Incompressible fluid model of electrically charged chemically reacting and heat conducting mixtures

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The system of PDE's

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary, $T > 0$, $L \geq 2$ be the number of the constituent. We consider in $\Omega_T = (0, T) \times \Omega$ the following system of equations

$$i = 1, 2, \ldots, L: \quad \partial_t c_i + \operatorname{div}(c_i v + q_c^i) - r_i = 0 \tag{1}$$

$$\partial_t v + \operatorname{div}(v \otimes v - \mathcal{T}) + Q\nabla\varphi = 0 \tag{2}$$

$$\operatorname{div} v = 0 \tag{3}$$

$$-\Delta\varphi = Q \tag{4}$$

$$\partial_t e + \operatorname{div}(ev + q_e) - \mathcal{T} : \nabla v + \sum_{i=1}^{L} z_i q_c^i \cdot \nabla\varphi = 0. \tag{5}$$

Above: $c := (c_1, \ldots, c_L)$ are the species concentrations, $v = (v_1, v_2, v_3)$ is the velocity field, φ is the electrostatic potential, $q_c^i = (q_c^1, \ldots, q_c^L)$ are the fluxes of the corresponding concentrations c_i, $q_e = (q_e^1, q_e^2, q_e^3)$ denotes the heat flux, $r = (r_1, \ldots, r_L)$ are reaction/productions terms for the concentration c, $z = (z_1, \ldots, z_L)$ are the specific electric charges of c, $Q := \sum_{i=1}^{L} c_i z_i$ is the total electric charge, $\mathcal{T}$ is the Cauchy stress tensor and e is the internal energy of the fluid.

Boundary and initial conditions

System (1)–(5) is completed by the the initial conditions

$$c(0) = c^0, \quad v(0) = v^0, \quad e(0) = e^0, \tag{6}$$

and by the following set of boundary conditions on $\Gamma_T := (0, T) \times \Omega$

$$v \cdot \nu = 0, \quad (I - \nu \otimes \nu)\mathcal{T}\nu = -\gamma v \tag{7}$$

$$q_c^i \cdot \nu = q_{c\Gamma}^i, \quad q_e \cdot \nu = q_{e\Gamma}, \quad \nabla\varphi \cdot \nu = q_{\varphi\Gamma}, \tag{8}$$

where ν denotes the unit outward normal vector to $\partial\Omega$.

Model

Simplifications:

- we consider the volume additivity and model the whole mixture as being incompressible (the density is set equal to one)
- the magnetic field and polarization is neglected, the Maxwell equations are reduced to (4) (the permittivity is equal to one)
- the Lorenz force is reduced to $Q\nabla\varphi$.

Model may include:

- the Peltier effect
- the Joule heat
- the Fourier law
- the Fick law
- the Ohm law
- the Soret effect
- the Dufour effect

Total energy formulation

The term $\mathcal{T} : \nabla v$ is sometimes difficult to treat. Therefore the internal energy balance is sometimes replaced by the total energy balance

$$\partial_t E + \mathrm{div} \left(\left(\frac{|v|^2}{2} + e + Q\varphi \right) v + \varphi \sum_{i=1}^{L} z_i q_c^i + q_e - \mathcal{T} v - \varphi \nabla \partial_t \varphi \right) = 0, \quad (9)$$

where $E = |v|^2/2 + e + |\nabla\varphi|^2/2$ is the specific total energy. The initial condition for $|\nabla\varphi|^2$ can be read from (4).
Advantage: the nonlinear term in velocity gradient disappeared
Disadvantage: the convective term $\sim |v|^3$ can be sometimes not defined, the weak formulation will contain also the pressure

Entropy equation

We assume that the entropy density associated to system (1)–(9) is a function of the internal energy e and the concentration vector c, i.e., $s := s^*(c, e)$. We define the chemical potential ζ and the temperature θ as

$$\zeta = \zeta^*(c, e) := -\partial_c s^*(c, e), \qquad \theta = \theta^*(c, e) := \frac{1}{\partial_e s^*(c, e)}.$$

We can deduce the entropy identity

$$\begin{aligned} &\partial_t s + \mathrm{div}\left(s v - \sum_{i=1}^{L} \zeta_i q_c^i + \frac{q_e}{\theta} \right) \\ &= -\zeta \cdot r + \frac{\mathcal{T} : \nabla v}{\theta} - \sum_{i=1}^{L} q_c^i \cdot \left(\nabla \zeta_i + \frac{z_i}{\theta} \nabla \varphi \right) + q_e \cdot \nabla \frac{1}{\theta}. \end{aligned} \tag{10}$$

The second principle of thermodynamics dictates that the right-hand side of (10) has to be non-negative. We introduce the constitutive relations for parameters that will be designed to satisfy this constraint.

Consistency

We assume

$$\sum_{i=1}^{L} r_i = \sum_{i=1}^{L} z_i r_i = 0.$$

and

$$\sum_{i=1}^{L} q_c^i = 0.$$

Then for $\ell = (1, \ldots, 1)$

$$\partial_t (c \cdot \ell) + \operatorname{div}((c \cdot \ell) v) = 0.$$

Constitutive assumptions I

Denote for $a \in \mathbb{R}^L$

$$P_a := I - \frac{a \otimes a}{|a|^2}.$$

Reaction term

$r := r^*(c, \theta, \zeta)$, where

$$|r^*(c,\theta,\zeta)| \le C_1, \qquad \zeta \cdot r^*(c,\theta,\zeta) \le 0,$$
$$r^*(c,\theta,\zeta) \cdot \ell = z \cdot r^*(c,\theta,\zeta) = 0.$$

Fluxes q_c and q_e

$$q_c^i := -\sum_{j=1}^{L} \mathfrak{M}^{ij}(c,\theta)\left(\nabla \zeta_j + \frac{z_j}{\theta}\nabla\varphi\right) - m^i(c,\theta)\nabla\frac{1}{\theta},$$

$$q_e := -\kappa(c,\theta)\nabla\theta - \sum_{i=1}^{L} m^i(c,\theta)\left(\nabla\zeta_i + \frac{z_i}{\theta}\nabla\varphi\right),$$

where for some $\beta > 1$

$$C_1 \le \frac{\kappa(c,\theta)}{1+\theta^{-\beta}} \le C_2.$$

Constitutive assumptions II

Further $\mathfrak{M}$ is a continuous symmetric matrices valued mapping and m is a continuous vector valued mapping fulfilling for all $(c, \theta) \in \mathbb{R}^L \times R_+$

$$\sum_{i=1}^{L} \mathfrak{M}^{ij}(c,\theta) = \sum_{i=1}^{L} m^i(c,\theta) = 0, \qquad \text{for all } j = 1, \ldots, L,$$

for all $w \in \mathbb{R}^L$

$$C_1 M(\theta)|P_\ell w|^2 \leq \sum_{i,j=1}^{L} \mathfrak{M}^{ij}(c,\theta) w_i w_j \leq C_2 M(\theta)|P_\ell w|^2$$

and for some $\alpha > 0$

$$C_1 \min(1, \theta^{\beta-\alpha}) \leq M(\theta) \leq C(1+\theta)^{\frac{5}{3}-\alpha},$$

$$|m(c,\theta)|^2 \leq C_2 \begin{cases} \min\{M(\theta)\theta^{-\beta+\alpha}, \theta^{-2(\beta-1)+\alpha}\} & \text{for } \theta < 1, \\ M(\theta)\theta & \text{for } \theta \geq 1. \end{cases}$$

Constitutive assumptions III

Cauchy stress

$$\mathcal{T} = -pI + S,$$

where $p : \Omega_T \to \mathbb{R}$ is the mean normal stress — the pressure, and S is the constitutively determined part given by

$$S = S^*(c, \theta, Dv)$$

with Dv denotes the symmetric part of the velocity gradient. The mapping $S^* : \mathbb{R}^L \times \mathbb{R} \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}^{d\times d}_{sym}$ is continuous and for all $(c, \theta, D, B) \in \mathbb{R}^L \times \mathbb{R}_+ \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d}$ and some $r > 3/2$

$$S^*(c, \theta, D) : D \geq C_1|D|^r - C_2,$$
$$|S^*(c, \theta, D)| \leq C_2(1 + |D|^{r-1}),$$
$$S^*(c, \theta, 0) = 0, \quad (S^*(c, \theta, D) - S^*(c, \theta, B)) : (D - B) \geq 0.$$

Constitutive assumptions IV

Entropy

The entropy s decomposes as the sum of two contributions, one from the internal energy e and another from the concentration vector c, i.e.,

$$s = s_e(e) + s_c(c),$$

where $s_e : \mathbb{R}_+ \to \mathbb{R}_+$ and $s_c : \mathbb{R}^L \to \mathbb{R}_+$ are strictly concave $\mathcal{C}^2$ functions. For s_c we assume that for all $c, x \in \mathbb{R}^L$

$$-\sum_{i,j=1}^{L} x_i x_j \partial^2_{c_i c_j} s_c(c) \geq C|x|^2$$

and that for all $K > 0$ there exists $\varepsilon > 0$ such that for all $c \in R^L$ and all $i = 1, \ldots, L$ we have

$$|\partial_c s_c(c)| \leq K \implies c_i \geq \varepsilon.$$

For s_e we assume that it is strictly increasing non-negative function fulfilling for all $e > 1$

$$C_1 \leq -\frac{s_e''(e)}{s_e'(e)^2} \leq C_2.$$

Constitutive assumptions V

In addition, concerning its behaviour near zero, we assume that

$$\lim_{e\to 0_+} \frac{1}{s_e(e)} = \lim_{e\to 0_+} s_e'(e) = \lim_{e\to 0_+} -\frac{s_e''(e)}{s_e'(e)^2} = \infty.$$

Further, for $e > 1$

$$C_1 \leq \frac{\theta^*(e)}{e} \leq C_2$$

and for all $e \geq 0$

$$e - 2s_e(e) + C_2 \geq 0.$$

A model example for s_c used frequently in praxis is e.g.

$$s_c(c) = \sum_{i=1}^{L} (c_i - c_i \log(c_i))$$

and a possible example for s_e is

$$s_e(e) = \begin{cases} C_1 + C_2 \log(e + C_3), & e > 1 \\ C_4 e^a, & 0 \leq e < 1, 0 < a < 1. \end{cases}$$

Constitutive assumptions VI

Second law of thermodynamics

We have

$$\partial_t s + \mathrm{div}\left(sv - \sum_{i=1}^{L} \zeta_i q_c^i + \frac{q_e}{\theta} \right) = -\zeta \cdot r^*(c, \theta, \zeta) + \frac{S^*(c, \theta, Dv) : Dv}{\theta}$$

$$+ \mathfrak{M}(c, \theta) \left(\nabla \zeta + \frac{z}{\theta} \nabla \varphi \right) \cdot \left(\nabla \zeta + \frac{z}{\theta} \nabla \varphi \right) + \frac{\kappa(c, \theta) |\nabla \theta|^2}{\theta^2} \geq 0.$$

Constitutive assumptions VII

Boundary conditions

We have

$$v \cdot \nu = 0, \quad (I - \nu \otimes \nu) S \nu = -\gamma(c, \theta) v \quad \text{on } \Gamma,$$

where γ is a non-negative continuous function fulfilling for all $(c, \theta) \in \mathbb{R}^L \times \mathbb{R}_+$

$$0 \leq \gamma(c, \theta) \leq C_2.$$

Next,

$$q^i_{c\Gamma} = \sum_{j=1}^{L} \mathfrak{D}_{ij}(x, c, \theta) \left(\zeta_j - \zeta^\Gamma_j + z_j(\varphi - \varphi^\Gamma) \right) \qquad \text{on } \Gamma,$$

$$q_{e\Gamma} = -\kappa^\Gamma(x, c, \theta) \left(\frac{1}{\theta} - \frac{1}{\theta^\Gamma} \right) \qquad \text{on } \Gamma,$$

$$q_{\varphi\Gamma} = -\lambda^\Gamma(x)(\varphi - \varphi^\Gamma) \qquad \text{on } \Gamma.$$

We assume $\sum_{i=1}^{L} \mathfrak{D}_{ij}(x, c, \theta) = 0$,

$$C_1 d(x) |P_\ell w|^2 \leq \sum_{i,j=1}^{L} \mathfrak{D}_{ij}(c, x, \theta) w_i w_j \leq C_2 d(x) |P_\ell w|^2$$

and

$$\int_{\partial\Omega} d(x) \, \mathrm{d}\sigma > 0.$$

Constitutive assumptions VIII

Further

$$C_1\overline{\kappa}(x) \leq \kappa^\Gamma(x, c, \theta) \leq C_2\overline{\kappa}(x)$$

with

$$\int_{\partial\Omega} \overline{\kappa}(x)\, \mathrm{d}\sigma > 0.$$

Finally $\lambda^\Gamma \in \mathcal{C}^1(\partial\Omega)$ is a non-negative function

$$\int_{\partial\Omega} \lambda^\Gamma(x)\, \mathrm{d}\sigma > 0.$$

We assume

$$(\theta^\Gamma)^{-1} \in L^2(\Gamma), P_\ell\zeta^\Gamma \in L^2(\Gamma; \mathbb{R}^L), \varphi^\Gamma \in W^{1,1}(0, T; \mathcal{C}^{1,1}(\partial\Omega)).$$

Initial conditions

$$\begin{aligned} &c^0 \in L^\infty([0,1]^L), \quad c^0 \cdot \ell = 1 \text{ a.e. in } \Omega, \\ &v^0 \in L^2_{0,\mathrm{div}}(\Omega) := \overline{\mathcal{C}^\infty_{0,\mathrm{div}}(\Omega; \mathbb{R}^3)}, \\ &e^0 \in L^1(\Omega). \end{aligned}$$

Weak solution I

Weak solution with internal energy

Species equations: for $i = 1, \dots, L$

$$\int_0^T \int_\Omega (c_i \partial_t \psi + (c_i v + q_c^i) \cdot \nabla \psi + r_i \psi) \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega c_i(0)\psi(0,\cdot) \, \mathrm{d}x = \int_0^T \int_{\partial\Omega} q_{c\Gamma}^i \psi \, \mathrm{d}\sigma \, \mathrm{d}t \tag{11}$$

for all $\psi \in C_0^\infty([0,T) \times \overline{\Omega})$

Momentum equation:

$$\begin{aligned} \int_0^T \int_\Omega (v \cdot \partial_t \psi + (v \otimes v - S) : D\psi - Q\nabla\varphi \cdot \psi) \, \mathrm{d}x \, \mathrm{d}t \\ + \int_\Omega v_0 \cdot \psi(0,\cdot) \, \mathrm{d}x = \int_0^T \int_{\partial\Omega} \gamma(c,\theta) v \cdot \psi \, \mathrm{d}\sigma \, \mathrm{d}t \end{aligned} \tag{12}$$

for all $\psi \in C_0^\infty([0,T) \times \overline{\Omega})$ with $\operatorname{div} \psi = 0$ and $\psi \cdot \nu = 0$ on $\partial\Omega$

Electrostatic potential:

$$\int_\Omega \nabla\varphi \cdot \nabla\psi \, \mathrm{d}x = \int_{\partial\Omega} q_{\varphi\Gamma} \psi \, \mathrm{d}\sigma + \int_\Omega Q\psi \, \mathrm{d}x \tag{13}$$

for all $\psi \in C^\infty(\overline{\Omega})$

Weak solution II

Internal energy:

$$\int_0^T \int_\Omega (e\partial_t\psi + (ev + q_e)\cdot\nabla\psi + S : D(v)\psi - \sum_{i=1}^L z_i q_c^i \cdot \nabla\varphi\psi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega e_0\psi(0,\cdot)\,\mathrm{d}x = \int_0^T \int_{\partial\Omega} q_{e\Gamma}\psi\,\mathrm{d}\sigma\,\mathrm{d}t \tag{14}$$

for all $\psi \in C_0^\infty([0,T)\times\overline{\Omega})$

Weak solution with total energy

The weak formulation for the species, momentum and electrostatic potential remain the same. Instead of the internal energy equation we consider

Total energy:

$$\begin{aligned} &\int_0^T \int_\Omega \Big(E\partial_t\psi + ((|v|^2/2 + e + Q\varphi + p)v - Sv)\cdot\nabla\psi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_\Omega \Big(\varphi\sum_{i=1}^L z_i q_c^i + q_e - \varphi\nabla\partial_t\varphi\Big)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_\Omega E_0\psi(0,\cdot)\,\mathrm{d}x = \int_0^T \int_{\partial\Omega} (q_{e\Gamma} + \varphi\sum_{i=1}^L z_i q_{c\Gamma}^i - \varphi\partial_t q_{\varphi\Gamma} + \gamma(c,\theta)|v|^2)\psi\,\mathrm{d}\sigma\,\mathrm{d}t \end{aligned} \tag{15}$$

for all $\psi \in C_0^\infty([0,T)\times\overline{\Omega})$, where $E_0 = e_0 + |v_0|^2/2 + |\nabla\varphi(0)|^2/2$

Variational energy solution

The weak formulation for the species, momentum and electrostatic potential remain the same. Instead of the internal energy equation we consider internal energy inequality:

$$\int_0^T \int_\Omega (e\partial_t\psi + (ev + q_e)\cdot\nabla\psi + S : D(v)\psi - \sum_{i=1}^{L} z_i q_c^i \cdot \nabla\varphi\psi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega e_0\psi(0,\cdot)\,\mathrm{d}x \le \int_0^T \int_{\partial\Omega} q_{e\Gamma}\psi\,\mathrm{d}\sigma\,\mathrm{d}t \tag{16}$$

for all non-negative $\psi \in C_0^\infty([0,T)\times\overline{\Omega})$ and the total energy balance integrated over Ω

$$\int_\Omega E(t)\,\mathrm{d}x + \int_0^t \int_{\partial\Omega} (q_{e\Gamma} + \sum_{i=1}^{L} z_i q_\Gamma^i - \varphi\partial_t q_{\varphi\Gamma} + \gamma(c,\theta)|v|^2)\,\mathrm{d}\sigma\,\mathrm{d}\tau = \int_\Omega E_0\,\mathrm{d}x$$

for all $t \in (0,T]$.

Main result

Theorem

Under the assumptions above, for any $r > 3/2$, *there exists a variational energy solution to our problem. If* $r > 9/5$, *the solution fulfills also the total energy balance and if* $r \geq 11/5$, *the solution fulfills also the internal energy balance.*

Comments:

- If $r > 3/2$, the convective term in the momentum equation makes sense ($v \in L^2(\Omega_T)$) and the convective term ev is integrable
- If $r > 9/5$, the convective in the total energy balance makes sense ($v \in L^3(\Omega_T)$)
- If $r \geq 11/5$, we prove strong convergence of ∇v in $L^r(Q_T)$ and the quadratic term in the internal energy balance is O.K.
- Weak strong compatibility holds, i.e. if we have smooth a variational energy solution, then it is a classical solution to our problem

Known result

This paper extends the results of the paper by Bulíček, Havrda (2015) treating larger interval for r and including the electrostatic field. Other similar results: e.g. Bulíček, Málek, Rajagopal (2009), Roubíček (2005,2006,2007) for incompressible fluid models, Feireisl, Petzeltová, Trivisa (2008) or Mucha, Pokorný, Zatorska (2015) and Xi, Xie (2016) for compressible fluid models.

A priori bounds I

Concentrations:
As $c_0 \cdot \ell = 1$ a.a. in Ω, $\operatorname{div} v = 0$ and $v \cdot \nu = 0$ on $\partial\Omega$, we get due to

$$\partial_t(c \cdot \ell) + \operatorname{div}((c \cdot \ell)v) = 0$$

that $c \cdot \ell \equiv 1$ a.a. in Ω_T. Moreover, due to the assumption on the entropy we have $c_i \geq 0$ a.a. in Ω_T for any $i = 1, \ldots, L$, thus $c_i \in L^\infty(\Omega_T)$.
Electrostatic potential:
As $Q = z \cdot c$ is bounded, we have that $\varphi \in L^\infty((0, T); W^{2,q}(\Omega))$ for any $q < \infty$.

A priori bounds II

Bounds from total energy balance and entropy inequality:

Total energy balance yields:

$$\frac{d}{dt}\left(\int_\Omega E \,\mathrm{d}x + \int_{\partial\Omega} \frac{\lambda^\Gamma |\varphi|^2}{2}\,\mathrm{d}\sigma\right) + \int_{\partial\Omega} \gamma(c,\theta)|v|^2\,\mathrm{d}\sigma$$

$$= \int_{\partial\Omega} \left(\kappa^\Gamma(c,\theta)\left(\frac{1}{\theta} - \frac{1}{\theta^\Gamma}\right) - \sum_{i,j=1}^{L} \mathfrak{D}_{ij}\varphi z_i \left(\zeta_j - \zeta_j^\Gamma + z_j(\varphi - \varphi^\Gamma)\right)\right)\mathrm{d}\sigma$$

$$+ \int_{\partial\Omega} \varphi\lambda^\Gamma \partial_t \varphi^\Gamma \,\mathrm{d}\sigma.$$

Entropy inequality yields:

$$\frac{d}{dt}\int_\Omega -s\,\mathrm{d}x + \int_\Omega \left(\frac{S^*(c,\theta,Dv) : Dv}{\theta} - \zeta\cdot r^*(c,\theta,\zeta)\right)\mathrm{d}x$$

$$+ \int_\Omega \left(\mathfrak{M}(c,\theta)\left(\nabla\zeta + \frac{z}{\theta}\nabla\varphi\right)\cdot\left(\nabla\zeta + \frac{z}{\theta}\nabla\varphi\right) + \frac{\kappa(c,\theta)|\nabla\theta|^2}{\theta^2}\right)\mathrm{d}x$$

$$= \int_{\partial\Omega}\left(-\sum_{i=1}^{L}\zeta_i q_c^i\cdot\nu + \frac{q_e\cdot\nu}{\theta}\right)\mathrm{d}\sigma$$

$$= -\int_{\partial\Omega}\left(\sum_{i,j=1}^{L}\mathfrak{D}_{ij}\left(\zeta_j - \zeta_j^\Gamma + z_j(\varphi-\varphi^\Gamma)\right)\zeta_i + \kappa^\Gamma(x,c,\theta)\left(\frac{1}{\theta} - \frac{1}{\theta^\Gamma}\right)\frac{1}{\theta}\right)\mathrm{d}\sigma.$$

A priori bounds III

Summing up and using our assumptions

$$\sup_{t\in(0,T)}\int_\Omega (E(t)-s(t)+C_2)\,\mathrm{d}x+\int_0^T\int_\Gamma(\gamma(c,\theta)|v|^2+\overline{\kappa}\theta^{-2}+d\,|P_\ell\zeta|^2)\,\mathrm{d}\sigma\,\mathrm{d}t$$
$$+\int_{\Omega_T}\left(\frac{|S:Dv|}{\theta}+M(\theta)\left|P_\ell\left(\nabla\zeta+\frac{z}{\theta}\nabla\varphi\right)\right|^2+|\nabla\ln\theta|^2+|\nabla\theta^{-\frac{\beta}{2}}|^2\right)\mathrm{d}x\,\mathrm{d}t\le C.$$

Hence:

$$\sup_{t\in(0,T)}(\|v(t)\|_2+\|e(t)\|_1+\|s(t)\|_1+\|\theta(t)\|_1)\le C, \tag{17}$$

$$\int_0^T(\|\ln\theta\|_{1,2}^2+\|\theta^{-\frac{\beta}{2}}\|_{1,2}+\|\theta^{-2}\|_{1,\partial\Omega})\,\mathrm{d}t\le C. \tag{18}$$

A priori bounds IV

Bounds from kinetic energy balance:
Using as test function in the momentum equation v:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\frac{|v|^2}{2}\,\mathrm{d}x + \int_{\partial\Omega}\gamma(c,\theta)|v|^2\,\mathrm{d}\sigma + \int_{\Omega} S : Dv\,\mathrm{d}x = -\int_{\Omega} Qv\cdot\nabla\varphi.$$

Thus

$$\int_0^T (\|\sqrt{\gamma}v\|^2_{L^2(\partial\Omega)} + \|v\|^{\frac{5r}{3}}_{\frac{5r}{3}} + \|v\|^r_{1,r} + \|S\|^{r'}_{r'})\,\mathrm{d}t \le C(T, v_0, c, z, \Omega). \qquad (19)$$

Due to the slip boundary conditions we have

$$p = p_1 + p_2 + p_3 + p_4,$$

where

$$\sup_{t\in(0,T)}\|p_4(t)\|_\infty + \int_{\Omega_T}(|p_1|^{r'} + |p_3|^2 + |p_2|^{\frac{5r}{6}})\,\mathrm{d}x\,\mathrm{d}t \le C. \qquad (20)$$

A priori bounds V

Bounds from internal energy balance:

We take $f(s) \in \mathcal{C}^\infty(0,\infty)$ such that $|f(s)| \le 1$, $f(s) = 0$ for $s \in (0,1)$ and $f(s) := (1+s)^{-\lambda}$ for $s \ge 2$, where $\lambda \in (0,1)$. Multiplying the internal energy balance by $f(e)$ and integrating over Ω gives

$$
\begin{aligned}
&-\frac{d}{dt}\int_\Omega F(e)\,\mathrm{d}x + \int_{\partial\Omega} f(e)\kappa^\Gamma(x,c,\theta)\left(\frac{1}{\theta}-\frac{1}{\theta^\Gamma}\right)\,\mathrm{d}\sigma + \int_\Omega f(e) S\cdot Dv\,\mathrm{d}x \\
&-\int_\Omega f'(e)\left(\kappa(c,\theta)\nabla\theta\cdot\nabla e + \sum_{i=1}^L m^i(c,\theta)\left(\nabla\zeta_i + \frac{z_i}{\theta}\nabla\varphi\right)\cdot\nabla e\right)\,\mathrm{d}x \\
&+\int_\Omega \left(f(e)\mathfrak{M}(c,\theta)\left(\nabla\zeta + \frac{z}{\theta}\nabla\varphi\right)\cdot(z\nabla\varphi) - \frac{f(e)(m(c,\theta)\cdot z)}{\theta^2}\nabla\theta\cdot\nabla\varphi\right)\,\mathrm{d}x = 0,
\end{aligned}
$$

where $F' = f$. We get

$$
\int_{\Omega_T} \frac{|\nabla\theta|^2}{(1+\theta)^{\lambda+1}}\,\mathrm{d}x\,\mathrm{d}t \le C(\lambda)\left(1 + \int_{\Omega_T}(1+\theta)^{\frac{5}{3}-\varepsilon_0}\,\mathrm{d}x\,\mathrm{d}t\right).
$$

Using Gagliardo–Nirenberg inequality we conclude

$$
\int_{\Omega_T} \frac{|\nabla\theta|^2}{(1+\theta)^{\lambda+1}}\,\mathrm{d}x\,\mathrm{d}t \le C(\lambda) \qquad \text{for all } 0<\lambda<1. \tag{21}
$$

and

$$
\int_{\Omega_T} \left(|\theta|^{\frac{5}{3}-\lambda} + |\nabla\theta|^{\frac{5}{4}-\lambda} + \frac{|\nabla\theta|^2}{(1+\theta)^{\lambda+1}}\right)\,\mathrm{d}x\,\mathrm{d}t \le C(\lambda) \qquad \text{for all } 0<\lambda<1.
$$

A priori bounds VI

Bounds for fluxes: Using assumptions on $m(\theta)$ and $M(\theta)$

$$\int_{\Omega_T} |q_c|^q \, \mathrm{d}x \, \mathrm{d}t \le C \int_{\Omega_T} \left(\left| \mathfrak{M}(c,\theta) P_\ell \left(\nabla\zeta + \frac{z}{\theta} \nabla\varphi \right) \right|^q + \frac{|m|^q |\nabla\theta|^q}{\theta^{2q}} \right) \mathrm{d}x \, \mathrm{d}t$$

$$\le C(\lambda) + C \int_{\Omega_T} |M(\theta)|^{\frac{q}{2-q}} \, \mathrm{d}x \, \mathrm{d}t + C \int_{\{\theta \ge 1\}} \left(\frac{|m|^{\frac{2q}{2-q}}}{(1+\theta)^{\frac{4q}{2-q} - \frac{q(1+\lambda)}{2-q}}} \right) \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_{\{\theta < 1\}} \frac{|m|^{\frac{2q}{2-q}}}{\theta^{\frac{q(2-\beta)}{2-q}}} \, \mathrm{d}x \, \mathrm{d}t.$$

Therefore

$$\int_{\Omega_T} |q_c|^q \, \mathrm{d}x \, \mathrm{d}t \le C + C \int_{\Omega_T} \left(\theta^{(\frac{5}{3} - \varepsilon_0)\frac{q}{2-q}} \chi_{\{\theta \ge 1\}} + \theta^{-\frac{q(\beta - \varepsilon_0)}{2-q}} \chi_{\{\theta \le 1\}} \right) \mathrm{d}x \, \mathrm{d}t$$

which yields that

$$\int_{\Omega_T} |q_c|^q \, \mathrm{d}x \, \mathrm{d}t \le C \qquad (23)$$

for some $q > 1$.

A priori bounds VII

Similarly

$$\int_{\Omega_T} |q_e|^q \,\mathrm{d}x\,\mathrm{d}t \le C \int_{\Omega_T} |\kappa(c,\theta)|^q |\nabla\theta|^q + |m(c,\theta)|^q \left| P_\ell \left(\nabla\zeta_i + \frac{z_i}{\theta}\nabla\varphi \right) \right|^q \,\mathrm{d}x\,\mathrm{d}t$$
$$\le C \int_{\Omega_T} |\nabla\theta|^q \chi_{\{\theta\ge 1\}} + \frac{|\nabla\theta|^q}{\theta^{\beta q}} \chi_{\{\theta\le 1\}} \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\Omega_T} \frac{|m(c,\theta)|^{\frac{2q}{2-q}}}{|M(\theta)|^{\frac{q}{2-q}}} + M(\theta) \left| P_\ell \left(\nabla\zeta_i + \frac{z_i}{\theta}\nabla\varphi \right) \right|^2 \,\mathrm{d}x\,\mathrm{d}t$$

Hence

$$\int_{\Omega_T} |q_e|^q \,\mathrm{d}x\,\mathrm{d}t \le C(q) + \int_{\Omega_T} \left(\frac{|\nabla\theta|^2}{\theta^{\beta+2}} \right)^{\frac{q}{2}} \left(\frac{1}{\theta^{\beta-2}} \right)^{\frac{q}{2}} \chi_{\{\theta\le 1\}} \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{\Omega_T} (\theta^{\frac{q}{2-q}} + \theta^{-\frac{q(\beta-\varepsilon_0)}{2-q}}) \,\mathrm{d}x\,\mathrm{d}t$$
$$\le C(q) + \int_{\Omega_T} (\frac{|\nabla\theta|^2}{\theta^{\beta+2}} + \theta^{\frac{q}{2-q}} + \theta^{-\frac{q(\beta-\varepsilon_0)}{2-q}} + \theta^{-\frac{q(\beta-2)}{2-q}}) \,\mathrm{d}x\,\mathrm{d}t.$$

Therefore

$$\int_{\Omega_T} |q_e|^q \,\mathrm{d}x\,\mathrm{d}t \le C(q). \tag{24}$$

for some $q > 1$.

A priori bounds VIII

Bounds for chemical potential:

We have

$$\int_{\Omega_T} |P_\ell \nabla\zeta|^q \,\mathrm{d}x\,\mathrm{d}t \leq C \int_{\Omega_T} \Big(\Big|P_\ell\left(\nabla\zeta + \frac{z}{\theta}\nabla\varphi\right)\Big|^q + \Big|\frac{z}{\theta}\nabla\varphi\Big|^q\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C + C\int_{\Omega_T} \Big(\frac{1}{(M(\theta))^{\frac{q}{2-q}}} + \frac{1}{\theta^q}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C + C\int_{\Omega_T} \Big(\frac{1}{\theta^{\frac{q(\beta-\varepsilon_0)}{2-q}}} + \frac{1}{\theta^q}\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

thus

$$\int_{\Omega_T} |P_\ell\nabla\zeta|^q \,\mathrm{d}x\,\mathrm{d}t \leq C + C\int_{\Omega_T} \frac{1}{\theta^\beta}\,\mathrm{d}x\,\mathrm{d}t \leq C,$$

provided $q \leq \beta$. Therefore we need $\beta > 1$. As we control the trace of $P_\ell\zeta$ from the total energy/entropy bounds, we have

$$\int_0^T \|P_\ell\zeta\|_{1,q}^q \,\mathrm{d}t \leq C$$

for some $q > 1$. Using the form of the entropy we finally get

$$\int_0^T \|\zeta\|_{1,q}^q \,\mathrm{d}t \leq C. \tag{25}$$

A priori bounds IX

Bounds for concentrations:

We have due to the form of the entropy

$$C_1|\partial_{x_k} c|^2 \le -\sum_{i,j=1}^{L} \partial^2_{c_i c_j} s_c(c) \partial_{x_k} c_i \partial_{x_k} c_j = \partial_{x_k} \zeta \cdot \partial_{x_k} c = \partial_{x_k}(P_\ell \zeta) \cdot \partial_{x_k} c.$$

Thus for some $q > 1$

$$\int_0^T \|c\|_{1,q}^q \, \mathrm{d}t \le C. \tag{26}$$

Existence of a solution I

Approximation:
We take $\varepsilon > 0$, $\delta > 0$ and introduce

$$s_c^{\varepsilon,\delta}(c) = \begin{cases} s_c^{\varepsilon}(c), & \delta < c_i < \frac{2}{\delta} \forall i = 1, \dots, L \\ \text{concave} & \text{otherwise,} \end{cases}$$

where

$$s_c^{\varepsilon}(c) = s_c(c) + \varepsilon \sum_{i=1}^{L} \log c_i .$$

We define

$$\zeta^{\varepsilon,\delta} = -\partial_c s_c^{\varepsilon,\delta}(c), \quad \zeta^{\varepsilon} = -\partial_c s_c^{\varepsilon}(c).$$

Similarly

$$s_e^{\delta}(e) = \begin{cases} s_e^{\delta}(e), & \delta < e < \frac{2}{\delta} \\ \text{concave} & \text{otherwise,} \end{cases}$$

and set

$$\theta^{*,\delta} = 1/\partial_e s_e^{\delta}(e)$$

Existence of a solution II

We introduce

$$T_\delta(s) = \begin{cases} 0 & 0 \le s \le \delta \\ 1 & 2\delta \le s \le \frac{1}{\delta} \\ 0 & \frac{2}{\delta} < s \\ \text{linear} & \text{otherwise} \end{cases}$$

and

$$\mathcal{T}_\delta(c) = \Pi_{i=1}^L T_\delta(c_i).$$

We define

$$\begin{aligned} q_c^\delta &= \mathcal{T}_\delta(c) T_\delta(e) q_c \\ q_e^\delta &= \mathcal{T}_\delta(c) T_\delta(e) q_e \\ r^\delta &= \mathcal{T}_\delta(c) T_\delta(e) r, \\ Q^\delta(c) &= \mathcal{T}_\delta(c) z \cdot c \end{aligned}$$

similarly for the boundary fluxes (no cut-off for the flux of electrostatic field). We regularize initial and boundary data. Then we introduce Galerkin approximation for the internal energy (dimension denoted by l), for the concentrations (dimension is m) and velocity (dimension is n). Furthermore, we replace the convective term in the momentum equation by a cut-off function $\xi_k(|v|) v \otimes v$. We finally set

$$\theta^{k,n,m,l,\varepsilon,\delta} = \max\{0, \theta^{*,\delta}(e^{k,n,m,l,\varepsilon,\delta})\}.$$

Existence of a solution III

Step 1: Existence of a solution for the approximation:
For fixed φ_0 we solve locally in time the system of nonlinear ODE's for the Galerkin approximation, via fixed point theorem find φ and extend the solution to $(0, T)$ using the a priori estimates.
Step 2: First limit passages
We first let $l \to \infty$ (internal energy) and then $m \to \infty$ (concentrations), the limit passages are relatively easy.
Step 3: Limit passage $\delta \to 0$
This is a relatively difficult part, we lose the regularity of all functions.
Step 4: Limit passage $\epsilon \to 0$ and $n \to \infty$
We set $\varepsilon_n = \frac{1}{n}$ and perform both limit passages simultaneously. Since the convective term is bounded, there is no problem in the limit passage for the velocity (the energy equality holds).
Step 5: Limit passage $k \to \infty$ We need to get the strong convergence of the velocity gradients for which we apply the Lipschitz truncation method (needed in fact only in the range $r \in (\frac{3}{2}, \frac{8}{5})$. If $r \geq 11/5$, then $D(v_n) \to D(v)$ strongly in $L^r(\Omega_T)$ and we may pass to the limit in the internal energy balance, if $r > \frac{9}{5}$, we have that $v_n \to v$ strongly in $L^3(\Omega_T)$ and we can pass to the limit in the total energy balance and if $r \in (\frac{3}{2}, \frac{9}{5}]$ we can pass only in the internal energy inequality and the total energy balance with a constant test function. This finishes the proof of Theorem 1. The reason for $r > \frac{3}{2}$ (and not $r > \frac{6}{5}$) is the term ev.

THANK YOU FOR THE ATTENTION!