\varepsilon \to 0^+$. Next $|I_3| \leq C(\delta) \to 0$ as $\delta \to 0$ uniformly with respect to ε, since $z \mapsto z^2 - z^{2-\delta}$ has in $(0,1)$ the maximum at $z_0 = 2^{\frac{-1}{\delta}}\sqrt{2-\delta}$, so the bound for I_3 is independent of ε.

We first pass in (25) with $\varepsilon \to 0^+$ and get

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 \,\mathrm{d}x \geq \int_\Omega |\nabla \log(\vartheta)|^{2-\delta} \,\mathrm{d}x + C(\delta),$$

due to the strong convergence of temperature, the fact that the integrand in the second term is bounded in $L^p(\Omega)$ for some $p > 1$ and the weak lower semicontinuity.

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Limit passage in the entropy inequality IV

Now we pass with $\delta \to 0$ and use Vitali convergence Theorem for I_2 (the sequence is equiintegrable, since it is bounded in $L^2(\Omega)$) and obtain

$$\liminf_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 \,\mathrm{d}x \geq \int_\Omega |\nabla \log(\vartheta)|^2 \,\mathrm{d}x. \tag{26}$$

We therefore end up with

$$\int_\Omega \Big(\frac{\mathbb{S}(\vartheta, \nabla \mathbf{u})}{\vartheta} + |\nabla \log(\vartheta)|^2 + \vartheta^{m-2}|\nabla\vartheta|^2\Big)\psi \,\mathrm{d}x + \int_{\partial\Omega} \frac{L\vartheta_0}{\vartheta}\psi \,\mathrm{d}S$$
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The missing point is the strong convergence of the density.

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Limit passage in the continuity equation I

Recall, we may extend the sequence of densities and velocities by zero outside of Ω_ε. Then the continuity equation is fulfilled in the sense of distributions in the whole $\mathbb{R}^3$ for $\mathbf{u}_\varepsilon$ and ϱ_ε and it is easy to pass to the limit to get that

$$\operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{holds in } \mathcal{D}'(\mathbb{R}^3).$$

Moreover, the equation is also satisfied in the renormalized sense (recall $\gamma > 2$)

$$\operatorname{div}\left(b(\varrho)\mathbf{u}\right) + \left(\varrho b'(\varrho) - b(\varrho)\right)\operatorname{div}\mathbf{u} = 0, \quad \text{holds in } \mathcal{D}'(\mathbb{R}^3),$$

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Limit passage in the momentum equation I

Here, the situation is slightly more complicated. The problem is that after the corresponding extension we cannot use directly as test functions smooth compactly supported functions in Ω.

Following

L. Diening, E. Feireisl, Y. Lu. The inverse of the divergence operator on perforated domains with applications to homogenization problems for the compressible Navier–Stokes system. ESAIM Control Optim. Calc. Var. **23** (2017), 851–868.

we can show

$$\operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p(\varrho, \vartheta)} - \operatorname{div} \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \varrho \mathbf{f}, \quad \text{in } \mathcal{D}'(\Omega).$$

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Strong convergence of the density I

The proof of the strong convergence is relatively standard, in the case when $\gamma > 2$ even easier than for small γ's. It is based on

- the validity of the effective viscous flux identity

$$\overline{\varrho^{\gamma+1}} + \overline{\varrho^2}\vartheta - \Big(\frac{4\mu(\vartheta)}{3} + \nu(\vartheta)\Big)\overline{\varrho\,\mathrm{div}\,\mathbf{u}} = \varrho\overline{\varrho^\gamma} + \varrho^2\vartheta - \Big(\frac{4\mu(\vartheta)}{3} + \nu(\vartheta)\Big)\varrho\,\mathrm{div}\,\mathbf{u}$$

  a.e. in Ω

- the validity of the renormalized continuity equation (which is immediate here)

This finishes the proof of the main Theorem for the steady problem.

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Original evolutionary problem

We consider in $(0, T) \times \Omega_\varepsilon$ given by (1) and (2):

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \tag{28}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\varrho, \vartheta) - \mathrm{div}\, \mathbb{S}(\vartheta, \nabla \mathbf{u}) = \varrho \mathbf{f}, \tag{29}$$

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where σ is the entropy production rate, together with the energy equality

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\right)(t, \cdot)\,\mathrm{d}x = \int_0^t \int_{\Omega_\varepsilon} \varrho \mathbf{u} \cdot \mathbf{f}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0)\right)\mathrm{d}x. \tag{31}$$

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Constitutive relations I

Pressure:

$$p(\varrho, \vartheta) = \varrho^{\gamma} + \varrho\vartheta + a\vartheta^4, \qquad \gamma > 6,\ a > 0. \tag{34}$$

Stress tensor:

$$\mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\Big(\nabla \mathbf{u} + \nabla^{\mathrm{T}} \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u}\, \mathbb{I}\Big) + \nu(\vartheta) \operatorname{div} \mathbf{u}\, \mathbb{I}, \tag{35}$$

where the viscosity coefficients are continuous functions of the temperature on $\mathbb{R}_+$, the shear viscosity $\mu(\cdot)$ is moreover globally Lipschitz continuous, and

$$C_1(1+\vartheta) \le \mu(\vartheta) \le C_2(1+\vartheta), \qquad 0 \le \nu(\vartheta) \le C_2(1+\vartheta). \tag{36}$$

Heat flux:

$$\mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{37}$$

where the heat conductivity is for simplicity

$$\kappa(\vartheta) = (1 + \vartheta^3). \tag{38}$$

Constitutive relations II

Total energy:

$$E = e + \frac{1}{2}|\mathbf{u}|^2,$$

and the specific internal energy e fulfils

$$e(\varrho, \vartheta) = c_v\vartheta + \frac{\varrho^{\gamma-1}}{\gamma - 1} + \frac{1}{\varrho}3a\vartheta^4. \tag{39}$$

Entropy:

$$s(\varrho, \vartheta) = c_v \log \vartheta - \log \varrho + \frac{1}{\varrho}4a\vartheta^3. \tag{40}$$

Entropy production rate:

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{u} + \frac{\kappa(\vartheta)}{\vartheta}|\nabla\vartheta|^2\right). \tag{41}$$

Main result

Theorem

Let $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, \theta_{0,\varepsilon})$ *be a sequence of functions defined in* Ω_ε. *Let* $\gamma > 6$ *and* α *fulfill*

$$\alpha > \max\left\{\frac{3(2\gamma-3)}{\gamma-6}, 7\right\}.$$

Let $\varrho_{0,\varepsilon} \geq 0$ *a.e. in* Ω_ε *be such that* $\int_{\Omega_\varepsilon} \varrho_{0,\varepsilon}\,\mathrm{d}x \geq M > 0$ *and being extended by zero to* Ω, $\varrho_{0,\varepsilon} \to \varrho_0$ *weakly in* $L^\gamma(\Omega)$ *and strongly in* $L^1(\Omega)$, $\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon} \to \varrho_0\mathbf{u}_0$ *weakly in* $L^1(\Omega;\mathbb{R}^3)$ *(extended again by zero to* Ω*)),* $\vartheta_{0,\varepsilon} > 0$ *a.e. in* Ω_ε *so that* $\vartheta_{0,\varepsilon}\chi_{\Omega_\varepsilon} \to \vartheta_0$ *weakly in* $L^4(\Omega)$ *with* $\theta_0 > 0$ *a.e. in* Ω. *Furthermore, let* $\varrho_{0,\varepsilon}s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}\chi_{\Omega_\varepsilon}) \to \varrho_0 s(\varrho_0, \vartheta_0)$ *weakly in* $L^1(\Omega)$ *and*

$$E_0^\varepsilon := \int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon}e(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon})\right)\mathrm{d}x$$
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THANK YOU FOR THE ATTENTION!