Basic notation

Here we present the most often used notation. For further details see Appendix or the corresponding place in the text.

$B_R(\mathbf{x})$	ball with diameter R and center at $\mathbf{x}$
Ω_R	intersection of the ball $B_R(\mathbf{0})$ with Ω
Ω^R	intersection of Ω with the exterior of the ball $B_R(\mathbf{0})$
$\Omega^{R_1}_{R_2} = \Omega^{R_1} \setminus \Omega_{R_2}$	for $R_1 < R_2$
$\nabla\varphi$, $\nabla\mathbf{v}$	gradient of a scalar and a vector field
$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}$	divergence of a vector field
$\nabla \cdot \mathbf{T} = \left\{\frac{\partial T_{ij}}{\partial x_j}\right\}_{i=1}^N$	divergence of a tensor field
$\nabla \times \mathbf{v}$	curl of a (threedimensional) vector field
$\mathbf{u} \cdot \mathbf{v} = u_i v_i$	scalar product of two vector fields
$\nabla\mathbf{u} : \nabla\mathbf{v} = \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}$	scalar product of two tensor fields
$\Delta\varphi = \sum_{i=1}^N \frac{\partial^2\varphi}{\partial x_i^2}$	Laplace operator
$A(\mathbf{u}) = -\Delta\mathbf{u} + \mu\frac{\partial^2\mathbf{u}}{\partial x_1^2}$	elliptic part of the modified Oseen operator, $\mu \in [0;1)$
$\mathcal{E}$	fundamental solution to the Laplace equation
$(\boldsymbol{\mathcal{O}}, \mathbf{e})$	fundamental solution to the Oseen problem
$(\boldsymbol{\mathcal{O}}^\mu, \mathbf{e})$	fundamental solution to the modified Oseen problem
$(\boldsymbol{\mathcal{S}}, \mathbf{e})$	fundamental solution to the Stokes problem
$f * g$	convolution of the functions f and g
$\mathcal{F}(f)$	Fourier transform of the function f
$\mathcal{F}^{-1}(f)$	inverse Fourier transform of a function f
$C^k(\overline{\Omega})$	space of k–times continuously differentiable functions with the norm $\\|\cdot\\|_{C^k}$
$C^k_0(\Omega)$	space of functions from $C^k(\overline{\Omega})$ with compact support in Ω
$\mathcal{D}(\Omega) = C_0^\infty(\Omega)$	
$L^q(\Omega)$	Lebesgue space with the norm $\\|\cdot\\|_q$
$L^q(\Omega)/\mathbb{R}$	factor Lebesgue space with the norm $\inf_{c\in\mathbb{R}} \\|\cdot + c\\|_q$
$L^q_{loc}(\Omega)$	space of locally integrable functions in the power q
$L^q_{loc}(\overline{\Omega})$	space of functions integrable over all Ω_R
$L^q_{(g)}(\Omega)$	space of functions integrable with the weight g and the norm $\\|\cdot\\|_{q,(g)}$

$W^{k,p}(\Omega)$	Sobolev space with the norm $\|\cdot\|_{k,p}$
$W_0^{k,p}(\Omega)$	closure of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_{k,p}$
$D^{k,p}(\Omega)$	homogeneous Sobolev space with the norm $|\cdot|_{k,p}$
$D_0^{k,p}(\Omega)$	closure of $\mathcal{D}(\Omega)$ in the seminorm $|\cdot|_{k,p}$
$\mathcal{D}'(\Omega)$	space of distributions on Ω
$\mathcal{S}(\mathbb{R}^N)$	Schwartz class of functions
$\mathcal{S}'(\mathbb{R}^N)$	space of tempered distributions
${}_0\mathcal{D}(\Omega)$	space of functions from $\mathcal{D}(\Omega)$ with zero divergence
$\widehat{H}_q^1(\Omega)$	space of functions from $W_0^{1,q}(\Omega)$ with zero divergence
$H_q^1(\Omega)$	closure of ${}_0\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_{1,q}$
$\widehat{\mathcal{D}}_0^{1,q}(\Omega)$	space of functions from $D_0^{1,q}(\Omega)$ with zero divergence
$\mathcal{D}_0^{1,q}(\Omega)$	closure of ${}_0\mathcal{D}(\Omega)$ in the norm of $|\cdot|_{1,q}$
$\tilde{H}_q(\Omega)$	subspace of $L^q(\Omega)$ with the divergence belonging to $L^q(\Omega)$
$s(\mathbf{x}) = |\mathbf{x}| - x_1$	
$\sigma_B^A(\mathbf{x}) = |\mathbf{x}|^A s(\mathbf{x})^B$	weight
$\eta_B^A(\mathbf{x}) = (1+|\mathbf{x}|)^A(1+s(\mathbf{x}))^B$	weight
$\nu_B^A(\mathbf{x}) = |\mathbf{x}|^A(1+s(\mathbf{x}))^B$	weight
$\mu_B^{A,\omega}(\mathbf{x}) = \eta_B^{A-\omega}(\mathbf{x})\nu_0^\omega(\mathbf{x})$	weight
$\eta_B^A(\mathbf{x};\beta) = \eta_B^A(\beta\mathbf{x})$	weight
$\nu_B^A(\mathbf{x};\beta) = |\mathbf{x}|^A(1+s(\beta\mathbf{x}))^B$	weight
$\mu_B^{A,\omega}(\mathbf{x};\beta) = \mu_B^{A,\omega}(\beta\mathbf{x})$	weight
η_R	usual cut–off function
ζ_R	Sobolev cut–off function

We finally note that

$$\frac{\partial f(\mathbf{x}-\mathbf{y})}{\partial y_i} = \frac{\partial f(\mathbf{z})}{\partial z_j}\Big|_{\mathbf{z}=\mathbf{x}-\mathbf{y}} \frac{\partial z_j}{\partial y_i} = -\frac{\partial f(\mathbf{z})}{\partial z_i}\Big|_{\mathbf{z}=\mathbf{x}-\mathbf{y}}.$$

Introduction

The method of decomposition, introduced recently by Novotný and Padula in [NoPa] (see also [No1] or [Du]), reveals to be powerful approach for studying the asymptotic structure of the steady slow flows, governed by the compressible Navier–Stokes equations. It has been shown that in the exterior domains, corresponding to the physical reality, there exists a wake region i.e. a parabolic domain in which the asymptotic behaviour is worse.[1] This corresponds exactly to the asymptotic structure of the Oseen fundamental tensor. The authors used the fact that the original problem can be decomposed into several linear problems which are standard and the solution of the original nonlinear problem has been constructed by means of a modified version of the Banach fixed point theorem. Let us also mention that due to the method one has to assume only small perturbations of the rest state.

It is known that several models of non–Newtonian fluids can be also decomposed into the Oseen (eventually Stokes) problem and the transport equations. A natural question appeared: is it possible to study the asymptotic structure of such models and show that, at least for small perturbations of the rest state, there exists similar wake region as in the case of Newtonian fluid? This problem has been used as a starting point for my studies. During the calculations it revealed that the classical results for the Oseen problem do not suffice for our purposes. We had to study in more details the L^p–weighted theory for convolutions with Oseen potentials and, moreover, we had to consider a certain modification of the classical Oseen problem. We call it the modified Oseen problem.

In Chapter I, after a short survey of results from fluid mechanics we list several models of fluids which will be studied later on. Chapters II and III contain a detailed study of the classical and modified Oseen problems. We first recall the asymptotic structure of the fundamental Oseen tensor and then present the weighted L^p–theory for the Oseen potentials, $p \in (1, \infty]$. We concentrate on the physically reasonable cases $N = 2, 3$; nevertheless many results can be extended to higher space dimensions and several results will be therefore presented generally in N space dimensions. Chapter III is then devoted to the detailed study of the modified Oseen problem. First we show the non–trivial but expected fact that its fundamental solution has similar asymptotic structure as the classical Oseen fundamental tensor and afterwards we give a detailed theory of the modified Oseen problem, including the L^p–theory and, in particular, the integral representation of solutions in exterior domains. Although many proofs are similar to those given for the classical Oseen problem in [Ga1] and [Ga2],

[1] It means that the velocity decreases more slowly than outside this region as $|\mathbf{x}| \to \infty$.

we shall, for the sake of completeness, reprove them here.

In Chapter IV, we shortly introduce the theory of the steady transport equations. The results as well as several modifications are presented in [No2], [No3] or [No4]. In the following chapter, we combine several results from the preceding chapters in order to show the existence of solutions to certain non–Newtonian models in exterior domains in Sobolev spaces. In Chapter VI, applying the weighted estimates on the integral representation of solutions to the (modified) Oseen problem together with the weighted estimates of solutions to the steady transport equation we show that, under certain assumptions on the data of the problem, our solutions constructed in Chapter V obey asymptotic properties which correspond to the asymptotic properties of the fundamental Oseen tensor. We consider only the physically interesting cases of two– and threedimensional flows; nevertheless, the generalizations to higher space dimensions are straightforward.

Chapter VII is devoted to a completely another problem. Unlike the first chapters, we study unsteady problems and concentrate ourselves on the axially symmetric flow of both ideal and viscous fluid in the whole space. It is well known that the problem of global regularity and uniqueness of Leray–Hopf weak solutions to the Navier–Stokes equations in three space dimensions is still an open problem. We show that, assuming the data to the Cauchy problem axially symmetric and regular, the Leray–Hopf weak solution is also axially symmetric, regular and therefore unique in the class of all weak solutions. The main contribution of this part is not the result itself but rather the method of proof which is very simple and uses the standard results on the (unsteady) Stokes problem. A similar method can be applied also for the ideal fluid, which can be considered as a limit when the viscosity tends to zero.

The last chapter contains a short survey of classical results from the theory of Sobolev and Lebesgue spaces, Fourier transform on the space of tempered distributions and finally some classical results on the Stokes problem and its modified version.

Parts of the results presented here has been published or are submitted for publishing, see [Po], [LeMaNePo] and [KrNoPo].

I

Preliminaries from fluid mechanics. Basic studied models

I.1 Fluid mechanics

The continuum mechanics studies the motion and deformation of bodies. A body $\mathcal{B}$ is an abstact set that consists of material points p, called usually particles. We assume that there exists $\mathbf{z}$, a smooth one to one mapping of $\mathcal{B}$ onto a region of the N–dimensional[1] Euclidean space $\mathcal{E}$,

$$\mathbf{X} = \mathbf{z}(p)\,. \tag{1.1}$$

The function $\mathbf{z}$ is called reference configuration. Next we assume that there exists a smooth one to one transformation of the Euclidean space $\mathcal{E}$ onto itself, called deformation, such that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})\,, \tag{1.2}$$

where $\mathbf{x}$ denotes the place occupied by the particle p, $\mathbf{X} = \mathbf{z}(p)$. The deformation gradient

$$\mathbf{F}(\mathbf{X}) = \nabla\boldsymbol{\chi}(\mathbf{X}) \tag{1.3}$$

plays a fundamental role in the continuum mechanics.

A motion of the body is one–parameter family of deformations

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)\,, \tag{1.4}$$

the real parameter t denotes the time. The function $\boldsymbol{\chi}(\mathbf{X}, t)$ is at each time instant invertable, i.e.

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) \tag{1.5}$$

and one can distinguish two approaches in continuum mechanics. Either we study the motion of each particle p in the Euclidean space during some time interval $(t_1; t_2)$ (the Lagrangean approach), or we fix a point in the Euclidean space and study the particles passing through the point $\mathbf{x}$ (the Eulerean approach). Both approaches are for $\boldsymbol{\chi}$ smooth equivalent and mutually connected by (1.4) and (1.5). In this thesis, as usually in fluid mechanics, we prefer the latter, i.e. the Eulerean approach.

By velocity we understand the (material) time derivative[2] of the function

[1] one usually assumes $N = 3$

[2] For a given spatial field $\Phi_s(\mathbf{x}, t)$, $\Phi_s(\mathbf{x}, t) = \Phi_s(\boldsymbol{\chi}(\mathbf{X}, t), t) = \Phi_m(\mathbf{X}, t)$ we distinguish between the material time derivative

$$\frac{d}{dt}\Phi_s = \frac{\partial}{\partial t}\Phi_m(\mathbf{X}, t)\Big|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)}$$

$\boldsymbol{\chi}$ in the spatial description, i.e.

$$\mathbf{v}(\mathbf{x},t) = \frac{d}{dt}\boldsymbol{\chi}(\mathbf{X},t)\Big|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)} \tag{1.6}$$

and by acceleration the material time derivative of the velocity,

$$\mathbf{a}(\mathbf{x},t) = \frac{d}{dt}\mathbf{v}(\mathbf{x},t)\,. \tag{1.7}$$

Let us recall that

$$\frac{d}{dt}\Phi(\mathbf{x},t) = \frac{\partial}{\partial t}\Phi(\mathbf{x},t) + \nabla_{\mathbf{x}}\Phi(\mathbf{x},t)\cdot\mathbf{v}(\mathbf{x},t)\,. \tag{1.8}$$

The velocity gradient $\mathbf{L}$ is the tensor

$$\mathbf{L}(\mathbf{x},t) = \nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x},t)\,. \tag{1.9}$$

We shall often use its symmetric part $\mathbf{D}$ (called also the rate of deformation) and the skew part $\mathbf{W}$ (called also the spin tensor),

$$\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^T)\,, \quad \mathbf{W} = \frac{1}{2}(\mathbf{L}-\mathbf{L}^T)\,, \tag{1.10}$$

$\mathbf{L}^T$ being the transpose of the tensor $\mathbf{L}$.

We assume that there exists a positive function $m(\mathcal{P})$, called mass, and mass density ϱ^0 such that for any Lebesgue measurable part $\mathcal{P}$ of the body $\mathcal{B}$

$$\begin{aligned} m(\mathcal{P}) &= \int_{\mathbf{z}(\mathcal{P})}\varrho^0\mathrm{d}\mathbf{X} = \int_{\boldsymbol{\chi}(\mathbf{z}(\mathcal{P}))}\varrho\mathrm{d}\mathbf{x}\,,\\ \varrho(\mathbf{x},t) &= \frac{\varrho^0(\mathbf{X})}{\det\mathbf{F}(\mathbf{X},t)}\,. \end{aligned} \tag{1.11}$$

One of the most important assumptions in the continuum mechanics are the empirically deduced balance laws. The balance of mass says that the mass of any part $\mathcal{P}$ of the body $\mathcal{B}$ conserves, i.e.

$$\frac{d}{dt}m(\mathcal{P}) = 0\,. \tag{1.12}$$

From (1.12), (1.11) and (1.8) one can deduce the following differential form of the balance of mass

$$\frac{\partial\varrho}{\partial t}(\mathbf{x},t) + \nabla\cdot(\varrho\mathbf{v})(\mathbf{x},t) = 0\,. \tag{1.13}$$

Let $\mathcal{F}(\mathcal{P})$ denotes the total force extended on the part $\mathcal{P}$ of $\mathcal{B}$. We have

$$\mathcal{F}(\mathcal{P}) = \int_{\mathbf{z}(\mathcal{P})}\varrho^0\mathbf{f}\mathrm{d}\mathbf{X} + \int_{\partial\mathbf{z}(\mathcal{P})}\mathbf{t}\mathrm{d}s\,, \tag{1.14}$$

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and the space time derivative

$$\frac{\partial}{\partial t}\Phi_s = \frac{\partial\Phi_s}{\partial t}(\mathbf{x},t)\,,$$

see e.g. [Gu] for more detailed description.

where $\mathbf{f}$ is the volume force and the stress vector $\mathbf{t} = \mathbf{t}(\mathbf{X}, \mathbf{n})$ depends on the point $\mathbf{X} \in \partial\mathbf{z}(\mathcal{P})$ and the exterior unit normal to $\partial\mathbf{z}(\mathcal{P})$ at this point. As a consequence of the balance of linear and angular[3] momentum we get that $\mathbf{t}$ depends on $\mathbf{n}$ linearly, i.e. there exists a symmetric tensor $\mathbf{T} = \mathbf{T}(\mathbf{x})$, called the stress tensor, such that

$$\mathbf{t}(\mathbf{X}, t, \mathbf{n})\Big|_{\mathbf{X}=\boldsymbol{\chi}^{-1}(\mathbf{x},t)} = \mathbf{T}(\mathbf{x}, t)\mathbf{n}(\mathbf{x}, t) \tag{1.15}$$

and we get the following differential form of the balance of linear momentum

$$\nabla \cdot \mathbf{T} + \varrho\mathbf{f} = \varrho\mathbf{a}\,, \tag{1.16}$$

$\mathbf{a}$ the acceleration (see (1.7)).

Assuming the exterior force $\mathbf{f}$ being given, the system (1.13), (1.16) has still more unknowns than the equations. We have to specify a certain dependence of the stress tensor $\mathbf{T}$ on the velocity, density and its gradients.[4]

The apriori general dependence of $\mathbf{T}$ on the functions $\mathbf{v}$, ϱ and their gradients will be reduced due to the material symmetry and some general physical assumptions. Let us start with the latter.

Consider a time dependent change of variables

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{q}(t) \tag{1.17}$$

with $\mathbf{Q}(t)$ rotation and $\mathbf{q}(t)$ a vector. Then we say that the motions are frame indifferent if the material relations remain the same in the sense that

$$\begin{aligned} \mathbf{T}^*(\mathbf{x}^*, t) &= \mathbf{Q}(t)\mathbf{T}(\mathbf{x}, t)\mathbf{Q}(t)^T\,, \\ \varrho^*(\mathbf{x}^*, t) &= \varrho(\mathbf{x}, t)\,, \end{aligned} \tag{1.18}$$

where $\mathbf{x}^*$ and $\mathbf{x}$ are connected by (1.17). We shall assume that all our motions are frame indifferent.

Let us consider a general relation between the stress tensor $\mathbf{T}$ and the history of the material, expressed by the response functional[5] $\mathcal{H}$

$$\mathbf{T}(\mathbf{X}, t) = \mathcal{H}_{s=0}^{\infty}(\boldsymbol{\chi}(\overline{\mathbf{X}}, t-s), \mathbf{X}, t)\,,\ \ \overline{\mathbf{X}} = \mathbf{z}(\overline{p})\,,\ \ \overline{p} \in \mathcal{B}\,. \tag{1.19}$$

We shall localize the dependence on the motions; the apriori general dependence on $\boldsymbol{\chi}(\overline{\mathbf{X}}, t-s)$ is reduced to a dependence on the gradient of $\boldsymbol{\chi}$ at the point $\mathbf{X}$,

$$\mathbf{T}(\mathbf{X}, t) = \mathcal{H}_{s=0}^{\infty}(\mathbf{F}(\mathbf{X}, t-s), \mathbf{X}, t)\,. \tag{1.20}$$

We say that the material is a fluid when the group of symmetry of the material is the whole unimodular group, i.e.

$$\mathcal{H}_{s=0}^{\infty}(\mathbf{F}(\mathbf{X}, t-s), \mathbf{X}, t) = \mathcal{H}_{s=0}^{\infty}(\mathbf{F}(\mathbf{X}, t-s)\mathbf{G}, \mathbf{X}, t) \tag{1.21}$$

[3]For the models studied here, the balance of angular momentum implies that the stress tensor defined below is symmetric. We shall therefore not write it explicitly. This is no more true in non–local theories like e.g. for the multipolar fluids.

[4]We shall not study processes when the internal energy changes; we therefore do not consider the balance of energy and restrictions coming from the second law of thermodynamics. See e.g. [Si] for a detailed descriptions of this phenomena.

[5]we assume the stress tensor in the reference configuration $\mathbf{X}$ for a while

for all $\mathbf{G}$, tensors of second order such that $|\det \mathbf{G}| = 1$. Next restrictions on (1.19) come from the so–called material constraints. Let us mention especially the volume preserving materials; in this case it is possible to decompose the stress tensor into two parts (see e.g. [Tr]),

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}^E \tag{1.22}$$

where the first part does zero work ($p\mathbf{I} : \mathbf{D} = p\,\mathrm{tr}\mathbf{D} = 0$) and the extra stress tensor $\mathbf{T}^E = \overline{\mathcal{H}}_{s=0}^{\infty}(\mathbf{F}(\mathbf{X}, t-s))$. We call the scalar function p the pressure. We shall now present several models of fluids studied later on.

I.2 Ideal fluid

We assume that the response functional is reduced to a simple function dependence

$$\mathcal{H}_{s=0}^{\infty}(\mathbf{F}(\mathbf{X}, t-s), \mathbf{X}, t) = \mathbf{T}(\mathbf{F}(\mathbf{X}, t), \varrho(\mathbf{X}, t))\,. \tag{2.1}$$

The dependence on the density ϱ follows from the fact that due to the symmetry the only possible change of the response functional is connected with the change of volume.

Combining the symmetry condition (1.21) with the material frame indifference it is possible to deduce (see e.g. [Lei]) that

$$\mathbf{T} = -p(\varrho)\mathbf{I}\,, \tag{2.2}$$

where the scalar function $p(\varrho)$ is again called pressure. From (1.8), (1.13), (1.16) and (2.2) we get the Euler equations

$$\begin{aligned} \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{v}) &= 0\,, \\ \varrho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) + \nabla p(\varrho) &= \varrho \mathbf{f}\,. \end{aligned} \tag{2.3}$$

We complete (2.3) by the constitutive relation

$$p = p(\varrho)\,, \tag{2.4}$$

by the initial conditions $\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x})$, $\varrho(\mathbf{x}, 0) = \varrho_0(\mathbf{x})$ and by the boundary conditions. The standard condition in this case is to assume that the fluid does not penetrate through the solid wall, i.e. the velocity field is tangential to the solid boundary

$$\mathbf{v}\cdot\mathbf{n} = 0\,.$$

In Chapter VII we shall study in particular Cauchy problem for the incompressible Euler equations, i.e. assuming the flow isochoric the continuity equation $(2.3)_1$ reduces to $\nabla\cdot\mathbf{v} = 0$. From (1.22) we get similarly as above that $T = -p\mathbf{I}$, $p = p(\mathbf{x}, t)$, i.e. $\overline{\mathcal{H}}_{s=0}^{\infty} = \mathbf{0}$. Recalling that we study the flow in the whole $\mathbb{R}^N$, we end up with

$$\left.\begin{aligned} \nabla\cdot\mathbf{v} &= 0 \\ \varrho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) + \nabla p &= \varrho\mathbf{f} \end{aligned}\right\} \text{ in } (0;T)\times\mathbb{R}^N \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \text{ in } \mathbb{R}^N \tag{2.5}$$

($\nabla \cdot \mathbf{v}_0 = 0$ in $\mathbb{R}^N$). In particular, we shall study this system in the threedimensional case, i.e. $N = 3$. See Section VII.2 for a short survey of known results about the system (2.5).

I.3 Newtonian fluid

We come back to (1.20), but localize the time dependence. Using the material frame indifference together with the symmetry properties, it is possible to show that assuming

$$\mathbf{T} = \mathbf{T}(\mathbf{F}, \frac{d}{dt}\mathbf{F}, \varrho)\,, \tag{3.1}$$

we end up with (see e.g. [Lei])

$$\mathbf{T} = \mathbf{T}^E(\mathbf{D}, \varrho) - p(\varrho)\mathbf{I}\,, \tag{3.2}$$

where $\mathbf{D}$ is the symmetric part of the velocity gradient (see (1.10)), $\mathbf{T}^E$ is the extra stress tensor and $p(\varrho)$ is the pressure. Applying once more the material frame indifference and the symmetry, the representation theorem for isotropic functions (see e.g. [Gu]) yields

$$\mathbf{T}^E = \varphi_0\mathbf{I} + \varphi_1\mathbf{D} + \varphi_2\mathbf{D}^2\,, \tag{3.3}$$

where φ_i, $i = 0, 1, 2$, are functions of ϱ and invariants of $\mathbf{D}$. Linearizing (3.3) we finally get

$$\mathbf{T}^E = \lambda(\mathrm{tr}\mathbf{D})\mathbf{I} + 2\mu\mathbf{D} \tag{3.4}$$

and (1.13), (1.16), (3.2) and (3.4) yield, under the assumption that λ and μ are independent of ϱ, the compressible Navier–Stokes equations

$$\begin{gathered} \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho\mathbf{v}) = 0 \\ \varrho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) - \mu\Delta\mathbf{v} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{v}) + \nabla p(\varrho) = \varrho\mathbf{f}\,. \end{gathered} \tag{3.5}$$

We again close the system by giving a constitutive equation of the type (2.4), the initial conditions on $\mathbf{v}$, ϱ and, (if $\Omega \neq \mathbb{R}^N$), the boundary conditions on $\mathbf{v}$. Usually, one assumes the Dirichlet ones, i.e. there exists $\mathbf{v}_*$ defined at $\partial\Omega$ such that $\mathbf{v} = \mathbf{v}_*$ at $(0;T) \times \partial\Omega$. In the case of a solid wall, the linearly viscous[6] fluids adhere, i.e. $\mathbf{v}_* \cdot \mathbf{n} = 0$ at such parts of the boundary.

In Chapter VII we shall study Cauchy problem for the incompressible Newtonian fluid, i.e. the system (3.5) reduces to (see (1.22))

$$\left.\begin{aligned} \nabla \cdot \mathbf{v} &= 0 \\ \varrho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) - \mu\Delta\mathbf{v} + \nabla p &= \varrho\mathbf{f} \end{aligned}\right\} \text{ in } \mathbb{R}^N \times (0,T)\,, \qquad \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}) \text{ in } \mathbb{R}^N\,, \tag{3.6}$$

where $\varrho =$ const and $\mathbf{v}_0$ is a given initial condition satisfying $\nabla \cdot \mathbf{v}_0 = 0$. In particular, we shall study the most interesting case $N = 3$, but under a special

[6]the constants λ and μ are called viscosities

symmetry condition, namely the axial one. See Section VII.1 for a short survey of known results on the system (3.6).

In this thesis we shall be mainly interested in the flow of certain classes of non–Newtonian fluids past an obstacle, i.e. we shall assume a more general relation between **T** and **L** than (3.4). Nevertheless, a very important role will be played by certain linearizations of the steady Navier–Stokes equations. Assuming the flow independent of time and the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ neglectibly small, we obtain the (stationary) Stokes system

$$\begin{aligned} \nabla\cdot\mathbf{v} &= 0 \\ -\mu\Delta\mathbf{v}+\nabla p &= \varrho\mathbf{f}\,, \end{aligned} \tag{3.7}$$

which must be completed in the case of $\Omega \neq \mathbb{R}^N$ by a boundary condition $\mathbf{v} = \mathbf{v}_*$ at $\partial\Omega$ and, if needed, by a condition at infinity. Due to $(3.7)_1$ in the case of Ω bounded one requires[7]

$$\int_{\partial\Omega} \mathbf{v}_*\cdot\mathbf{n}\,\mathrm{d}S = 0\,. \tag{3.8}$$

While for Ω being a bounded, simply connected domain, the system (3.7) seems to correspond quite satisfactorily to a slow flow of a viscous fluid, for Ω an exterior domain due to the Stokes paradox (see e.g. [Ga1]) one needs another linearization. The Oseen linearization, presented below, acquitted itself quite well for steady flow with $\mathbf{v}_\infty$, non–zero constant velocity prescribed at infinity (see [Os])

$$\begin{aligned} \left.\begin{aligned} \nabla\cdot\mathbf{v} &= 0 \\ -\mu\Delta\mathbf{v}+\varrho(\mathbf{v}_\infty\cdot\nabla)\mathbf{v}+\nabla p &= \varrho\mathbf{f} \end{aligned}\right\} &\text{ in } \Omega\,, \\ \mathbf{v}(\mathbf{x}) = \mathbf{v}_*(\mathbf{x}) &\text{ at } \partial\Omega\,, \\ \mathbf{v}\to\mathbf{0} &\text{ as } |\mathbf{x}|\to\infty\,. \end{aligned} \tag{3.9}$$

As will be seen in the next chapter, this linearization corresponds much better to a real slow flow of the viscous fluid past an obstacle (e.g. the existence of the wake region). In this thesis, we shall study the Oseen linearization and its modification in detail (see Chapters II and III), while for the Stokes problem, we present only a short overview of the results needed in the text in Appendix. We refer e.g. [Ga1], Chapters IV–VI for a detailed study of the properties of the Stokes problem.

I.4 Some models of non–Newtonian fluids

The aim of this section is to present some models of non–Newtonian fluids which will be studied in the following chapters. We shall be particularly interested in the models of viscoelastic fluids.

[7] For Ω exterior domain, this condition for $N \geq 3$ can be easily skipped — see Chapter III. Nevertheless, if $N = 2$, the technique proposed e.g. in [Ga1] seems inapplicable. We shall mention this problem later on.

I.4.1 Maxwell and Oldroyd–type fluids

We start with a general differential model of a viscoelastic fluid. As announced in the previous section, we study only steady flow of incompressible fluids. The continuity equation which expresses the balance of mass reduces to

$$\nabla \cdot \mathbf{v} = 0 \tag{4.1}$$

and the balance of (linear) momentum reads

$$\varrho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \nabla \cdot \mathbf{T}^E + \varrho \mathbf{f}\,. \tag{4.2}$$

The symmetric extra stress tensor $\mathbf{T}^E$ obeys the constitutive equations

$$\mathbf{T}^E = \mathbf{T}^S + \mathbf{T}^P \tag{4.3}$$

$$\mathbf{T}^S = 2\eta^S \mathbf{D} \tag{4.4}$$

$$\mathbf{T}^P = \sum_{i=1}^{n} \mathbf{T}_i \tag{4.5}$$

$$\mathbf{T}_i + \lambda_i \frac{\mathcal{D}_a \mathbf{T}_i}{\mathcal{D}t} + \mathbf{B}_i(\mathbf{D}, \mathbf{T}_i) = 2\eta_i \mathbf{D}\,, \quad 1 \le i \le n \tag{4.6}$$

(see e.g. [Jo] or [BaGuSa]). The constants $\eta_i(> 0)$, $1 \le i \le n$ and $\eta^S(\ge 0)$ are called viscosities and the constants $\lambda_i(> 0)$ are called relaxation times. The symbol $\frac{\mathcal{D}_a \mathbf{T}_i}{\mathcal{D}t}$ represents objective derivative of a symmetric tensor and in the stationary case is given by

$$\frac{\mathcal{D}_a \mathbf{T}_i}{\mathcal{D}t} = (\mathbf{v} \cdot \nabla)\mathbf{T}_i + \mathbf{T}_i \mathbf{W} - \mathbf{W}\mathbf{T}_i - a(\mathbf{D}\mathbf{T}_i + \mathbf{T}_i \mathbf{D})\,, \tag{4.7}$$

where $a \in [-1; 1]$ is a given real parameter. The tensor–valued smooth functions $\mathbf{B}_i(\mathbf{D}, \mathbf{T}_i)$ are at least quadratic in their two arguments in a neighborhood of $\mathbf{T}_i = \mathbf{0}$ and submitted to certain restrictions due to the material frame indifference.

First, let us assume that $\eta^S \equiv 0$. For the sake of simplicity, let us assume that $n = 1$, i.e. we have only one relaxation time; the more general case $n > 1$ can be treated in a very analogous way. We also restrict ourselves to the cases when $\mathbf{B}_1(\mathbf{D}, \mathbf{T}) = \mathbf{B}(\mathbf{D}, \mathbf{T})$ is either zero or bilinear near $\mathbf{T} = \mathbf{0}$ and $\mathbf{D} = \mathbf{0}$. Some generalizations in the sense that $\mathbf{B}$ has one part bilinear and another (at least) quadratic in $\mathbf{D}$ are possible, but we shall not study them. The technique used in Chapter VI does not allow to study a general polynomial function in $\mathbf{T}$. Nevertheless, let us mention that our restrictions still involve several physically reasonable models like lower–convected, corotational and upper convected Maxwell fluid, certain special cases of 8–constant Oldroyd model etc.; see [BaGuSa]. In order to rewrite the system (4.1)–(4.7) into a more appropriate form, we follow Renardy ([Re]). We multiply the i-th component of (4.2) by v_j, apply the divergence with respect to j and get

$$\nabla \cdot [\varrho((\mathbf{v} \cdot \nabla)\mathbf{v}) \otimes \mathbf{v} + \nabla p \otimes \mathbf{v} - \varrho \mathbf{f} \otimes \mathbf{v}] = \nabla \cdot [(\nabla \cdot \mathbf{T}) \otimes \mathbf{v}]\,. \tag{4.8}$$

Thanks to the incompressibility condition (4.1) we have

$$\nabla \cdot [(\nabla \cdot \mathbf{T}) \otimes \mathbf{v}] = (\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{T}) . \tag{4.9}$$

Combining (4.7) and (4.6) and inserting them into (4.2) we have

$$\varrho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \varrho \mathbf{f} + \\ + \nabla \cdot [2\eta \mathbf{D} - \mathbf{B}(\mathbf{D}, \mathbf{T}) - \lambda(\mathbf{v} \cdot \nabla)\mathbf{T} - \lambda(\mathbf{TW} - \mathbf{WT}) + \lambda a(\mathbf{DT} + \mathbf{TD})] .$$

Observing that thanks to (4.1)

$$\nabla \cdot ((\mathbf{v} \cdot \nabla)\mathbf{T}) = \nabla \mathbf{T} : (\nabla \mathbf{v})^T + (\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{T}) = \nabla \cdot (\mathbf{T}(\nabla \mathbf{v})^T) + (\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{T})$$

and using (4.9) and (4.8) we finally get

$$\varrho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \varrho \mathbf{f} + \nabla \cdot [2\eta \mathbf{D} - \mathbf{B}(\mathbf{D}, \mathbf{T}) - \lambda \mathbf{T}(\nabla \mathbf{v})^T - \lambda \varrho((\mathbf{v} \cdot \nabla)\mathbf{v}) \otimes \mathbf{v} - \\ - \lambda \nabla p \otimes \mathbf{v} + \lambda \varrho \mathbf{f} \otimes \mathbf{v} - \lambda(\mathbf{TW} - \mathbf{WT}) + \lambda a(\mathbf{DT} + \mathbf{TD})] .$$

Denoting

$$\pi = p + \lambda(\mathbf{v} \cdot \nabla)p \tag{4.10}$$

$$\mathbf{F}(\nabla \mathbf{v}, \mathbf{T}) = -\lambda \mathbf{T}(\nabla \mathbf{v})^T - \lambda(\mathbf{TW} - \mathbf{WT}) + \lambda a(\mathbf{DT} + \mathbf{TD}) - \mathbf{B}(\mathbf{D}, \mathbf{T}) \tag{4.11}$$

$$\mathbf{G}(\nabla \mathbf{v}, \mathbf{T}) = \lambda(\mathbf{TW} - \mathbf{WT}) - \lambda a(\mathbf{DT} + \mathbf{TD}) + \mathbf{B}(\mathbf{D}, \mathbf{T}) \tag{4.12}$$

we end up with the following system

$$\begin{gathered} -\eta \Delta \mathbf{v} + \varrho(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \pi = \varrho \mathbf{f} + \nabla \cdot [\mathbf{F}(\nabla \mathbf{v}, \mathbf{T}) - \lambda \varrho((\mathbf{v} \cdot \nabla)\mathbf{v}) \otimes \mathbf{v} + \\ + \lambda \varrho \mathbf{f} \otimes \mathbf{v} + \lambda p(\nabla \mathbf{v})^T] \\ \nabla \cdot \mathbf{v} = 0 \\ p + \lambda(\mathbf{v} \cdot \nabla)p = \pi \\ \mathbf{T} + \lambda(\mathbf{v} \cdot \nabla)\mathbf{T} + \mathbf{G}(\nabla \mathbf{v}, \mathbf{T}) = 2\eta \mathbf{D}(\mathbf{v}) . \end{gathered} \tag{4.13}$$

Renardy used this procedure in the study of existence of strong solutions to (4.1) – (4.7) for small data in bounded domains. Another results to such models can be found e.g. in [Ha], [GuSa], [Tal1] and [Tal2]. See also [BaGuSa] and the references therein. For threedimensional exterior domains see [MaSeVi] for compressible model. To the knowledge of the author there are no results concerning the asymptotic structure of solutions for such models.

Let us now formulate the boundary value problem for the system (4.13). Let us assume that $\mathcal{O} \subset \mathbb{R}^N$, $N = 2, 3$ is a compact, simply connected set and $\Omega = \mathbb{R}^N \setminus \mathcal{O}$. Then $\mathcal{O}$ represents the obstacle while Ω the part filled in by the fluid flowing past the obstacle. We add to the system (4.13) conditions at the boundary of Ω and at infinity.

Let us assume that there is a prescribed, non–zero constant velocity at infinity $\mathbf{v}_\infty$. We may always rotate the system of coordinates in such a way that $\mathbf{v}_\infty = \beta \mathbf{e}_1$, where $\mathbf{e}_1$ is the unit vector in the direction of x_1.

Let $\mathbf{v}_0$ be a prescribed velocity on $\partial\Omega$ satisfying $\mathbf{v}_0 \cdot \mathbf{n} = 0$; $\mathbf{n}$ the outer normal to $\partial\Omega$. (This condition is obvious since we suppose that the velocity does not penetrate the solid boundary.) Denoting $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$ we rewrite (4.13) as follows

$$\begin{gathered}
-\eta\Delta\mathbf{u} + \varrho\beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla\pi = \\
= \varrho\mathbf{f} + \nabla\cdot\Big[\mathbf{F}(\nabla\mathbf{u},\mathbf{T}) - \lambda\varrho((\mathbf{u}\cdot\nabla)\mathbf{u})\otimes\mathbf{u} - \varrho\mathbf{u}\otimes\mathbf{u} - \lambda\varrho\beta^2\frac{\partial \mathbf{u}}{\partial x_1}\otimes\mathbf{e}_1 - \\
-\lambda\varrho\beta\Big(\frac{\partial \mathbf{u}}{\partial x_1}\otimes\mathbf{u} + ((\mathbf{u}\cdot\nabla)\mathbf{u})\otimes\mathbf{e}_1\Big) + \lambda\varrho\mathbf{f}\otimes(\mathbf{u}+\beta\mathbf{e}_1) + \lambda p(\nabla\mathbf{u})^T\Big] \\
\nabla\cdot\mathbf{u} = 0 \qquad (4.14) \\
\pi = p + \lambda((\mathbf{u}+\mathbf{v}_\infty)\cdot\nabla)p \\
\mathbf{T} + \lambda((\mathbf{u}+\mathbf{v}_\infty)\cdot\nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{u},\mathbf{T}) = 2\eta\mathbf{D}(\mathbf{u})\,.
\end{gathered}$$

The system (4.14) holds in Ω and we have furthermore

$$\begin{gathered}
\mathbf{u} = \mathbf{v}_0 - \beta\mathbf{e}_1 \text{ at } \partial\Omega \\
\mathbf{u} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty\,.
\end{gathered} \qquad (4.15)$$

We easily check that $\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n}\mathrm{d}S = 0$.

Remark 4.1 We shall assume throughout this thesis $\mathbf{v}_0 = \mathbf{0}$. If $\mathbf{v}_0 \neq \mathbf{0}$, the natural assumption is $\mathbf{v}_0\cdot\mathbf{n} = 0$ pointwise at $\partial\Omega$ as we study the flow past an obstacle. Under this assumption, everything proved here remains true for $\mathbf{v}_0$ nonzero, but small. We namely extend $\mathbf{v}_0$ to Ω in such a way that $\nabla\cdot\mathbf{v}_0 = 0$ in Ω and $\mathbf{v}_0$ has bounded support. We then search $\mathbf{v}$ in the form $\mathbf{v}_0 + \mathbf{u} + \mathbf{v}_\infty$ and assuming $\mathbf{v}_0$ sufficiently small we proceed exactly in the same way as in the following chapters.

We shall construct solutions to (4.14)–(4.15) with an appropriate asymptotic structure by means of a fixed point argument. First, let us note that we do not study the precise estimates on the data, under which the solution exists (due to the technique, the data have to be sufficiently small). For the sake of simplicity, we assume all constant except to $|\mathbf{v}_\infty| = \beta$ to be equal to 1, i.e. $\varrho = \lambda = \eta = 1$. We denote by

$$A(\mathbf{u}) = -\Delta\mathbf{u} + \beta^2\frac{\partial^2\mathbf{u}}{\partial x_1^2}\,. \qquad (4.16)$$

Let $\mathbf{w}, s$ be a fixed pair of a vector and a scalar function. Requiring that $\mathbf{w} + \mathbf{v}_\infty = \mathbf{0}$ on $\partial\Omega$ we consider the operator

$$\mathcal{M}: \quad (\mathbf{w}, s) \to (\mathbf{u}, \pi)\,, \qquad (4.17)$$

where

$$\begin{gathered}
A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla\pi = \\
= \mathbf{f} + \nabla\cdot\Big[\mathbf{F}(\nabla\mathbf{w},\mathbf{T}) - ((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{w} - \mathbf{w}\otimes\mathbf{w} - \\
-\beta\Big(\frac{\partial \mathbf{w}}{\partial x_1}\otimes\mathbf{w} + ((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{e}_1\Big) + \mathbf{f}\otimes(\mathbf{w}+\beta\mathbf{e}_1) + p(\nabla\mathbf{w})^T\Big] \qquad (4.18) \\
\nabla\cdot\mathbf{u} = 0 \\
p + ((\mathbf{w}+\mathbf{v}_\infty)\cdot\nabla)p = s \\
\mathbf{T} + ((\mathbf{w}+\mathbf{v}_\infty)\cdot\nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{w},\mathbf{T}) = 2\mathbf{D}(\mathbf{w})\,,
\end{gathered}$$

$$\begin{aligned} \mathbf{u} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty \\ \mathbf{u} = -\beta \mathbf{e}_1 \text{ at } \partial\Omega\,. \end{aligned} \tag{4.19}$$

Therefore searching a solution to (4.1)–(4.7) (or, equivalently to (4.14)–(4.15)) means in fact searching a fixed point of the operator $\mathcal{M}$. We shall specify the correct spaces on which works $\mathcal{M}$ later on. Let us only note that for $\beta < 1$ the operator $A(\mathbf{u})$ defined by (4.16) is strongly elliptic. The necessity of using such an operator instead of the laplacian comes from the fact that the weighted estimates for the Oseen kernels, obtained in Chapter II, are unlike the singular integral operators not "optimal" and we loose "epsilon" in the weight. Therefore the linear term must be involved in the left hand side of (4.18).

We have decomposed the original problem into two kinds of linear problems. The equations $(4.18)_{3,4}$ are scalar and tensor steady transport equations, respectively. We shall study this kind of linear problems in Chapter IV. Next, the linear problem $(4.18)_{1,2}$ is similar to the classical Oseen problem. For $\beta < 1$ it can be expected that it will have similar properties as the Oseen problem. We shall call the system $(4.18)_{1,2}$ together with the "boundary" conditions (4.19) modified Oseen problem and the detailed study of this problem is performed in Chapter III. We shall verify that it has very similar (or almost the same) properties as the classical Oseen problem which will be studied in Chapter II. Chapters V and VI will be devoted to the study of existence and asymptotic structure of solutions to the nonlinear problem.

Next, let us present the formulation of the problem in the case of $\eta^S \neq 0$. We shall study the Oldroyd type models (i.e. $\mathbf{B} \equiv 0$) or their slight generalization (i.e. $\mathbf{B}$ bilinear). Unlike the previous case, it is not necessary (and, unfortunately, also impossible) to reformulate the original problem as between (4.8)–(4.12) for the Maxwell type fluids; we start directly from (4.2). We have

$$\begin{aligned} -\eta^S \Delta \mathbf{v} + \varrho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= \varrho \mathbf{f} + \nabla\cdot\mathbf{T} \\ \nabla\cdot\mathbf{v} &= 0 \\ \mathbf{T} + \lambda(\mathbf{v}\cdot\nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{v},\mathbf{T}) &= 2\eta^P \mathbf{D}\,, \end{aligned} \tag{4.20}$$

where $\mathbf{G}(\nabla\mathbf{v},\mathbf{T})$ satisfies (4.12). Such models were studied by the above mentioned authors. Let us also mention the work of Videman [Vi] and the references therein, where also problems on unbounded domains are studied, both with compact and noncompact boundary. Nevertheless, the asymptotic structure at infinity for such models in exterior domains has been studied neither in [Vi] nor anywhere else[8] and it seems to be more difficult due to the "epsilon" loss in the weighted estimates, similarly to the model of the second grade fluid presented below.

We shall proceed now as above. We add to the system (4.20) the boundary conditions at $\partial\Omega$ ($\mathbf{v}_0 = \mathbf{0}$) and at infinity ($\mathbf{v}_\infty \neq \mathbf{0}$), denote by $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$,

[8] at least to the knowledge of the author

assume all constants (up to β and η^P) equal to one and end up with the system

$$\begin{aligned} -\Delta \mathbf{u} + \beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla p &= \mathbf{f} + \nabla \cdot (\mathbf{T} - \mathbf{u} \otimes \mathbf{u}) \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{T} + ((\mathbf{u} + \mathbf{v}_\infty) \cdot \nabla)\mathbf{T} + \mathbf{G}(\nabla \mathbf{u}, \mathbf{T}) &= 2\eta^P \mathbf{D}(\mathbf{u}) \end{aligned} \tag{4.21}$$

in Ω and

$$\begin{aligned} \mathbf{u} &\to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty \\ \mathbf{u} &= -\beta \mathbf{e}_1 \text{ at } \partial\Omega\,. \end{aligned} \tag{4.22}$$

We introduce again the operator $\mathcal{M}$, which assigns to a vector function $\mathbf{w}$ the vector function $\mathbf{u}$, solution to

$$\begin{aligned} -\Delta \mathbf{u} + \beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla p &= \mathbf{f} + \nabla \cdot [\mathbf{T}(\mathbf{w}) - \mathbf{w} \otimes \mathbf{w}] \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{T} + ((\mathbf{w} + \mathbf{v}_\infty) \cdot \nabla)\mathbf{T} + \mathbf{G}(\nabla \mathbf{w}, \mathbf{T}) &= 2\eta^P \mathbf{D}(\mathbf{w})\,, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \mathbf{u} &\to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty \\ \mathbf{u} &= -\beta \mathbf{e}_1 \text{ at } \partial\Omega\,. \end{aligned} \tag{4.24}$$

We are therefore left with the classical Oseen problem and a (tensor) steady transport equation. Due to the linear dependence of $\mathbf{T}$ on $\nabla \mathbf{w}$ we shall only be able to reobtain the result presented in [Vi], i.e. the existence of solutions in Sobolev spaces, for β and η^P sufficiently small; our technique does not allow to study the asymptotic structure of solution to (4.21)–(4.22). We shall only mention this results in Chapter V without proving them explicitly.

I.4.2 Second grade fluid

Before starting the study of the linear problems we shall briefly introduce another model of non–Newtonian fluid — the second grade fluid — and show that the system of equations describing its stationary flow can be after a proper linearization rewritten into a similar kind of linear problems; in this case we obtain the classical Oseen problem and the scalar transport equation.

The constitutive law characterizing the second-grade fluid has the form (see e.g. [TrNo])

$$\mathbf{T}^E = 2\mu \mathbf{D} + 2\alpha_1 \mathbf{A_1} + 4\alpha_2 \mathbf{D}^2\,, \tag{4.25}$$

where μ is viscosity, α_1 and α_2 are stress moduli and

$$\mathbf{A_1} = \frac{d}{dt}\mathbf{D} + (\nabla \mathbf{v})^T \mathbf{D} + \mathbf{D} \nabla \mathbf{v}\,. \tag{4.26}$$

We use the condition of thermodynamical stability $\alpha_1 + \alpha_2 = 0$, see [DuFo], and get from (4.1), (4.2), (4.25) and (4.26) in the case of the steady flow past an obstacle

$$
\begin{aligned}
-\mu\Delta\mathbf{v} - \alpha_1(\mathbf{v}\cdot\nabla)\Delta\mathbf{v} + \nabla p &= -\varrho(\mathbf{v}\cdot\nabla)\mathbf{v} + \varrho\mathbf{f} + \\
&+ \alpha_1\nabla\cdot[(\nabla\mathbf{v})^T(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)] \\
\nabla\cdot\mathbf{v} &= 0 \\
\mathbf{v} = \mathbf{0} &\qquad \text{at } \partial\Omega = \partial\mathcal{O} \\
\mathbf{v}\to\mathbf{v}_\infty &\qquad \text{as } |\mathbf{x}|\to\infty\,.
\end{aligned}
\tag{4.27}
$$

Again, assuming $\mathbf{v}_\infty \neq \mathbf{0}$ we can rotate the coordinate system in such a way that $\mathbf{v}_\infty = \beta\mathbf{e}_1$ and denoting $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$ we get from (4.27)

$$
\begin{aligned}
-\mu\Delta\mathbf{u} - \alpha_1(\mathbf{u}\cdot\nabla)\Delta\mathbf{u} - \alpha_1\beta\Delta\frac{\partial\mathbf{u}}{\partial x_1} + \varrho\beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla p &= \\
= -\varrho(\mathbf{u}\cdot\nabla)\mathbf{u} + \varrho\mathbf{f} + \alpha_1\nabla\cdot[(\nabla\mathbf{u})^T(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)]& \\
\nabla\cdot\mathbf{u} = 0& \\
\mathbf{u} = -\mathbf{v}_\infty = -\beta\mathbf{e}_1 \qquad \text{at } \partial\Omega& \\
\mathbf{u}\to\mathbf{0} \qquad \text{as } |\mathbf{x}|\to\infty\,.&
\end{aligned}
$$

Using the decomposition procedure proposed by Mogilevskij and Solonnikov (see [MoSo]) we consider formally the mapping

$$
\mathcal{M} : \mathbf{g} \mapsto (\mathbf{u}, s) \mapsto \mathbf{z}\,,
$$

where

$$
\begin{aligned}
-\Delta\mathbf{u} + \varrho\frac{\beta}{\mu}\frac{\partial\mathbf{u}}{\partial x_1} + \nabla s &= \mathbf{g} \\
\nabla\cdot\mathbf{u} &= 0 \\
\mathbf{u} = -\beta\mathbf{e}_1 &\qquad \text{at } \partial\Omega \\
\mathbf{u}\to\mathbf{0} &\qquad \text{as } |\mathbf{x}|\to\infty\,,
\end{aligned}
\tag{4.28}
$$

it means that the pair $(\mathbf{u}, s)$ satisfies the Oseen problem with the right hand side $\mathbf{g}$, and

$$
\begin{aligned}
\mu\mathbf{z} + \alpha_1((\mathbf{u}+\mathbf{v}_\infty)\cdot\nabla)\mathbf{z} &= -\varrho(\mathbf{u}\cdot\nabla)\mathbf{u} + \varrho\mathbf{f} + \alpha_1\frac{\varrho\beta^2}{\mu}\frac{\partial^2\mathbf{u}}{\partial x_1^2} + \\
&+ \alpha_1\nabla\cdot\Big[(\nabla\mathbf{u})^T(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) + \frac{\varrho\beta}{\mu}\frac{\partial\mathbf{u}}{\partial x_1}\otimes\mathbf{u} - s\,(\nabla\mathbf{u})^T\Big]\,,
\end{aligned}
\tag{4.29}
$$

it means that $\mathbf{z}$ satisfies the transport equation with the right hand side depending on $(\mathbf{u}, s)$.

We meet again the same problem as in the case of the problem (4.23), (4.24). The presence of the linear term on the right hand side of (4.29) does not allow us to investigate the asymptotic structure of the solution.

The model of second grade fluid was studied for different types of domains by several authors, see e.g. [DuFo], [DuRa], [GaSe], [NoSeVi], [PiSeVi], [Vi]. In Chapter V, we shall present the proof of existence of strong solutions to (4.27) for the plane flow which is taken from [Po]. The existence was also shown in [Vi] for both plane and threedimensional flows, under slightly more restrictive conditions on the right-hand side.

II

Oseen problem

In this chapter we would like to present some classical but also some new results concerning the Oseen flow in unbounded domains. First we recall the notion of the fundamental Oseen tensor (a little bit more precisely as it is essential for our following study) and very briefly some existence and uniqueness theorems; similar results will be shown in the following chapter for the modified Oseen problem and the classical Oseen problem can be considered as a special case. The asymptotic properties of the fundamental solution enable us to obtain the integral representation of solutions to the Oseen problem. The last subchapter is devoted to the study of some convolution coming from the integral representation. We give a detailed theory of L^p–weighted estimates of convolutions with Oseen kernels and apply it for a very trivial case. Such estimates will be then essential in Chapter VI.

Let us recall that we study the following problem ($\Omega = \mathbb{R}^N \setminus \mathcal{O}$, exterior domain)

$$\left.\begin{aligned} -\Delta \mathbf{v} + \beta \frac{\partial \mathbf{v}}{\partial x_1} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{v} = 0 \end{aligned}\right\} \text{in } \Omega \qquad (0.1)$$
$$\mathbf{v} = \mathbf{v}_* \qquad \text{at } \partial\Omega$$
$$\mathbf{v} \to 0 \qquad \text{as } |\mathbf{x}| \to \infty\,.$$

Definition 0.1 *A vector field $\mathbf{v} \in W^{1,q}_{loc}(\Omega)$ is called a q-weak*[1] *solution to (0.1) if for some $q \in (1;\infty)$*

(i) $\mathbf{v}$ *is (weakly) divergence free in* Ω

(ii) $\mathbf{v}$ *assumes the value* $\mathbf{v}_*$ *at* $\partial\Omega$ *(in the trace sense)*

(iii)

$$\lim_{R\to\infty} \int_{S_N} |\mathbf{v}(R,\omega)|\, d\omega = 0$$

(see Subsection VIII.1.4)

(iv) the equality

$$\int_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\varphi} d\mathbf{x} - \beta \int \mathbf{v} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_1} d\mathbf{x} = < \mathbf{f}, \boldsymbol{\varphi} >$$

is satisfied for all

$$\boldsymbol{\varphi} \in {}_0\mathcal{D}(\Omega) = \{\boldsymbol{\varphi} \in C_0^\infty(\Omega)\,, \nabla \cdot \boldsymbol{\varphi} = 0\}\,.$$

[1] We use also a q–generalized solution. If $q = 2$, we shall usually speak only about a weak (generalized) solution.

II.1 Oseen fundamental tensor and its asymptotic properties

In this part we follow [Ga1] or more originally, [Os]. We denote by

$$\mathcal{O}_{ij}(\mathbf{x},\mathbf{y}) = -\Big(\delta_{ij}\Delta - \frac{\partial^2}{\partial y_i \partial y_j}\Big)\Phi(\mathbf{x},\mathbf{y}) \tag{1.1}$$

$$e_j(\mathbf{x},\mathbf{y}) = -\frac{\partial}{\partial y_j}\Big(\Delta + 2\lambda \frac{\partial}{\partial y_1}\Big)\Phi(\mathbf{x},\mathbf{y})\,, \tag{1.2}$$

where i, $j = 1, 2, \ldots, N$, $\lambda = \frac{\beta}{2}$ and $\Phi(\mathbf{x},\mathbf{y})$ is a smooth function for $\mathbf{x} \neq \mathbf{y}$. We easily check that

$$\begin{aligned} -\Big(\Delta + 2\lambda\frac{\partial}{\partial y_1}\Big)\mathcal{O}_{ij}(\mathbf{x},\mathbf{y}) - \frac{\partial}{\partial y_i} e_j(\mathbf{x},\mathbf{y}) &= \delta_{ij}\Delta\Big(\Delta + 2\lambda\frac{\partial}{\partial y_1}\Big)\Phi(\mathbf{x},\mathbf{y}) \\ \frac{\partial}{\partial y_l}\mathcal{O}_{lj}(\mathbf{x},\mathbf{y}) &= 0\,. \end{aligned} \tag{1.3}$$

We want (1.1) and (1.2) to be a singular solution to $(0.1)_{1,2}$. We require therefore

$$\Delta\Big(\Delta + 2\lambda\frac{\partial}{\partial y_1}\Big)\Phi(\mathbf{x},\mathbf{y}) = \Delta\mathcal{E}(|\mathbf{x}-\mathbf{y}|)\,, \tag{1.4}$$

where $\mathcal{E}(|\mathbf{x}-\mathbf{y}|)$ is the fundamental solution to the Laplace equation. So, particularly, for $N = 2$ $\mathcal{E}(|\mathbf{x}-\mathbf{y}|) = \frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{y}|$, for $N = 3$ $\mathcal{E}(|\mathbf{x}-\mathbf{y}|) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}$ and the right hand side of (1.3) is equal to $\delta_{ij}\delta_{\mathbf{x}}$.[2] The system (1.3) must be considered in the sense of distributions (see Section VIII.4).

We search the solution to (1.4) in the form

$$\begin{aligned} \Phi(\mathbf{x},\mathbf{y}) = \frac{1}{2\lambda}\int^{y_1-x_1} & [\Phi_2(\tau, y_2 - x_2\,, \ldots, y_n - x_n) - \\ & -\Phi_1(\tau, y_2 - x_2, \ldots, y_n - x_n)]\mathrm{d}\tau\,, \end{aligned} \tag{1.5}$$

where Φ_1 and Φ_2 must be selected suitably. From (1.4) we get formally

$$\Delta\Big(\Delta + 2\lambda\frac{\partial}{\partial y_1}\Big)(\Phi_2 - \Phi_1) = 2\lambda\Delta\Big(\frac{\partial\mathcal{E}}{\partial y_1}\Big)\,. \tag{1.6}$$

Let us choose $\Phi_2(\mathbf{x},\mathbf{y}) = \mathcal{E}(|\mathbf{x}-\mathbf{y}|)$; now it is sufficient to take

$$\Big(\Delta + 2\lambda\frac{\partial}{\partial y_1}\Big)\Phi_1 = \Delta\mathcal{E}\,. \tag{1.7}$$

Moreover, from (1.4) and (1.2) we have

$$e_j(\mathbf{x},\mathbf{y}) = -\frac{\partial}{\partial y_j}\mathcal{E}(|\mathbf{x}-\mathbf{y}|)\,. \tag{1.8}$$

We take

$$\Phi_1(\mathbf{x},\mathbf{y}) = \frac{e^{-\lambda(y_1-x_1)}}{|\mathbf{x}-\mathbf{y}|^{\frac{N-2}{2}}} f(\lambda|\mathbf{x}-\mathbf{y}|)\,,$$

[2] It means for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\langle\delta_{\mathbf{x}},\varphi\rangle = \varphi(\mathbf{x})$.

so by direct calculations we deduce

$$\Big(\Delta + 2\lambda \frac{\partial}{\partial y_1}\Big)\Phi_1(\mathbf{x},\mathbf{y}) = \frac{e^{-\lambda(y_1-x_1)}}{|\mathbf{x}-\mathbf{y}|^{\frac{N+2}{2}}}\Big[z^2 f''(z) + zf'(z) - \\ -\Big(\Big[\frac{N-2}{2}\Big]^2 + z^2\Big)f(z)\Big] \equiv \frac{e^{-\lambda(y_1-x_1)}}{|\mathbf{x}-\mathbf{y}|^{\frac{N+2}{2}}}\mathcal{L}(f)\,, \tag{1.9}$$

where $z = \lambda|\mathbf{x}-\mathbf{y}|$ and the prime denotes differentiation with respect to z. The equation $\mathcal{L}(f) = 0$ is the Bessel modified equation which admits two independent solutions $I_{\frac{N-2}{2}}(z)$ and $K_{\frac{N-2}{2}}(z)$ called modified Bessel's functions. While $I_{\frac{N-2}{2}}(z)$ is regular for all values of the argument, $K_{\frac{N-2}{2}}(z)$ is singular at $z = 0$. We have for $z > 0$ (see [KoKo])

$$\text{(i) } N=2 \;\; K_0(z) = -\ln z + \ln 2 - \gamma - \ln\Big(\frac{z}{2}\Big)\sum_{k=1}^{\infty}\frac{1}{(k!)^2}\Big(\frac{z}{2}\Big)^{2k} + \\ + \sum_{k=1}^{\infty}\frac{1}{(k!)^2}\Big(\sum_{j=1}^{k}\Big(\frac{1}{j}\Big) - \gamma\Big)\Big(\frac{z}{2}\Big)^{2k} \tag{1.10}$$

$$\text{(ii) } N=3 \;\; K_{\frac{1}{2}}(z) = \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z}\,,$$

while for $N \geq 4$ we get the following asymptotic expansion for z sufficiently small

$$K_{\frac{N-2}{2}}(z) = \frac{2^{\frac{N-2}{2}}\Gamma(\frac{N}{2})}{N-2}\frac{1}{z^{\frac{N-2}{2}}} + \sigma(z)\,, \tag{1.11}$$

where γ is the Euler constant, Γ is the gamma function and the remainder σ satisfies

$$\frac{d^k\sigma}{dz^k} = o(z^{\frac{2-N}{2}-k})\,, \qquad k \geq 0\,, \quad |z| \to 0.$$

In what follows we shall treat separately the cases $N = 2$ and $N = 3$.

II.1.1 Fundamental solution in two dimensions

We start with the more complicated situation in two spatial dimensions. As follows from (1.7), Φ_1 must be in the neighborhood of $z = 0$ as singular as $\mathcal{E}$ is. So we get

$$\Phi_1(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}K_0(\lambda|\mathbf{x}-\mathbf{y}|)e^{-\lambda(y_1-x_1)} \tag{1.12}$$

and finally

$$\Phi(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi\lambda}\int^{y_1-x_1}\Big\{\log\sqrt{\tau^2+(x_2-y_2)^2} + \\ + K_0\Big(\lambda\sqrt{\tau^2+(x_2-y_2)^2}\Big)e^{-\lambda\tau}\Big\}\mathrm{d}\tau. \tag{1.13}$$

The problem consists in the right choice of the constants which corresponds to the right choice of the lower bound of the integral on the right hand side of (1.13). Formally we need it to be equal to ∞. Unfortunately the integral in

(1.13) does not converge, as K_0 behaves regularly at infinity (see (1.23)). We calculate formally the derivatives of (1.13) and put

$$G(\mathbf{x}-\mathbf{y};2\lambda)=\frac{1}{4\pi\lambda}\Big[\log\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}+ + K_0\Big(\lambda\sqrt{(y_1-x_1)^2+(x_2-y_2)^2}\Big)e^{-\lambda(y_1-x_1)}\Big]\mathrm{d}\tau, \tag{1.14}$$

$$H(\mathbf{x}-\mathbf{y};2\lambda)=\frac{1}{4\pi\lambda}\int_\infty^{y_1-x_1}\Big[\frac{\tau^2-(y_2-x_2)^2}{(\tau^2+(y_2-x_2)^2)^2}+ + \Big(K_0''\Big(\lambda\sqrt{\tau^2+(y_2-x_2)^2}\Big)\frac{\lambda^2(y_2-x_2)^2}{\tau^2+(y_2-x_2)^2}+ + K_0'(\lambda\sqrt{\tau^2+(y_2-x_2)^2})\frac{\lambda\tau^2}{(\tau^2+(y_2-x_2)^2)^{\frac{3}{2}}}\Big)e^{-\lambda\tau}\Big]d\tau\,, \tag{1.15}$$

i.e G is formally taken derivative of (1.13) with respect to y_1, H the second derivative of (1.13) with respect to y_2^2. First we express (1.15) in a more appropriate way (without the integrals).

Denoting $q=y_2-x_2$ we calculate: $(\frac{d}{d\tau}K_0(z)=K_0'(z)\frac{\lambda\tau}{\sqrt{\tau^2+q^2}})$

$$\begin{aligned}
H(\mathbf{x}-\mathbf{y}\,;2\lambda) &= \frac{1}{4\pi\lambda}\int_\infty^{y_1-x_1}\Big[\frac{\tau^2-q^2}{(\tau^2+q^2)^2}+\Big(K_0''(\lambda\sqrt{\tau^2+q^2})\frac{\lambda^2q^2}{\tau^2+q^2}+\\
&+K_0'(\lambda\sqrt{\tau^2+q^2})\frac{\lambda\tau^2}{(\tau^2+q^2)^{\frac{3}{2}}}\Big)e^{-\lambda\tau}\Big]\mathrm{d}\tau=\\
&=\frac{1}{4\pi\lambda}\int_\infty^{y_1-x_1}\Big\{\frac{d}{d\tau}\Big(\frac{-\tau}{\tau^2+q^2}\Big)+\Big[K_0''(\lambda\sqrt{\tau^2+q^2})\frac{\lambda^2q^2}{\tau^2+q^2}+\\
&+K_0'(\lambda\sqrt{\tau^2+q^2})\lambda\tau\frac{d}{d\tau}\Big(\frac{-1}{\sqrt{\tau^2+q^2}}\Big)\Big]e^{-\lambda\tau}\Big\}\mathrm{d}\tau=\\
&=\frac{1}{4\pi\lambda}\Big[\frac{-\tau}{\tau^2+q^2}\Big]_\infty^{y_1-x_1}-\frac{1}{4\pi\lambda}\Big[K_0'(\lambda\sqrt{\tau^2+q^2})\frac{\lambda\tau}{\sqrt{\tau^2+q^2}}e^{-\lambda\tau}\Big]_\infty^{y_1-x_1}+\\
&+\frac{1}{4\pi\lambda}\int_\infty^{y_1-x_1}\Big[K_0''(\lambda\sqrt{\tau^2+q^2})\lambda^2+K_0'(\lambda\sqrt{\tau^2+q^2})\frac{\lambda^2}{\lambda\sqrt{\tau^2+q^2}}-\\
&-\frac{d}{d\tau}K_0(\lambda\sqrt{\tau^2+q^2})\lambda\Big]e^{-\lambda\tau}d\tau=-\frac{1}{4\pi\lambda}\Big\{\frac{y_1-x_1}{(y_1-x_1)^2+(y_2-x_2)^2}+\\
&+\Big[K_0'\Big(\lambda\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}\Big)\frac{\lambda(y_1-x_1)}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}}+\\
&+K_0\Big(\lambda\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}\Big)\lambda\Big]\cdot e^{-\lambda(y_1-x_1)}\Big\}\,,
\end{aligned}$$

where we used that $\mathcal{L}(K_0)=0$ outside $\mathbf{x}=\mathbf{y}$ (see (1.9)).

We define $\Big(r=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}\Big)$

$$
\begin{aligned}
\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda) &= -H(\mathbf{x}-\mathbf{y};2\lambda) \;= \\
&= \frac{1}{4\pi\lambda}\Big[\frac{y_1-x_1}{r^2}+\Big(K_0'(\lambda r)\frac{\lambda(y_1-x_1)}{r}+K_0(\lambda r)\lambda\Big)e^{-\lambda(y_1-x_1)}\Big] \\
\mathcal{O}_{12}(\mathbf{x}-\mathbf{y};2\lambda) &= \mathcal{O}_{21}(\mathbf{x}-\mathbf{y};2\lambda) \;=\; \frac{\partial}{\partial y_2}G(x-y;2\lambda) = \\
&= \frac{1}{4\pi\lambda}\Big[\frac{y_2-x_2}{r^2}+K_0'(\lambda r)\lambda\frac{y_2-x_2}{r}e^{-\lambda(y_1-x_1)}\Big] \\
\mathcal{O}_{22}(\mathbf{x}-\mathbf{y};2\lambda) &= -\frac{\partial}{\partial y_1}G(\mathbf{x}-\mathbf{y};2\lambda) \;= \\
&= -\frac{1}{4\pi\lambda}\Big[\frac{y_1-x_1}{r^2}+\Big(K_0'(\lambda r)\lambda\frac{y_1-x_1}{r}-K_0(\lambda r)\lambda\Big)e^{-\lambda(y_1-x_1)}\Big]
\end{aligned}
\tag{1.16}
$$

$$
e_j(\mathbf{x},\mathbf{y}) = -\frac{\partial\mathcal{E}(|\mathbf{x}-\mathbf{y}|)}{\partial y_j} = -\frac{1}{2\pi}\frac{y_j-x_j}{r^2}\,. \tag{1.17}
$$

We shall now study the asymptotic structure of $\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};2\lambda)$ defined in (1.16) near zero. Using it we then verify that $(\boldsymbol{\mathcal{O}},\mathbf{e})$ solves (1.3) in the sense of distributions. From (1.16) we easily deduce the following homogeneity property

$$
\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};2\lambda) = \mathcal{O}_{ij}(2\lambda(\mathbf{x}-\mathbf{y});1) \tag{1.18}
$$

which will play an important role later on.

Now, let $\lambda r\to 0$. Then we have from (1.10) (i)

$$
\begin{aligned}
K_0(z) &= -\ln z+\ln 2-\gamma+O(z^2\ln z) \\
K_0'(z) &= -\frac{1}{z}+O(z\ln z) \\
K_0''(z) &= \frac{1}{z^2}+O(\ln z) \\
K_0^{(k)}(z) &= \frac{C(k)}{z^k}+O(z^{-k+2})\,,\quad k\ge 3\,,\quad |z|\to 0
\end{aligned}
\tag{1.19}
$$

and therefore

$$
\begin{aligned}
\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda) = \frac{1}{4\pi\lambda}\Big\{\frac{y_1-x_1}{r^2}-\frac{y_1-x_1}{r^2}e^{-\lambda(y_1-x_1)}-\lambda\ln(\lambda r)e^{-\lambda(y_1-x_1)}\Big\}+ \\
+\frac{\ln 2-\gamma}{4\pi}e^{-\lambda(y_1-x_1)}+\nu_{11}(\lambda r)
\end{aligned}
$$

with

$$
\begin{aligned}
\nu_{11}(\lambda r) &= \nu_{11}(z) = O(z\ln z) \\
\nu_{11}'(z) &= O(\ln z) \\
\nu_{11}^{(k)}(z) &= O(z^{-k+1})\,,\quad k\ge 2\,,\quad |z|\to 0\,.
\end{aligned}
$$

Thus

$$
\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda) = \frac{1}{4\pi}\Big(\frac{(y_1-x_1)^2}{r^2}+\ln\frac{1}{2\lambda r}\Big)+\tilde{\nu}_{11}(\lambda r)
$$

with

$$
\begin{aligned}
\tilde{\nu}_{11}(z) &= O(1) \\
\tilde{\nu}_{11}'(z) &= O(\ln z) \\
\tilde{\nu}_{11}^{(k)}(z) &= O(z^{-k+1})\,,\quad k\ge 2\,,\quad z\to 0\,.
\end{aligned}
$$

Similarly we proceed for other terms:

$$
\begin{aligned}
\mathcal{O}_{12}(\mathbf{x}-\mathbf{y};2\lambda) &= \frac{1}{4\pi\lambda}\Big(\frac{y_2-x_2}{r^2}-\frac{y_2-x_2}{r^2}e^{-\lambda(y_1-x_1)}\Big)+\tilde{\nu}_{12}(\lambda r) = \\
&= \frac{1}{4\pi\lambda}\frac{(y_2-x_2)(y_1-x_1)}{r^2}+\tilde{\nu}_{12}(\lambda r) \\
\mathcal{O}_{22}(\mathbf{x}-\mathbf{y};2\lambda) &= -\frac{1}{4\pi\lambda}\Big(\frac{y_1-x_1}{r^2}-\frac{y_1-x_1}{r^2}e^{-\lambda(y_1-x_1)}+ \\
&\quad + \ln(\lambda r)\lambda e^{-\lambda(y_1-x_1)}\Big)+\frac{\ln 2-\gamma}{4\pi}e^{-\lambda(y_1-x_1)}+\nu_{22}(\lambda r) = \\
&= -\frac{1}{4\pi}\Big(\frac{(y_1-x_1)^2}{r^2}+\ln(\lambda r)e^{-\lambda(y_1-x_1)}\Big)+\tilde{\nu}_{22}(\lambda r) = \\
&= \frac{1}{4\pi}\Big(\frac{(y_2-x_2)^2}{r^2}+\ln\frac{1}{2\lambda r}\Big)+\tilde{\nu}_{22}(\lambda r)\,.
\end{aligned}
$$

Summarizing we get

$$
\begin{aligned}
\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};2\lambda) &= \frac{1}{4\pi}\left[\delta_{ij}\log\frac{1}{2\lambda r}+\frac{(y_i-x_i)(y_j-x_j)}{r^2}\right]+\tilde{\nu}_{ij}(\lambda r) = \\
&= \mathcal{S}_{ij}(\mathbf{x}-\mathbf{y})+\frac{1}{4\pi}\delta_{ij}\log\frac{1}{2\lambda}+\tilde{\nu}_{ij}(\lambda r)\,,
\end{aligned}
\tag{1.20}
$$

where $\mathcal{S}_{ij}(\mathbf{x}-\mathbf{y})$ is the fundamental Stokes tensor (see e.g. [Ga1]) and

$$
\begin{aligned}
\tilde{\nu}_{ij}(z) &= \begin{cases} O(1) & (i=j) \\ O(z\ln z) & (i\neq j) \end{cases} \\
\tilde{\nu}'_{ij}(z) &= O(\ln z) \\
\tilde{\nu}^{(k)}_{ij} &= O(z^{-k+1}) \qquad k\geq 2\,, \qquad z\to 0\,.
\end{aligned}
$$

We are going to verify that $(\boldsymbol{\mathcal{O}},\mathbf{e})$ solves (1.3) in the sense of distributions. Let us observe from (1.16) that

$$
\begin{aligned}
\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda) &= \frac{1}{2\lambda}\frac{\partial\mathcal{E}}{\partial y_1}(\mathbf{x}-\mathbf{y})-\frac{1}{2\lambda}\frac{\partial\Phi_1}{\partial y_1}(\mathbf{x}-\mathbf{y};2\lambda)- \\
&\quad -\Phi_1(\mathbf{x}-\mathbf{y};2\lambda) \\
\mathcal{O}_{12}(\mathbf{x}-\mathbf{y};2\lambda) &= \mathcal{O}_{21}(\mathbf{x}-\mathbf{y};2\lambda) = \frac{1}{2\lambda}\frac{\partial\mathcal{E}}{\partial y_2}(\mathbf{x}-\mathbf{y})- \\
&\quad -\frac{1}{2\lambda}\frac{\partial\Phi_1}{\partial y_2}(\mathbf{x}-\mathbf{y};2\lambda) \\
\mathcal{O}_{22}(\mathbf{x}-\mathbf{y};2\lambda) &= -\frac{1}{2\lambda}\frac{\partial\mathcal{E}}{\partial y_1}(\mathbf{x}-\mathbf{y})+\frac{1}{2\lambda}\frac{\partial\Phi_1}{\partial y_1}(\mathbf{x}-\mathbf{y};2\lambda)\,,
\end{aligned}
\tag{1.21}
$$

where $\Phi_1(\mathbf{x}-\mathbf{y};2\lambda)=\Phi_1(\mathbf{x},\mathbf{y})$ is defined in (1.12). We therefore easily see that O_{ij},e_j solves $(1.3)_1$ for $\mathbf{x}\neq\mathbf{y}$. Moreover

$$
\frac{\partial\mathcal{O}_{12}}{\partial y_1}+\frac{\partial\mathcal{O}_{22}}{\partial y_2}=0 \qquad \text{for } \mathbf{x}\neq\mathbf{y}
$$

and

$$\frac{\partial \mathcal{O}_{11}}{\partial y_1} + \frac{\partial \mathcal{O}_{21}}{\partial y_2} = \frac{1}{2\lambda}\Delta_y \mathcal{E} - \frac{1}{2\lambda}\left(\frac{\partial^2 \Phi_1}{\partial y_1^2} + \frac{\partial^2 \Phi_1}{\partial y_2^2}\right) - \frac{\partial \Phi_1}{\partial y_1} =$$
$$= -\frac{1}{2\lambda}\left(\frac{\partial^2 \Phi_1}{\partial y_1^2} - \frac{\partial^2 \Phi_1}{\partial y_2^2}\right) - \frac{\partial \Phi_1}{\partial y_1}$$

for $\mathbf{x} \neq \mathbf{y}$. Let us verify that last expression is equal zero. Evidently

$$-\frac{1}{2\lambda}\left(\frac{\partial^2 \Phi_1}{\partial y_1^2} + \frac{\partial^2 \Phi_1}{\partial y_2^2}\right) + \frac{\partial \Phi_1}{\partial y_1} =$$
$$= \frac{1}{2\lambda}\Big(K_0''(\lambda r)\lambda^2 + K_0'(\lambda r)\frac{\lambda}{r} - \lambda^2 K_0(\lambda r)\Big)\mathrm{e}^{-\lambda(y_1 - x_1)} = 0$$

for $r \neq 0$ (see (1.9)). Moreover from (1.20) we have that $|\boldsymbol{\mathcal{O}}(\mathbf{x}-\mathbf{y})| \leq C \ln |\mathbf{x}-\mathbf{y}|$ for $|\mathbf{x}-\mathbf{y}| \ll 1$ (see (1.20)) and therefore

$$\frac{\partial \mathcal{O}_{ij}}{\partial y_i} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^2)\,.$$

It remains to verify that

$$-\left(\Delta + 2\lambda\frac{\partial}{\partial y_1}\right)\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda) - \frac{\partial}{\partial y_i} e_j(\mathbf{x}-\mathbf{y}) = \delta_{ij}\delta_{\mathbf{x}}\,.$$

Due to the asymptotic behaviour of $\boldsymbol{\mathcal{O}}$ and due to the fact that

$$\lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} e_j(\mathbf{x}-\mathbf{y}) n_i(\mathbf{y}) F_j(\mathbf{y}) \; d_{\mathbf{y}}S = \frac{1}{2}F_j(\mathbf{x})\delta_{ij}$$

we have to check that

$$\lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} \frac{\partial \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y})}{\partial n} F_j(\mathbf{y}) \; \mathrm{d}S = \frac{1}{2}F_j(\mathbf{x})\delta_{ij}\,. \tag{1.22}$$

Namely, then we easily get

$$\int_{\mathbb{R}^2} \left[\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda)[(-\Delta_{\mathbf{y}} + 2\lambda\frac{\partial}{\partial y_1})F_j(\mathbf{y})] + e_j(\mathbf{x}-\mathbf{y})\frac{\partial F_j}{\partial y_i}(\mathbf{y})\right] d\mathbf{y} =$$
$$= \text{v.p.} \int_{\mathbb{R}^2} \left[(-\Delta_{\mathbf{y}} - 2\lambda\frac{\partial}{\partial y_1})\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda) - \frac{\partial e_j}{\partial y_i}(\mathbf{x}-\mathbf{y})\right] F_j(\mathbf{y})\, d\mathbf{y} +$$
$$+ \lim_{\varepsilon \to 0^+} \Big[2\lambda \int_{\partial B^\varepsilon(\mathbf{x})} \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda) n_1(\mathbf{y}) F_j(\mathbf{y})\, d_{\mathbf{y}}S -$$
$$- \int_{\partial B^\varepsilon(\mathbf{x})} \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda) n_k(\mathbf{y}) \frac{\partial F_j}{\partial y_k}(\mathbf{y}) \mathrm{d}_{\mathbf{y}}S +$$
$$+ \int_{\partial B^\varepsilon(\mathbf{x})} \frac{\partial \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda)}{\partial n} F_j(\mathbf{y})\, d_{\mathbf{y}}S + \int_{\partial B^\varepsilon(\mathbf{x})} e_j(\mathbf{x}-\mathbf{y}) n_i(\mathbf{y}) F_j(\mathbf{y})\, d_{\mathbf{y}}S\Big] =$$
$$= \lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} \left[\frac{\partial \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda)}{\partial n} + e_j(\mathbf{x}-\mathbf{y}) n_i(\mathbf{y})\right] F_i(\mathbf{y})\, d_{\mathbf{y}}S = F_j(\mathbf{x})\delta_{ij}\,.$$

From (1.20) we get after straightforward calculations that

$$\begin{aligned}
\frac{\partial \mathcal{O}_{11}}{\partial n} &= \frac{1}{4\pi r} + O(\ln r)\\
\frac{\partial \mathcal{O}_{12}}{\partial n} = \frac{\partial \mathcal{O}_{21}}{\partial n} &= O(\ln r)\\
\frac{\partial O_{22}}{\partial n} &= \frac{1}{4\pi r} + O(\ln r)
\end{aligned}$$

and therefore (1.22) follows. Thus $(\boldsymbol{\mathcal{O}}, \mathbf{e})$, defined by (1.16) and (1.17) solves (1.3) in the sense of distributions, i.e. it is the fundamental solution to (0.1) in $\mathbb{R}^2$.

The next part is devoted to the asymptotic properties for $\lambda r \to \infty$. Unlike the Stokes fundamental tensor we get anisotropic structure here. First, let us recall that (see [KoKo]):

$$K_0(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\Big[\sum_{k=0}^{\nu-1} \frac{\Gamma(k+\frac{1}{2})}{k!\Gamma(\frac{1}{2}-k)}(2z)^{-k} + \sigma_\nu(z)\Big]\,, \tag{1.23}$$

where

$$\frac{d^k\sigma_\nu}{dz^k} = O(z^{k-\nu}) \qquad \text{as } z \to \infty\,,\ k \geq 0\,.$$

Especially

$$\begin{aligned}
K_0(z) &= \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z}\Big[1 - \frac{1}{8z} + \frac{9}{2!(8z)^2} + O(z^{-3})\Big]\\
K_0'(z) &= \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z}\Big[-1 - \frac{3}{8z} + \frac{15}{128z^2} + O(z^{-3})\Big]\\
K_0''(z) &= \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z}\Big[1 + \frac{7}{8z} + \frac{57}{128z^2} + O(z^{-3})\Big]\\
K_0'''(z) &= \Big(\frac{\pi}{2z}\Big)^{\frac{1}{2}} e^{-z}\Big[-1 - \frac{11}{8z} - \frac{225}{128z^2} + O(z^{-3})\Big]
\end{aligned} \tag{1.24}$$

We pass to the polar coordinates with the origin at (x_1, x_2). So

$$\begin{aligned}
y_1 - x_1 &= r\cos\varphi\\
y_2 - x_2 &= r\sin\varphi\,.
\end{aligned}$$

Denoting

$$s = (r + y_1 - x_1) = r(1+\cos\varphi)$$

formulas (1.16) furnish:

$$\begin{aligned}
\mathcal{O}_{11}(r,\varphi;2\lambda) &= \frac{1}{4\pi\lambda}\Big[\frac{\cos\varphi}{r} + e^{-\lambda s}\sqrt{\frac{\pi}{2\lambda r}}\Big\{\lambda\cos\varphi\,(-1-\frac{3}{8\lambda r}) + \lambda(1-\frac{1}{8\lambda r}) +\\
&\quad + \lambda\nu(\lambda r)\Big\}\Big]\\
\mathcal{O}_{12}(r,\varphi;2\lambda) &= \frac{1}{4\pi\lambda}\Big[\frac{\sin\varphi}{r} + \lambda\sin\varphi\; e^{-\lambda s}\sqrt{\frac{\pi}{2\lambda r}}\Big\{-1-\frac{3}{8\lambda r} + \lambda\nu(\lambda r)\Big\}\Big] \qquad (1.25)\\
\mathcal{O}_{22}(r,\varphi;2\lambda) &= -\frac{1}{4\pi\lambda}\Big[\frac{\cos\varphi}{r} + e^{-\lambda s}\sqrt{\frac{\pi}{2\lambda r}}\Big\{\lambda\cos\varphi\,(-1-\frac{3}{8\lambda r}) - \lambda(1-\frac{1}{8\lambda r}) +\\
&\quad + \lambda\nu(\lambda r)\Big\}\Big]\,,
\end{aligned}$$

where $D^k\nu(z) = O(z^{-k-2})$ as $z \to \infty$, $k \geq 0$.

Before starting the study of asymptotic behaviour at infinity, let us recall several more or less trivial facts. We have (see also Lemma 3.1)

$$\frac{\partial s}{\partial y_1} = \frac{s}{r}\,, \qquad \frac{\partial s}{\partial y_2} = \frac{(y_2 - x_2)}{r} = \sin\varphi\,,$$

$$\frac{\partial \varphi}{\partial y_2} = \frac{\cos\varphi}{r}\,, \qquad \frac{\partial \varphi}{\partial y_1} = -\frac{\sin\varphi}{r}\,,$$

$$s \sim r \quad \text{for} \quad y_1 - x_1 \geq 0 \quad \text{but} \quad s \sim \frac{(y_2 - x_2)^2}{r} \quad \text{for} \quad y_1 - x_1 < 0$$

$$e^{-\lambda s} \sin^2\varphi\, \lambda r = \lambda s e^{-\lambda s}(1 - \cos\varphi) \leq 2e^{-1}\,.$$

So from (1.25) we get the following asymptotic expansion of $\boldsymbol{\mathcal{O}}$

$$\begin{aligned}
\mathcal{O}_{11}(r,\varphi;2\lambda) &= \frac{1}{4\pi\lambda r}\cos\varphi - \frac{1}{4\sqrt{2\pi\lambda r}}e^{-\lambda s}\Big[\cos\varphi - 1 + \\
&\quad + \frac{1}{\lambda r}\big(\frac{3}{8}\cos\varphi + \frac{1}{8}\big) + \nu(\lambda r)\Big] \\
\mathcal{O}_{12}(r,\varphi;2\lambda) &= \frac{1}{4\pi\lambda r}\sin\varphi - \frac{1}{4\sqrt{2\pi\lambda r}}e^{-\lambda s}\sin\varphi\,\Big[1 + \frac{3}{8\lambda r} + \nu(\lambda r)\Big] \\
\mathcal{O}_{22}(r,\varphi;2\lambda) &= -\frac{1}{4\pi\lambda}\cos\varphi + \frac{1}{4\sqrt{2\pi\lambda r}}e^{-\lambda s}\Big[(1 + \cos\varphi) + \\
&\quad + \frac{1}{\lambda r}\big(\frac{3}{8}\cos\varphi - \frac{1}{8}\big) + \nu(\lambda r)\Big]\,.
\end{aligned} \tag{1.26}$$

In particular we have the following uniform behaviour

$$\left.\begin{aligned}
|O_{12}(\mathbf{x}-\mathbf{y};2\lambda)|, |O_{22}(\mathbf{x}-\mathbf{y};2\lambda)| &\leq \frac{C}{\lambda r} \\
|O_{11}(\mathbf{x}-\mathbf{y};2\lambda)| &\leq \frac{C}{\sqrt{\lambda r}}
\end{aligned}\right\} \text{as } \lambda r \to \infty\,. \tag{1.27}$$

Moreover, from $(1.26)_1$ we may deduce the following anisotropic structure

$$|O_{11}(\mathbf{x}-\mathbf{y};2\lambda)| \leq \frac{C}{\sqrt{\lambda r}\sqrt{1+\lambda s}}\,. \tag{1.28}$$

Now we might calculate the derivatives and get the asymptotic expansions of them. As we are interested only in the estimates of the type (1.27) – (1.28), we do not do it. We rather observe that

$$\Big|\frac{\partial}{\partial y_1}(e^{-\lambda s})\Big| = \Big|e^{-\lambda s}\frac{\lambda s}{r}\Big| \leq \frac{C}{r}\,,$$

while

$$\Big|\frac{\partial}{\partial y_2}(e^{-\lambda s})\Big| = |e^{-\lambda s}\lambda\sin\varphi\,| \leq \frac{C\sqrt{\lambda}}{\sqrt{r}\sqrt{1+\lambda s}}\,.$$

Moreover, as the derivatives of φ with respect to y_1 produce a sinus (unlike the derivate with respect to y_2), we get easily

$$\begin{aligned}
&\Big|\frac{\partial}{\partial y_2}\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda)\Big| \le \frac{C}{r(1+\lambda s)}\\
\Big|\frac{\partial}{\partial y_2}\mathcal{O}_{12}(\mathbf{x}-\mathbf{y};2\lambda)\Big|, &\Big|\frac{\partial}{\partial y_1}\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda)\Big| \le \frac{C}{\sqrt{\lambda r}\, r\sqrt{1+\lambda s}} \qquad (1.29)\\
\Big|\frac{\partial}{\partial y_1}\mathcal{O}_{12}(\mathbf{x}-\mathbf{y};2\lambda)\Big| &\Big|\frac{\partial}{\partial y_1}\mathcal{O}_{22}(\mathbf{x}-\mathbf{y};2\lambda)\Big|, \\
&\Big|\frac{\partial}{\partial y_2}\mathcal{O}_{22}(\mathbf{x}-\mathbf{y};2\lambda)\Big| \le \frac{C}{\lambda r^2},
\end{aligned}$$

where for the last term we used the fact that $\frac{\partial}{\partial y_2}\cos\varphi = \frac{-\sin\varphi\cos\varphi}{r}$.

For higher derivatives we do not need such precise estimates. We therefore only observe that

$$\Big|\frac{\partial^2}{\partial y_2^2}\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda)\Big| \le \frac{C\sqrt{\lambda}}{r^{\frac{3}{2}}(1+\lambda s)^{\frac{3}{2}}} \qquad (1.30)$$

while for the other terms we have the following uniform estimate

$$|\tilde{D}^2\boldsymbol{\mathcal{O}}(\mathbf{x}-\mathbf{y};2\lambda)| \le \frac{C}{r^2}, \qquad (1.31)$$

where $\tilde{D}^2\boldsymbol{\mathcal{O}}$ contains all second derivatives of $\boldsymbol{\mathcal{O}}$ except of $\frac{\partial^2\mathcal{O}_{11}}{\partial y_2^2}$. For higher derivatives we then easily see that

$$|D^k\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};2\lambda)| \le \frac{C\lambda^{\frac{k}{2}-\frac{1}{2}}}{r^{\frac{k}{2}+\frac{1}{2}}} \quad \text{for } k \ge 3. \qquad (1.32)$$

The proofs are the same as for the first derivatives. We could, certainly, get a more precise structure of higher gradients of $\boldsymbol{\mathcal{O}}$ including its anisotropic structure. But as we do not need it, we skip such a study.

Another interesting task is the local and global integrability of the fundamental Oseen tensor. We shall mention it in the next chapter.

II.1.2 Fundamental solution in three dimensions

The study of the threedimensional Oseen fundamental tensor is from several point of view easier than the twodimensional one. It also indicates that we may expect the same for our problem in exterior domain.

Using again the fact that Φ_1 must behave around $z=0$ in the same way as $\mathcal{E}$, we get from (1.10) (ii) that

$$\Phi_1 = -\frac{1}{2\pi}\sqrt{\frac{\lambda}{2\pi|\mathbf{x}-\mathbf{y}|}}K_{1/2}(\lambda|\mathbf{x}-\mathbf{y}|)e^{-\lambda(y_1-x_1)}. \qquad (1.33)$$

Inserting (1.33) into (1.5)

$$\Phi(\mathbf{x},\mathbf{y}) = -\frac{1}{8\pi\lambda}\int^{y_1-x_1}\frac{1-\exp\{-\lambda(\sqrt{\tau^2+(y_2-x_2)^2+(y_3-x_3)^2}+\tau)\}}{\sqrt{\tau^2+(y_2-x_2)^2+(y_3-x_3)^2}}\,\mathrm{d}\tau\,.$$

Unlike the twodimensional case we may fix the constant up to which Φ is defined by taking $\Phi(\mathbf{x},\mathbf{x}) = 0$. Therefore

$$\Phi(\mathbf{x},\mathbf{y}) = -\frac{1}{8\pi\lambda}\int_0^{\lambda(|\mathbf{x}-\mathbf{y}|+y_1-x_1)} \frac{1-e^{-\tau}}{\tau}\,d\tau \tag{1.34}$$

and from (1.8) we have

$$e_j = -\frac{1}{4\pi}\frac{y_j - x_j}{|\mathbf{x}-\mathbf{y}|^3}\,. \tag{1.35}$$

We next start to calculate $\mathcal{O}_{ij}$. Similarly as in the twodimensional case we use the anisotropic function ($r = |\mathbf{x}-\mathbf{y}|$)

$$s = (r + y_1 - x_1)\,.$$

By direct calculation we can verify (compare also with the twodimensional case)

$$\frac{\partial s}{\partial y_1} = \frac{s}{r}\,, \qquad \frac{\partial s}{\partial y_i} = \frac{y_i - x_i}{r} \qquad i = 2,3\,.$$

Evidently

$$\begin{aligned}
\frac{\partial \Phi}{\partial y_1} &= -\frac{1}{8\pi\lambda}\frac{1-e^{-\lambda s}}{r}\\
\frac{\partial \Phi}{\partial y_2} &= -\frac{1}{8\pi\lambda}\frac{1-e^{-\lambda s}}{s}\frac{y_2-x_2}{r}\\
\frac{\partial \Phi}{\partial y_3} &= -\frac{1}{8\pi\lambda}\frac{1-e^{-\lambda s}}{s}\frac{y_3-x_3}{r}\,.
\end{aligned} \tag{1.36}$$

We easily see that $\mathcal{O}_{ij}$ is a smooth function outside the origin. This can be verified using the fact that $f(y) = \frac{1-e^{-y}}{y}$ is smooth on $[0,\infty)$ what follows either by induction or by regarding its Taylor expansion around zero.

Taking derivatives in (1.36) and using (1.1) we finally get

$$\begin{aligned}
\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda) &= \frac{1}{8\pi\lambda}\Big\{\frac{(y_2-x_2)^2+(y_3-x_3)^2}{r^2}\Big(\frac{\lambda e^{-\lambda s}}{s} - \frac{(1-e^{-\lambda s})}{s^2} - \\
&\quad - \frac{1-e^{-\lambda s}}{rs}\Big) + \frac{2(1-e^{-\lambda s})}{rs}\Big\}\\
\mathcal{O}_{22}(\mathbf{x}-\mathbf{y};2\lambda) &= \frac{1}{8\pi\lambda}\Big\{\frac{(y_3-x_3)^2}{r^2}\Big(\frac{\lambda e^{-\lambda s}}{s} - \frac{1-e^{-\lambda s}}{s^2} - \frac{1-e^{-\lambda s}}{rs}\Big) + \\
&\quad + \frac{e^{-\lambda s}\lambda s}{r^2} - \frac{1-e^{-\lambda s}}{r^2}\frac{y_1-x_1}{r} + \frac{1-e^{-\lambda s}}{rs}\Big\}\\
\mathcal{O}_{33}(\mathbf{x}-\mathbf{y};2\lambda) &= \frac{1}{8\pi\lambda}\Big\{\frac{(y_2-x_2)^2}{r^2}\Big(\frac{\lambda e^{-\lambda s}}{s} - \frac{1-e^{-\lambda s}}{s^2} - \frac{1-e^{-\lambda s}}{rs}\Big) + \\
&\quad + \frac{e^{-\lambda s}\lambda s}{r^2} - \frac{1-e^{-\lambda s}}{r^2}\frac{y_1-x_1}{r} + \frac{1-e^{-s}}{rs}\Big\}
\end{aligned} \tag{1.37}$$

$$\mathcal{O}_{12}(\mathbf{x}-\mathbf{y};2\lambda)=-\frac{1}{8\pi\lambda}\Big[\frac{\lambda e^{-\lambda s}}{r}\frac{y_2-x_2}{r}-\frac{1-e^{-\lambda s}}{r^2}\frac{y_2-x_2}{r}\Big]$$

$$\mathcal{O}_{13}(\mathbf{x}-\mathbf{y};2\lambda)=-\frac{1}{8\pi\lambda}\Big[\frac{\lambda e^{-\lambda s}}{r}\frac{y_3-x_3}{r}-\frac{1-e^{-\lambda s}}{r^2}\frac{y_3-x_3}{r}\Big]$$

$$\mathcal{O}_{23}(\mathbf{x}-\mathbf{y};2\lambda)=-\frac{1}{8\pi\lambda}\Big[\frac{\lambda e^{-\lambda s}}{s}\frac{(y_2-x_2)(y_3-x_3)}{r^2}-\frac{1-e^{-\lambda s}}{s^2}\cdot$$

$$\cdot\ \frac{(y_2-x_2)(y_3-x_3)}{r^2}-\frac{1-e^{-\lambda s}}{s}\frac{(y_2-x_2)(y_3-x_3)}{r^3}\Big].$$

As for $N=2$, we study separately the asymptotic properties of $\mathcal{O}_{ij}$ if $\lambda r\to 0$ and $\lambda r\to\infty$. We have the following homogeneity property which is a consequence of (1.37)

$$\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};2\lambda)=2\lambda\,\mathcal{O}_{ij}(2\lambda(\mathbf{x}-\mathbf{y});1)\,. \tag{1.38}$$

We have in the case of $\lambda r\to 0$:

$$\begin{aligned}\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda)&=\frac{1}{8\pi r}\Big[-\frac{(y_2-x_2)^2+(y_3-x_3)^2}{r^2}+2+2\frac{1-e^{-\lambda s}-\lambda s}{\lambda s}+\\&+\frac{(y_2-x_2)^2+(y_3-x_3)^2}{r^2}\Big\{\frac{r}{s}\Big(\frac{\lambda s e^{-\lambda s}-1+e^{-\lambda s}}{\lambda s}\Big)+\frac{\lambda s+e^{-\lambda s}-1}{\lambda s}\Big\}\Big]\\&=\frac{1}{8\pi r}\Big(1+\frac{(y_1-x_1)^2}{r^2}\Big)+R_{11}(\lambda|\mathbf{x}-\mathbf{y}|)\,,\end{aligned}$$

where

$$\nabla^k R_{11}(\lambda r)=\lambda^{k+1}O\Big(\frac{1}{(\lambda r)^k}\Big),\ \ k\geq 0 \text{ as } \lambda r\to 0\,.$$

We proceed analogously for other terms and recalling that the fundamental Stokes tensor in three dimensions has the form (see e.g. [Ga1])

$$\mathcal{S}_{ij}(\mathbf{x}-\mathbf{y})=\frac{1}{8\pi}\Big(\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{y}|}+\frac{(y_i-x_i)(y_j-x_j)}{|\mathbf{x}-\mathbf{y}|^3}\Big) \tag{1.39}$$

we get

$$\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};2\lambda)=S_{ij}(\mathbf{x}-\mathbf{y})+R_{ij}(\mathbf{x}-\mathbf{y};\lambda) \tag{1.40}$$

with

$$D^k_{\mathbf{y}}R_{ij}(\mathbf{x}-\mathbf{y};\lambda)=\lambda^{k+1}O\Big(\frac{1}{(\lambda|\mathbf{x}-\mathbf{y}|)^k}\Big)\,,\ k\geq 0 \text{ as } \lambda|\mathbf{x}-\mathbf{y}|\to 0\,.$$

Next part is devoted to the study at infinity. We have

$$\begin{aligned}\mathcal{O}_{11}(\mathbf{x}-\mathbf{y};2\lambda)&=\frac{1}{8\pi r}\Big[\frac{(y_2-x_2)^2+(y_3-x_3)^2}{rs}\Big(e^{-\lambda s}-\frac{1-e^{-\lambda s}}{\lambda s}\Big)\Big]+\\&+\frac{1}{4\pi r}\frac{1-e^{-\lambda s}}{\lambda s}+\frac{1}{8\pi r}\frac{(y_2-x_2)^2+(y_3-x_3)^2}{r^2}\frac{1-e^{-\lambda s}}{\lambda s}\,.\end{aligned} \tag{1.41}$$

Now as $e^{-\lambda s} \le \frac{1-e^{-\lambda s}}{\lambda s} \quad \forall s > 0$ and

$$\frac{(y_j - x_j)^2}{rs} \le C \quad \text{for} \quad j = 2,3 \begin{pmatrix} s \sim \dfrac{(x_2-y_2)^2 + (x_3-y_3)^2}{r} & \text{if}\, y_1 - x_1 < 0 \end{pmatrix} \\ \begin{pmatrix} s \sim r & \text{if}\, y_1 - x_1 \ge 0 \end{pmatrix},$$

we get that

$$|\mathcal{O}_{11}(\mathbf{x}-\mathbf{y}; 2\lambda)| \le \frac{C}{r}\frac{1-e^{-\lambda s}}{\lambda s}\,.$$

Analogously we proceed for other terms and finally we get the following uniform and anisotropic decay as $\lambda|\mathbf{x}-\mathbf{y}| \to \infty$:

$$\begin{aligned} |\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda)| &\le \frac{C}{r} \\ |\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda)| &\le \frac{C}{r(1+\lambda s)} \end{aligned} \qquad i,j = 1,2,3\,. \tag{1.42}$$

Moreover, let us observe that from (1.41) and $\nabla(e^{-\lambda s}) \le C\nabla(\frac{1-e^{-\lambda s}}{\lambda s})$ it follows that for $k \ge 1$

$$|\nabla^k \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y}; 2\lambda)| \le C\Big|\nabla^k\Big(\frac{1}{r}\frac{1-e^{-\lambda s}}{\lambda s}\Big)\Big| \tag{1.43}$$

Recalling that

$$\Big|\frac{\partial s}{\partial y_j}\Big| = \Big|\frac{y_j - x_j}{r}\Big| \le C\sqrt{\frac{s}{r}} \qquad j = 2,3$$

we get

$$\begin{aligned} \Big|\frac{\partial \mathcal{O}_{ij}}{\partial y_1}(\mathbf{x}-\mathbf{y}; 2\lambda)\Big| &\le \frac{C}{r^2}\frac{1-e^{-\lambda s}}{\lambda s} + \frac{C}{r}\Big(\frac{1-e^{-\lambda s} - \lambda s e^{-\lambda s}}{\lambda s^2}\frac{\partial s}{\partial y_1}\Big) \le \\ &\le \frac{C}{r^2}\Big(\frac{1-e^{-\lambda s}}{\lambda s} + e^{-\lambda s}\Big) \le \frac{C}{r^2}\frac{1}{(1+\lambda s)} \\ \Big|\frac{\partial \mathcal{O}_{ij}}{\partial y_j}(\mathbf{x}-\mathbf{y}; 2\lambda)\Big| &\le \frac{C}{r^2}\frac{1-e^{-\lambda s}}{\lambda s} + \frac{C}{r}\Big(\frac{1-e^{-\lambda s} - \lambda s e^{-\lambda s}}{\lambda s^2}\frac{\partial s}{\partial y_j}\Big) \le \\ &\le \frac{C}{r^{\frac{3}{2}}}\Big(\frac{1-e^{-\lambda s}}{\lambda s\sqrt{r}} + \frac{1-e^{-\lambda s} - \lambda s e^{-\lambda s}}{\lambda s^{\frac{3}{2}}}\Big) \le \\ &\le \frac{C\sqrt{\lambda}}{r^{\frac{3}{2}}(1+\lambda s)^{\frac{3}{2}}} \qquad j = 2,3\,. \end{aligned} \tag{1.44}$$

Analogously we proceed for higher derivatives and get

$$\Big|\frac{\partial^2 \mathcal{O}_{ij}}{\partial y_1^2}(\mathbf{x}-\mathbf{y}; 2\lambda)\Big| \le \frac{C}{r^3(1+\lambda s)} \tag{1.45}$$

$$\Big|\frac{\partial^2 \mathcal{O}_{ij}}{\partial y_1 \partial y_k}(\mathbf{x}-\mathbf{y}; 2\lambda)\Big| \le \frac{\sqrt{\lambda}C}{r^{\frac{5}{2}}(1+\lambda s)^{\frac{3}{2}}}\,,\ k = 2,3 \tag{1.46}$$

$$\Big|\frac{\partial^2 \mathcal{O}_{ij}}{\partial y_k \partial y_l}(\mathbf{x}-\mathbf{y}; 2\lambda)\Big| \le \frac{\lambda C}{r^2(1+\lambda s)^2}\,,\ k,l = 2,3\,. \tag{1.47}$$

For higher derivatives we need only uniform estimates,

$$|\nabla^k O_{ij}(\mathbf{x}-\mathbf{y};2\lambda)| \le \frac{C\lambda^{\frac{k}{2}}}{r^{1+\frac{k}{2}}} \qquad k \ge 3\,. \tag{1.48}$$

The local and global integrability properties will be studied in the next chapter.

II.2 Oseen flow in exterior domains

This section contains several existence and uniqueness lemmas as well as the integral representation of solutions to the Oseen problem in exterior domains. The results are given without proof; they can be found e.g. in [Ga1] and in [No2], or (even for a more general problem) will be given in Chapter III.

Lemma 2.1 (Existence of pressure)
Let Ω be an exterior domain in $\mathbb{R}^N$, $N \ge 2$. Let $f \in W_0^{-1,q}(\Omega')$, $1 < q < \infty$, for any bounded subdomain Ω' with $\overline{\Omega}' \subset \Omega$. Then, to every q-weak solution $\mathbf{v}$ we can associate a pressure field $p \in L^q_{loc}(\Omega)$ such that

$$\int_\Omega (\nabla\mathbf{v}\nabla\boldsymbol{\psi} - \beta\mathbf{v}\frac{\partial\boldsymbol{\psi}}{\partial y_1})\,dx = \int_\Omega p\,\nabla\cdot\boldsymbol{\psi}\,dx + \langle\mathbf{f},\boldsymbol{\psi}\rangle \tag{2.1}$$

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Furthermore, if Ω is locally lipschitzian and $\mathbf{f} \in W_0^{-1,q}(\Omega_R)$, $R > diam\,\Omega^c$, then $p \in L^q(\Omega_R)$.

Lemma 2.2 (Regularity)
Let $\mathbf{f} \in W^{m,q}_{loc}(\Omega)$, $m \ge 0$, $1 < q < \infty$ and let

$$\mathbf{v} \in W^{1,q}_{loc}(\Omega)\,, \quad p \in L^q_{loc}(\Omega)$$

with $\mathbf{v}$ weakly divergence free, satisfy (2.1) for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$. Then

$$\mathbf{v} \in W^{m+2,q}_{loc}(\Omega)\,, \quad p \in W^{m+1,q}_{loc}(\Omega)\,.$$

In particular, if $\mathbf{f} \in C^\infty(\Omega)$, then $\mathbf{v}$, $p \in C^\infty(\Omega)$. Furthermore, if Ω is of class C^{m+2} and

$$\mathbf{f} \in W^{m,q}_{loc}(\overline{\Omega})\,, \quad \mathbf{v}_* \in W^{m+2-\frac{1}{q},q}(\partial\Omega)\,,$$

then

$$\mathbf{v} \in W^{m+2,q}_{loc}(\overline{\Omega})\,, \quad p \in W^{m+1,q}_{loc}(\overline{\Omega})\,.$$

In particular, if Ω is of class C^∞ and $\mathbf{f} \in C^\infty(\Omega)$, $\mathbf{v}_* \in C^\infty(\partial\Omega)$, then $\mathbf{v}$, $p \in C^\infty(\overline{\Omega}')$ for all bounded $\Omega' \subset \Omega$.

The proof of existence of the weak solution to (0.1) is much simpler for the threedimensional exterior domains than for the plane flow (see also Chapter III). We have

Lemma 2.3 (Existence of 3–D flow)
Let Ω be a threedimensional exterior, locally lipschitzian domain. Given

$$\mathbf{f} \in D_0^{-1,2}(\Omega)\,, \quad \mathbf{v}_* \in W^{\frac{1}{2},2}(\partial\Omega)\,,$$

there exists one and only one generalized solution to (0.1). This solution satisfies the estimates

$$\begin{aligned} \|\mathbf{v}\|_{2,\Omega_R} + |\mathbf{v}|_{1,2} &\le c_1(|\mathbf{f}|_{-1,2} + (1+\beta)\|\mathbf{v}_*\|_{\frac{1}{2},2,(\partial\Omega)}) \\ \int_{S_3} |\mathbf{v}(R,\omega)d\omega &= O\Big(\frac{1}{R}\Big) \text{ as } R\to\infty \\ \|p\|_{2,\Omega_R/\mathbb{R}} &\le c_2(|\mathbf{f}|_{-1,2} + (1+\beta)|\mathbf{v}|_{1,2}) \end{aligned} \tag{2.2}$$

for all $R > \mathrm{diam}(\Omega^c)$. In (2.2)$_3$ p is the pressure associated to $\mathbf{v}$ by Lemma 2.1, while $c_i = c_i(R,\Omega)$ and $c_i \to \infty$ as $R \to \infty$.

In the twodimensional case the study is much more delicate. Using the procedure proposed by Finn and Smith in [FiSm] we finally get

Lemma 2.4 (Existence of 2–D flow)
Let Ω be a twodimensional exterior, locally lipschitzian domain. Given[3]

$$\begin{aligned} &\mathbf{f} \in D_0^{-1,2}(\Omega) \cap L^q(\Omega)\,, \quad 1 < q < \frac{3}{2} \\ &\mathbf{v}_* \in W^{\frac{1}{2},2}(\partial\Omega)\,, \quad \int_{\partial\Omega} \mathbf{v}_*\cdot\mathbf{n}dS = 0 \end{aligned}$$

there exists one and only one generalized solution to (0.1). Moreover, for all $R > \mathrm{diam}(\Omega^c)$ this solution verifies

$$\begin{aligned} \mathbf{v} &\in D^{2,q}(\Omega^R) \cap D^{1,\frac{3q}{3-q}}(\Omega^R) \cap L^{\frac{3q}{3-2q}}(\Omega) \\ v_2 &\in L^{\frac{2q}{2-q}}(\Omega) \cap D^{1,q}(\Omega) \\ \frac{\partial v_1}{\partial x_1} &\in L^q(\Omega) \\ p &\in D^{1,q}(\Omega^R)\,, \end{aligned} \tag{2.3}$$

where p is the pressure field associated to $\mathbf{v}$ by Lemma 2.1. Finally the following estimate holds

$$\begin{aligned} \|\mathbf{v}\|_{2,\Omega_R} + |\mathbf{v}|_{1,2} + \beta\Big(\|v_2\|_{\frac{2q}{2-q}} + |v_2|_{1,q} + \Big\|\frac{\partial v_1}{\partial x_1}\Big\|_q\Big) + b\|\mathbf{v}\|_{\frac{3q}{3-2q}} + \beta^{\frac{1}{3}}|\mathbf{v}|_{1,\frac{3q}{3-q},\Omega^R} \\ +|\mathbf{v}|_{2,q,\Omega^R} + |p|_{1,q,\Omega^R} \le c(\|\mathbf{f}\|_q + |\mathbf{f}|_{-1,2} + (1+\beta)^2\|\mathbf{v}_*\|_{\frac{1}{2},2,(\partial\Omega)})\,, \end{aligned}$$

where $b = \min(1,\beta^{\frac{2}{3}})$ and $c = c(q,\Omega,R)$.

[3] see Chapter III for the discussion of the necessity of the compatibility condition (zero flux of $\mathbf{v}_*$ through $\partial\Omega$) in two space dimensions

Lemma 2.5 (L^q–estimates)
Let Ω be an exterior domain in $\mathbb{R}^N$ of class C^{m+2}, $m \geq 0$. Given

$$\mathbf{f} \in W^{m,q}(\Omega)\,, \mathbf{v}_* \in W^{m+2-\frac{1}{q},q}(\partial\Omega)\,, 1 < q < \frac{N+1}{2}\,,$$

there exists one and only one solution corresponding to the Oseen problem (0.1) such that

$$\mathbf{v} \in W^{m,s_2}(\Omega) \bigcap \Big\{ \bigcap_{l=0}^{m} \Big[D^{l+1,s_1}(\Omega) \bigcap D^{l+2,q}(\Omega) \Big] \Big\}$$

$$p \in \bigcap_{l=0}^{m} D^{l+1,q}(\Omega)$$

with $s_1 = \frac{(N+1)q}{N+1-q}$, $s_2 = \frac{(N+1)q}{N+1-2q}$. If $N = 2$, we also have

$$v_2 \in W^{m,\frac{2q}{2-q}}(\Omega) \bigcap \Big(\bigcap_{l=0}^{m} D^{l+1,q}(\Omega) \Big)\,.$$

Moreover, $\mathbf{v}$, p verify

$$\begin{aligned} a_1\|\mathbf{v}\|_{m,s_2} + \beta\Big\|\frac{\partial \mathbf{v}}{\partial x_1}\Big\|_{m,q} + \sum_{l=0}^{m}(a_2|\mathbf{v}|_{l+1,s_1} + |\mathbf{v}|_{l+2,q} + |p|_{l+1,q}) \leq \\ \leq c(\|\mathbf{f}|_{m,q} + \|\mathbf{v}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)}) \end{aligned} \tag{2.4}$$

and, if $N = 2$,

$$\begin{aligned} \beta(\|v_2\|_{m,\frac{2q}{2-q}} + \|\nabla v_2\|_{m+1,q}) + a_1\|\mathbf{v}\|_{m,\frac{3q}{3-2q}} + \beta\Big\|\frac{\partial v_1}{\partial x_1}\Big\|_{m,q} + \\ + \sum_{l=0}^{m}(a_2|\mathbf{v}|_{l+1,\frac{3q}{3-q}} + |\mathbf{v}|_{l+2,q} + |p|_{l+1,q}) \leq c(\|\mathbf{f}\|_{m,q} + \|\mathbf{v}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)}) \end{aligned} \tag{2.5}$$

with $a_1 = \min\{1, \beta^{\frac{2}{N+1}}\}$, $a_2 = \min\{1, \beta^{\frac{1}{N+1}}\}$. The constant c depends on m, q, N, Ω and β. However, if $q \in (1, \frac{N}{2})$ and $\beta \in (0; B]$ for some $B > 0$, c depends solely on m, q, N, Ω and B.

We denote

$$\begin{aligned} \mathcal{T}(\mathbf{e}) &= 2\mathbf{D}(\mathbf{e}) + \beta e_1 \mathbf{I} \\ \mathbf{T}(\mathbf{v}, p) &= 2\mathbf{D}(\mathbf{v}) - p\mathbf{I}\,. \end{aligned} \tag{2.6}$$

Let us recall that the fundamental Oseen tensor $\boldsymbol{\mathcal{O}} = \boldsymbol{\mathcal{S}} + \boldsymbol{\mathcal{N}}$, where $D^2\boldsymbol{\mathcal{S}}$ (the fundamental Stokes tensor) is the Calderón–Zygmung singular kernel while $D^2\boldsymbol{\mathcal{N}}$ is locally integrable. Assuming $f_i = \frac{\partial \mathcal{F}_{ik}}{\partial x_k}$ with $\mathcal{F} \in C_0^\infty(\overline{\Omega})$, we get the following set of integral representation formulas:

Lemma 2.6 (Integral representation)
Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ be an exterior domain of class C^2, $f_i = \frac{\partial \mathcal{F}_{ik}}{\partial x_i}$ with

$\mathcal{F} \in C_0^\infty(\overline{\Omega})$. Let $\mathbf{w}_j = \{\mathcal{O}_{ij}(\cdot\,;\beta)\}$, $i = 1,2,\dots,N$, $\mathbf{e}_j$ be the fundamental Oseen solution. Let $\alpha \in \mathbb{N}^N$. Then we have

$$D^\alpha v_j(\mathbf{x}) = \int_\Omega \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)\frac{\partial}{\partial y_k}D^\alpha \mathcal{F}_{ik}(\mathbf{y})\,d\mathbf{y}+ \\ +\int_{\partial\Omega}\Big[D^\alpha v_i(\mathbf{y})T_{il}(\mathbf{w}_j,\mathbf{e}_j)(\mathbf{x}-\mathbf{y};\beta) - \beta\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)D^\alpha v_i(\mathbf{y})\delta_{1l}+ \\ +\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{il}(D^\alpha\mathbf{v},D^\alpha p)(\mathbf{y})\Big]n_l(\mathbf{y})\,dS \tag{2.7}$$

$$D^\alpha v_j(\mathbf{x}) = \int_\Omega \frac{\partial\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k}D^\alpha \mathcal{F}_{ik}(\mathbf{y})\,d\mathbf{y}+ \\ +\int_{\partial\Omega}\Big[D^\alpha v_i(\mathbf{y})T_{il}(\mathbf{w}_j,\mathbf{e}_j)(\mathbf{x}-\mathbf{y};\beta) - \beta\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)D^\alpha v_i(\mathbf{y})\delta_{1l}+ \\ +\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{il}(D^\alpha\mathbf{v},D^\alpha p)(\mathbf{y}) + \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)D^\alpha\mathcal{F}_{il}(\mathbf{y})\Big]n_l(\mathbf{y})\,dS \tag{2.8}$$

$$\frac{\partial}{\partial x_r}D^\alpha v_j(\mathbf{x}) = \int_\Omega \frac{\partial\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r}\frac{\partial}{\partial y_k}D^\alpha \mathcal{F}_{ik}(\mathbf{y})\,d\mathbf{y}+ \\ +\int_{\partial\Omega}\Big[\frac{\partial\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r}T_{il}(D^\alpha\mathbf{v},D^\alpha p)(\mathbf{y})+ \\ +D^\alpha v_i(\mathbf{y})\frac{\partial}{\partial x_r}T_{il}(\mathbf{w}_j,\mathbf{e}_j)(\mathbf{x}-\mathbf{y};\beta) - \beta\frac{\partial\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r}D^\alpha v_i(\mathbf{y})\delta_{1l}\Big]n_l(\mathbf{y})\,dS \tag{2.9}$$

$$\frac{\partial}{\partial x_r}D^\alpha v_j(\mathbf{x}) = \text{v.p.}\int_\Omega \frac{\partial^2\mathcal{S}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k\partial x_r}D^\alpha \mathcal{F}_{ik}(\mathbf{y})\,d\mathbf{y}+ \\ +\int_\Omega \frac{\partial^2\mathcal{N}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k\partial x_r}D^\alpha \mathcal{F}_{ik}(\mathbf{y})\,d\mathbf{y} + c_{ijkr}D^\alpha\mathcal{F}_{ik}(\mathbf{x})+ \\ +\int_{\partial\Omega}\Big[D^\alpha v_i(\mathbf{y})\frac{\partial T_{il}(\mathbf{w}_j,\mathbf{e}_j)(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r} - \beta\frac{\partial\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r}D^\alpha v_i(\mathbf{y})\delta_{1l}+ \\ +\frac{\partial\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r}T_{il}(D^\alpha\mathbf{v},D^\alpha p)(\mathbf{y}) + \frac{\partial\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r}D^\alpha\mathcal{F}_{il}(\mathbf{y})\Big]n_l(\mathbf{y})\,dS \tag{2.10}$$

$$\frac{\partial^2}{\partial x_r\partial x_s}D^\alpha v_j(\mathbf{x}) = \text{v.p.}\int_\Omega \frac{\partial^2\mathcal{S}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r\partial x_s}\frac{\partial}{\partial y_k}D^\alpha \mathcal{F}_{ik}(\mathbf{y})\,d\mathbf{y}+ \\ +\int_\Omega \frac{\partial^2\mathcal{N}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r\partial x_s}\frac{\partial}{\partial y_k}D^\alpha \mathcal{F}_{ik}(\mathbf{y})d\mathbf{y} + c_{ijrs}\frac{\partial}{\partial x_k}D^\alpha\mathcal{F}_{ik}(\mathbf{x})+ \\ +\int_{\partial\Omega}\Big[D^\alpha v_i(\mathbf{y})\frac{\partial^2}{\partial x_r\partial x_s}T_{il}(\mathbf{w}_j,\mathbf{e}_j)(\mathbf{x}-\mathbf{y};\beta) - \beta\frac{\partial^2\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r\partial x_s}D^\alpha v_i(\mathbf{y})\delta_{1l}+ \\ +\frac{\partial^2\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_r\partial x_s}T_{il}(D^\alpha\mathbf{v},D^\alpha p)(\mathbf{y})\Big]n_l(\mathbf{y})\,dS \tag{2.11}$$

$$D^\alpha p(\mathbf{x}) = \int_\Omega e_i(\mathbf{x}-\mathbf{y})\frac{\partial}{\partial y_k}D^\alpha \mathcal{F}_{ik}(\mathbf{y})\,d\mathbf{y}+ \\ +\int_{\partial\Omega}\Big[D^\alpha v_i(\mathbf{y})\mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + e_i(\mathbf{x}-\mathbf{y})T_{il}(D^\alpha\mathbf{v},D^\alpha p)(\mathbf{y})- \\ -\beta e_i(\mathbf{x}-\mathbf{y})D^\alpha v_i(\mathbf{y})\delta_{1l}\Big]n_l(\mathbf{y})\,dS \tag{2.12}$$

$$
\begin{aligned}
D^\alpha p(\mathbf{x}) = \text{v.p.} \int_\Omega \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_k} D^\alpha \mathcal{F}_{ik}(\mathbf{y})\, d\mathbf{y} + c_{ik} D^\alpha \mathcal{F}_{ik}(\mathbf{x}) + \\
+ \int_{\partial\Omega} \Big[D^\alpha v_i(\mathbf{y}) \mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + e_i(\mathbf{x}-\mathbf{y}) T_{il}(D^\alpha \mathbf{v}, D^\alpha p)(\mathbf{y}) - \\
-\beta e_i(\mathbf{x}-\mathbf{y}) D^\alpha v_i(\mathbf{y}) \delta_{1l} + e_i(\mathbf{x}-\mathbf{y}) D^\alpha \mathcal{F}_{il}(\mathbf{y}) \Big] n_l(\mathbf{y})\, dS
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x_r} D^\alpha p(\mathbf{x}) = \text{v.p.} \int_\Omega \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_r} \frac{\partial}{\partial y_k} D^\alpha \mathcal{F}_{ik}(\mathbf{y})\, d\mathbf{y} + c_{ir} \frac{\partial}{\partial y_k} D^\alpha \mathcal{F}_{ik}(\mathbf{x}) + \\
+ \int_{\partial\Omega} \Big[D^\alpha v_i(\mathbf{y}) \frac{\partial}{\partial x_r} \mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_r} T_{il}(D^\alpha \mathbf{v}, D^\alpha p)(\mathbf{y}) - \\
-\beta \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_r} D^\alpha v_i(\mathbf{y}) \delta_{1l} \Big] n_l(\mathbf{y})\, dS\,.
\end{aligned} \tag{2.14}
$$

Remark 2.1 By standard density argument we can show that the integral representation remains valid even for a much larger class of function. Namely, assuming that $D^\alpha \mathbf{v} \in W^{2,q}_{loc}(\overline{\Omega})$, $D^\alpha p \in W^{1,q}_{loc}(\overline{\Omega})$ we get the following conditions:

- (2.7) $\nabla D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < \frac{N+1}{2}$
- (2.8) $D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < N+1$
- (2.9) $\nabla D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < N+1$
- (2.10) $\nabla D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < \infty$
- (2.11) $\nabla D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < \infty$
- (2.12) $\nabla D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < N$
- (2.13) $D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < \infty$
- (2.14) $\nabla D^\alpha \mathcal{F} \in L^q(\Omega)$, $1 < q < \infty$

The proof is a special case of the integral representation given in Chapter III. Another method of proof (under slightly different assumptions) can be found in [Ga1].

II.3 Weighted estimates for Oseen kernels

The aim of this section is to obtain weighted estimates for convolutions with Oseen kernels i.e. estimates for both singular and weakly singular convolutions which appear in the integral representation for solutions to the Oseen problem (see Lemma 2.6). There are two kinds of kernels in (2.7)–(2.14). Those which are locally integrable will be treated in Subsections 2.3.2 and 2.3.3. The other ones which represent the singular integrals (i.e., in our case, integrals in the sense of the principal value) will be studied in Subsection II.3.1. Finally, the last subsection is devoted to an easy application of these estimates. As will be shown in the next chapter, all these estimates are applicable also for the modified Oseen problem and therefore will be our main tool in Chapter VI.

We shall use the following non–negative weights ($A, B \in \mathbb{R}$, $\beta > 0$, $s(\mathbf{x}) = |\mathbf{x}| - x_1$):

$$
\begin{aligned}
\sigma^A_B(\mathbf{x}) &= |\mathbf{x}|^A s(\mathbf{x})^B\,,\\
\eta^A_B(\mathbf{x}) &= (1+|\mathbf{x}|)^A (1+s(\mathbf{x}))^B\,,\\
\nu^A_B(\mathbf{x}) &= |\mathbf{x}|^A (1+s(\mathbf{x}))^B\,,\\
\mu^{A,\omega}_B(\mathbf{x}) &= \eta^{A-\omega}_B(\mathbf{x})\,\nu^\omega_0(\mathbf{x})\,,\\
\eta^A_B(\mathbf{x};\beta) &= (1+|\beta\mathbf{x}|)^A (1+s(\beta\mathbf{x}))^B\,,\\
\nu^A_B(\mathbf{x};\beta) &= |\mathbf{x}|^A (1+s(\beta\mathbf{x}))^B\,,\\
\mu^{A,\omega}_B(\mathbf{x};\beta) &= \eta^{A-\omega}_B(\mathbf{x};\beta)\,\nu^\omega_0(\mathbf{x};\beta)\,.
\end{aligned}
\tag{3.1}
$$

Let us note that outside the unit ball centered at the origin the weights $\eta^A_B(\mathbf{x})$, $\nu^A_B(\mathbf{x})$ and $\mu^{A,\omega}_B(\mathbf{x})$ are equivalent. The reason for using different kinds of weights will be seen later; it is essentially connected with the fact that we shall assume β small.

Before starting this studies, let us show some properties of the function $s(\mathbf{x})$. We take $N \geq 2$.

Lemma 3.1 *If* $x_1 > 0$ *then* $s(\mathbf{x}) \sim \frac{|\mathbf{x}'|^2}{|\mathbf{x}|}$*; otherwise* $s(\mathbf{x}) \sim |\mathbf{x}|$*. Here* $\mathbf{x}' = (x_2, \dots, x_N)$*.*

Proof: Introducing the generalized spherical coordinates ($N \geq 3$)

$$
\begin{aligned}
x_1 &= R\cos\theta_1\\
x_2 &= R\sin\theta_1\cos\theta_2\\
\cdots&\\
x_{N-1} &= R\sin\theta_1\cdots\sin\theta_{N-2}\cos\theta_{N-1}\\
x_N &= R\sin\theta_1\cdots\sin\theta_{N-2}\sin\theta_{N-1}\,,
\end{aligned}
\tag{3.2}
$$

where $\theta_1, \cdots, \theta_{N-2} \in (0,\pi)$, $\theta_{N-1} \in (0, 2\pi)$, we have

$$
s = R(1-\cos\theta_1) = 2\frac{(R\sin\theta_1)^2}{R}\Big(\frac{\sin\frac{\theta_1}{2}}{\sin\theta_1}\Big)^2\,.
$$

For $x_1 > 0$

$$
\theta_1 \in (0,\pi/2)\,,\ \text{i.e.}\ \ 2\Big(\frac{\sin\frac{\theta_1}{2}}{\sin\theta_1}\Big)^2 \in \Big(\frac{1}{2}, 1\Big)
$$

what implies

$$
\frac{1}{2}\frac{|\mathbf{x}'|^2}{R} \leq s(\mathbf{x}) \leq \frac{|\mathbf{x}'|^2}{R}\,.
$$

Analogously we proceed for $x_1 < 0$ where $\theta_1 \in (\pi/2, \pi)$ and $|\mathbf{x}| \leq s(\mathbf{x}) \leq 2|\mathbf{x}|$. If $N = 2$ we use the polar coordinates and the only change consists in the fact that $\varphi = \theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for $x_1 > 0$ and $\varphi \in [\frac{\pi}{2}, \frac{3}{2}\pi]$ for $x_1 \leq 0$.

□

Next we study the integral of $\eta^{-a}_{-b}(\mathbf{x})$ over the sphere for sufficiently large $R = |\mathbf{x}|$.

Lemma 3.2 *Let $N \geq 2$. Then for the exponents a, $b \in \mathbb{R}$ we have*

$$\int_{\partial B_R} \eta_{-b}^{-a}(\mathbf{x})\mathrm{d}S \sim R^{N-1-a-\min\{\frac{N-1}{2},b\}} \cdot \Big(\ln R \text{ if } b = \frac{N-1}{2}\Big) \tag{3.3}$$

as $R \to \infty$. Consequently, $\int_{\mathbb{R}^N} \eta_{-b}^{-a}(\mathbf{x})\mathrm{d}\mathbf{x} < \infty \iff a + \min\{\frac{N-1}{2}, b\} > N$.

Proof: Using the generalized spherical coordinates (if $N \geq 3$ — see (3.2)) or the polar ones ($N = 2$) we get

$$\int_{\partial B_R} \eta_{-b}^{-a}(\mathbf{x})\mathrm{d}S = C\int_0^\pi (1+R)^{-a}(1+s)^{-b}R^{N-1}(\sin\theta_1)^{N-2}\mathrm{d}\theta_1 =$$
$$= C\int_0^\pi (1+R)^{-a}(1+R(1-\cos\theta_1))^{-b}R^{N-1}(\sin\theta_1)^{N-2}\mathrm{d}\theta_1 .$$

Changing the variable $s = R(1-\cos\theta_1)$ we estimate the last integral by

$$C(1+R)^{1-a}\int_0^{2R}(1+s)^{-b}(\sqrt{2sR-s^2})^{N-3}\mathrm{d}s \tag{3.4}$$

We divide the integral (3.4) into three integrals and study them separately. Let us also note that for $N = 3$ (3.4) can be calculated explicitly. We have

$$\int_0^1 (1+s)^{-b}(2sR-s^2)^{\frac{N-3}{2}}\mathrm{d}s \sim R^{\frac{N-3}{2}}\int_0^1 s^{\frac{N-3}{2}}\mathrm{d}s \sim R^{\frac{N-3}{2}} ,$$
$$\int_1^R (1+s)^{-b}(2sR-s^2)^{\frac{N-3}{2}}\mathrm{d}s \sim R^{\frac{N-3}{2}}\int_1^R s^{-b+\frac{N-3}{2}}\mathrm{d}s \sim$$
$$\sim R^{N-2-\min(b,\frac{N-1}{2})} \cdot \Big(\ln R \text{ if } b = \frac{N-1}{2}\Big),$$
$$\int_R^{2R} (1+s)^{-b}(2sR-s^2)^{\frac{N-3}{2}}\mathrm{d}s \sim R^{\frac{N-3}{2}-b}\int_R^{2R}(2R-s)^{\frac{N-3}{2}}\mathrm{d}s \sim R^{N-2-b}$$

which shows (3.3). As $\eta_{-b}^{-a}(\cdot) \in C(\mathbb{R}^N)$, the condition implying the global integrability follows trivially.

□

II.3.1 Singular integrals

As was shown in Section II.1, we may write the Oseen fundamental tensor in the form

$$\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\beta) = \mathcal{S}_{ij}(\mathbf{x}-\mathbf{y}) + \mathcal{N}_{ij}(\mathbf{x}-\mathbf{y};\beta) , \tag{3.5}$$

where

$$\begin{aligned} \mathcal{S}_{ij}(\mathbf{x}-\mathbf{y}) &= \frac{1}{8\pi}\Big(\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{y}|} + \frac{(y_i-x_i)(y_j-x_j)}{|\mathbf{x}-\mathbf{y}|^3}\Big) \quad (N=3) \\ \mathcal{S}_{ij}(\mathbf{x}-\mathbf{y}) &= \frac{1}{4\pi}\Big(\delta_{ij}\ln\frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{(y_i-x_i)(y_j-x_j)}{|\mathbf{x}-\mathbf{y}|^2}\Big) \quad (N=2) \end{aligned} \tag{3.6}$$

and the other part $\mathcal{N}_{ij}(\mathbf{x}-\mathbf{y};\beta)$ is locally integrable up to the second gradient. But it is easy to see that the second gradient of $\mathcal{S}_{ij}(\mathbf{x}-\mathbf{y})$[4] are non–integrable functions and therefore we must apply completely another approach as for $\nabla^2\mathcal{N}_{ij}(\mathbf{x}-\mathbf{y};\beta)$.

[4]taken in the classical sense outside $\mathbf{x} = \mathbf{y}$

Analogously, recalling that

$$e_i(\mathbf{x}-\mathbf{y}) = -\frac{\partial}{\partial y_i}\mathcal{E}(|\mathbf{x}-\mathbf{y}|)\,, \quad i=1,2,\ldots,N\,, \tag{3.7}$$

where $\mathcal{E}(|\mathbf{x}-\mathbf{y}|)$ is the fundamental solution to the Laplace equation, it is an easy matter to see that the gradients of $e_i(\mathbf{x}-\mathbf{y})$ represent again a kernel which is not $L^1_{loc}(\mathbb{R}^N)$. There are two (in our case equivalent) approaches how to treat integrals of the type

$$\frac{\partial^2}{\partial x_i \partial x_j} I(\mathbf{x}) = \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^N} \mathcal{R}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \mathrm{d}\mathbf{y}\,,$$

where $\mathcal{R}$ stays either for $\mathcal{E}(|\mathbf{x}-\mathbf{y}|)$ or for $\mathcal{S}_{ij}(|\mathbf{x}-\mathbf{y}|)$. Let us assume for a moment that $f \in C_0^\infty(\mathbb{R}^N)$. Then we have ($\nabla\mathcal{R} \in L^1_{loc}(\mathbb{R}^N)$)

$$\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_j} I(\mathbf{x}) &= \int_{\mathbb{R}^N} \frac{\partial \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_j}\frac{\partial f(\mathbf{y})}{\partial y_i}\mathrm{d}\mathbf{y} = \\
&= \lim_{\varepsilon\to 0^+}\int_{B^\varepsilon(\mathbf{x})} \frac{\partial \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_j}\frac{\partial f(\mathbf{y})}{\partial y_i}\mathrm{d}\mathbf{y} = \\
= \lim_{\varepsilon\to 0^+}\Big[\int_{B^\varepsilon(\mathbf{x})} &\frac{\partial^2 \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_i \partial x_j} f(\mathbf{y})\mathrm{d}\mathbf{y} + \int_{\partial B^\varepsilon(\mathbf{x})} \frac{\partial \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_j} f(\mathbf{y}) n_i(\mathbf{y})\mathrm{d}S\Big]\,.
\end{aligned}$$

It can be shown for our kernels (see e.g. [Ga1]) that

$$\lim_{\varepsilon\to 0^+}\int_{\partial B^\varepsilon(\mathbf{x})} \frac{\partial \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_j} f(\mathbf{y}) n_i(\mathbf{y})\mathrm{d}S = c_{ij}(\mathcal{R}) f(\mathbf{x})\,,$$

where $c_{ij}(\mathcal{R})$ are constants depending only on i, j and $\mathcal{R}$ and can be (eventually) equal to zero. The volume integrals of our type (i.e. in the principal value sense), noted by

$$\text{v.p.}\int_{\mathbb{R}^N} \frac{\partial^2 \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_i \partial x_j} f(\mathbf{y})\mathrm{d}\mathbf{y}\,,$$

were intensively studied by several authors, see e.g. [CaZy], [St] and we have the following result

Theorem 3.1 *Let* $Tf = \text{v.p.}\int_{\mathbb{R}^N} K(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$, *where*

$$K(\mathbf{x},\mathbf{y}) = \Omega\Big(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\Big)|\mathbf{x}-\mathbf{y}|^{-N}\,.$$

Let $\int_{S_N}\Omega\, dS = 0$ *and let*

$$\omega(t) = \sup_{\substack{|\mathbf{x}-\mathbf{x}'|\le t\\ |\mathbf{x}|=|\mathbf{x}'|=1}} |\Omega(\mathbf{x})-\Omega(\mathbf{x}')| \tag{3.8}$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty\,. \tag{3.9}$$

Then the operator T *maps* $\mathcal{D}(\mathbb{R}^N)$ *into* $L^p(\mathbb{R}^N)$ *for all* $p\in(1,\infty)$ *and there exists a constant* $c(p)>0$ *such that*

$$\|Tf\|_p \le c(p)\|f\|_p\,. \tag{3.10}$$

Remark 3.1 Using the density argument, the operator T can be continuously extended onto $L^p(\mathbb{R}^N)$.

Proof: see e.g. [St].

□

Another approach uses the Fourier transform (see Section VIII.4). Let us denote ($f \in C_0^\infty(\mathbb{R}^N)$, as above)

$$v(\mathbf{x}) = \frac{\partial^2}{\partial x_i \partial x_j} I(\mathbf{x}) = \int_{\mathbb{R}^N} \mathcal{R}(\mathbf{x}-\mathbf{y}) \frac{\partial^2}{\partial y_i \partial y_j} f(\mathbf{y}) \mathrm{d}\mathbf{y} \,, \tag{3.11}$$

$\mathcal{R}$ as above. The kernels $\mathcal{R}$ and $\nabla\mathcal{R}$ belong to $L^1_{loc}(\mathbb{R}^N)$ and so

$$\mathcal{F}(v)(\xi) = \mathcal{F}\Big(\int_{\mathbb{R}^N} \frac{\partial}{\partial x_i}\mathcal{R}(\mathbf{x}-\mathbf{y}) \frac{\partial}{\partial y_j} f(\mathbf{y})\mathrm{d}\mathbf{y}\Big) = (2\pi)^{\frac{N}{2}} \mathcal{F}\Big(\frac{\partial \mathcal{R}}{\partial x_i}\Big)(\xi) \mathcal{F}\Big(\frac{\partial f}{\partial y_i}\Big)(\xi) =$$
$$= (2\pi)^{\frac{N}{2}} \mathcal{F}\Big(\frac{\partial \mathcal{R}}{\partial x_i}\Big)(\xi)(-i\xi_j)\mathcal{F}(f)(\xi)\,,$$

the product is to be understood in the sense of $\mathcal{S}'$, see Section VIII.4.

We have from Lemmas VIII.4.13 and VIII.4.14

$$\begin{aligned} \mathcal{F}(v)(\xi) &= \frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(f)(\xi) \quad (\text{if } \mathcal{R}=\mathcal{E}) \\ \mathcal{F}(v)(\xi) &= \frac{\xi_i\xi_j}{|\xi|^2}\frac{\xi_k\xi_l - \delta_{kl}|\xi|^2}{|\xi|^2}\mathcal{F}(f)(\xi) \quad (\text{if } \mathcal{R}=\mathcal{S}_{kl})\,. \end{aligned} \tag{3.12}$$

Let us denote

$$\begin{aligned} m_1(\xi) &= \frac{\xi_i\xi_j}{|\xi|^2} \\ m_2(\xi) &= \frac{\xi_i\xi_j}{|\xi|^2}\frac{\xi_k\xi_l - \delta_{kl}|\xi|^2}{|\xi|^2} \end{aligned} \tag{3.13}$$

with $i,j,k,l = 1,2,\ldots,N$. We can rewrite (3.12)

$$\mathcal{F}(v)(\xi) = m_r(\xi)\mathcal{F}(f)(\xi)\,, \quad r=1,2 \tag{3.14}$$

with $m_r(\xi)$ bounded functions. In such situations we can apply the following theorems on multipliers

Theorem 3.2 *Let $m \in L^\infty(\mathbb{R}^N) \cap C^s(\mathbb{R}^N \setminus \{\mathbf{0}\})$, $s = \left[\frac{N}{2}\right]+1$, be such that*

$$\sup_{\substack{\xi\in\mathbb{R}^N \\ |\alpha|\le s}} |D^\alpha m(\xi)||\xi|^{|\alpha|} \le C_1\,. \tag{3.15}$$

Then m is a L^p–multiplier for $1<p<\infty$; it means that the operator $T : \mathcal{S} \mapsto \mathcal{S}'$ defined

$$Tf = \mathcal{F}^{-1}(m\mathcal{F}(f)) \tag{3.16}$$

maps $\mathcal{S}$ into $L^p(\mathbb{R}^N)$ and it holds

$$\|Tf\|_p \le C(C_1,p,N)\|f\|_p\,. \tag{3.17}$$

T can be therefore continuously extended onto $L^p(\mathbb{R}^N)$.

Proof: see e.g. [St].

□

Remark 3.2 It is an easy matter to see that our kernels (3.13) satisfy (3.15).

This theorem is usually called the Marcinkiewicz multiplier theorem. The following theorem is due to Lizorkin and is in some sense a generalization of the Marcinkiewicz theorem and will be used in order to study the L^p–estimates for both classical and modified Oseen problem in the whole $\mathbb{R}^N$. Let us emphasize that even for $\beta = 0$ the functions need not to satisfy (3.15) as e.g. the second derivative of $\boldsymbol{\mathcal{O}}$.

Theorem 3.3 (Lizorkin)
Let $m \in C^{N-1}(\mathbb{R}^N \setminus \{\mathbf{0}\})$, $\frac{\partial^N m}{\partial\xi_1 \ldots \partial\xi_N} \in C(\mathbb{R}^N \setminus \{\mathbf{0}\})$ and let there exists $\beta \in [0,1)$ such that

$$|D^\alpha m(\xi)| |\xi_1|^{\alpha_1+\beta} \ldots |\xi_N|^{\alpha_N+\beta} \leq C_2 \tag{3.18}$$

for any multiindex $\alpha = (\alpha_1, \ldots, \alpha_N)$ such that $\alpha_i \in \{0,1\}$ and $|\alpha| \leq N$. Let $1 < q < \frac{1}{\beta}$ and $r = \frac{q}{1-\beta q}$ i.e. $\frac{1}{r} = \frac{1}{q} - \beta$. Then the operator T defined in (3.16) maps $\mathcal{S}$ into $L^r(\mathbb{R}^N)$ and there exists a constant $C = C(C_2, q, \beta, N)$ such that

$$\|Tf\|_r \leq C\|f\|_q\,. \tag{3.19}$$

T can be therefore continuously extended onto $L^q(\mathbb{R}^N)$.

Proof: see [Liz].

□

The aim of this chapter is to develop a general L^p–weighted theory for certain operators of the type (3.16) and certain class of weights and then to apply it on the integral operators with kernels $\frac{\partial^2 \mathcal{E}}{\partial x_i \partial x_j}$ and $\frac{\partial^2 \mathcal{S}_{ij}}{\partial x_i \partial x_j}$ and weights introduced in (3.1).

There are again two approaches, one using the potential theory and the other one using the Fourier transform. We only shortly mention the former but we shall use the latter. The reason for this will be clear in the following chapter, where we get some information only on the Fourier transform of the fundamental solution to the modified Oseen problem, not on the solution itself. Both approaches are taken from [KuWh], where also the proofs of Theorems 3.4 and 3.5 can be found.

Definition 3.1 *The non–negative weight g belongs to the Muckenhoupt class A_r, $1 \leq r < +\infty$, if there is a constant C such that*

$$\sup_Q \Big[\Big(\frac{1}{|Q|}\int_Q g(\mathbf{x})\,d\mathbf{x}\Big)\Big(\frac{1}{|Q|}\int_Q g(\mathbf{x})^{-\frac{1}{r-1}}\,d\mathbf{x}\Big)^{r-1}\Big] \leq C < \infty \tag{3.20}$$

if $r \in (1;\infty)$ and

$$\sup_Q \frac{1}{|Q|}\int_Q g(\mathbf{x})\,d\mathbf{x} \leq C g(\mathbf{x}_0), \quad \forall \mathbf{x}_0 \in \mathbb{R}^N \tag{3.21}$$

if $r = 1$. *In the first case, the supremum is taken over all cubes* Q *in* $\mathbb{R}^N$, *in the second case only over all cubes which contain* $\mathbf{x}_0$; $|Q|$ *denotes the Lebesgue measure of* Q . *The constant does not depend on* $\mathbf{x}_0$.

Remark 3.3

a) For $r = 1$, the condition (3.21) can be replaced by

$$Mg(\mathbf{x}) \leq Cg(\mathbf{x}) \quad \text{for a.a. } \mathbf{x} \in \mathbb{R}^N \,, \tag{3.22}$$

where $Mg(\mathbf{x})$ is the Hardy–Littlewood maximal function (see e.g. [St]).

b) In (3.20) and (3.21) it is enough to take the supremum over all cubes with edges parallel to an arbitrary chosen cartesian system X.

Indeed, let X be a cartesian system in $\mathbb{R}^N$ and X' another one arisen from X by any rotation. Then we have

$$\frac{1}{N^{\frac{N}{2}}} \frac{1}{|Q_1|} \int_{Q_1} w(\mathbf{x}) \mathrm{d}\mathbf{x} \leq \frac{1}{|Q'|} \int_{Q'} w(\mathbf{x}) \mathrm{d}\mathbf{x} \leq \frac{N^{\frac{N}{2}}}{|Q_2|} \int_{Q_2} w(\mathbf{x}) \mathrm{d}x \tag{3.23}$$

for any $w \geq 0$ locally integrable function. In (3.23) Q' is a cube with edges parallel to the axes of X', Q_1 is the greatest cube with edges parallel to the axes of X such that $Q_1 \subset Q'$ and Q_2 is the smallest cube with edges parallel to the axes of X such that $Q' \subset Q_2$, see Fig.1 below.

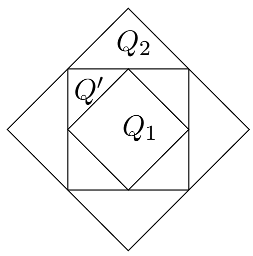

Fig. 1

Remark 3.4 The weighted L^p–spaces are defined in Subsection VIII.1.2. Let us only note that in general there is a difference between $L^p_{(g)}(\Omega)$ and

$$\mathcal{L}^p_{(g)}(\Omega) = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{p,(g)}} \,,$$

where $\|f\|_{p(g)} = (\int_{\mathbb{R}^N} |f(\mathbf{x})|^p g(\mathbf{x}) \mathrm{d}\mathbf{x})^{\frac{1}{p}}$, $1 < p < \infty$. Nevertheless, for our weights defined in (3.1) and $f \equiv 0$ in $B_{\frac{1}{2}}(\mathbf{0})$ both spaces coincide. Generally, this is true when e.g. $g \geq C > 0$ in $\mathbb{R}^N$.

Theorem 3.4 *Let* $N \in \mathbb{N}$, $N \geq 2$, $\Omega \in L^\infty(\partial B_1(\mathbf{0}))$ *satisfies the Dini condition (3.9),* $\int_{\partial B_1(\mathbf{0})} \Omega dS = 0$, Ω *is positively homogeneous function of degree zero,*

$$R(\mathbf{x}) = |\mathbf{x}|^{-N} \Omega\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big) \,.$$

.

Let T be an operator with the kernel $\mathcal{R}$, i.e. $Tf(\mathbf{x}) = (\mathcal{R} * f)(\mathbf{x})$ in principal value sense and $g \in A_p$ in $\mathbb{R}^N$, $p > 1$. Then T is continuous operator $\mathcal{L}^p(\mathbb{R}^N; g) \mapsto L^p(\mathbb{R}^N; g)$.

As announced above, we shall prefer another approach. Namely

Definition 3.2 *We say that a bounded function m defined on $\mathbb{R}^N \setminus \{\mathbf{0}\}$ belongs to a class $M(a, l)$ for some $a \geq 1$, $l = 1, 2, \ldots$ if*

$$\sup_{R>0} \Big(R^{a|\alpha|-N} \int_{R<|\xi|<2R} |D^\alpha m(\xi)|^a d\xi \Big)^{\frac{1}{a}} < \infty\,, \quad \forall |\alpha| \leq l\,. \tag{3.24}$$

Remark 3.5 Let us observe that if m satisfies (3.15) for $s = l$, then $m \in M(a, l)$ for all $a \geq 1$. So, in particular, our functions $m_i(\xi)$ defined in (3.13) satisfy (3.24) for any $l \in \mathbb{N}$ and they belong to any $M(a, l)$, $a \geq 1$, $l \in \mathbb{N}$.

Theorem 3.5 *Let $1 < a \leq 2$, $\frac{N}{a} < l \leq N$ and $m \in M(a, l)$, $g \in A_{\frac{pl}{N}}$. Let $\frac{N}{l} < p < \infty$. Let T be the operator defined by (3.16). Then there exists C, independent of f, such that*

$$\|Tf\|_{p,(g)} \leq C \|f\|_{p,(g)}$$

for all $f \in C_0^\infty(\mathbb{R}^N)$. Therefore, T can be continuously extended onto $\mathcal{L}^p_{(g)}(\mathbb{R}^N)$, $\frac{N}{l} < p < \infty$.

Remark 3.6 As follows from Remark 3.5, we may take for our kernels $l = N$.

Next part is devoted to the investigation under which conditions the weights defined in (3.1) belong to A_r for some $1 \leq r < +\infty$. Firstly we recall several general results:

Lemma 3.3

(i) If $g_1, g_2 \in A_1$, then for any $1 \leq r < +\infty$ the weight $g_1\, g_2^{1-r} \in A_r$.

(ii) If g_1, $g_2 \in A_r$ for some $1 \leq r < +\infty$, then for any $\hbar \in [0; 1]$, $g_1^{\hbar} g_2^{1-\hbar} \in A_r$.

(iii) If $g \in A_r$ for some $1 \leq r < +\infty$, then for any $\hbar \in [0; 1]$, $g^{\hbar} \in A_r$.

(iv) If $g \in A_r$ for some $1 \leq r < +\infty$, then $g \in A_p$ for all $r \leq p < \infty$.

Proof: (i) follows directly from the definition of A_1 and A_r as

$$\begin{aligned}
\sup_Q \Big\{ \Big(\frac{1}{|Q|} \int_Q g_1 g_2^{1-r} \mathrm{d}\mathbf{x} \Big) \Big(\frac{1}{|Q|} \int_Q g_1^{-\frac{1}{r-1}} g_2 \mathrm{d}\mathbf{x} \Big)^{r-1} \Big\} \leq \\
\leq C \sup_Q \Big(\frac{1}{|Q|} \int_Q g_1 \mathrm{d}\mathbf{x} \Big) \sup_Q \Big(\frac{1}{|Q|} \int_Q g_2 \mathrm{d}\mathbf{x} \Big)^{1-r} \cdot \\
\cdot \sup_Q \Big(\frac{1}{|Q|} \int_Q g_1 \mathrm{d}\mathbf{x} \Big)^{-1} \sup_Q \Big(\frac{1}{|Q|} \int_Q g_2 \mathrm{d}\mathbf{x} \Big)^{r-1} = C\,.
\end{aligned}$$

(ii) is a direct consequence of Hölder inequality and (iii) follows from (ii) and the fact that $1 \in A_p \,\forall p \in [1,\infty)$. The last assertion follows again from the Hölder inequality.

□

Definition 3.3 *Let μ be a non-negative Borel measure. We define the maximal function*

$$M\mu(\mathbf{x}) = \sup_Q \frac{1}{|Q|}\int_Q d\mu(\mathbf{y})\,, \tag{3.25}$$

where the supremum is taken over all cubes Q such that $\mathbf{x} \in Q$. Analogously we define $Mf(\mathbf{x})$ for $f \in L^1_{loc}(\mathbb{R}^N)$, replacing $d\mu(\mathbf{y})$ by $|f|d\mathbf{y}$. (See also Remark 3.3).

Lemma 3.4

(i) If $M\mu$ is finite for a.a. $\mathbf{x} \in \mathbb{R}^N$, then for any $\hbar \in [0;1)$, $(M\mu)^\hbar \in A_1$.

(ii) Let $w \in A_1$. Then there exists a function $f \in L^1_{loc}(\mathbb{R}^N)$ such that for $d\mu(\mathbf{y}) = |f|d\mathbf{y}$ we have $w \sim (M\mu)^\hbar$ for some $\hbar \in [0,1)$.

Proof: see e.g. [To]

□

Lemma 3.5 *The weights $|\mathbf{x}|^{-a}$ and $(1+|\mathbf{x}|)^{-a}$ satisfy the A_1-condition on $\mathbb{R}^N$ for each $a \in [0,N)$.*

Proof: We have that for $\mu = \delta_0$ the maximal function $M\mu(\mathbf{x}) \sim |\mathbf{x}|^{-N}$ and so $|\mathbf{x}|^{-N\hbar} \in A_1, \quad \forall \hbar \in [0,1)$. Further, if we define $\mu(A) = |A \cap B_1(\mathbf{0})|$, then $M\mu(\mathbf{x}) \sim (1+|\mathbf{x}|)^{-N}$ and again Lemma 3.4 (i) furnishes the result.

□

Using Lemmas 3.3–3.5 we shall study conditions under which our weights defined in (3.1) belong to the Muckenhoupt classes A_r, $1 \le r < \infty$. The three-dimensional case was intensively studied by Farwig (see [Fa1] or [Fa2]). Nevertheless, as we need some generalizations of this result, we shall repeat them here. See also [KrNoPo], where both three– and twodimensional cases are studied.

Lemma 3.6 *For $b \in (-1;2]$ and $\hbar \in [0,1)$ the function $w_0^{(3)}(\mathbf{x}) = \left(\frac{|\mathbf{x}|^{b-1}}{s(\mathbf{x})}\right)^\hbar$, $\mathbf{x} \in \mathbb{R}^3$ is a weight of the class A_1 in $\mathbb{R}^3$.*

Proof: Let $b \in (-1;2]$. We define the measure μ by

$$\mu(A) \equiv \int_{A^+} x_1^b \mathrm{d}x_1\,, \tag{3.26}$$

where $A^+ = \{x_1 \in \mathbb{R};\ x_1 > 0 : \ (x_1, 0, 0) \in A\}$ for $A \subset \mathbb{R}^3$, measurable. Evidently, μ is a non-negative Borel measure on $\mathbb{R}^3$. We shall show that $M\mu \sim \frac{r^{b-1}}{s^{\frac{1}{2}}}$; then the assertion of the lemma follows from Lemma 3.4 (i).

Let Q_a be a closed cube containing $\mathbf{x}$ with sides parallel to the axes (see Remark 3.3 b)) with the side length $a > |\mathbf{x}'|$, $q(a) = \frac{\mu(Q_a)}{|Q_a|}$ and $|\mathbf{x}|$ denotes the maximum norm of $\mathbf{x} = (x_1, x_2, x_3) = (x_1, \mathbf{x}') \in \mathbb{R}^3$. We have to distinguish several cases:

A) Let $x_1 > 0$, $\ |\mathbf{x}'| \le x_1$ and $b \in (-1; 0]$. We first consider cubes Q_a with $|\mathbf{x}'| \le a \le x_1$. It is enough to take such cubes that $Q_a^+ = [x_1 - a; x_1]$. Then we have

$$q(a) = \frac{1}{b+1} \frac{x_1^{b+1} - (x_1 - a)^{b+1}}{a^3} = \frac{1}{b+1} x_1^{b-2} \varphi\Big(\frac{a}{x_1}\Big)$$

with

$$\varphi(y) = \frac{1 - (1-y)^{b+1}}{y^3}\,, \qquad \frac{|\mathbf{x}'|}{x_1} \le y \le 1.$$

Let $b < 0$. Then $\varphi(0^+) = +\infty$, $\varphi(1) = 1$, $\varphi'(1^-) = +\infty$. We show that there exists exactly one point $y_0 \in (0,1)$ such that $\varphi'(y_0) = 0$. We have namely

$$\varphi'(y) = \frac{(1-y)^b[(b+1)y + 3(1-y)] - 3}{y^4}\,. \tag{3.27}$$

Denoting the nominator in (3.27) by $F(y)$, we easily see that $F \in C[0;1)$, $F(0) = 0$, $F(1^-) = \infty$ and

$$F'(y) = (1-y)^{b-1}(b+1)[(2-b)y - 2]\,.$$

As $F'(y) > 0$ for $y \in (\frac{2}{2-b}; 1)$ and $F'(y) < 0$ for $y \in (0; \frac{2}{2-b})$, there exists exactly one point $y_0 \in (\frac{2}{2-b}; 1)$ such that $F(y_0) = \varphi'(y_0) = 0$.

Now, recalling the properties of $\varphi(y)$ we observe that the point y_0 is the only local (and global) minimum of φ on the interval $(0;1)$. Therefore there exists exactly one point $y_b \in (0;1)$ such that $\varphi(y_b) = 1$.

If $|\mathbf{x}'| \le y_b \cdot x_1$, then φ attains its maximum on $[\frac{|\mathbf{x}'|}{x_1}; 1)$ at the point $\frac{|\mathbf{x}'|}{x_1}$ and

$$\max\{q(a); |\mathbf{x}'| \le a \le x_1\} = q(|\mathbf{x}'|) \sim \frac{x_1^b}{|\mathbf{x}'|^2} \sim \frac{r^b}{|\mathbf{x}'|^2}\,.$$

If $|\mathbf{x}'| > y_b \cdot x_1$ then φ is maximal in 1 what yields

$$\max\{q(a); |\mathbf{x}'| \le a \le x_1\} = q(x_1) \sim x_1^{b-2} \sim \frac{r^b}{|\mathbf{x}'|^2} \qquad \text{as } x_1 \sim |\mathbf{x}'|\,.$$

Now, let us consider cubes Q_a with $a \ge x_1$ and $Q_a^+ = [0; a)$. Then $q(a) = \frac{a^{b-2}}{b+1}$ is strictly decreasing in $a \ge x_1$. Therefore

$$\max\{q(a); x_1 \le a\} \sim x_1^{b-2} \le C \frac{r^b}{|\mathbf{x}'|^2}.$$

Combining this with the fact that $s(\mathbf{x}) \sim \frac{|\mathbf{x}'|^2}{r}$ (see Lemma 3.1) we get

$$M\mu \sim \frac{r^{b-1}}{s}\,.$$

If $b = 0$, then easily $q(a) = a^{-2}$; the maximum is attained for a minimal, i.e.

$$M\mu \sim \frac{1}{|\mathbf{x}'|^2} \sim \frac{1}{rs}\,.$$

B) Let $x_1 > 0$, $|\mathbf{x}'| > x_1$, and $b \in (-1; 0]$. It suffices to consider cubes Q_a with $a \geq |\mathbf{x}'|$ and $Q_a^+ = [0; a]$. But then obviously

$$\max\{q(a); a \geq |\mathbf{x}'|\} \sim q(|\mathbf{x}'|) \sim |\mathbf{x}'|^{b-2} \sim \frac{r^{b-1}}{s}\,.$$

C) Let $x_1 > 0$ and $b \in (0; 2]$. We can consider only cubes Q_a such that $a \geq |\mathbf{x}'|$ but now, as $b > 0$, with $Q_a^+ = [x_1; x_1 + a]$. Therefore

$$q(a) = \frac{1}{b+1}\frac{(x_1 + a)^{b+1} - x_1^{b+1}}{a^3}$$

and since $q(a)$ is evidently decreasing, we get

$$\max\{q(a); a \geq |\mathbf{x}'|\} = q(|\mathbf{x}'|).$$

Now, if $|\mathbf{x}'| < x_1$, then $q(|\mathbf{x}'|) \sim \frac{x_1^b}{|\mathbf{x}'|^2} \sim \frac{r^{b-1}}{s}$. On the other side, if $|\mathbf{x}'| > x_1$, then $q(|\mathbf{x}'|) \sim |\mathbf{x}'|^{b-2} \sim \frac{r^{b-1}}{s}$.

D) Let $x_1 < 0$. We can restrict ourselves on cubes Q_a with $a \geq \max(|\mathbf{x}'|, |x_1|)$ and $Q_a^+ = [0; a - |x_1|]$. Then

$$q(a) = \frac{|x_1|^{b-2}}{b+1}\varphi\Big(\frac{a}{|x_1|}\Big) \quad \text{with} \quad \varphi(y) = \frac{(y-1)^{b+1}}{y^3}.$$

The function $\varphi(y)$ vanishes in 1 and, if $b < 2$, at infinity. Thus for $b < 2$ there is a point $y_b > 1$ such that φ is maximal in y_b for $y \geq 1$. If $|\mathbf{x}'| \leq y_b \cdot |x_1|$, then

$$\max\{q(a); a \geq |\mathbf{x}'|\} = q(y_b|x_1|) \sim x_1^{b-2} \sim \frac{r^{b-1}}{s} \quad \text{as } s \sim r \qquad \text{for } x_1 < 0.$$

But if $|\mathbf{x}'| > y_b \cdot |x_1|$, then

$$\max\{q(a); a \geq |\mathbf{x}'|\} \sim q(|\mathbf{x}'|) \sim |\mathbf{x}'|^{b-2} \sim \frac{r^{b-1}}{s}\,.$$

The case $b = 2$ is trivial since q is maximal for $a \to \infty$ yielding

$$\max\{q(a); a \geq |\mathbf{x}'|\} \sim 1 \sim \frac{r}{s}\,.$$

□

A similar result holds in two space dimensions:

Lemma 3.7 *For $b \in (-1;1]$ and $\hbar \in [0;1)$ the function $w_0^{(2)}(\mathbf{x}) = \Big(\frac{|\mathbf{x}|^{b-\frac{1}{2}}}{s(\mathbf{x})^{\frac{1}{2}}}\Big)^{\hbar}$, $\mathbf{x} \in \mathbb{R}^2$, is a weight of the class A_1 in $\mathbb{R}^2$.*

Sketch of the proof: We proceed similarly as in the threedimensional case. We again introduce the measure (3.26) and study the behaviour of $q(a) = \frac{\mu(Q_a)}{|Q_a|}$ in four different cases.

A) Let $x_1 > 0$, $|x_2| \le x_1$ and $b \in (-1;0]$. We start again with $|x_2| \le a \le x_1$ and $Q_a^+ = [x_1 - a; x_1]$. So we get

$$q(a) = \frac{1}{b+1}\frac{x_1^{b+1} - (x_1-a)^{b+1}}{a^2} = \frac{1}{b+1}x_1^{b-1}\varphi\Big(\frac{a}{x_1}\Big)$$

with

$$\varphi(y) = \frac{1-(1-y)^{b+1}}{y^2}\,, \qquad \frac{|x_2|}{x_1} \le y \le 1.$$

Similarly as in Lemma 3.6 we first assume $b < 0$. Then we can show that there exists a unique point $y_b \in (0;1)$ such that $\varphi(y_b) = 1$.

If $|x_2| \le y_b \cdot x_1$, then

$$\max\{q(a); |x_2| \le a \le x_1\} = q(|x_2|) \sim \frac{x_1^b}{|x_2|} \sim \frac{r^{b-\frac{1}{2}}}{s^{\frac{1}{2}}}\,.$$

If $|x_2| > y_b \cdot x_1$, then

$$\max\{q(a); |x_2| \le a \le x_1\} = q(x_1) \sim x_1^{b-1} \sim \frac{r^b}{|x_2|} \sim \frac{r^{b-\frac{1}{2}}}{s^{\frac{1}{2}}}\,.$$

The case $a \ge x_1$ and $Q_a^+ = [0;a)$ is again trivial as well as the case $b = 0$.

B) Let $x_1 > 0$, $|x_2| > x_1$ and $b \in (-1;0]$. Assuming Q_a such that $Q_a^+ = [0;a]$ we have

$$\max\{q(a); a \ge |x_2|\} \sim q(|x_2|) \sim |x_2|^{b-1} \sim \frac{r^{b-\frac{1}{2}}}{s^{\frac{1}{2}}} \qquad \text{as } s(\mathbf{x}) \sim |x_2| \sim r\,.$$

C) Let $x_1 > 0$ and $b \in (0;1]$. We consider again squares Q_a with $a \ge |x_2|$ and $Q_a^+ = [x_1; x_1 + a]$. We can easily verify that

$$q(a) = \frac{1}{b+1}\frac{(x_1+a)^{b+1} - x_1^{b+1}}{a^2}$$

is decreasing, and therefore $\max q(a) = q(|x_2|)$. Now, if $|x_2| < x_1$, then $q(|x_2|) \sim \frac{x_1^b}{|x_2|} \sim \frac{r^{b-\frac{1}{2}}}{s^{\frac{1}{2}}}$, while for $|x_2| > x_1$ we get $q(|x_2|) \sim |x_2|^{b-1} \sim \frac{r^{b-\frac{1}{2}}}{s^{\frac{1}{2}}}$.

D) Let $x_1 < 0$. It is enough to consider $a \geq \max\{|x_2|, |x_1|\}$ and the squares $Q_a^+ = [0; a - |x_1|]$. Then

$$q(a) = \frac{|x_1|^{b-1}}{b+1}\varphi\left(\frac{a}{|x_1|}\right) \qquad \text{with} \ \ \varphi(y) = \frac{(y-1)^{b+1}}{y^2}.$$

As in the threedimensional case we can show that for $b < 1$ there exists a point $y_b > 1$ such that φ is maximal in y_b. So if $|x_2| \leq y_b \cdot |x_1|$, then

$$\max\{q(a); a \geq |x_2|\} = q(y_b|x_1|) \sim x_1^{b-1} \sim \frac{r^{b-\frac{1}{2}}}{s^{\frac{1}{2}}}$$

and if $|x_2| > y_b \cdot |x_1|$, then

$$\max\{q(a); a \geq |x_2|\} \sim q(|x_2|) \sim |x_2|^{b-1} \sim \frac{r^{b-\frac{1}{2}}}{s^{\frac{1}{2}}}.$$

The case $b = 1$ is trivial since

$$\max\{q(a); a \geq |x_2|\} \sim 1 \sim \frac{r^{\frac{1}{2}}}{s^{\frac{1}{2}}}.$$

□

The results of Lemmas 3.6 and 3.7 are applicable for the weight σ_B^A. Since we are rather interested in the weights η_B^A and ν_B^A, we need the following results.

Lemma 3.8 *For* $b \in (-1; 2]$ *and* $\hbar \in [0; 1)$ *the function* $w_1^{(3)} = \left(\eta_{-1}^{b-1}\right)^{\hbar}$ *is a weight of class* A_1 *in* $\mathbb{R}^3$.

Proof: We have to verify that $(M\, w_1^{(3)})(\mathbf{x}) \leq C w_1^{(3)}(\mathbf{x})$ a.e. in $\mathbb{R}^3$ (see Remark 3.3). Let Q_a denotes, similarly as in the previous lemmas, a closed cube with sides parallel to the axes and with the side length a; R will be a sufficiently large constant. We again distinguish several cases:

A) $s(\mathbf{x}) \leq 1, \quad r = |\mathbf{x}| \geq R$

α) $a \leq \frac{1}{2} r^{\frac{1}{2}}$

Then for all $\mathbf{y} \in Q_a$ we have that $w_1^{(3)}(\mathbf{y}) \sim w_1^{(3)}(\mathbf{x})$ and

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\, \mathrm{d}\mathbf{y} \leq C|Q_a| w_1^{(3)}(\mathbf{x}).$$

β) $a = \frac{1}{2} r^{\frac{1}{2}+\sigma}, \quad \sigma \in (0; \frac{1}{2}]$

Now $Q_a \subset \{\mathbf{y} \in \mathbb{R}^3;\ ||\mathbf{y}| - r| \leq c r^{\frac{1}{2}+\sigma};\ s(\mathbf{y}) \leq c r^{2\sigma}\}$ and proceeding analogously as in Lemma 3.2 we get

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\, \mathrm{d}\mathbf{y} \leq C \int_{r - cr^{\frac{1}{2}+\sigma}}^{r + cr^{\frac{1}{2}+\sigma}} \varrho^{1+(b-1)\hbar} \mathrm{d}\varrho \int_0^{Cr^{2\sigma}} \frac{\mathrm{d}s}{(1+s)^{\hbar}} \leq$$
$$\leq C|Q_a| r^{\frac{3}{2}+(b-1)\hbar+(3-2\hbar)\sigma} \leq C|Q_a| r^{(b-1)\hbar} \leq C w_1^{(3)}(\mathbf{x})$$

as $r \geq R \gg 1$, $s(\mathbf{x}) \leq 1$.

γ) $a \geq \frac{r}{2}$
In this case all Q_a such that $\mathbf{x} \in Q_a$ are contained in the ball B_{4a} and therefore similarly as above

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\mathrm{d}\mathbf{y} \leq C \int_0^{4a} (1+\varrho)^{2-\hbar+(b-1)\hbar}\mathrm{d}\varrho \leq C|Q_a|a^{(b-2)\hbar}\,.$$

Evidently,

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\,\mathrm{d}\mathbf{y} \leq C|Q_a|r^{(b-2)\hbar} \leq C|Q_a|r^{(b-1)\hbar}\,.$$

B) $s(\mathbf{x}) \geq 1, \quad r \geq R$
Now, as $b \leq 2$ and $s \leq r$, we have

$$\Big(1+\frac{1}{r}\Big)^{b-1} \leq C\left(1+\frac{1}{r}\right) \leq C\left(1+\frac{1}{s}\right)$$

and so

$$w_1^{(3)}(\mathbf{x}) \leq \Big(\frac{(1+r)^{b-1}}{(1+s)}\Big)^{\hbar} \leq C\,\frac{r^{(b-1)\hbar}}{s^{\hbar}} = C\,w_0^{(3)}(\mathbf{x})\,.$$

So, for all $\mathbf{y} \in \mathbb{R}^3$ we have that $w_1^{(3)}(\mathbf{y}) \leq w_0^{(3)}(\mathbf{y})$. But then, as $w_0^{(3)} \in A_1$, we have

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\,\mathrm{d}\mathbf{y} \leq C\int_{Q_a} w_0^{(3)}(\mathbf{y})\,\mathrm{d}y \leq C\,|Q_a|\,w_0^{(3)}(\mathbf{x})\,.$$

As $|\mathbf{x}| \geq 1,\ s(\mathbf{x}) \geq 1\,$, we have $w_0^{(3)}(\mathbf{x}) \leq C\,w_1^{(3)}(\mathbf{x})$ and the required inequality follows.

C) $r \leq R$

α) If $\frac{a}{2} \leq R$, then trivially

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\,\mathrm{d}\mathbf{y} \leq C\,|Q_a|\,w_1^{(3)}(\mathbf{x}).$$

β) If $\frac{a}{2} > R$, then $Q_a \subset B_{3a}$ and analogously as in Aγ)

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\mathrm{d}\mathbf{y} \leq \int_{B_{3a}} w_1^{(3)}(\mathbf{y})\,\mathrm{d}\mathbf{y} \leq C\int_0^{3a}(\varrho+1)^{2-\hbar+(b-1)\hbar}\mathrm{d}\varrho \leq$$
$$\leq C\,a^{3+(b-2)\hbar} \leq C_1\,|Q_a|\,a^{(b-2)\hbar}\,.$$

As $b \leq 2$,

$$\int_{Q_a} w_1^{(3)}(\mathbf{y})\mathrm{d}\mathbf{y} \leq C|Q_a| \leq C_2|Q_a|w_1^{(3)}(\mathbf{x})\,.$$

$\square$

In the case of the weight $\nu_B^A(\mathbf{x})$ we have

Lemma 3.9 *For $b \in (-1; 1]$ and $\hbar \in [0; 1)$ the function $w_2^{(3)} = \left(\nu_{-1}^{b-1}\right)^{\hbar}$ is a weight of class A_1 in $\mathbb{R}^3$.*

Proof: We proceed analogously as in Lemma 3.8. The part `A)` remains the same; in the part `B)` we use that $(1 + \frac{1}{s(\mathbf{y})})^{-\hbar} \leq 1$ for all $\mathbf{y} \in \mathbb{R}^3$. The only difference is in part `C` where we have to assume that $w_2^{(3)}(\mathbf{x}) \geq C > 0$ for $|\mathbf{x}| \leq R$. This is true only for $b \leq 1$.

□

Next we continue with the twodimensional case.

Lemma 3.10

(i) For $b \in (-1; 1]$ and $\hbar \in [0; 1)$ the function $w_1^{(2)} = \left(\eta_{-\frac{1}{2}}^{b-\frac{1}{2}}\right)^{\hbar}$ is a weight of class A_1 in $\mathbb{R}^2$.

(ii) For $b \in (-1; \frac{1}{2}]$ and $\hbar \in [0; 1)$ the function $w_2^{(2)} = \left(\nu_{-\frac{1}{2}}^{b-\frac{1}{2}}\right)^{\hbar}$ is a weight of class A_1 in $\mathbb{R}^2$.

Proof: The demonstration of the first assertion is completely analogous to the proof of Lemma 3.8. Only in part `A`β) the estimates are slightly more technical, similarly as in Lemma 3.2.

The proof of the other part is then the same as the proof of Lemma 3.9.

□

Using Lemma 3.3 one can now show

Theorem 3.6

(i) Let $-1 < B < p-1$, $-3 < A + B < 3(p-1)$. Then the weights η_B^A and σ_B^A are A_p-weights in $\mathbb{R}^3$ for $p \in (1; \infty)$.

(ii) Let $-1 < B < p-1$, $-3 < A + B < 3(p-1)$ and $-3 < A < 3(p-1)$. Then the weight ν_B^A is a A_p-weight in $\mathbb{R}^3$ for $p \in (1; \infty)$.

Proof: We start with the weight η_B^A; the proof for σ_B^A is exactly the same. We have that for $b \in (-1; 2]$, $a \in [0; 3)$ and $\hbar \in [0; 1)$

$$\begin{array}{lcl} \eta_{\hbar(p-1)}^{-a-(p-1)(b-1)\hbar} \in A_p & & B \in [0; p-1),\ A = -a - (b-1)B \\ & \text{i.e. } \eta_B^A \in A_p \text{ for} & \\ \eta_{-\hbar}^{a(p-1)+(b-1)\hbar} \in A_p & & B \in [-1; 0],\ A = a(p-1) - (b-1)B\,. \end{array}$$

We have for $B \in [0; p-1)$

$$A \leq 0 - (b-1)B < 2B$$
$$A > -3 - (b-1)B \geq -3 - B$$

and for $B \in (-1; 0]$

$$A < 3(p-1) - (b-1)B \le 3(p-1) - B$$
$$A \ge -(b-1)B > 2B\,.$$

Now, using Lemma 3.3 (ii) we get easily the statement. For the weight ν_B^A only weaker results are available. Namely

$$\left(\nu_{-1}^{b-1}\right)^{\hbar} \in A_1 \quad \text{for } b \in (-1; 1]\,, \quad \hbar \in [0; 1)\,.$$

Using that $|\mathbf{x}|^{-a} \in A_1$ for $a \in [0; 3)$ and applying Lemma 3.3 (i) we have

$$\begin{array}{lcl} \nu_{\hbar(p-1)}^{-a-(p-1)(b-1)\hbar} \in A_p & & B \in [0; p-1),\ A = -a-(b-1)B \\ & \text{i.e. } \nu_B^A \in A_p \text{ for} & \\ \eta_{-\hbar}^{a(p-1)+(b-1)\hbar} \in A_p & & B \in [-1; 0], A = a(p-1)-(b-1)B\,. \end{array}$$

We have for $B \in [0; p-1)$

$$A \le -(b-1)B < 2B$$
$$A > -3 - (b-1)B \ge -3$$

and for $B \in (-1; 0]$

$$A < 3(p-1) - (b-1)B \le 3(p-1)$$
$$A \ge -(b-1)B > 2B\,.$$

Again, Lemma 3.3 (ii) finishes the proof.

□

Corollary 3.1 *Let $0 \le \omega \le A$, $B \in (-1; p-1)$, $A < \min\{3(p-1), 3(p-1)-B\}$. Then the weight $\mu_B^{A,\omega}$ is a A_p–weight in $\mathbb{R}^3$, $p \in (1; \infty)$.*

Proof: It follows from Theorem 3.6 by means of Lemma 3.3 (ii).

□

In two space dimensions we have

Theorem 3.7

(i) Let $-\frac{1}{2} < B < \frac{1}{2}(p-1)$, $-2 < A + B < 2(p-1)$. Then the weights η_B^A and σ_B^A are A_p-weights in $\mathbb{R}^2$ for $p \in (1; \infty)$.

(ii) Let $-\frac{1}{2} < B < \frac{1}{2}(p-1)$, $-2 < A + B < 2(p-1)$, $-2 < A < 2(p-1)$. Then the weight ν_B^A is a A_p-weight in $\mathbb{R}^2$ for $p \in (1; \infty)$.

Proof: We proceed analogously as in the threedimensional case. Using Lemma 3.5 and Lemma 3.3 (i) we see that ($a \in [0;2)$, $b \in (-1;1]$)

$$\eta_B^A(\mathbf{x}) \in A_p \text{ for } \begin{array}{ll} B \in [0; \frac{1}{2}(p-1)), & A = -a - 2B(b - \frac{1}{2}) \\ B \in (-\frac{1}{2}; 0], & A = a(p-1) - 2B(b - \frac{1}{2})\,. \end{array}$$

We have for $B \in [0; \frac{1}{2}(p-1))$ that $-2 - B < A < 3B$ and for $B \in (-\frac{1}{2}; 0]$ that $3B < A < 2(p-1) - B$. Lemma 3.3 (ii) finishes the proof of the first assertion.

To study the weight ν_B^A we start again from Lemma 3.5, use Lemma 3.3 (i) to get

$$\nu_B^A \in A_p \text{ for } \begin{array}{ll} B \in [0; \frac{1}{2}(p-1)), & -2 < A < 3B \\ B \in [-\frac{1}{2}; 0], & 3B < A < 2(p-1) \end{array}$$

and finish the proof by Lemma 3.3 (ii).

□

Corollary 3.2 *Let* $0 \le \omega \le A$, $B \in (-\frac{1}{2}; \frac{1}{2}(p-1))$, $A < \min\{2(p-1), 2(p-1) - B\}$. *Then the weight* $\mu_B^{A,\omega}$ *is a* A_p*–weight in* $\mathbb{R}^2$, $p \in (1;\infty)$.

Proof: It is a consequence of Theorem 3.7 and Lemma 3.3 (ii).

□

We finish this subsection by summarizing

Theorem 3.8 *Let*

$$Tf(\mathbf{x}) = \frac{\partial}{\partial x_i} \int_{\mathbb{R}^N} \frac{\partial \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_j} f(\mathbf{y})\, d\mathbf{y}\,, \quad i,j = 1,2,\ldots,N\,, N = 2,3\,,$$

$f \in C_0^\infty(\mathbb{R}^N)$. *Let* $\mathcal{R}$ *stand either for* $\mathcal{E}$ *or* $\boldsymbol{\mathcal{S}}$. *Let* $1 < p < \infty$ *and let* g *stand for one of the weights* η_B^A, ν_B^A, σ_B^A *or* $\mu_B^{A,A-\omega}$. *Let* A, B, ω *be such that* g *is a* A_p *weight in* $\mathbb{R}^N$. *Then* T *maps* $C_0^\infty(\mathbb{R}^N)$ *into* $L^p_{(g)}(\mathbb{R}^N)$ *and we have*

$$\|Tf\|_{p,(g),\mathbb{R}^N} \le C\|f\|_{p,(g),\mathbb{R}^N}\,. \tag{3.28}$$

T can be therefore continuously extended onto $\mathcal{L}^p_{(g)}(\mathbb{R}^N)$.

Proof: As follows from $(3.15)_{1,2}$, we have that[5] $D^2\mathcal{R}$ belongs to $M(a,l)$ for any $a \ge 1$ and $l \in \mathbb{N}$. Putting $a = 2$ and $l = N$ in Theorem 3.5, we have that (3.28) holds whenever $g \in A_p$. The proof is complete.

□

[5]the derivative is taken in the sense of distributions

Remark 3.7

(i) As the operator Tf differs only up to the term $cf(\mathbf{x})$ from

$$\widetilde{T}f(\mathbf{x}) \equiv \text{v.p.} \int_{\mathbb{R}^N} \frac{\partial^2 \mathcal{R}(\mathbf{x}-\mathbf{y})}{\partial x_i \partial x_j} f(\mathbf{y}) \mathrm{d}\mathbf{y}\,,$$

we easily see that Theorem 3.8 holds also for the operator $\widetilde{T}$.

(ii) The kernels $\mathcal{E}$ and $\boldsymbol{\mathcal{S}}$ are β–independent and their second gradients are homogeneous functions of degree N; moreover $\eta^A_B(\cdot\,;\beta) \in A_p$ whenever $\eta^A_B \in A_p$ (analogously for other weights). We can replace the weight g by the corresponding β–dependent weight and get the same estimate as (3.28) with C independent of β.

II.3.2 Weakly singular operators. Weighted L^∞–estimates

The following two subsections are devoted to the L^p–weighted theory of the Oseen volume potentials. We first start with $p=\infty$. The aim of this subsection is twofold. We shall not only develop the L^∞–weighted theory for weakly singular Oseen potentials but we also prepare several estimates for the next subsection, where the situation $p \in (1,\infty)$ will be studied.

Let us recall that similar estimates were for the first time studied by Finn (see [Fi1], [Fi2]) in the threedimensional and by Smith (see [Sm]) in the two-dimensional cases. Their approach, generalized by Farwig (see [Fa1]) in the threedimensional situation, has been modified by Dutto (see [Du]) also to the twodimensional case. For the sake of completeness we shall repeat here the calculation; we shall study the general situation $N \geq 2$.

Following [Fa1] we first find a constant K such that

$$(\eta^{-c}_{-d} * \eta^{-a}_{-b})(\mathbf{x}) \leq K \eta^{-e}_{-f}(\mathbf{x})\,, \tag{3.29}$$

where a, b, c, d, e, $f \in \mathbb{R}$. This result is evidently applicable only for the weights of the type η^A_B. In order to extend the results also for other weight, we shall use the following lemmas.

First, let $I_{\alpha,\gamma}(\mathbf{x})$ denote the following integral

$$I^{(N)}_{\alpha,\gamma}(\mathbf{x}) := \int_{B_1(\mathbf{0})} \nu_0^{-\alpha}(\mathbf{y}) \nu_0^{-\gamma}(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{B_1(\mathbf{x})} \nu_0^{-\alpha}(\mathbf{x}-\mathbf{y}') \nu_0^{-\gamma}(\mathbf{y}')\,\mathrm{d}\mathbf{y}'\,,$$

$\mathbf{x}, \mathbf{y}, \mathbf{y}' \in \mathbb{R}^N$. For notational convenience we denote for $\mathbf{x} \neq \mathbf{0}$

$$\ln_- |\mathbf{x}| := \max\{1, -\ln|\mathbf{x}|\}\,.$$

Lemma 3.11 *For $\alpha < N$, $\gamma < N$ there exists a positive constant $C_1 = C_1(\alpha,\gamma,N)$ such that for $\mathbf{x} \in B_2(\mathbf{0}) \setminus \{\mathbf{0}\} \subset \mathbb{R}^N$*

$$I^{(N)}_{\alpha,\gamma}(\mathbf{x}) \leq C_1 \begin{cases} \nu_0^{-(\alpha+\gamma-N)}(\mathbf{x}), & \textit{if} \ \ \alpha+\gamma > N \\ \ln_- |\mathbf{x}|, & \textit{if} \ \ \alpha+\gamma = N \\ 1, & \textit{if} \ \ \alpha+\gamma < N. \end{cases}$$

Moreover, there exists a positive constant $C_2 = C_2(\alpha, \gamma, N)$ *such that for* $\mathbf{x} \in B^2(\mathbf{0}) \subset \mathbb{R}^N$

$$I^{(N)}_{\alpha,\gamma}(\mathbf{x}) \le C_2\, \nu_0^{-\gamma}(\mathbf{x}).$$

Proof: We divide the proof into two parts:
a) Firstly we assume $|\mathbf{x}| \le 2$. We will estimate integrals over sets, which union contains unit ball $B_1(\mathbf{0})$.

$$B_1(\mathbf{0}) \subset B_{\frac{|\mathbf{x}|}{2}}(\mathbf{0}) \cup B_{\frac{|\mathbf{x}|}{2}}(\mathbf{x}) \cup \Big\{ B_{2|\mathbf{x}|}(\mathbf{0}) \setminus \Big(B_{\frac{|\mathbf{x}|}{2}}(\mathbf{0}) \cup B_{\frac{|\mathbf{x}|}{2}}(\mathbf{x}) \Big) \Big\} \cup \cup \Big\{ B_4(\mathbf{0}) \setminus B_{2|\mathbf{x}|}(\mathbf{0}) \Big\}$$

$$\int_{B_{\frac{|\mathbf{x}|}{2}}(\mathbf{0})} \frac{1}{|\mathbf{y}|^\alpha} \frac{1}{|\mathbf{x}-\mathbf{y}|^\gamma}\, \mathrm{d}\mathbf{y} \le \frac{c_1}{|\mathbf{x}|^\gamma} \int_0^{\frac{|\mathbf{x}|}{2}} r^{N-1-\alpha} \mathrm{d}r \le \frac{c_2}{|\mathbf{x}|^{\alpha+\gamma-N}}, \text{ if } \alpha < N$$

$$\int_{B_{\frac{|\mathbf{x}|}{2}}(\mathbf{x})} \frac{1}{|\mathbf{y}|^\alpha} \frac{1}{|\mathbf{x}-\mathbf{y}|^\gamma}\, \mathrm{d}\mathbf{y} \le \frac{c_3}{|\mathbf{x}|^\alpha} \int_0^{\frac{|\mathbf{x}|}{2}} r^{N-1-\gamma} \mathrm{d}r \le \frac{c_4}{|\mathbf{x}|^{\alpha+\gamma-N}}, \text{ if } \gamma < N$$

$$\int_{B_{2|\mathbf{x}|}(\mathbf{0}) \setminus (B_{\frac{|\mathbf{x}|}{2}}(\mathbf{0}) \cup B_{\frac{|\mathbf{x}|}{2}}(\mathbf{x}))} \frac{1}{|\mathbf{y}|^\alpha} \frac{1}{|\mathbf{x}-\mathbf{y}|^\gamma}\, \mathrm{d}\mathbf{y} \le \frac{c_5}{|\mathbf{x}|^{\alpha+\gamma}} \int_0^{2|\mathbf{x}|} r^{N-1} \mathrm{d}r \le \frac{c_6}{|\mathbf{x}|^{\alpha+\gamma-N}}$$

$$\int_{B_4(\mathbf{0}) \setminus B_{2|\mathbf{x}|}(\mathbf{0})} \frac{1}{|\mathbf{y}|^\alpha} \frac{1}{|\mathbf{x}-\mathbf{y}|^\gamma}\, \mathrm{d}\mathbf{y} \le c_7 \int_{2|\mathbf{x}|}^4 \frac{r^{N-1-\alpha}}{|r - |\mathbf{x}||^\gamma} \mathrm{d}r \le$$

$$\le c_8 \int_{2|\mathbf{x}|}^4 \Big(\frac{r}{|r - |\mathbf{x}||} \Big)^\gamma r^{-\alpha-\gamma+N-1} \mathrm{d}r \le c_9 \int_{2|\mathbf{x}|}^4 r^{-\alpha-\gamma+N-1} \mathrm{d}r\,;$$

here we used the inequality $\frac{r}{|r-|\mathbf{x}||} \le 2$.

The last integral can be estimated by $c_{10}|\mathbf{x}|^{-\alpha-\gamma+N}$ if $\alpha + \gamma > N$, by $c_{11} \ln_- |\mathbf{x}|$ if $\alpha + \gamma = N$ and by some constant if $\alpha + \gamma < N$.

b) Now let $|\mathbf{x}| \ge 2$.

$$\int_{B_1(\mathbf{0})} \frac{1}{|\mathbf{y}|^\alpha} \frac{1}{|\mathbf{x}-\mathbf{y}|^\gamma}\, \mathrm{d}\mathbf{y} \le \frac{c_{11}}{|\mathbf{x}|^\gamma} \int_{B_1(\mathbf{0})} \frac{1}{|\mathbf{y}|^\alpha}\, \mathrm{d}\mathbf{y} \le \frac{c_{12}}{|\mathbf{x}|^\gamma}.$$

We get the assertion of Lemma 3.11 from these five estimates of convolution integrals.

□

We define for pairs of real numbers $[a,b] \le [c,d]$: $a \le c$ and $a + b \le c + d$. It is evident that $\eta^a_b(\mathbf{x}) \le C \eta^c_d(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^N$ if $[a,b] \le [c,d]$.

Lemma 3.12 *Let* $a, b, c, d, e, f \in \mathbb{R}$ *and positive constant* C *be such that for all* $\mathbf{x} \in \mathbb{R}^N$*:*

$$\int_{\mathbb{R}^N} \eta^{-c}_{-d}(\mathbf{x}-\mathbf{y})\, \eta^{-a}_{-b}(\mathbf{y})\, d\mathbf{y} \le C\, \eta^{-e}_{-f}(\mathbf{x}), \ N \in \mathbb{N}, \ N \ge 2.$$

Let $g < N$*,* $h < N$*,* $[e,f] \le [a,b]$*,* $[e,f] \le [c,d]$*. Then there exists a positive* *constant C', such that the following inequality is satisfied for* $\mathbf{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$:

$$\int_{\mathbb{R}^N} \mu_{-d}^{-c,-h}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\,d\mathbf{y} \le C' \begin{cases} \mu_{-f}^{-e,-g-h+N}(\mathbf{x}), & \text{if } g+h>N, \\ \mu_{-f}^{-e,-\delta}(\mathbf{x}),\ \delta>0 & \text{if } g+h=N, \\ \mu_{-f}^{-e,0}(\mathbf{x}) \equiv \eta_{-f}^{-e}(\mathbf{x}), & \text{if } g+h<N. \end{cases}$$

Proof: Evidently, for $\alpha, \beta \in \mathbb{R}$ there exist positive constants c_1, c_2, c_3, and c_4 such that

$$\begin{aligned} &a) \quad c_1\,\eta_B^A(\mathbf{x}) \le \nu_B^A(\mathbf{x}) \le c_2\,\eta_B^A(\mathbf{x}) \quad \text{for all } \mathbf{x} \in B^1(\mathbf{0}) \\ &b) \quad c_3\,\eta_B^A(\mathbf{x}) \le \eta_B^A(\mathbf{y}) \le c_4\,\eta_B^A(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}^N,\ \mathbf{y} \in B_1(\mathbf{x})\,. \end{aligned}$$

Let us start to deal with the convolution integral.

$$\begin{aligned} (\mu_{-d}^{-c,-h} * \mu_{-b}^{-a,-g})(\mathbf{x}) &= \int_{\mathbb{R}^N} \mu_{-d}^{-c,-h}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\mathrm{d}\mathbf{y} \le \\ \le C_1 \int_{B_1(\mathbf{x})} \nu_0^{-h}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\,\mathrm{d}\mathbf{y} &+ C_2 \int_{B^1(\mathbf{x})} \eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\mathrm{d}\mathbf{y}. \end{aligned}$$

We will study these two integrals separately. Using Lemma 3.11 we get

$$\int_{B_1(\mathbf{x})} \nu_0^{-h}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{B_1(\mathbf{0})} \mu_{-b}^{-a,-g}(\mathbf{x}-\mathbf{y})\,\nu_0^{-h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \le$$

$$\le \max_{\mathbf{y}\in B_1(\mathbf{x})} \eta_{-b}^{-a+g}(\mathbf{y}) \int_{B_1(\mathbf{0})} \nu_0^{-g}(\mathbf{x}-\mathbf{y})\,\nu_0^{-h}(\mathbf{y})\,\mathrm{d}\mathbf{y} \le$$

$$\le C_3 \eta_{-b}^{-a+g}(\mathbf{x}) \left\{ \begin{array}{ll} \left.\begin{array}{ll} \nu_0^{-g-h+N}(\mathbf{x}), & g+h>N \\ \ln_-|\mathbf{x}|, & g+h=N \\ 1, & g+h<N \end{array}\right\} & \mathbf{x} \in B_1 \setminus \{\mathbf{0}\} \\ \eta_0^{-g}(\mathbf{x}) & \mathbf{x} \in B^1 \end{array} \right\} \le$$

$$\le C_5 \left\{ \begin{array}{l} \mu_{-b}^{-a,-g-h+N}(\mathbf{x}), \\ \mu_{-b}^{-a,-\delta}(\mathbf{x}),\ \delta>0, \\ \mu_{-b}^{-a,0}(\mathbf{x}), \end{array} \right\} \le C_4 \left\{ \begin{array}{ll} \mu_{-f}^{-e,-g-h+N}(\mathbf{x}), & g+h>N \\ \mu_{-f}^{-e,-\delta}(\mathbf{x}),\ \delta>0, & g+h=N \\ \mu_{-f}^{-e,0}(\mathbf{x}) \equiv \eta_{-f}^{-e}(\mathbf{x}), & g+h<N. \end{array} \right.$$

In the second inequality we used Lemma 3.11 and the relation b), in the last inequality we took into account the assumption $[e,f] \le [a,b]$.

The remaining integral $\int_{B^1(\mathbf{x})} \eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\,\mathrm{d}\mathbf{y}$ we estimate for $\mathbf{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ in the following way:

$$\begin{aligned} \int_{B^1(\mathbf{x})} \eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\,\mathrm{d}\mathbf{y} &\le \int_{\mathbb{R}^N} \eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})\,\mu_{-b}^{-a,-g}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \\ = \int_{B_1} \eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})\,\nu_0^{-g}(\mathbf{y})\,\mathrm{d}\mathbf{y} &+ \int_{B^1} \eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})\,\eta_{-b}^{-a}(\mathbf{y})\mathrm{d}\mathbf{y} \le \\ \le C_5 \max_{\mathbf{y}\in B_1(\mathbf{x})} \eta_{-d}^{-c}(\mathbf{y}) \int_{B_1(\mathbf{0})} \nu_0^{-g}(\mathbf{y})\mathrm{d}\mathbf{y} &+ C_6 \eta_{-f}^{-e}(\mathbf{x}) \le C_7 \eta_{-f}^{-e}(\mathbf{x})\,. \end{aligned}$$

From these estimates follows the proof of Lemma 3.12.

□

Let us start to deal with the convolutions of the type

$$\left(\eta_{-d}^{-c} * \eta_{-b}^{-a}\right)(\mathbf{x}) \tag{3.30}$$

in order to get conditions under which (3.29) holds. Let us remark that the conditions do not change if we replace the kernel $\eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})$ by

$$K(\mathbf{z}) \sim \begin{cases} |\mathbf{z}|^{-\gamma} & \mathbf{z} \in B_1(\mathbf{0}) \\ \eta_{-d}^{-c}(\mathbf{z}) & \mathbf{z} \in B^1(\mathbf{0}) \end{cases} \sim \mu_{-d}^{-c,-\gamma}(\mathbf{z})\,, \gamma < N \tag{3.31}$$

(see also I_1 below).

In the sequel we shall use the following notation

$$\begin{array}{lll} \mathbf{x} = (x_1, \mathbf{x}') & \mathbf{y} = (y_1, \mathbf{y}') & \\ R = |\mathbf{x}| & r = |\mathbf{y}| & \tilde{r} = |\mathbf{x}-\mathbf{y}| \\ s = s(\mathbf{x}) & t = y_1 & \tilde{t} = x_1 - y_1 \\ & \varrho = |\mathbf{y}'| & \tilde{\varrho} = |\mathbf{x}'-\mathbf{y}'|\,. \end{array} \tag{3.32}$$

In order to capture the anisotropic structure of the function $\eta_{-b}^{-a}(\cdot)$ we shall study the convolution (3.30) in four different situations:

A) $R \le R_0$

B) $x_1 > 0$, $|\mathbf{x}'| \le \sqrt{x_1}$, $R > R_0$

C) $x_1 > 0$, $|\mathbf{x}'| = \frac{1}{2}R^{\frac{1}{2}+\sigma}$, $R > R_0$, $\sigma \in [0, \frac{1}{2}]$

D) $x_1 > 0$, $|\mathbf{x}'| \ge \frac{R}{2}$, $R > R_0$ or $x_1 < 0$, $R > R_0$.

Using Lemma 3.1 we easily verify that

$$\begin{aligned} \eta_{-b}^{-a}(\mathbf{y}) &\sim \begin{cases} 1\,, & r \le 1 \\ r^{-a}\,, & r > 1\,, \quad t > 0\,, \quad \varrho < \sqrt{t} \\ r^{-a+b}\varrho^{-2b}\,, & r > 1\,, \quad t > 0\,, \quad \varrho \ge \sqrt{t} \\ r^{-a-b}\,, & r > 1\,, \quad t \le 0\,. \end{cases} \\ \eta_{-d}^{-c}(\mathbf{x}-\mathbf{y}) &\sim \begin{cases} 1\,, & \tilde{r} \le 1 \\ \tilde{r}^{-c}\,, & \tilde{r} > 1\,, \quad \tilde{t} > 0\,, \quad \tilde{\varrho} < \sqrt{\tilde{t}} \\ \tilde{r}^{-c+d}\tilde{\varrho}^{-2d}\,, & \tilde{r} > 1\,, \quad \tilde{t} > 0\,, \quad \tilde{\varrho} \ge \sqrt{\tilde{t}} \\ \tilde{r}^{-c-d}\,, & \tilde{r} > 1\,, \quad \tilde{t} \le 0\,. \end{cases} \end{aligned} \tag{3.33}$$

For notational convenience we denote $\nu = \sigma + \frac{1}{2}$ and $b^* = \min\{\frac{N-1}{2}, b\}$; analogously $d^* = \min\{\frac{N-1}{2}, d\}$.

We start with the case A). Applying Lemma 3.2 to the halfspaces $y_1 > 0$ and $y_1 < 0$ we get that the convolution is uniformly bounded if

$$a + b^* + c + d > N \text{ and } a + b + c + d^* > N\,. \tag{3.34}$$

Next we continue with the most complicated case C). We follow Farwig (see [Fa1]) and divide $\mathbb{R}^N$ into 16 subdomains as shown in Fig.2 below. If $N = 2$, the subdomains are plane, otherwise they are cylindrical.

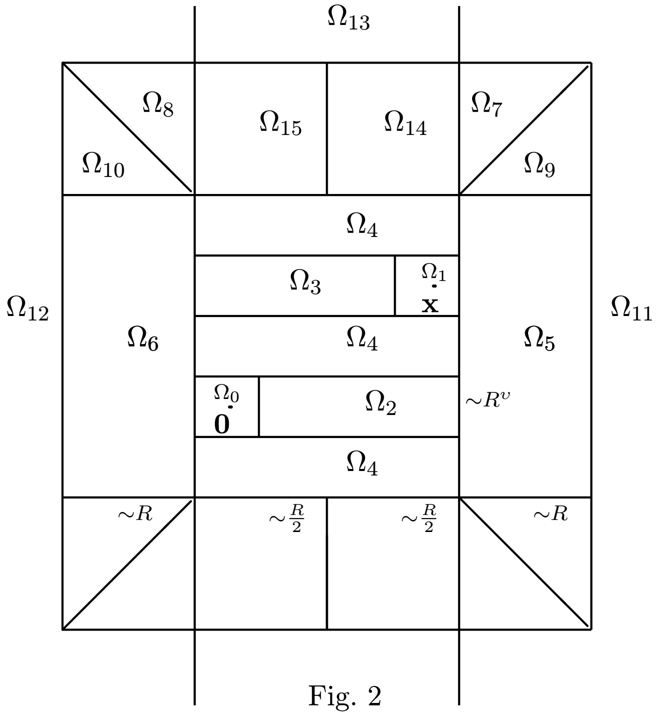

Fig. 2

We calculate the convolutions separately on each subdomain. For the readers convenience, the results are summarized in Tab.1, Tab.2 (for $N = 3$) and Tab.3, Tab.4 (for $N = 2$). We denote by I_k the corresponding part of the integral (3.30) over Ω_k, $k = 0, 1, \cdots, 15$. We shall get

$$I_i(\mathbf{x}) \le KR^{-e_i+2\sigma f_i} \sim \eta^{-e_i}_{-f_i}(\mathbf{x})\,.$$

Unfortunately in many cases additionally logarithmic terms will appear which will cause some losses in the weighted estimates later on.

Remark 3.8 Let A be a positive function. We denote

$$\ln_+ A = \max\{\ln A, 1\}\,.$$

We start to estimate the convolutions over the sixteen subdomains.

$\underline{I_0}$ We have $\Omega_0 = \{\mathbf{y} \in \mathbb{R}^N : |t| \le \frac{1}{8}R^\nu; \varrho \le \frac{1}{8}R^\nu\}$ and therefore $\eta^{-a}_{-b}(\mathbf{y}) \sim r^{-a}(1+s(\mathbf{y}))^{-b}$, $\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y}) \sim R^{-c-2d\sigma}$. Applying Lemma 3.2 we get

$$I_0 \sim R^{-c-2\sigma d} \int_0^{R^\nu} (1+r)^{N-1-a-b^*} \cdot (\ln_+ r\,,\ \text{if}\ \ b = \tfrac{N-1}{2})\mathrm{d}r \sim$$
$$\sim R^{-c-2\sigma d} \begin{cases} 1 & \text{if}\ a+b^* \geq N \\ R^{\nu(N-a-b^*)} & \text{if}\ a+b^* < N \end{cases} .$$
$$\cdot \begin{cases} \ln R & \text{if}\ a+b^* = N\,,\ b \neq \frac{N-1}{2} \\ & \text{or}\ \ a+b^* < N\,,\ b = \frac{N-1}{2} \\ \ln^2 R & \text{if}\ a+b^* = N\,,\ b = \frac{N-1}{2} \end{cases} \sim$$
$$\sim R^{-A} \begin{cases} \ln R & \text{if}\ a+b^* = N\,,\ b \neq \frac{N-1}{2} \\ & \text{or}\ a+b^* < N\,,\ b = \frac{N-1}{2} \\ \ln^2 R & \text{if}\ a+b^* = N\,,\ b = \frac{N-1}{2}\,, \end{cases}$$

where $A = c + \frac{1}{2}\min\{0, a+b^*-N\} - 2\sigma(d + \min\{0, a+b^*-N\})$. The results are summarized in Tab.1—Tab.4.

$\underline{\mathrm{I}_1}$ The integral can be estimated in the same way by exchanging a, b for c, d. Assuming the kernel (3.31) instead of η_{-c}^{-d} we have

$$\tilde{I}_1(\mathbf{x}) = \int_{B_1(\mathbf{0})} K(\mathbf{x}-\mathbf{y})\eta_{-b}^{-a}(\mathbf{x})\mathrm{d}\mathbf{x} \sim R^{-a-2\sigma b}$$

since $\gamma < N$. Again, the summarized results can be found in Tab.1—Tab.4.

$\underline{\mathrm{I}_2}$ We have in Ω_2 that $r \sim t \in (R^\nu; R)$. So $\eta_{-b}^{-a}(\mathbf{y}) \sim r^{-a}$ for $\varrho < \sqrt{t}$ and $\eta_{-b}^{-a}(\mathbf{y}) \sim r^{-a+b}\varrho^{-2b}$ for $\varrho > \sqrt{t}$. Further $\tilde{\varrho} \sim R^\nu$ and $\tilde{r} \sim R + R^\nu - r$; therefore $\eta_{-d}^{-c}(\mathbf{x}-\mathbf{y}) \sim \tilde{r}^{-c+d}R^{-2\nu d}$. Thus

$$I_2 \sim R^{-2d\nu} \int_{R^\nu}^R \mathrm{d}r(R+R^\nu-r)^{d-c}\Big[r^{-a}\int_0^{\sqrt{r}} \varrho^{N-2}\mathrm{d}\varrho +$$
$$+ r^{b-a}\int_{\sqrt{r}}^{R^\nu} \varrho^{N-2-2b}\mathrm{d}\varrho\Big] = R^{-2\nu d}\int_{R^\nu}^R \mathrm{d}r(R+R^\nu-r)^{d-c} \cdot$$
$$\cdot \Big[r^{\frac{N}{2}-\frac{1}{2}-a} + r^{b-a}(R^{\nu(N-1-2b)} - r^{\frac{N}{2}-\frac{1}{2}-b})\Big](\ln\tfrac{R^\nu}{\sqrt{r}}\ \text{if}\ \ b = \tfrac{N-1}{2}) \sim$$
$$\sim R^{-2\nu(b^*-\frac{N-1}{2}+d)}\int_{R^\nu}^R \mathrm{d}r(R+R^\nu-r)^{d-c}r^{b^*-a} \cdot$$
$$\cdot(\ln\tfrac{R^\nu}{\sqrt{r}}\ \text{if}\ b = \tfrac{N-1}{2}) \equiv J\,.$$

Let us verify the last equivalence.

i) $b \geq \frac{N-1}{2}$, i.e. $b^* = \frac{N-1}{2}$

We must control the term

$$\int_{R^\nu}^R (R+R^\nu-r)^{-c+d}r^{b-a}R^{\nu(N-1-2b)}\mathrm{d}r \leq$$
$$\leq R^{(b-\frac{N-1}{2})+\nu(N-1-2b)}\int_{R^\nu}^R (R+R^\nu-r)^{-c+d}r^{b^*-a}\mathrm{d}r\,.$$

But $(b-\frac{N-1}{2})(1-2\nu) < 0$ and the significant term is

$$\int_{R^\nu}^R (R+R^\nu-r)^{-c+d}r^{\frac{N}{2}-\frac{1}{2}-a}\mathrm{d}r = \int_{R^\nu}^R (R+R^\nu-r)^{-c+d}r^{b^*-a}\mathrm{d}r\,.$$

ii) $b < \frac{N-1}{2}$, i.e. $b^* = b$

In this case, the other term must be controlled. But we have

$$\begin{aligned}\int_{R^\nu}^{R} (R+R^\nu-r)^{-c+d} r^{\frac{N-1}{2}-a} \mathrm{d}r = \\ = \int_{R^\nu}^{R} (R+R^\nu-r)^{-c+d} r^{b-a} r^{\frac{N-1}{2}-b} \mathrm{d}r \leq \\ \leq R^{-2\nu(b-\frac{N-1}{2})} R^{2\sigma(b-\frac{N-1}{2})} \int_{R^\nu}^{R} (R+R^\nu-r)^{-c+d} r^{b-a} \mathrm{d}r \leq \\ \leq R^{-2\nu(b-\frac{N-1}{2})} \int_{R^\nu}^{R} (R+R^\nu-r)^{-c+d} r^{b-a} \mathrm{d}r\,,\end{aligned}$$

as $b < \frac{N-1}{2}$. Therefore the significant term is the other one and we have shown the validity of the last equivalence for R sufficiently large.

Let us divide J into two parts J_1 and J_2 — the integral over $(R^\nu; \frac{R}{2})$ and the other one over $(\frac{R}{2}; R)$. We estimate these two parts separately.

$$\begin{aligned}J_1 = R^{-c+d-2\nu(b^*-\frac{N-1}{2}+d)} \int_{R^\nu}^{\frac{R}{2}} \mathrm{d}r\, r^{b^*-a} \cdot (\ln \tfrac{R^\nu}{\sqrt{r}} \text{ if } b = \tfrac{N-1}{2}) \sim \\ \sim R^{-(c+a-\frac{N+1}{2})-2\sigma(b^*-\frac{N-1}{2}+d)+(\sigma-\frac{1}{2})\min(0,b^*-a+1)} \cdot \\ \cdot \begin{cases} \ln_+ \frac{R}{1+s} \text{ if } b \neq \frac{N-1}{2}\,,\ a = b^*+1 \\ (\ln_+ s\,,\ a < \frac{N+1}{2}) \cdot (\ln R\,,\ a > \frac{N+1}{2}) \text{ if } b = \frac{N-1}{2} \\ \ln R \cdot \ln_+ \frac{R}{1+s}\,,\ \text{ if } a = \frac{N+1}{2}\,,\ b = \frac{N-1}{2}\,. \end{cases}\end{aligned}$$

We used the fact that $s(\mathbf{x}) \sim \frac{|\mathbf{x}'|^2}{|\mathbf{x}|} \sim R^{2\sigma}$ and $\ln_+ \frac{R}{R^\nu} = \frac{1}{2} \ln_+ \frac{R}{R^{2\sigma}} \sim \ln_+ \frac{R}{1+s}$. Analogously

$$\begin{aligned}J_2 \sim R^{-2\nu(b^*-\frac{N-1}{2}+d)+b^*-a} \int_{\frac{R}{2}}^{R} \mathrm{d}r (R+R^\nu-r)^{d-c} \cdot \\ \cdot (\ln \tfrac{R^\nu}{\sqrt{r}} \text{ if } b = \tfrac{N-1}{2}) \sim \\ \sim R^{-(c+a-\frac{N+1}{2})-2\sigma(b^*-\frac{N-1}{2}+d)+(\sigma-\frac{1}{2})\min(0,d-c+1)} \cdot \\ \cdot (\ln_+ \tfrac{R}{1+s} \text{ if } c = d+1) \cdot (\ln_+ s \text{ if } b = \tfrac{N-1}{2})\,.\end{aligned}$$

The results can be again found in Tab.1—Tab.4.

<u>I3</u> We proceed analogously as for I_2 exchanging a, b for c, d.

<u>I4</u> Ω_4 can be considered as a subset of Ω_2 and Ω_3. Therefore I_4 can be estimated by I_2 and I_3.

<u>I5</u> We have in Ω_5 $t \sim r \sim R$, so $\eta_{-b}^{-a}(\mathbf{y}) \sim R^{-a}$ $(\varrho < \sqrt{t})$ or $\eta_{-b}^{-a}(\mathbf{y}) \sim R^{-a+b}\varrho^{-2b}$ $(\varrho > \sqrt{t})$, where ϱ varies between 0 and R^ν. Further $\tilde{r} \sim |\tilde{t}| \in$

$(R^\nu;R)$. As $\tilde t<0$ we have $\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y})\sim|\tilde t|^{-c-d}$.

$$I_5\sim\int_{R^\nu}^{R}\mathrm{d}\tau\,\tau^{-c-d}\Big(R^{-a}\int_0^{\sqrt{R}}\varrho^{N-2}\mathrm{d}\varrho+R^{b-a}\int_{\sqrt{R}}^{R^\nu}\varrho^{N-2-2b}\mathrm{d}\varrho\Big)\sim$$
$$\sim R^{1-c-d+(\sigma-\frac12)\min(0,1-c-d)}\Big[R^{\frac{N-1}{2}-a}+$$
$$+R^{b-a}(R^{(N-2-2b)\nu}-R^{\frac{N-2-2b}{2}})(\ln_+ s\,,\ \text{if}\ b=\tfrac{N-1}{2})\Big]\cdot$$
$$\cdot(\ln\tfrac{R}{R^\nu}\ \text{ if }\ c+d=1)\sim$$
$$\sim R^{1-c-d+\frac{N-1}{2}-a+(\sigma-\frac12)\min(0,1-c-d)+2\sigma(\frac{N-1}{2}-b^*)}\cdot$$
$$\cdot(\ln_+ s\ \text{ if}\ b=\tfrac{N-1}{2})\,(\ln_+\tfrac{R}{1+s}\ \text{ if}\ c+d=1)\,.$$

$\underline{I_6}$ It is sufficient to exchange a, b for c, d and use the result for Ω_5.

$\underline{I_7}$ Denoting $\tau=|\tilde t|\in(R^\nu;R)$ we have in Ω_7 that $t\sim r\sim R$, $\varrho\sim\tilde\varrho\sim\tilde r\in(\tau;R)$. Therefore $\eta^{-a}_{-b}(\mathbf{y})\sim R^{-a+b}\rho^{-2b}$, $\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y})\sim\rho^{-c-d}$ and

$$I_7\sim R^{-a+b}\int_{R^\nu}^{R}\mathrm{d}\tau\Big(\int_\tau^R\varrho^{N-2-d-c-2b}\mathrm{d}\varrho\Big)\sim$$
$$\sim R^{b-a}\int_{R^\nu}^{R}\mathrm{d}\tau(R^{N-1-c-d-2b}-\tau^{N-1-c-d-2b})\cdot$$
$$\cdot(\ln\tfrac{R}{\tau}\ \text{ if }\ c+d+2b=N-1)\sim R^{N-a-b-c-d}+$$
$$+R^{N-a-b-c-d+(\sigma-\frac12)\min(0,N-c-d-2b)}(\ln\tfrac{R}{1+s}\ \text{ if }\ c+d+2b=N)\sim$$
$$\sim R^{N-a-b-c-d+(\sigma-\frac12)\min(0,N-c-d-2b)}(\ln\tfrac{R}{1+s}\ \text{ if }\ c+d+2b=N)\,.$$

$\underline{I_8}$ We get the result exchanging a, b for c, d and using the result for Ω_7.

$\underline{I_9}$ Analogously as in Ω_7 we have $t\sim r\sim R$, $\varrho\sim\tilde\varrho\in(R^\nu;\tau)$, $\tilde r\sim\tau=|\tilde t|\in(R^\nu;R)$; so $\eta^{-a}_{-b}(\mathbf{y})\sim R^{-a+b}\rho^{-2b}$, $\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y})\sim\rho^{-c-d}$ and

$$I_9\sim R^{-a+b}\int_{R^\nu}^{R}\mathrm{d}\tau\,\tau^{-d-c}\Big(\int_{R^\nu}^{\tau}\varrho^{N-2-2b}\mathrm{d}\varrho\Big)\,.$$

If $b>\frac{N-1}{2}$, the significant term in the inner integral will be the lower bound and we can use I_5. If $b<\frac{N-1}{2}$, the significant term in the inner integral will be the upper bound and we can use I_7. If $b=\frac{N-1}{2}$, then

$$I_9\sim R^{-a+\frac{N-1}{2}}\int_{R^\nu}^{R}\tau^{-d-c}\ln\tfrac{\tau}{R^\nu}\mathrm{d}\tau\,.$$

In comparison with I_5 we get some additional logarithmic factors

$$b=\tfrac{N-1}{2}\ \ :(\ln_+\tfrac{R}{1+s}\ \text{ if }\ c+d<1)(\ln^2_+\tfrac{R}{1+s}\ \text{ if }\ c+d=1)\,.$$

$\underline{I_{10}}$ As in I_9, we may use I_6 for $d>\frac{N-1}{2}$, I_8 for $d>\frac{N-1}{2}$ and get some additional logarithmic factors to I_8 for $d=\frac{N-1}{2}$.

I$_{11}$ The domain Ω_{11} is unbounded. We have $\tilde{r} \sim r \in (R;\infty)$. Therefore $\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y}) \sim r^{-c-d}$, $\eta^{-a}_{-b}(\mathbf{y}) \sim r^{-a}(1+s(\mathbf{y}))^{-b}$ and applying Lemma 3.2 we get under the assumption $a+b^*+c+d>N$

$$I_{11} \sim \int_R^\infty \mathrm{d}r\, r^{N-1-a-b^*-c-d} \cdot (\ln r \text{ if } b = \tfrac{N-1}{2}) \sim$$
$$\sim R^{N-a-b^*-c-d} \cdot (\ln R \text{ if } b = \tfrac{N-1}{2})\,.$$

I$_{12}$ We proceed as in the previous case and get under the assumption $a+b+c+d^*>N$

$$I_{12} \sim R^{N-a-b-c-d^*} \cdot (\ln R \text{ if } d = \tfrac{N-1}{2})\,.$$

I$_{13}$ The domain Ω_{13} can be considered as a subset of Ω_{11} and Ω_{12}. Therefore I_{13} can be bounded by I_{11} and I_{12}.

I$_{14}$ In this subregion we have $r \sim R$, $\varrho \sim \tilde{\varrho} \in (R^\nu;R)$. Moreover $\tilde{r} \sim |\tilde{t}| + \tilde{\varrho}$, where $\tilde{t} \in (-\frac{1}{8}R^\nu;\frac{R}{2})$. Then $\eta^{-a}_{-b}(\mathbf{y}) \sim R^{-a+b}\varrho^{-2b}$, $\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y}) \sim (\tilde{t}+\tilde{\varrho})^{-c+d}\varrho^{-2d}$ if $\tilde{t}>0$ and $\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y}) \sim \tilde{\varrho}^{-c-d}$ if $\tilde{t}<0$. Let us note that the strip $\tilde{t} \in (-\frac{1}{8}R;0)$ has no influence on the asymptotic behaviour since $\tilde{\varrho} > |\tilde{t}|$ there.

$$I_{14} \sim R^{b-a} \int_{R^\nu}^R \mathrm{d}\varrho\, \varrho^{N-2-2b-2d} \int_0^{R/2} (\tilde{t}+\rho)^{d-c} \mathrm{d}\tilde{t} \sim$$
$$\sim R^{b-a} \int_{R^\nu}^R \mathrm{d}\varrho\, \varrho^{N-2-2b-2d} \cdot \begin{cases} R^{1+d-c} & 1+d-c>0 \\ \varrho^{1+d-c} & 1+d-c<0 \\ \ln\frac{R}{\varrho} & 1+d-c=0. \end{cases}$$

Now we distinguish three cases.

ad a) $1+d-c>0$

If $b+d \le \frac{N-1}{2}$, then

$$I_{14} \sim R^{N-a-b-c-d} \cdot (\ln_+ \tfrac{R}{1+s} \text{ if } b+d = \tfrac{N-1}{2})\,,$$

while for $b+d > \frac{N-1}{2}$ we have

$$I_{14} \sim R^{-a-c+\frac{N+1}{2}+2\sigma(\frac{N-1}{2}-b-d)} \qquad (\text{see } I_2\,,\ I_3)\,.$$

ad b) $1+d-c<0$

$$I_{14} \sim R^{b-a} \int_{R^\nu}^R \varrho^{-2b-c-d+N-1} \mathrm{d}\varrho$$

and the integral can be estimated by I_7.

ad c) $1+d-c=0$

$$I_{14} \sim R^{b-a} \int_{R^\nu}^R \varrho^{N-2-2b-2d} \ln\tfrac{R}{\varrho} \mathrm{d}\varrho =$$
$$= R^{N-a-b-c-d} \int_1^{R^{1-\nu}} z^{2b+2d-N} \ln z\, \mathrm{d}z\,.$$

Now for $b+d \le \frac{N-1}{2}$

$$I_{14} \sim R^{N-a-b-c-d} \cdot (\ln_+^2 \tfrac{R}{1+s} \text{ if } b+d = \tfrac{N-1}{2})$$

and for $b+d > \frac{N-1}{2}$

$$I_{14} \sim R^{\frac{N+1}{2}-a-c-2\sigma(b+d-\frac{N-1}{2})} \ln_+ \tfrac{R}{1+s}$$

which can be estimated by I_2.

$\underline{I_{15}}$ Interchanging a, b and c, d we can use the results from I_{14}.

Thus we completed investigation of the situation C). The results are summarized in Tab.1,2 ($N = 3$) and Tab.3,4 ($N = 2$).

The situation D) is almost trivial since we are left with subdomains of the type Ω_1, Ω_2, Ω_{11}, Ω_{12} and Ω_{13}. The integrals can be estimated by the corresponding integrals in C) taking $\sigma = \frac{1}{2}$ i.e. $\nu = 1$.

Finally in the case B) we proceed as in case C) but the subdomains Ω_2, Ω_3 and Ω_4 coincide. The other integrals can be again estimated by the corresponding ones from the part C) taking $\sigma = 0$ i.e. $\nu = \frac{1}{2}$.

The study of the convolution (3.30) is therefore completed.

We now apply the results from Tab.1—Tab.4 in the study of L^∞–weighted estimates for Oseen potentials. We will use the following notation.

$$\overline{\eta}_F^E(\mathbf{x};\lambda) := \eta_F^E(\mathbf{x};\lambda) \quad \text{if no logarithmic factor appears}$$

$$\overline{\eta}_F^E(\mathbf{x};\lambda) := \eta_F^E(\mathbf{x};\lambda) \cdot \begin{cases} P(\ln_+^{-1} |\lambda\mathbf{x}|) \\ P(\ln_+^{-1} s(\lambda\mathbf{x})) \end{cases} \quad \text{if there are some logarithmic factors,}$$

where function $P(\cdot)$ is a polynomial of the first or the second order, see also Remark 3.9. Similarly we define $\overline{\nu}_F^E(\cdot\,;\lambda)$. Then we have

Tab.1 $\quad N=3$

Dom.	t	ϱ	r	$\tilde t$	$\tilde\varrho$	$\tilde r$	$\eta^{-a}_{-b}(\mathbf{y})$	$\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y})$	e	f	log. factors
Ω_0	$(-\frac18 R^\upsilon;\frac18 R^\upsilon)$	$(0;\frac18 R^\upsilon)$	$(0;\frac18 R^\upsilon)$	$\sim x_1$ R	$\sim \lvert\mathbf{x}'\rvert$ R^υ	$\sim \lvert\mathbf{x}\rvert$ R	$r^{-a}(1+s(\mathbf{y}))^{-b}$	$R^{-c-2\sigma d}$	$c+\frac12\min(0,a+b^*-3)$	$d+\frac12\min(0,a+b^*-3)$	$\ln R(b=1\wedge a<2)\vee$ $\vee(b\neq 1\wedge a+b^*=3)$ $\ln^2 R(b=1\wedge a=2)$
Ω_1	$\sim x_1$ R	$\sim \lvert\mathbf{x}'\rvert$ R^υ	$\sim \lvert\mathbf{x}\rvert$ R	$(-\frac18 R^\upsilon;\frac18 R^\upsilon)$	$(0;\frac18 R^\upsilon)$	$(0;\frac18 R^\upsilon)$	$R^{-a-2\sigma b}$	$\tilde r^{-c}(1+s(\mathbf{x}-\mathbf{y}))^{-d}$	$a+\frac12\min(0,c+d^*-3)$	$b+\frac12\min(0,c+d^*-3)$	$\ln R(d=1\wedge c<2)\vee$ $\vee(d\neq 1\wedge c+d^*=3)$ $\ln^2 R\ (c=2\wedge d=1)$
Ω_2	$\sim r$ $(R^\upsilon;R)$	$(0;\frac18 R^\upsilon)$	$\sim t$ $(R^\upsilon;R)$	$(-\frac18 R^\upsilon;$ $R-R^\upsilon)$	$\sim \lvert\mathbf{x}'\rvert$ R^υ	$R+R^\upsilon-r$	$r^{-a}\quad \varrho<\sqrt{t}$ $r^{-a+b}\varrho^{-2b}\quad \varrho>\sqrt{t}$	$\tilde r^{-c+d}R^{-2d\upsilon}$	$a+c-2+\frac12\min$ $(0,1+b^*-a,1+d-c)$	$b^*+d-1-\frac12\min$ $(0,1+b^*-a,1+d-c)$	$\ln\frac{R}{1+s}(\min(1+b^*-a,$ $1+d-c)=0\wedge b\neq 1)$ $\ln_+ s\cdot\ln\frac{R}{1+s}$ $(b=1\wedge 1+d=0)$ $(\ln_+ s\quad a<2)(\ln R\quad a>2)$ $(\ln R\ \ln\frac{R}{1+s}\ a=2)\wedge b=1$
Ω_3	$(-\frac18 R^\upsilon;$ $R-R^\upsilon)$	$\sim \lvert\mathbf{x}'\rvert$ R^υ	$R+R^\upsilon-\tilde r$	$\sim \tilde r$ $(R^\upsilon;R)$	$(0;\frac18 R^\upsilon)$	$\sim \tilde t$ $(R^\upsilon;R)$	$r^{-a+b}R^{-2b\upsilon}$	$\tilde r^{-c}\quad \tilde\varrho<\sqrt{\tau}$ $\tilde r^{-c+d}\tilde\varrho^{-2d}\quad \tilde\varrho>\sqrt{\tau}$	$a+c-2+\frac12\min$ $(0,1+b-a,1+d^*-c)$	$b+d^*-1-\frac12\min$ $(0,1+b-a,1+d^*-c)$	$\ln\frac{R}{1+s}(\min(1+b-a,$ $1+d^*-c)=0\wedge d\neq 1)$ $\ln_+ s\cdot\ln\frac{R}{1+s}(d=1\wedge$ $\wedge 1+b-a=0)$ $(\ln_+ s\ c<2)(\ln R\ c>2)$ $(\ln R\frac{R}{1+s}\ c=2)\wedge d=1$
Ω_4									see Ω_2,Ω_3	see Ω_2,Ω_3	see Ω_2,Ω_3
Ω_5	$\sim r$ R	$(0;R^\upsilon)$	$\sim t$ R	$\sim -\tilde r$ $(-R;-R^\upsilon)$	$(0;R^\upsilon)$	$\sim \lvert\tilde t\rvert$ $(R^\upsilon;R)$	$R^{-a}\quad \varrho<\sqrt{t}$ $R^{-a+b}\varrho^{-2b}\quad \varrho>\sqrt{t}$	$\lvert\tilde t\rvert^{-c-d}$	$a+c+d-2+\frac12\min$ $(0,1-c-d)$	$b^*-1-\frac12\min$ $(0,1-c-d)$	$(\ln_+ s\quad b=1)\cdot$ $\cdot(\ln\frac{R}{1+s}\ c+d=1)$
Ω_6	$\sim -r$ $(-R;-R^\upsilon)$	$(0;R^\upsilon)$	$\sim \lvert t\rvert$ $(R^\upsilon;R)$	$\sim \tilde r$ R	$(0;R^\upsilon)$	$\sim \tilde t$ R	$\lvert t\rvert^{-a-b}$	$R^{-c}\quad \tilde\varrho<\sqrt{\tilde t}$ $R^{-c+d}\tilde\varrho^{-2d}\quad \tilde\varrho>\sqrt{\tilde t}$	$a+b+c-2+\frac12\min$ $(0,1-a-b)$	$d^*-1-\frac12\min$ $(0,1-a-b)$	$(\ln_+ s\quad d=1)\cdot$ $\cdot(\ln\frac{R}{1+s}\quad a+b=1)$
Ω_7	R	$\sim\tilde\varrho$ $(\lvert\tilde t\rvert;R)$	R	$(-R;-R^\upsilon)$	$\sim \varrho,\tilde r$ $(\lvert\tilde t\rvert;R)$	$\sim\tilde\varrho$ $(\lvert\tilde t\rvert;R)$	$R^{-a+b}\varrho^{-2b}$	ϱ^{-c-d}	$a+b+c+d-3+$ $\frac12\min(0,3-2b-c-d)$	$-\frac12\min(0,3-2b-$ $-c-d)$	$\ln\frac{R}{1+s}\quad 2b+c+d=3$
Ω_8	$(-R;-R^\upsilon)$	$\sim\tilde\varrho,r$ $(\lvert t\rvert;R)$	$\sim\varrho$ $(\lvert t\rvert;R)$	R	$\sim\varrho$ $(\lvert t\rvert;R)$	R	$\tilde\varrho^{-a-b}$	$R^{-c+d}\tilde\varrho^{-2d}$	$a+b+c+d-3+$ $\frac12\min(0,3-a-b-2d)$	$-\frac12\min(0,3-a-b-2d)$	$\ln\frac{R}{1+s}\quad a+b+2d=3$

Tab.2 $N = 3$

Dom.	t	ϱ	r	$\tilde t$	$\tilde\varrho$	$\tilde r$	$\eta^{-a}_{-b}(\mathbf{y})$	$\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y})$	e	f	log. factors
Ω_9	$\sim r$ R	$\sim \tilde\varrho$ $(R^v; \lvert\tilde t\rvert)$	$\sim \tilde t$ R	$\sim -\tilde r$ $(-R; -R^v)$	$\sim \varrho$ $(R^v; \lvert\tilde t\rvert)$	$\sim \lvert\tilde t\rvert$ $(R^v; R)$	$R^{-a+b} \varrho^{-2b}$	$\lvert\tilde t\rvert^{-c-d}$	$b > 1$ see Ω_5 $b < 1$ see Ω_7 $b = 1$ see Ω_5	$b > 1$ see Ω_5 $b < 1$ see Ω_7 $b = 1$ see Ω_5	$\Big(\ln \frac{R}{1+s}\ c+d<1)$ $(\ln^2 \frac{R}{1+s}\ c+d=1)\Big) \wedge b=1$
Ω_{10}	$\sim -r$ $(-R; -R^v)$	$\sim \tilde\varrho$ $(R^v; \lvert t\rvert)$	$\sim \lvert t\rvert$ $(R^v; R)$	$\sim \tilde r$ R	$\sim \varrho$ $(R^v; \lvert t\rvert)$	$\sim \tilde t$ R	$\lvert t\rvert^{-a-b}$	$R^{-c+d}\ \tilde\varrho^{-2d}$	$d > 1$ see Ω_6 $d < 1$ see Ω_8 $d = 1$ see Ω_6	$d > 1$ see Ω_6 $d < 1$ see Ω_8 $d = 1$ see Ω_6	$\Big(\ln \frac{R}{1+s}\ a+b<1)$ $(\ln^2 \frac{R}{1+s}\ a+b=1)\Big) \wedge d=1$
Ω_{11}	$(R; \infty)$	$(0; \infty)$	$\sim \tilde r$ $(R; \infty)$	$(-\infty; -R^v)$	$(0; \infty)$	$\sim r$ $(R; \infty)$	$(1+r)^{-a}(1+s(\mathbf{y}))^{-b}$	r^{-c-d}	$a+b^*+c+d-3>0$	0	$\ln R \quad b=1$
Ω_{12}	$(-\infty; -R^v)$	$(0; \infty)$	$\sim \tilde r$ $(R; \infty)$	$(R; \infty)$	$(0; \infty)$	$\sim r$ $(R; \infty)$	$\tilde r^{-a-b}$	$(1+\tilde r)^{-c}(1+s(\mathbf{x}-\mathbf{y}))^{-d}$	$a+b+c+d^*-3>0$	0	$\ln R \quad d=1$
Ω_{13}		$\sim r, \tilde r$ $(R; \infty)$	$\sim \tilde r$ $(R; \infty)$		$\sim \varrho$ $(R; \infty)$	$\sim r$ $(R; \infty)$			see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}
Ω_{14}	$\sim r$ R	$\sim \tilde\varrho$ $(R^v; R)$	$\sim t$ R	$(-\frac{1}{8}R^v; \frac{R}{2})$	$\sim \varrho$ $(R^v; R)$	$\sim \lvert\tilde t\rvert + \tilde\varrho$	$R^{-a+b}\ \varrho^{-2b}$	$(\tilde t+\tilde\varrho)^{-c+d}\ \tilde\varrho^{-2d}\ \tilde t>0$ $\tilde\varrho^{-c-d}\ \tilde t<0$	see Ω_2, Ω_3 Ω_7 Ω_2 $a+b+c+d-3$	$b+d>1 \wedge 1+d-c>0$ $1+d-c<0$ $1+d-c=0 \wedge b+d>1$ 0 otherwise	$\ln \frac{R}{1+s}\ b+d=1 \wedge 1+d-c>0$ $\ln^2 \frac{R}{1+s}\ b+d=1 \wedge 1+d-c=0$
Ω_{15}	$(-R^v; \frac{R}{2})$	$\sim \tilde\varrho$ $(R^v; R)$	$\sim \lvert t\rvert + \varrho$	$\sim r$ R	$\sim \varrho$ $(R^v; R)$	$\sim t$ R	$(t+\varrho)^{-a+b}\ \varrho^{-2b}\ t>0$ $\varrho^{-a-b}\ t<0$	$R^{-c+d}\ \tilde\varrho^{-2d}$	see Ω_2, Ω_3 Ω_8 Ω_3 $a+b+c+d-3$	$b+d>1 \wedge 1+b-a>0$ $1+b-a<0$ $1+b-a=0 \wedge b+d>1$ 0 otherwise	$\ln \frac{R}{1+s}\ b+d=1 \wedge 1+b-a>0$ $\ln^2 \frac{R}{1+s}\ b+d=1 \wedge 1+b-a=0$

Tab.3 $N = 2$

Dom.	t	ϱ	r	$\tilde{t}$	$\tilde{\varrho}$	$\tilde{r}$	$\eta^{-a}_{-b}(\mathbf{y})$	$\eta^{-c}_{-d}(\mathbf{x}-\mathbf{y})$	e	f	log. factors
Ω_0	$(-\frac{1}{8}R^{\upsilon}; \frac{1}{8}R^{\upsilon})$	$(0; \frac{1}{8}R^{\upsilon})$	$(0; \frac{1}{8}R^{\upsilon})$	$\sim x_1$ R	$\sim \lvert x_2\rvert$ R^{υ}	$\sim \lvert\mathbf{x}\rvert$ R	$r^{-a}(1+s(\mathbf{y}))^{-b}$	$R^{-c-2\sigma d}$	$c + \frac{1}{2}\min(0, a+b^*-2)$	$d + \frac{1}{2}\min(0, a+b^*-2)$	$\ln R(b = \frac{1}{2} \wedge a < \frac{3}{2})\vee$ $\vee(b \neq \frac{1}{2} \wedge a + b^* = 2)$ $\ln^2 R(b = \frac{1}{2} \wedge a = \frac{3}{2})$
Ω_1	$\sim x_1$ R	$\sim \lvert x_2\rvert$ R^{υ}	$\sim \lvert\mathbf{x}\rvert$ R	$(-\frac{1}{8}R^{\upsilon}; \frac{1}{8}R^{\upsilon})$	$(0; \frac{1}{8}R^{\upsilon})$	$(0; \frac{1}{8}R^{\upsilon})$	$R^{-a-2\sigma b}$	$\tilde{r}^{-c}(1+s(\mathbf{x}-\mathbf{y}))^{-d}$	$a + \frac{1}{2}\min(0, c+d^*-2)$	$b + \frac{1}{2}\min(0, c+d^*-2)$	$\ln R(d = \frac{1}{2} \wedge c < \frac{3}{2})\vee$ $\vee(d \neq \frac{1}{2} \wedge c + d^* = 2)$ $\ln^2 R\ (c = \frac{3}{2} \wedge d = \frac{1}{2})$
Ω_2	$\sim r$ $(R^{\upsilon}; R)$	$(0; \frac{1}{8}R^{\upsilon})$	$\sim t$ $(R^{\upsilon}; R)$	$(-\frac{1}{8}R^{\upsilon};$ $R - R^{\upsilon})$	$\sim \lvert\mathbf{x}'\rvert$ R^{υ}	$R + R^{\upsilon} - r$	$r^{-a} \quad \varrho < \sqrt{t}$ $r^{-a+b}\varrho^{-2b} \quad \varrho > \sqrt{t}$	$\tilde{r}^{-c+d}R^{-2d\upsilon}$	$a + c - \frac{3}{2} + \frac{1}{2}\min$ $(0, 1+b^*-a, 1+d-c)$	$b^* + d - \frac{1}{2} - \frac{1}{2}\min$ $(0, 1+b^*-a, 1+d-c)$	$\ln\frac{R}{1+s}(\min(1+b^*-a,$ $1+d-c) = 0 \wedge b \neq \frac{1}{2})$ $\ln_+ s \cdot \ln\frac{R}{1+s}$ $(b = \frac{1}{2} \wedge 1+d = c)$ $(\ln_+ s \quad a < \frac{3}{2})(\ln R\ a > \frac{3}{2})$ $(\ln R\ \ln\frac{R}{1+s}\ a = \frac{3}{2}) \wedge b = \frac{1}{2}$
Ω_3	$(-\frac{1}{8}R^{\upsilon};$ $R - R^{\upsilon})$	$\sim \lvert x_2\rvert$ R^{υ}	$R + R^{\upsilon} - \tilde{r}$	$\sim \tilde{r}$ $(R^{\upsilon}; R)$	$(0; \frac{1}{8}R^{\upsilon})$	$\sim \tilde{t}$ $(R^{\upsilon}; R)$	$r^{-a+b}R^{-2b\upsilon}$	$\tilde{r}^{-c} \quad \tilde{\varrho} < \sqrt{\tau}$ $\tilde{r}^{-c+d}\tilde{\varrho}^{-2d} \quad \tilde{\varrho} > \sqrt{\tau}$	$a + c - \frac{3}{2} + \frac{1}{2}\min$ $(0, 1+b-a, 1+d^*-c)$	$b + d^* - \frac{1}{2} - \frac{1}{2}\min$ $(0, 1+b-a, 1+d^*-c)$	$\ln\frac{R}{1+s}(\min(1+b-a,$ $1+d^*-c) = 0 \wedge d \neq 1)$ $\ln_+ s \cdot \ln\frac{R}{1+s}(d = \frac{1}{2}\wedge$ $\wedge 1+b-a = 0)$ $(\ln_+ s\ c < \frac{3}{2})(\ln R\ c > \frac{3}{2})$ $(\ln R\frac{R}{1+s}\ c = \frac{3}{2}) \wedge d = \frac{1}{2}$
Ω_4									see Ω_2, Ω_3	see Ω_2, Ω_3	see Ω_2, Ω_3
Ω_5	$\sim r$ R	$(0; R^{\upsilon})$	$\sim t$ R	$\sim -\tilde{r}$ $(-R; -R^{\upsilon})$	$(0; R^{\upsilon})$	$\sim \lvert\tilde{t}\rvert$ $(R^{\upsilon}; R)$	$R^{-a} \quad \varrho < \sqrt{t}$ $R^{-a+b}\varrho^{-2b} \quad \varrho > \sqrt{t}$	$\lvert\tilde{t}\rvert^{-c-d}$	$a + c + d - \frac{3}{2} + \frac{1}{2}\min$ $(0, 1-c-d)$	$b^* - \frac{1}{2} - \frac{1}{2}\min$ $(0, 1-c-d)$	$(\ln_+ s \quad b = \frac{1}{2})\cdot$ $\cdot(\ln\frac{R}{1+s}\ c+d = 1)$
Ω_6	$\sim -r$ $(-R; -R^{\upsilon})$	$(0; R^{\upsilon})$	$\sim \lvert t\rvert$ $(R^{\upsilon}; R)$	$\sim \tilde{r}$ R	$(0; R^{\upsilon})$	$\sim \tilde{t}$ R	$\lvert t\rvert^{-a-b}$	$R^{-c} \quad \tilde{\varrho} < \sqrt{\tilde{t}}$ $R^{-c+d}\tilde{\varrho}^{-2d} \quad \tilde{\varrho} > \sqrt{\tilde{t}}$	$a + b + c - \frac{3}{2} + \frac{1}{2}\min$ $(0, 1-a-b)$	$d^* - \frac{1}{2} - \frac{1}{2}\min$ $(0, 1-a-b)$	$(\ln_+ s \quad d = \frac{1}{2})\cdot$ $\cdot(\ln\frac{R}{1+s} \quad a+b = 1)$
Ω_7	R	$\sim \tilde{\varrho}$ $(\lvert\tilde{t}\rvert; R)$	R	$(-R; -R^{\upsilon})$	$\sim \varrho, \tilde{r}$ $(\lvert\tilde{t}\rvert; R)$	$\sim \tilde{\varrho}$ $(\lvert\tilde{t}\rvert; R)$	$R^{-a+b}\varrho^{-2b}$	ϱ^{-c-d}	$a + b + c + d - 2 +$ $\frac{1}{2}\min(0, 2-2b-c-d)$	$-\frac{1}{2}\min(0, 2-2b-$ $-c-d)$	$\ln\frac{R}{1+s} \quad 2b+c+d = 2$
Ω_8	$(-R; -R^{\upsilon})$	$\sim \tilde{\varrho}, r$ $(\lvert t\rvert; R)$	$\sim \varrho$ $(\lvert t\rvert; R)$	R	$\sim \varrho$ $(\lvert t\rvert; R)$	R	$\tilde{\varrho}^{-a-b}$	$R^{-c+d}\tilde{\varrho}^{-2d}$	$a + b + c + d - 2 +$ $\frac{1}{2}\min(0, 2-a-b-2d)$	$-\frac{1}{2}\min(0, 2-a-b-2d)$	$\ln\frac{R}{1+s} \quad a+b+2d = 2$

Tab.4 $\quad N=2$

Dom.	t	ϱ	r	$\tilde{t}$	$\tilde{\varrho}$	$\tilde{r}$	$\eta_{-b}^{-a}(\mathbf{y})$	$\eta_{-d}^{-c}(\mathbf{x}-\mathbf{y})$	e	f	log. factors
Ω_9	$\sim r$ R	$\sim \tilde{\varrho}$ $(R^{\upsilon}; \lvert\tilde{t}\rvert)$	$\sim t$ R	$\sim -\tilde{r}$ $(-R; -R^{\upsilon})$	$\sim \varrho$ $(R^{\upsilon}; \lvert\tilde{t}\rvert)$	$\sim \lvert\tilde{t}\rvert$ $(R^{\upsilon}; R)$	$R^{-a+b}\varrho^{-2b}$	$\lvert\tilde{t}\rvert^{-c-d}$	$b > \frac{1}{2}$ see Ω_5 $b < \frac{1}{2}$ see Ω_7 $b = \frac{1}{2}$ see Ω_5	$b > \frac{1}{2}$ see Ω_5 $b < \frac{1}{2}$ see Ω_7 $b = \frac{1}{2}$ see Ω_5	$\Big(\ln \frac{R}{1+s}\ c+d<1)$ $(\ln^2 \frac{R}{1+s}\ c+d=1)\Big) \wedge b=\frac{1}{2}$
Ω_{10}	$\sim -r$ $(-R; -R^{\upsilon})$	$\sim \tilde{\varrho}$ $(R^{\upsilon}; \lvert t\rvert)$	$\sim \lvert t\rvert$ $(R^{\upsilon}; R)$	$\sim \tilde{r}$ R	$\sim \varrho$ $(R^{\upsilon}; \lvert t\rvert)$	$\sim \tilde{t}$ R	$\lvert t\rvert^{-a-b}$	$R^{-c+d}\ \tilde{\varrho}^{-2d}$	$d > \frac{1}{2}$ see Ω_6 $d < \frac{1}{2}$ see Ω_8 $d = \frac{1}{2}$ see Ω_6	$d > \frac{1}{2}$ see Ω_6 $d < \frac{1}{2}$ see Ω_8 $d = \frac{1}{2}$ see Ω_6	$\Big(\ln \frac{R}{1+s}\ a+b<1)$ $(\ln^2 \frac{R}{1+s}\ a+b=1)\Big) \wedge d=\frac{1}{2}$
Ω_{11}	$(R; \infty)$	$(0; \infty)$	$\sim \tilde{r}$ $(R; \infty)$	$(-\infty; -R^{\upsilon})$	$(0; \infty)$	$\sim r$ $(R; \infty)$	$(1+r)^{-a}(1+s(\mathbf{y}))^{-b}$	r^{-c-d}	$a+b^*+c+d-2>0$	0	$\ln R \quad b=\frac{1}{2}$
Ω_{12}	$(-\infty; -R^{\upsilon})$	$(0; \infty)$	$\sim \tilde{r}$ $(R; \infty)$	$(R; \infty)$	$(0; \infty)$	$\sim r$ $(R; \infty)$	$\tilde{r}^{-a-b}$	$(1+\tilde{r})^{-c}(1+s(\mathbf{x}-\mathbf{y}))^{-d}$	$a+b+c+d^*-2>0$	0	$\ln R \quad d=\frac{1}{2}$
Ω_{13}		$\sim r, \tilde{r}$ $(R; \infty)$	$\sim \tilde{r}$ $(R; \infty)$		$\sim \varrho$ $(R; \infty)$	$\sim r$ $(R; \infty)$			see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}	see Ω_{11} and Ω_{12}
Ω_{14}	$\sim r$ R	$\sim \tilde{\varrho}$ $(R^{\upsilon}; R)$	$\sim t$ R	$(-\frac{1}{8}R^{\upsilon}; \frac{R}{2})$	$\sim \varrho$ $(R^{\upsilon}; R)$	$\sim \lvert\tilde{t}\rvert + \tilde{\varrho}$	$R^{-a+b}\ \varrho^{-2b}$	$(\tilde{t}+\tilde{\varrho})^{-c+d}\ \tilde{\varrho}^{-2d}\ \tilde{t}>0$ $\tilde{\varrho}^{-c-d}\ \tilde{t}<0$	see Ω_2, Ω_3 Ω_7 Ω_2 $a+b+c+d-2$	$b+d>\frac{1}{2} \wedge 1+d-c>0$ $1+d-c<0$ $1+d-c=0 \wedge b+d>\frac{1}{2}$ 0 otherwise	$\ln \frac{R}{1+s}\ b+d=\frac{1}{2} \wedge 1+d-c>0$ $\ln^2 \frac{R}{1+s}\ b+d=\frac{1}{2} \wedge 1+d-c=0$
Ω_{15}	$(-R^{\upsilon}; \frac{R}{2})$	$\sim \tilde{\varrho}$ $(R^{\upsilon}; R)$	$\sim \lvert t\rvert + \varrho$	$\sim r$ R	$\sim \varrho$ $(R^{\upsilon}; R)$	$\sim t$ R	$(t+\varrho)^{-a+b}\ \varrho^{-2b}\ t>0$ $\varrho^{-a-b}\ t<0$	$R^{-c+d}\ \tilde{\varrho}^{-2d}$	see Ω_2, Ω_3 Ω_8 Ω_3 $a+b+c+d-2$	$b+d>\frac{1}{2} \wedge 1+b-a>0$ $1+b-a<0$ $1+b-a=0 \wedge b+d>\frac{1}{2}$ 0 otherwise	$\ln \frac{R}{1+s}\ b+d=\frac{1}{2} \wedge 1+b-a>0$ $\ln^2 \frac{R}{1+s}\ b+d=\frac{1}{2} \wedge 1+b-a=0$

Theorem 3.9 *Let $A + B^* > 1$, $i, j = 1, 2, 3$. Let $f \in L^\infty(\mathbb{R}^3; \eta^A_B(\cdot\,;\lambda))$. Then $\mathcal{O}_{ij} * f \in L^\infty(\mathbb{R}^3; \overline{\eta}^E_F(\cdot\,;\lambda))$, where*

$$E = \begin{cases} A-1 & \text{for } A \le B^* + 1 \\ \frac{A+B-1}{2} & \text{for } A \ge B+1,\ A+B \le 3 \\ 1 & \text{for } A + B^* \ge 3 \end{cases} \tag{i}$$

$$E + F = \begin{cases} A + B^* - 1 & \text{for } A + B^* \le 3 \\ 2 & \text{for } A + B^* \ge 3 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \quad \text{for} \quad \begin{cases} A + B^* = 3 \\ A = B + 1,\ 0 \le B \le 1 \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(\mathbf{x})) \quad \text{for} \quad A + B < 3,\ B \le 1, \tag{iv}$$

(see Remark 3.9). Moreover we have

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-2} \|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.35}$$

Let moreover

$$1 \le A < 3, \quad B > 0, \quad \text{or} \quad A \le B + 5,\ 1 < A + B \le 3,\ B \le 0. \tag{v}$$

Then for $f \in L^\infty(\mathbb{R}^3; \nu^A_B(\cdot\,;\lambda))$ we have $\mathcal{O}_{ij}(\ ;\lambda) * f \in L^\infty(\mathbb{R}^3; \overline{\nu}^E_F(\cdot\,;\lambda))$ and

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-2+A-E} \|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.36}$$

Remark 3.9 The inequalities (3.35), (3.36) must be understood in the following sense. If no logarithmic terms appear, then

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\eta^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-2} \|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,,$$

analogously for the weight $\nu^E_F(\ ;\lambda)$. But for $A+B^* = 3$ or $A = B+1$, $0 \le B \le 1$ we have

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\eta^E_F(\cdot\,;\lambda)\,P(\ln_+^{-1}(\lambda|\cdot|))),\mathbb{R}^3} \le C\lambda^{-2} \|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3} \tag{3.37}$$

and for $A + B < 3$, $B \le 1$

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\eta^E_F(\cdot\,;\lambda)\,P(\ln_+^{-1}(s(\lambda\cdot)))),\mathbb{R}^3} \le C\lambda^{-2} \|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,, \tag{3.38}$$

where $P(\cdot)$ is a polynomial. Analogously for the weights $\nu^E_F(\cdot\,;\lambda)$. We can use instead of (3.37),(3.38) for $\varepsilon > 0$

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\eta^{E-\varepsilon}_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-2} \|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,, \tag{3.37'}$$

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\eta^E_{F-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-2} \|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,, \tag{3.38'}$$

respectively.

Finally, in the case of $f = 0$ in $B_{1/2}(\mathbf{0})$ (this is e.g. the case for $\Omega \subset \mathbb{R}^N$, an exterior domain) we can get for the weight $\nu^A_B(\cdot\,;\lambda)$

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{\infty,(\nu^{E-\varepsilon}_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-2+A-E+\varepsilon} \|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,.$$

Proof of Theorem 3.9: Let $f \in L^\infty(\mathbb{R}^3; \eta^A_B(\cdot\,;1))$. Recalling that $\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};1) \sim \nu^{-1}_{-1}(\mathbf{x}-\mathbf{y};1)$ we have

$$\begin{aligned} |\mathcal{O}_{ij}(\cdot\,;1) * f(\mathbf{x})| &\le C\nu^{-1}_{-1}(\cdot\,;1) * \eta^{-A}_{-B}(\cdot\,;1)(\mathbf{x}) \le \\ &\le C\eta^{-1}_{-1}(\cdot\,;1) * \eta^{-A}_{-B}(\cdot\,;1)(\mathbf{x})\,. \end{aligned} \tag{3.39}$$

We have therefore to study the convolution (3.39); we apply Tab.1 and Tab.2 with $c = d = 1$, $a = A$, $b = B$ and we get, under the condition $A + B^* > 1$ that (we skip the logarithmic factors, for a moment)

$$\eta^{-1}_{-1}(\cdot\,;1) * \eta^{-A}_{-B}(\cdot\,;1)(\mathbf{x}) \le C\eta^{-E}_{-F}(\mathbf{x};1)$$

with

$$\begin{aligned} E &\le \min\Big\{1,\ \frac{A+B^*-1}{2},\ A-\frac{1}{2},\ A-1,\ A+B-1,\ \frac{A+B-1}{2},\ A+B^*-1\Big\} = \\ &= \min\Big\{1,\ \frac{A+B^*-1}{2},\ A-1\Big\} \\ E+F &\le \min\{2,\ A+B-1,\ A+B^*-1\} = \min\{2, A+B^*-1\}\,. \end{aligned} \tag{3.40}$$

We therefore easily get (i) and (ii), see Fig. 3 below:

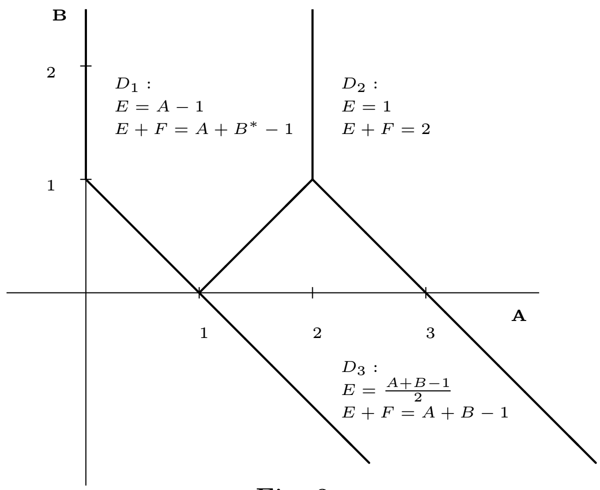

Fig. 3

Let us now regard the logarithmic factors. From Ω_0 we have $\ln_+(\lambda|\mathbf{x}|)$ whenever $B = 1, A \le 2$ or $A + B^* = 3$ and $e_0 = 1 + \frac{1}{2}\min(0,\ A + B^* - 3)$, $e_0 + f_0 = 2 + \min(0,\ A + B^* - 3)$. Therefore, if $A + B^* = 3$ the factor $\ln_+(\lambda|\mathbf{x}|)$ must be taken into account. But for $B = 1,\ 0 < A < 2$ we have $e_0 = \frac{A}{2} > A - 1$, $e_0 + f_0 = A + B^* - 1 = A$ and therefore, we can assume only $\ln_+(\lambda s(\mathbf{x}))$ here. Next in Ω_1 $\ln_+(\lambda|\mathbf{x}|)$ due to $d = 1$. But $e_1 = A - \frac{1}{2} > \min\{1, \frac{A+B^*-1}{2}, A-1\}$ and $e_1 + f_1 = A + B - 1$. So for $B > 1$ we easily see $A + B - 1 > A + B^* - 1$, but for $1 < A + B < 3$, $B \le 1$ we have $e_1 + f_1 = A + B^* - 1$ and therefore we must assume $\ln_+(\lambda s(\mathbf{x}))$ here.

Analogically we proceed in other subdomains and we get (iii) and (iv). The estimate (3.35) for $\lambda = 1$ is therefore shown. In order to show (3.35) for $\lambda \neq 1$, let us recall the homogeneity property of $\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\lambda)$. Namely, for $N = 3$ we have $\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\lambda) = \lambda\, \mathcal{O}_{ij}(\lambda(\mathbf{x}-\mathbf{y});1)$ and therefore

$$\Big| \int_{\mathbb{R}^3} \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\lambda)\, f(\mathbf{y})\, \mathrm{d}\mathbf{y} \Big| = \lambda^{-2} \Big| \int_{\mathbb{R}^3} \mathcal{O}_{ij}(\lambda\mathbf{x}-\mathbf{z}) f(\frac{\mathbf{z}}{\lambda})\, \mathrm{d}\mathbf{z} \Big| \le$$
$$\le \lambda^{-2} \sup_{\mathbf{y}\in\mathbb{R}^3} \Big| f(\mathbf{y})\, \eta^A_B(\lambda\,\mathbf{y};1) \Big| \eta^{-E}_{-F}(\lambda\,\mathbf{x};1)\, P_1(\ln_+(\lambda|\mathbf{x}|))\, P_2(\ln_+ s(\lambda\,\mathbf{x}))$$

and so, as $\eta^A_B(\lambda\,\mathbf{x};1) = \eta^A_B(\mathbf{x};\lambda)$ we have (3.35).

Let us study the weight $\nu^A_B(\mathbf{x};\lambda)$. From Lemma 3.11 we have the conditions $E \ge \max(0, A-2)$ and $A < 3$ and therefore we get on D_1 that $A \ge 1$, on D_2 that $A < 3$ and on D_3 $\frac{A+B-1}{2} \ge A-2$ i.e. $A \le B+5$. Finally, to show (3.36) we proceed as in the case of the estimate (3.35). Evidently, (3.36) holds for $\lambda = 1$. Therefore

$$\Big| \int_{\mathbb{R}^3} \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\lambda)\, f(\mathbf{y})\, \mathrm{d}\mathbf{y} \Big| \le$$
$$\le \lambda^{-2} \sup_{\mathbf{y}\in\mathbb{R}^3} |f(\mathbf{y})\nu^A_B(\lambda\mathbf{y};1)|\, \nu^{-E}_{-F}(\lambda\mathbf{x};1)\, P_1(\ln_+(\lambda|\mathbf{x}|))\, P_2(\ln_+ s(\lambda\mathbf{x})) =$$
$$= \lambda^{-2+A-E}\, \|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\, \nu^{-E}_{-F}(\mathbf{x};\lambda)\, P_1(\ln_+(\lambda|\mathbf{x}|))\, P_2(\ln_+ s(\lambda\mathbf{x})).$$

□

Theorem 3.10 *Let $A + B > 1/2$, $A > -1$ and $i, j = 1, 2, 3$. Then for $f \in L^\infty(\mathbb{R}^3; \eta^A_B(\cdot\,;\lambda))$ we have $\nabla\mathcal{O}_{ij}(\cdot\,;\lambda) * f \in L^\infty(\mathbb{R}^3; \overline{\eta}^E_F(\cdot\,;\lambda))$, where*

$$E = \begin{cases} A - \frac{1}{2} & \textit{for } -1 < A \le 2,\ A \le B+1,\ B \ge 0 \\ \frac{3}{2} & \textit{for } A + B^* \ge 3 \\ A + B - \frac{1}{2} & \textit{for } B < 0,\ A + B \le 1 \\ \frac{A+B}{2} & \textit{for } B \le A-1,\ 1 \le A+B \le 3 \end{cases} \tag{i}$$

$$E + F = \begin{cases} A + B^* & \textit{for } -1 < A \le 2,\ B \ge \frac{3}{2} \\ A + B - \frac{1}{2} & \textit{for } A + B \le \frac{7}{2},\ B \le \frac{3}{2} \\ 3 & \textit{for } A + B \ge \frac{7}{2},\ A \ge 2 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \quad \textit{for} \quad \begin{cases} A + B^* = 3 \\ A = B+1,\ 0 \le B \le 1 \\ A + B = 1,\ B \le 0 \\ B = -\frac{1}{2},\ \frac{3}{4} \le A \le \frac{5}{4} \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(\mathbf{x})) \quad \textit{for} \quad \begin{cases} A = B+1,\ 1 < B \le \frac{5}{4} \\ A + B = 1,\ 0 < B \le \frac{3}{2}. \end{cases} \tag{iv}$$

Moreover we have

$$\|\nabla\mathcal{O}_{ij}(\cdot\,;\lambda)*f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-1}\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.41}$$

Let in addition for A, B *following conditions are satisfied:*

$$\frac{1}{2}\le A\le\frac{5}{2},\quad B\ge-\frac{1}{2},\quad A\le B+2. \tag{v}$$

Then for $f\in L^\infty(\mathbb{R}^3;\nu^A_B(\cdot\,;\lambda))$ *we have* $\nabla_k\mathcal{O}_{ij}*f\in L^\infty(\mathbb{R}^3;\overline{\nu}^E_F(\cdot\,;\lambda))$ *and*

$$\|\nabla\mathcal{O}_{ij}(\cdot\,;\lambda)*f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-1+A-E}\,\|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.42}$$

Sketch of the proof: The theorem (as the following theorems of this subsection) can be proved analogically as Theorem 4.1. We shall therefore not give the details of the proof but we only mention the most important steps.

From Tab.1 and Tab.2 we have for $A+B^*>0$ and $A+B>\frac{1}{2}$

$$E\le\min\left\{\frac{3}{2},\,\frac{A+B^*}{2},\,A-\frac{1}{4},\,A-\frac{1}{2},\,A,\,A+B-\frac{1}{2},\,\frac{A+B}{2},\,A+B^*\right\}=$$

$$=\min\left\{\frac{3}{2},\,\frac{A+B^*}{2},\,A-\frac{1}{2},\,A+B-\frac{1}{2}\right\}, \tag{3.43}$$

$$E+F\le\min\left\{3,\,A+B^*,\,A+B-\frac{1}{2}\right\}.$$

For the weight ν^A_B we use that $E\ge\max\{A-1,0\}$. Finally, the estimates (3.41) and (3.42) follow from the rescaling argument.

□

Theorem 3.11 *Let* $A+B^*>0$. *Let* $R=|\nabla^2\boldsymbol{\mathcal{O}}-\nabla^2\boldsymbol{\mathcal{S}}|$ *or* $R=\nabla_1\boldsymbol{\mathcal{O}}$. *Then for* $f\in L^\infty(\mathbb{R}^3;\eta^A_B(\cdot\,;\lambda))$ *we have* $R*f\in L^\infty(\mathbb{R}^3;\overline{\eta}^E_F(\cdot\,;\lambda))$, *where*

$$E=\begin{cases}A & \text{for } -1<A\le2,\ A\le B+1,\ B\ge0\\ 2 & \text{for } A+B^*\ge3\\ A+B & \text{for } B\le0,\ 0<A+B<1\\ \frac{A+B+1}{2} & \text{for } B\le A-1,\ 1\le A+B\le3\end{cases} \tag{i}$$

$$E+F=\begin{cases}A+B^* & \text{for } A+B^*\le3\\ 3 & \text{for } A+B^*\ge3\end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|)\quad\text{for}\quad A+B^*\le3. \tag{iii}$$

Moreover we have

$$\||\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda)-\nabla^2\boldsymbol{\mathcal{S}}(\cdot)|*f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3} \tag{3.44}$$

$$\|\nabla_1 \mathcal{O}(\cdot\,;\lambda) * f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-1}\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.45}$$

Let in addition for A, B *following conditions are satisfied*

$$0 \le A < 3, \quad B \ge -1, \quad A < B+3. \tag{iv}$$

Then for $f \in L^\infty(\mathbb{R}^3; \nu^A_B(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\mathbb{R}^3; \overline{\nu}^E_F(\cdot\,;\lambda))$ *and*

$$\| |\nabla^2 \mathcal{O}(\cdot\,;\lambda) - \nabla^2 \mathcal{S}(\cdot)| * f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{A-E}\,\|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,, \tag{3.46}$$

$$\|\nabla_1 \mathcal{O}(\cdot\,;\lambda) * f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-1+A-E}\,\|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.47}$$

Sketch of the proof: From Tab.1 and Tab.2 we have for $A + B^* > 0$

$$E \le \min\Big\{2,\ \frac{A+B^*+1}{2},\ A,\ \frac{A+B+1}{2},\ A+B,\ A+\frac{1}{2},$$

$$A+B+\frac{1}{2},\ A+B^*\Big\} = \min\Big\{2,\ \frac{A+B^*+1}{2},\ A,\ A+B^*\Big\} \tag{3.48}$$

$$E+F \le \min\{3,\ A+B^*,\ A+B\}$$

$$= \min\{3,\ A+B^*\}\,.$$

For the weight ν^A_B we use that $E \ge \max\{A-1, 0\}$. Finally, the estimates (3.44)—(3.47) follow from the rescaling argument.

□

Theorem 3.12 *Let* $A + B^* > 1$, $i = 1,2,3$. *Then for* $f \in L^\infty(\mathbb{R}^3; \eta^A_B(\cdot\,;\lambda))$ *we have* $e_i * f \in L^\infty(\mathbb{R}^3; \overline{\eta}^E_F(\cdot\,;\lambda))$, *where*

$$E = \begin{cases} 2 & \text{for } A+B^* \ge 3,\ A \ge \frac{5}{2} \\ A - \frac{1}{2} & \text{for } A \le \frac{5}{2},\ B \ge \frac{1}{2} \\ A+B^*-1 & \text{for } B \le \frac{1}{2},\ A+B^* \le 3 \end{cases} \tag{i}$$

$$E+F = \begin{cases} 2 & \text{for } A+B^* \ge 3 \\ A+B^*-1 & \text{for } A+B^* \le 3 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \quad \text{for} \quad \begin{cases} B = \frac{1}{2},\ \frac{1}{2} < A \le \frac{5}{2} \\ B = 1,\ 2 \le A \le \frac{5}{2} \\ A+B^* = 3,\ A \ge \frac{5}{2} \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(\mathbf{x})) \quad \text{for} \quad \begin{cases} A+B^* = 3,\ 2 \le A < \frac{5}{2} \\ B = 1,\ 0 < A < 2. \end{cases} \tag{iv}$$

Moreover we have

$$\|e_i * f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-1}\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.49}$$

Let in addition for A, B *following conditions are satisfied*

$$\frac{1}{2} \le A < 3, \quad B \ge 0. \tag{v}$$

Then for $f \in L^\infty(\mathbb{R}^3; \nu_B^A(\cdot\,;\lambda))$ *we have* $e_i * f \in L^\infty(\mathbb{R}^3; \overline{\nu}_F^E(\cdot\,;\lambda))$ *and*

$$\|e_i * f\|_{\infty,(\overline{\nu}_F^E(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{-1+A-E}\, \|f\|_{\infty,(\nu_B^A(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.50}$$

Sketch of the proof: From Tab.1 and Tab.2 we have for $A + B^* > 1$

$$E \le \min\Big\{2,\ \frac{A+B^*+1}{2},\ A - \frac{1}{2},\ \frac{A+B+1}{2},\ A+B-1,$$
$$A+B^*-1\Big\} = \min\Big\{2,\ A-\frac{1}{2},\ A+B^*-1\Big\} \tag{3.51}$$
$$E + F \le \min\{2,\ A+B-1,\ A+B^*-1\} =$$
$$= \min\{2,\ A+B^*-1\}\,. \tag{3.52}$$

For the weight ν_B^A we use that $E \ge \max\{A-1, 0\}$. Finally, the estimates (3.49)—(3.50) follow from the rescaling argument.

□

Theorem 3.13 *Let* $A + B^* > 0$, $k = 2, 3$. *Let* $R = |\nabla_{1k}^2 \boldsymbol{\mathcal{O}} - \nabla_{1k}^2 \boldsymbol{\mathcal{S}}|$. *Then for* $f \in L^\infty(\mathbb{R}^3; \eta_B^A(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\mathbb{R}^3; \overline{\eta}_F^E(\cdot\,;\lambda))$, *where*

$$E = \begin{cases} A & \text{for } A \le \frac{5}{2},\ A \le B+2,\ B \ge 0 \\ \frac{5}{2} & \text{for } A + \min\{B, \frac{1}{2}\} \ge 3 \\ A + B^* & \text{for } B \le 0,\ A + B \le 2 \\ \frac{A+B+2}{2} & \text{for } B \le A-2,\ 2 \le A+B \le 3 \end{cases} \tag{i}$$

$$E + F = \begin{cases} A + B^* & \text{for } A + B^* \le 3 \\ 3 & \text{for } A + B^* \ge 3 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \quad \text{for} \quad \begin{cases} A + B^* = 3, B \le \frac{1}{2} \\ A \le \frac{5}{2}, B \ge 0, A \ge B+2 \\ A + B = 2, B \le 0 \end{cases} \tag{iii}$$

$$\ln_+(\lambda s|\mathbf{x}|) \quad \text{for} \quad \begin{cases} A + B^* < 2, B \le 0 \\ 2 < A + B^* < 3, A > B+2\,. \end{cases} \tag{iv}$$

Moreover we have

$$\||\nabla_{1k}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla_{1k}^2 \boldsymbol{\mathcal{S}}(\cdot)| * f\|_{\infty,(\overline{\eta}_F^E(\cdot\,;\lambda)),\mathbb{R}^3} \le C\, \|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.53}$$

Let in addition for A, B *following conditions are satisfied*

$$0 < A < 3, \quad B \ge -1. \tag{v}$$

Then for $f \in L^\infty(\mathbb{R}^3; \nu_B^A(\cdot\,;\lambda))$ *we have* $R * f \in L^\infty(\mathbb{R}^3; \overline{\nu}_F^E(\cdot\,;\lambda))$ *and*

$$\||\nabla_{1k}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla_{1k}^2 \boldsymbol{\mathcal{S}}(\cdot)| * f\|_{\infty,(\overline{\nu}_F^E(\cdot\,;\lambda)),\mathbb{R}^3} \le C\lambda^{A-E}\, \|f\|_{\infty,(\nu_B^A(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.54}$$

Sketch of the proof: From Tab.1 and Tab.2 we have for $A+B^*>0$

$$E \le \min\left\{\frac{5}{2},\ \frac{A+B^*+2}{2},\ A,\ \frac{A+B+2}{2},\ A+B+\frac{1}{2},\ A+B,\ A+B^*\right\} =$$

$$= \min\left\{\frac{5}{2},\ \frac{A+B^*+2}{2},\ A,\ A+B^*\right\} \tag{3.55}$$

$$E+F \le \min\{3,\ A+B^*,\ A+B\} =$$

$$= \min\{3,\ A+B^*\}\,.$$

For the weight ν_B^A we use that $E \ge \max\{A-1, 0\}$. Finally, the estimates (3.53)—(3.54) follow from the rescaling argument.

□

Theorem 3.14 *Let $A+B^*>0$. Let $R=|\nabla_{11}^2\boldsymbol{\mathcal{O}}-\nabla_{11}^2\boldsymbol{\mathcal{S}}|$. Then for $f\in L^\infty(\mathbb{R}^3;\eta_B^A(\cdot\,;\lambda))$ we have $R*f\in L^\infty(\mathbb{R}^3;\overline{\eta}_F^E(\cdot\,;\lambda))$, where*

$$E=\begin{cases} A & \textit{for}\ A\le 3,\ B\ge 0\\ 3 & \textit{for}\ A+\min\{B,0\}\ge 3\\ A+B & \textit{for}\ B\le 0,\ A+B\le 3\end{cases} \tag{i}$$

$$E+F=\begin{cases} A+B^* & \textit{for}\ A+B^*\le 3\\ 3 & \textit{for}\ A+B^*\ge 3\end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|)\quad \textit{for}\quad \begin{cases} A\le 3, B\ge 0\\ A+B=3, B\le 0\,.\end{cases} \tag{iii}$$

$$\ln_+(\lambda s|\mathbf{x}|)\quad \textit{for}\quad A+B^*<3, B\le 0\,. \tag{iv}$$

Moreover we have

$$\||\nabla_{11}^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda)-\nabla_{11}^2\boldsymbol{\mathcal{S}}(\cdot)|*f\|_{\infty,(\overline{\eta}_F^E(\cdot\,;\lambda)),\mathbb{R}^3}\le C\,\|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.56}$$

Let in addition for A, B following conditions are satisfied

$$A<3,\quad B\ge -1\,. \tag{v}$$

Then for $f\in L^\infty(\mathbb{R}^3;\nu_B^A(\cdot\,;\lambda))$ we have $R*f\in L^\infty(\mathbb{R}^3;\overline{\nu}_F^E(\cdot\,;\lambda))$ and

$$\||\nabla_{11}^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda)-\nabla_{11}^2\boldsymbol{\mathcal{S}}(\cdot)|*f\|_{\infty,(\overline{\nu}_F^E(\cdot\,;\lambda)),\mathbb{R}^3}\le C\lambda^{A-E}\,\|f\|_{\infty,(\nu_B^A(\cdot\,;\lambda)),\mathbb{R}^3}\,. \tag{3.57}$$

Sketch of the proof: From Tab.1 and Tab.2 we have for $A+B^*>0$

$$E \le \min\left\{3,\ \frac{A+B^*+3}{2},\ A,\ \frac{A+B+3}{2},\ A+B+1,\ A+B,\ A+B^*\right\} =$$

$$= \min\{3,\ A,\ A+B^*\} \tag{3.58}$$

$$E + F \leq \min\{3,\ A + B^*,\ A + B\} = \min\{3,\ A + B^*\}.$$

For the weight ν_B^A we use that $E \geq \max\{A - 1, 0\}$. Finally, the estimates (3.56)—(3.57) follow from the rescaling argument.

□

We continue with the theorems in the twodimensional case. Again, the proofs are only sketched.

Theorem 3.15 *Let $A + B^* > 1$. Then for $f \in L^\infty(\mathbb{R}^2; \eta_B^A(\cdot\,;\lambda))$ we have $\mathcal{O}_{11} * f \in L^\infty(\mathbb{R}^2; \overline{\eta}_F^E(\cdot\,;\lambda))$, where*

$$E = \begin{cases} A - 1 & \text{for } A \leq B^* + 1 \\ \frac{A+B-1}{2} & \text{for } A \geq B + 1\ A + B \leq 2 \\ \frac{1}{2} & \text{for } A + B^* \geq 2 \end{cases} \tag{i}$$

$$E + F = \begin{cases} 1 & \text{for } A + B^* \geq 2 \\ A + B^* - 1 & \text{for } A + B^* \leq 2 \end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \quad \text{for} \quad \begin{cases} A + B^* = 2 \\ A = B + 1,\ 0 < B \leq \frac{1}{2} \end{cases} \tag{iii}$$

$$\ln_+(\lambda s(\mathbf{x})) \quad \text{for} \quad \left\{ A + B < 2,\ B \leq \frac{1}{2}. \right. \tag{iv}$$

Moreover we have

$$\|\mathcal{O}_{11}(\cdot\,;\lambda) * f\|_{\infty,(\overline{\eta}_F^E(\cdot\,;\lambda)),\mathbb{R}^2} \leq C\lambda^{-2}\, \|f\|_{\infty,(\eta_B^A(\cdot\,;\lambda)),\mathbb{R}^2}\,. \tag{3.59}$$

Let in addition for A following conditions are satisfied

$$1 \leq A < 2. \tag{v}$$

Then for $f \in L^\infty(\mathbb{R}^2; \nu_B^A(\cdot\,;\lambda))$ we have $\mathcal{O}_{11}(\cdot\,;\lambda) * f \in L^\infty(\mathbb{R}^2; \overline{\nu}_F^E(\cdot\,;\lambda))$ and

$$\|\mathcal{O}_{11}(\cdot\,;\lambda) * f\|_{\infty,(\overline{\nu}_F^E(\cdot\,;\lambda)),\mathbb{R}^2} \leq C\lambda^{-2+A-E}\, \|f\|_{\infty,(\nu_B^A(\cdot\,;\lambda)),\mathbb{R}^2}\,. \tag{3.60}$$

Sketch of the proof: From Tab.3 and Tab.4 we have for $A + B^* > 1$

$$E \leq \min\Big\{\frac{1}{2},\ \frac{A + B^* - 1}{2},\ A - \frac{1}{2},\ A - 1,\ \frac{A + B - 1}{2},\ A + B - 1,$$
$$A + B^* - 1\Big\} = \min\Big\{\frac{1}{2},\ \frac{A + B^* - 1}{2},\ A - 1\Big\} \tag{3.61}$$
$$E + F \leq \min\{1,\ A + B^* - 1,\ A + B - 1\} = \min\{1,\ A + B^* - 1\}\,.$$

For the weight ν_B^A we use that $E \geq \max\{A - 2, 0\}$. Finally, the estimates (3.59) and (3.60) follow from the rescaling argument.

□

Theorem 3.16 *Let $A+B^*>1$, $i,j=1,2$, $i\cdot j\neq 1$, $R=\mathcal{O}_{ij}$ or $R=e_i$. Then for $f\in L^\infty(\mathbb{R}^2;\eta^A_B(\cdot\,;\lambda))$ we have $R*f\in L^\infty(\mathbb{R}^2;\overline{\eta}^E_F(\cdot\,;\lambda))$, where*

$$E=E+F=\begin{cases}1 & \textit{for } A+B^*\geq 2\\ A+B^*-1 & \textit{for } A+B^*\leq 2\end{cases} \tag{i}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|)\quad \textit{for}\quad \begin{cases}A+B^*=2\\ 1<A+B^*\leq 2,\ B\geq \frac{1}{2}.\end{cases} \tag{ii}$$

Moreover, we have

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda)*f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^2}\leq C\lambda^{-2}\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^2}\,, \tag{3.62}$$

$$\|e_i*f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^2}\leq C\lambda^{-1}\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^2}\,. \tag{3.63}$$

Let in addition for A, B following conditions are satisfied

$$A<2,\ B^*\geq 0\ (\ \textit{if } R=\mathcal{P}_i)\qquad \textit{or}\qquad A<2,\ (\ \textit{if } R=\mathcal{O}_{ij}). \tag{iii}$$

Then for $f\in L^\infty(\mathbb{R}^2;\nu^A_B(\cdot\,;\lambda))$ we have $R*f\in L^\infty(\mathbb{R}^2;\overline{\nu}^E_F(\cdot\,;\lambda))$ and

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda)*f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^2}\leq C\lambda^{-2+A-E}\,\|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^2}\,, \tag{3.64}$$

$$\|e_i*f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^2}\leq C\lambda^{-1+A-E}\,\|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^2}\,. \tag{3.65}$$

Sketch of the proof: From Tab.3 and Tab.4 we have for $A+B^*>1$

$$E\leq \min\left\{1,\ \frac{A+B^*}{2},\ A-\frac{1}{2},\ \frac{A+B}{2},\ A+B^*-1,\ A+B-1\right\}=$$

$$=\min\{1,\ A+B^*-1\} \tag{3.66}$$

$$E+F\leq \min\{1,\ A+B^*-1,\ A+B-1\}=\min\{1,\ A+B^*-1\}\ .$$

For the weight ν^A_B we use that $E\geq\max\{A-2,0\}$ and $E\geq\max\{A-1,0\}$ for the kernels $\mathcal{O}_{ij}$ and e_i, respectively. Finally, the estimates (3.62)–(3.65) follow from the rescaling argument.

□

Theorem 3.17 *Let $A+B^*>0$, $A+B>\frac{1}{2}$. Then for $f\in L^\infty(\mathbb{R}^2;\eta^A_B(\cdot\,;\lambda))$ we have $\nabla_2\mathcal{O}_{11}(\cdot\,;\lambda)*f\in L^\infty(\mathbb{R}^2;\overline{\eta}^E_F(\cdot\,;\lambda))$, where*

$$E=\begin{cases}1 & \textit{for } A+B^*\geq 2\\ A-\frac{1}{2} & \textit{for } -\frac{1}{2}<A\leq\frac{3}{2},\ B\geq 0,\ A\leq B+1\\ \frac{A+B^*}{2} & \textit{for } 1\leq A+B\leq 2,\ A\geq B+1\\ A+B-\frac{1}{2} & \textit{for } B\leq 0,\ A+B\leq 1\end{cases} \tag{i}$$

$$E+F=\begin{cases}2 & \textit{for } A+B\geq\frac{5}{2},\ A\geq\frac{3}{2}\\ A+B^{*} & \textit{for } -\frac{1}{2}<A\leq\frac{3}{2},\ B\geq 1\\ A+B-\frac{1}{2} & \textit{for } A+B\leq\frac{5}{2},\ B\leq 1\end{cases}\tag{ii}$$

with logarithmic factors

$$\ln_{+}(\lambda|\mathbf{x}|)\quad \textit{for}\quad \begin{cases}A+B^{*}=2\\ B=A-1,\ 0\leq B\leq\frac{1}{2}\\ A+B=1,\ B\leq 0\end{cases}\tag{iii}$$

$$\ln_{+}(\lambda s(\mathbf{x}))\quad \textit{for}\quad A+B=1,\ 0<B\leq 1.\tag{iv}$$

Moreover, we have

$$\|\nabla_2\mathcal{O}_{11}(\cdot\,;\lambda)*f\|_{\infty,(\overline{\eta}^{E}_{F}(\cdot\,;\lambda)),\mathbb{R}^2}\leq C\lambda^{-1}\,\|f\|_{\infty,(\eta^{A}_{B}(\cdot\,;\lambda)),\mathbb{R}^2}\,.\tag{3.67}$$

Let in addition for A, B following conditions are satisfied

$$\frac{1}{2}<A<2,\ B\geq-\frac{1}{2},\ B\geq A-2.\tag{v}$$

Then for $f\in L^{\infty}(\mathbb{R}^2;\nu^{A}_{B}(\cdot\,;\lambda))$ we have $\nabla_2\mathcal{O}_{11}(\cdot\,;\lambda)*f\in L^{\infty}(\mathbb{R}^2;\overline{\nu}^{E}_{F}(\cdot\,;\lambda))$ and

$$\|\nabla_2\mathcal{O}_{11}(\cdot\,;\lambda)*f\|_{\infty,(\overline{\nu}^{E}_{F}(\cdot\,;\lambda)),\mathbb{R}^2}\leq C\lambda^{-1+A-E}\,\|f\|_{\infty,(\nu^{A}_{B}(\cdot\,;\lambda)),\mathbb{R}^2}\,.\tag{3.68}$$

Sketch of the proof: From Tab.3 and Tab.4 we have for $A+B^{*}>0$ and $A+B>\frac{1}{2}$

$$E\leq\min\left\{1,\ \frac{A+B^{*}}{2},\ A-\frac{1}{4},\ A-\frac{1}{2},\ \frac{A+B}{2},\ A,\ A+B-\frac{1}{2},\ A+B^{*}\right\}=$$
$$=\min\left\{1,\ \frac{A+B^{*}}{2},\ A-\frac{1}{2},\ A+B-\frac{1}{2}\right\}\tag{3.69}$$
$$E+F\leq\min\left\{2,\ A+B^{*},\ A+B-\frac{1}{2},\ A+B\right\}=$$
$$=\min\left\{2,\ A+B^{*},\ A+B-\frac{1}{2}\right\}\,.$$

For the weight ν^{A}_{B} we use that $E\geq\max\{A-1,0\}$. Finally, the estimates (3.67) and (3.68) follow from the rescaling argument.

□

Theorem 3.18 *Let $A+B^{*}>0$ and $R=|\nabla^2\boldsymbol{\mathcal{O}}-\nabla^2\boldsymbol{\mathcal{S}}|$ or $R=|\nabla_i\mathcal{O}_{jk}|$, $(i,j,k)\neq(2,1,1)$. Then for $f\in L^{\infty}(\mathbb{R}^2;\eta^{A}_{B}(\cdot\,;\lambda))$ $R*f\in L^{\infty}(\mathbb{R}^2;\overline{\eta}^{E}_{F}(\cdot\,;\lambda))$, where*

$$E=\begin{cases}\frac{3}{2} & \textit{for } A+B^{*}\geq 2\\ A & \textit{for } A\leq\frac{3}{2},\ B\geq 0,\ A\leq B+1\\ \frac{A+B+1}{2} & \textit{for } 1\leq A+B\leq 2,\ A\geq B+1\\ A+B & \textit{for } B\leq 0,\ A+B\leq 1\end{cases}\tag{i}$$

$$E+F=\begin{cases}2 & \textit{for } A+B^* \geq 2\\ A+B^* & \textit{for } A+B^* \leq 2,\end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \quad \textit{for} \quad A+B^* \leq 2. \tag{iii}$$

Moreover, we have

$$\left\| |\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2 \boldsymbol{\mathcal{S}}(\cdot)| * f \right\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \leq C \left\| f \right\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^2} \tag{3.70}$$

$$\left\| |\nabla_i \mathcal{O}_{jk}(\cdot\,;\lambda)| * f \right\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \leq C\lambda^{-1} \left\| f \right\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.71}$$

Let in addition for A, B following conditions are satisfied

$$0 \leq A < 2, \;\; B \geq -1. \tag{iv}$$

Then for $f \in L^\infty(\mathbb{R}^2;\nu^A_B(\cdot\,;\lambda))$ we have $R * f \in L^\infty(\mathbb{R}^2;\overline{\nu}^E_F(\cdot\,;\lambda))$ and

$$\left\| |\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2 \boldsymbol{\mathcal{S}}(\cdot)| * f \right\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \leq C\lambda^{A-E} \left\| f \right\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^2} \tag{3.72}$$

$$\left\| |\nabla_i \mathcal{O}_{jk}(\cdot\,;\lambda)| * f \right\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \leq C\lambda^{-1+A-E} \left\| f \right\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.73}$$

Sketch of the proof: From Tab.3 and Tab.4 we have for $A+B^*>0$

$$\begin{aligned} E &\leq \min\left\{\frac{3}{2},\, \frac{A+B^*+1}{2},\, A,\, A+B,\, \frac{A+B+1}{2},\, A+B^*\right\} = \\ &= \min\left\{\frac{3}{2},\, \frac{A+B+1}{2},\, A,\, A+B\right\} \end{aligned} \tag{3.74}$$

$$E+F \leq \min\{2,\, A+B,\, A+B^*\} = \min\{2,\, A+B^*\}\,.$$

For the weight ν^A_B we use that $E \geq \max\{A-1,0\}$. Finally, the estimates (3.70)–(3.73) follow from the rescaling argument.

□

Theorem 3.19 *Let $A+B^*>0$ and $R = |\nabla^2_{ij}\mathcal{O}_{kl} - \nabla^2_{ij}\mathcal{S}_{kl}|$, $(i,j,k,l) \neq (2,2,1,1)$ or $R = |\nabla_i \mathcal{O}_{jk}|$, $(j,k) \neq (1,1)$, $(i,j,k) \neq (1,1,2)$. Then for $f \in L^\infty(\mathbb{R}^2;\eta^A_B(\cdot\,;\lambda))$ we have $R*f \in L^\infty(\mathbb{R}^2;\overline{\eta}^E_F(\cdot\,;\lambda))$, where*

$$E=\begin{cases}2 & \textit{for } A+B^* \geq 2\\ A & \textit{for } A \leq 2,\; B \geq 0\\ A+B & \textit{for } B \leq 0,\; A+B \leq 2\end{cases} \tag{i}$$

$$E+F=\begin{cases}2 & \textit{for } A+B^* \geq 2\\ A+B^* & \textit{for } A+B^* \leq 2\end{cases} \tag{ii}$$

with logarithmic factors

$$\ln_+(\lambda|\mathbf{x}|) \quad \textit{for} \quad A+B^* \leq 2. \tag{iii}$$

Moreover, we have

$$\||\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda) - \nabla^2_{ij}\mathcal{S}_{kl}(\cdot)| * f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^2} \tag{3.75}$$

$$\||\nabla_i\mathcal{O}_{jk}(\cdot\,;\lambda)| * f\|_{\infty,(\overline{\eta}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \le C\lambda^{-1}\,\|f\|_{\infty,(\eta^A_B(\cdot\,;\lambda)),\mathbb{R}^2}\,. \tag{3.76}$$

Let in addition for A, B following conditions are satisfied

$$0 \le A < 2,\;\; B \ge -1. \tag{iv}$$

Then for $f \in L^\infty(\mathbb{R}^2;\nu^A_B(\cdot\,;\lambda))$ we have $R * f \in L^\infty(\mathbb{R}^2;\overline{\nu}^E_F(\cdot\,;\lambda))$ and

$$\||\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda) - \nabla^2_{ij}\mathcal{S}_{kl}(\cdot)| * f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \le C\lambda^{A-E}\,\|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^2} \tag{3.77}$$

$$\||\nabla_i\mathcal{O}_{jk}(\cdot\,;\lambda)| * f\|_{\infty,(\overline{\nu}^E_F(\cdot\,;\lambda)),\mathbb{R}^2} \le C\lambda^{-1+A-E}\,\|f\|_{\infty,(\nu^A_B(\cdot\,;\lambda)),\mathbb{R}^2}\,. \tag{3.78}$$

Sketch of the proof: From Tab.3 and Tab.4 we have for $A + B^* > 0$

$$E \le \min\left\{2,\,\frac{A+B^*+2}{2},\,A,\,A+B+\frac{1}{2},\,\frac{A+B+2}{2},\,A+B,\,A+B^*\right\} =$$
$$= \min\{2,\,A,\,A+B^*\} \tag{3.79}$$
$$E + F \le \min\{2,\,A+B^*,\,A+B\} = \min\{2,\,A+B^*\}\,.$$

For the weight ν^A_B we use that $E \ge \max\{A-1, 0\}$. Finally, the estimates (3.75)–(3.78) follow from the rescaling argument.

□

II.3.3 Weakly singular integrals. Weighted L^p–estimates

This subsection is devoted to the L^p–theory for the weakly singular Oseen potentials; combining the results with those concerning the singular potentials we then get the L^p–theory for integral operators with kernels formed by second gradients of the fundamental Oseen tensor.

Similarly as in the case of the L^∞–theory, we give detailed proof only for one case. The other theorems can be shown following the same lines.

Theorem 3.20 *Let T be an integral operator with the kernel $\mathcal{O}_{ij}(\cdot\,;\lambda)$, $T : f \mapsto \mathcal{O}_{ij}(\cdot\,;\lambda) * f$, $i,j = 1,2,3$ and let $1 < p < \infty$. Then T is well defined continuous operator:*

a) $$L^p(\mathbb{R}^3;\eta^{\alpha+p/2}_\beta(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3;\eta^{\alpha-p/2}_{\beta-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta \le p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$, $-3p/2+1 < \alpha-\beta < p/2+1$, $\varepsilon > 0$

b) $$L^p(\mathbb{R}^3;\nu^{\alpha+p/2}_\beta(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3;\nu^{\alpha-p/2}_{\beta-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta \le p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$, $-3p/2+1 < \alpha-\beta < p/2+1$, $p/2-3 < \alpha < 5p/2-3$, $\varepsilon > 0$.

Moreover, we have for α, β specified in a) and b), respectively *ad a)*

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{p,(\eta_{\beta-\varepsilon}^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\lambda^{-2}\|f\|_{p,(\eta_{\beta}^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.80}$$

ad b)

$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{p,(\nu_{\beta-\varepsilon}^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\lambda^{-1}\|f\|_{p,(\nu_{\beta}^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \tag{3.81}$$

Proof: We proceed similarly as in the case of L^∞–weighted estimates. Studying first $\eta_\beta^\alpha(\cdot\,;\lambda)$ weights we show (3.80) for $\lambda = 1$, applying the homogeneity properties of $\boldsymbol{\mathcal{O}}(\cdot\,;\lambda)$ we get (3.80) in the general situation $\lambda \neq 1$. Next, using the results from a) together with Lemma 3.12 we show (3.81). Let us denote

$$K(\mathbf{x},\mathbf{y}) = \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};1)\left(\eta_{\beta-\varepsilon}^{\alpha-p/2}(\mathbf{x})\right)^{1/p}\left(\eta_{\beta}^{\alpha+p/2}(\mathbf{y})\right)^{-1/p}$$
$$F(\mathbf{y}) = f(\mathbf{y})\left(\eta_{\beta}^{\alpha+p/2}(\mathbf{y})\right)^{1/p}.$$

We easily observe that, in order to verify (3.80) with $\lambda = 1$, it is sufficient to show that there exists $C > 0$, independent of f, such that

$$\Big\|\int_{\mathbb{R}^3} K(\cdot\,,\mathbf{y})\,F(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big\|_p \le C\,\|F\|_p\,. \tag{3.82}$$

Let $L(\cdot)$, $M(\cdot)$ be non-negative functions defined on $\mathbb{R}^3$ such that for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^3$

$$\begin{aligned} J_0(\mathbf{x}) &:= \int_{\mathbb{R}^3} |K(\mathbf{x},\mathbf{y})|\;L(\mathbf{y})^q\,\mathrm{d}\mathbf{y} \le C^q\,M(\mathbf{x})^q,\\ J_1(\mathbf{y}) &:= \int_{\mathbb{R}^3} |K(\mathbf{x},\mathbf{y})|\;M(\mathbf{x})^p\,\mathrm{d}\mathbf{x} \le C^p\,L(\mathbf{y})^p, \end{aligned} \tag{3.83}$$

where $C > 0$, $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Then relation (3.82) is satisfied. Indeed,

$$\begin{aligned} &\Big\|\int_{\mathbb{R}^3} K(\cdot\,,\mathbf{y})\,F(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big\|_p^p \le\\ &\le \int_{\mathbb{R}^3}\Big\{\Big(\int_{\mathbb{R}^3}|K(\mathbf{x},\mathbf{y})|\;|F(\mathbf{y})|^p\,L(\mathbf{y})^{-p}\,\mathrm{d}\mathbf{y}\Big)^{\frac{1}{p}}\;J_0(\mathbf{x})^{\frac{1}{q}}\Big\}^p\mathrm{d}\mathbf{x} \le\\ &\le C^p\int_{\mathbb{R}^3} M(\mathbf{x})^p\int_{\mathbb{R}^3}|K(\mathbf{x},\mathbf{y})|\;|F(\mathbf{y})|^p\,L(\mathbf{y})^{-p}\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{x} =\\ &C^p\int_{\mathbb{R}^3}|F(\mathbf{y})|^p\,J_1(\mathbf{y})L(\mathbf{y})^{-p}\mathrm{d}\mathbf{y} \le C^{2p}\,\|F\|_p^p\,, \end{aligned}$$

i.e. we get (3.83). We shall suppose the functions $L(\cdot), M(\cdot)$ in the form $L(\mathbf{x}) = M(\mathbf{x}) = \eta_{-B}^{-A}(\mathbf{x})$, $A, B \in \mathbb{R}^1$. Denoting

$$\begin{aligned} a_0 &= qA + \tfrac{\alpha}{p} + \tfrac{1}{2} & a_1 &= pA - \tfrac{\alpha}{p} + \tfrac{1}{2}\\ b_0 &= qB + \tfrac{\beta}{p} & b_1 &= pB - \tfrac{\beta}{p} \end{aligned} \tag{3.84}$$

we get that in order to verify $(3.83)_{1,2}$ we have to find a_i, b_i, $i = 0, 1$, such that

$$\int_{\mathbb{R}^3}\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};1)\,\eta_{-b_i}^{-a_i}(\mathbf{y})\,\mathrm{d}\mathbf{y} \le C\,\eta_{-b_i+\varepsilon}^{-a_i+1}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^3$. Applying Theorem 3.9 with $f = \eta_{-b_i}^{-a_i}(\cdot)$ we get the following set of conditions:

$$a_i < b_i + 1, \qquad b_i \le 1, \qquad a_i + b_i > 1, \qquad i = 1,2, \tag{3.85}$$

a_i, b_i defined in (3.84). We take δ and κ in such a way that

$$A = \frac{-\alpha}{pq} + \frac{2\delta}{pq} \quad B = \frac{-\beta}{pq} + \frac{2\kappa}{pq}\,.$$

From (3.84) we get

$$\begin{array}{ll} \kappa \le \frac{p}{2} & \kappa < (1+\beta)\frac{q}{2} \\ \delta + \kappa < \frac{p}{4} + 2\kappa & \delta + \kappa \le 2\kappa + (\alpha - \beta + \frac{1}{2})\frac{q}{2} \\ \delta + \kappa > \frac{p}{4} & \delta + \kappa \ge (\frac{1}{2} + \alpha + \beta)\frac{q}{2}\,, \end{array} \tag{3.86}$$

it means that

$$\begin{gathered} \max\Big\{\frac{p}{4},\, (\frac{1}{2} + \alpha + \beta)\frac{q}{2}(*)\Big\} < \min\Big\{\frac{5}{4}p, \\ \frac{p}{4} + (1+\beta)q,\, (\alpha - \beta + \frac{1}{2})\frac{q}{2} + p(*),\, (1+\beta)q + (\alpha - \beta + \frac{1}{2})\frac{q}{2}\Big\}\,, \end{gathered} \tag{3.87}$$

where the sign $(*)$ denotes that the corresponding inequality can be taken non–sharp. From here we easily see that the conditions on a_i, b_i can be satisfied for some $A, B \in \mathbb{R}$ if, for sufficiently small $\varepsilon > 0$, we have $-1 < \beta \le p - 1$, $p/2 - 3 < \alpha + \beta < 5p/2 - 3$, $-3p/2 + 1 < \alpha - \beta < p/2 + 1$. Now, recalling the obvious fact that $(1 + s(\mathbf{x}))^\beta \le (1 + s(\mathbf{x}))^{\beta+\varepsilon}$ for all $\varepsilon > 0$ and $\mathbf{x} \in \mathbb{R}^3$ we prove (3.80) with $\lambda = 1$ for any $\varepsilon > 0$.

Next let $\lambda \ne 1$. As $\mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}; \lambda) = \lambda \mathcal{O}_{ij}(\lambda(\mathbf{x} - \mathbf{y}); 1)$ we easily have

$$\begin{aligned} \int_{\mathbb{R}^3} \Big| \int_{\mathbb{R}^3} \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}; \lambda)\, f(\mathbf{y})\, \mathrm{d}\mathbf{y} \Big|^p \eta_{\beta-\varepsilon}^{\alpha-p/2}(\mathbf{x}; \lambda)\, \mathrm{d}\mathbf{x} = \\ = \lambda^{-2p} \int_{\mathbb{R}^3} \Big| \int_{\mathbb{R}^3} \mathcal{O}_{ij}(\lambda\mathbf{x} - \mathbf{z}; 1)\, f\Big(\frac{\mathbf{z}}{\lambda}\Big)\, \mathrm{d}\mathbf{z} \Big|^p \eta_{\beta-\varepsilon}^{\alpha-p/2}(\lambda\mathbf{x}; 1)\, \mathrm{d}\mathbf{x} \le \\ \le C\, \lambda^{-2p-3} \int_{\mathbb{R}^3} \Big| f\Big(\frac{\mathbf{z}}{\lambda}\Big) \Big|^p \eta_\beta^{\alpha+p/2}(\mathbf{z}; 1)\, \mathrm{d}\mathbf{z} = C\, \lambda^{-2p} \int_{\mathbb{R}^3} |f(\mathbf{y})|^p\, \eta_\beta^{\alpha+p/2}(\mathbf{y}; \lambda)\, \mathrm{d}\mathbf{y} \end{aligned}$$

and we have (3.80) with $\lambda \ne 1$.

In order to prove (3.81) we redefine the functions $K(\cdot,\cdot)$, and $F(\cdot)$

$$\begin{gathered} K(\mathbf{x}, \mathbf{y}) = \mathcal{O}_{ij}(\mathbf{x} - \mathbf{y}; 1) \left(\nu_{\beta-\varepsilon}^{\alpha-p/2}(\mathbf{x})\right)^{1/p} \left(\nu_\beta^{\alpha+p/2}(\mathbf{y})\right)^{-1/p} \\ F(\mathbf{y}) = f(\mathbf{y}) \left(\nu_\beta^{\alpha+p/2}(\mathbf{y})\right)^{1/p}. \end{gathered}$$

We will now proceed as in first part of the proof but now we search the functions $L(\cdot), M(\cdot)$ in the form $L(\mathbf{x}) = \mu_{-B}^{-A,-G}(\mathbf{x})$, $M(\mathbf{x}) = \mu_{-B}^{-A,-H}(\mathbf{x})$. Denoting

$$\begin{array}{ll} c_0 = qG + \frac{\alpha}{p} + \frac{1}{2} & c_1 = pG - \frac{\alpha}{p} + \frac{1}{2} \\ d_0 = qH + \frac{\alpha}{p} - \frac{1}{2} & d_1 = pH - \frac{\alpha}{p} - \frac{1}{2} \end{array} \tag{3.88}$$

we see that in order to verify $(3.83)_{1,2}$ we have to find a_i, b_i (see (3.84), (3.85)) and c_i, d_i such that

$$\int_{\mathbb{R}^3} \mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};1)\,\mu_{-b_i}^{-a_i,-c_i}(\mathbf{y})\,\mathrm{d}\mathbf{y} \le C\,\mu_{-b_i+\varepsilon}^{-a_i+1,-d_i}(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^3\setminus\{\mathbf{0}\}.$$

Recalling that $\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};1) \sim \nu_{-1}^{-1}(\mathbf{x}-\mathbf{y};1)$ we get from Lemma 3.12 the following two possible sets of conditions for c_i, d_i:

$$(i)\quad \begin{array}{l} c_i < 3 \\ c_i + 1 > 3 \\ d_i \ge c_i - 2 \end{array} \qquad\qquad (ii)\quad \begin{array}{l} c_i < 2 \\ d_i \ge 0\,, \end{array}$$

where in both cases $i = 0, 1$. Conditions for a_i, b_i are the same as in the first part of proof. From the conditions (i) we get following additional restriction[6]

$$p/2 - 3 < \alpha < 5p/2 - 3.$$

Case (ii) gives more restrictive conditions on α, and therefore no extension of the result. So, (3.81) is proved in the case $\lambda = 1$.

Finally to get (3.81) with $\lambda \ne 1$ we proceed as in the case of the weights $\eta_B^A(\cdot\,;\lambda)$. We have

$$\begin{aligned} &\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};\lambda)\,f(\mathbf{y})\,\mathrm{d}\mathbf{y}\Big|^p \nu_{\beta-\varepsilon}^{\alpha-p/2}(\mathbf{x};\lambda)\,\mathrm{d}\mathbf{x} = \\ &= \lambda^{-2p-\alpha+p/2}\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3}\mathcal{O}_{ij}(\lambda\mathbf{x}-\mathbf{z};1)\,f\Big(\frac{\mathbf{z}}{\lambda}\Big)\,\mathrm{d}\mathbf{z}\Big|^p \nu_{\beta-\varepsilon}^{\alpha-p/2}(\lambda\mathbf{x};1)\,\mathrm{d}\mathbf{x} \le \\ &\le C\,\lambda^{-2p-\alpha+p/2-3}\int_{\mathbb{R}^3}\Big|f\Big(\frac{\mathbf{z}}{\lambda}\Big)\Big|^p \nu_{\beta}^{\alpha+p/2}(\mathbf{z};1)\,\mathrm{d}\mathbf{z} = \\ &C\,\lambda^{-p}\int_{\mathbb{R}^3}|f(\mathbf{y})|^p\,\nu_\beta^{\alpha+p/2}(\mathbf{y};\lambda)\,\mathrm{d}\mathbf{y} \end{aligned}$$

and we have (3.81) with $\lambda \ne 1$. The proof is finished.

□

The following theorems can be shown using the same technique as above.

Theorem 3.21 *Let T be an integral operator with the kernel $|\nabla\mathcal{O}|$, $T : f \mapsto |\nabla\mathcal{O}| * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a) $$L^p(\mathbb{R}^3;\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3;\eta_\beta^{\alpha}(\cdot\,;\lambda))$$

for $-3/2 < \beta < 3p/2 - 3/2$, $-7/2 < \alpha+\beta < 3p - 7/2$, $-3p/2 - 1/2 < \alpha < 3p/2 - 1/2$, $-3p+2 < \alpha-\beta < p/2+2$

b) $$L^p(\mathbb{R}^3;\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3;\nu_\beta^{\alpha}(\cdot\,;\lambda))$$

for $-3/2 < \beta < 3p/2 - 3/2$, $-7/2 < \alpha+\beta < 3p - 7/2$, $\max\{-3p/2 - 1/2, -3\} < \alpha < \min\{3p/2 - 1/2, 5p/2 - 3\}$, $-3p+2 < \alpha-\beta < p/2+2$.

Moreover, we have for α, β specified in $a)$ and $b)$, respectively

[6] The procedure is more or less the same as above; G and H play now the role of δ and κ.

ad a)

$$\| |\nabla \boldsymbol{\mathcal{O}}(\cdot\,;\lambda)| * f \|_{p,(\eta_\beta^\alpha(\cdot\,;\lambda)),\mathbb{R}^3} \leq C\,\lambda^{-1} \|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.89}$$

ad b)

$$\| |\nabla \boldsymbol{\mathcal{O}}(\cdot\,;\lambda)| * f \|_{p,(\nu_\beta^\alpha(\cdot\,;\lambda)),\mathbb{R}^3} \leq C\,\lambda^{-\frac{1}{2}} \|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \tag{3.90}$$

Theorem 3.22 *Let* $R = |\nabla^2 \boldsymbol{\mathcal{O}} - \nabla^2 \boldsymbol{\mathcal{S}}|$ *or* $R = \nabla_1 \boldsymbol{\mathcal{O}}$. *Let* T *be an integral operator with the kernel* R, $T : f \mapsto R * f$, *and let* $1 < p < \infty$. *Then* T *is well defined continuous operator:*

a) $$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 \leq \beta \leq p-1$, $-p/2-3 < \alpha+\beta \leq 5p/2-3$, $-5p/2+1 < \alpha-\beta \leq p/2+1$, $\varepsilon > 0$

b) $$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 \leq \beta \leq p-1$, $-p/2-3 < \alpha+\beta < 5p/2-3$, $-p/2-3 < \alpha < 5p/2-3$, $-5p/2+1 < \alpha-\beta \leq p/2+1$, $0 < \varepsilon < p/2+3+\alpha$.

Moreover, we have for α, β *specified in* a) *and* b), *respectively*

ad a)

$$\begin{aligned} \| |\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2 \boldsymbol{\mathcal{S}}(\cdot)| * f \|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} &\leq \\ \leq C\,\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \end{aligned} \tag{3.91}$$

$$\|\nabla_1 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \leq C\,\lambda^{-1} \|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.92}$$

ad b)

$$\begin{aligned} \| |\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2 \boldsymbol{\mathcal{S}}(\cdot)| * f \|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} &\leq \\ \leq C\,\lambda^{\frac{\varepsilon}{p}} \|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \end{aligned} \tag{3.93}$$

$$\|\nabla_1 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \leq C\,\lambda^{\frac{\varepsilon}{p}-1} \|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \tag{3.94}$$

Corollary 3.3 *Let* T *be an integral operator in the principal value sense with the kernel* $\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda)$, $T : f \mapsto R * f$, *and let* $1 < p < \infty$. *Then* T *is well defined continuous operator:*

a) $$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $-p/2-3 < \alpha+\beta < 5p/2-3$, $-5p/2+1 < \alpha-\beta \leq p/2+1$, $\varepsilon > 0$

b) $$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3, \mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot\,;\lambda))$$

$$L^p(\mathbb{R}^3 \setminus \Omega, \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $-p/2-3 < \alpha+\beta < 5p/2-3$, $-p/2-3 < \alpha < 5p/2-3$, $-5p/2+1 < \alpha-\beta \le p/2+1$, $0 < \varepsilon < p/2+3+\alpha$, $\Omega \subset \mathbb{R}^3$ *– an arbitrary domain,* $\mathbf{0} \in \Omega$ *.*

Moreover, we have for α, β *specified in* $a)$ *and* $b)$, *respectively*
ad a)

$$\|v.p.\,(\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.95}$$

ad b)

$$\|v.p.\,(\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot;\lambda)),\mathbb{R}^3} \le C\,\|f\|_{p,(\mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot;\lambda)),\mathbb{R}^3}, \tag{3.96}$$

$$\|v.p.\,(\nabla^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot;\lambda)),\mathbb{R}^3\setminus\Omega} \le C\,\lambda^{\frac{\varepsilon}{p}}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \tag{3.97}$$

Theorem 3.23 *Let* T *be an integral operator with the kernel* e_i, $T : f \mapsto e_i * f$, $i = 1, 2, 3$, *and let* $1 < p < \infty$. *Then* T *is well defined continuous operator:*

$a_1)$
$$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_{\beta-p/2}^{\alpha}(\cdot\,;\lambda))$$

for $p/2-1 < \beta < p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$

$a_2)$
$$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$

$b_1)$
$$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_{\beta-p/2}^{\alpha}(\cdot\,;\lambda))$$

for $p/2-1 < \beta < p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$, $-3 < \alpha < 5p/2-3$

$b_2)$
$$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $p/2-3 < \alpha+\beta < 5p/2-3$, $p/2-3 < \alpha < 5p/2-3$.

Moreover, we have for α, β *specified in* $a)$ *and* $b)$, *respectively*

ad $a_1)$
$$\|e_i * f\|_{p,(\eta_{\beta-p/2}^{\alpha}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.98}$$

ad $a_2)$
$$\|e_i * f\|_{p,(\eta_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.99}$$

ad $b_1)$
$$\|e_i * f\|_{p,(\nu_{\beta-p/2}^{\alpha}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\lambda^{-\frac{1}{2}}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.100}$$

ad $b_2)$
$$\|e_i * f\|_{p,(\nu_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \tag{3.101}$$

Theorem 3.24 *Let* $R = |\nabla^2_{1k}\boldsymbol{\mathcal{O}} - \nabla^2_{1k}\boldsymbol{\mathcal{S}}|$ $k = 2,3$. *Let* T *be an integral operator with the kernel* R, $T : f \mapsto R * f$, *and let* $1 < p < \infty$. *Then* T *is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 \le \beta \le p-1$, $-p/2 - 3 < \alpha + \beta \le 5p/2 - 3$, $-5p/2 < \alpha - \beta \le 3p/2$, $\varepsilon > 0$

b)
$$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 \le \beta \le p-1$, $-p/2-3 < \alpha+\beta \le 5p/2-3$, $-p/2-3 < \alpha < 5p/2-3$, $-5p/2 < \alpha - \beta \le 3p/2$, $0 < \varepsilon < p/2 + 3 + \alpha$.

Moreover, we have for α, β, *specified in* $a)$ *and* $b)$, *respectively*

ad a)
$$\begin{aligned} \big\| |\nabla^2_{1k}\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2_{1k}\boldsymbol{\mathcal{S}}(\cdot)| * f \big\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le \\ \le C\, \|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \end{aligned} \tag{3.102}$$

ad b)
$$\begin{aligned} \big\| |\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2\boldsymbol{\mathcal{S}}(\cdot)| * f \big\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le \\ \le C\, \lambda^{\frac{\varepsilon}{p}} \|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \end{aligned} \tag{3.103}$$

Corollary 3.4 *Let* T *be an integral operator in the principal value sense with the kernel* $\nabla^2_{1k}\boldsymbol{\mathcal{O}}(\cdot\,;\lambda)$, $k = 2,3$, $T : f \mapsto R * f$, *and let* $1 < p < \infty$. *Then* T *is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $-p/2 - 3 < \alpha + \beta < 5p/2 - 3$, $-5p/2 < \alpha - \beta \le 3p/2$, $\varepsilon > 0$

b)
$$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3, \mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot\,;\lambda))$$

$$L^p(\mathbb{R}^3 \setminus \Omega, \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $-p/2-3 < \alpha+\beta < 5p/2-3$, $-p/2-3 < \alpha < 5p/2-3$, $-5p/2 < \alpha - \beta \le 3p/2$, $0 < \varepsilon < p/2 + 3 + \alpha$, $\Omega \subset \mathbb{R}^3$ *– an arbitrary domain,* $\mathbf{0} \in \Omega$

Moreover, we have for α, β *specified in* $a)$ *and* $b)$, *respectively*
ad a)
$$\|v.p.\,(\nabla^2_{1k}\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.104}$$

ad b)
$$\|v.p.\,(\nabla^2_{1k}\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le C\,\|f\|_{p,(\mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot;\lambda)),\mathbb{R}^3}, \tag{3.105}$$
$$\|v.p.\,(\nabla^2_{1k}\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot;\lambda)),\mathbb{R}^3\setminus\Omega} \le C\,\lambda^{\frac{\varepsilon}{p}} \|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \tag{3.106}$$

Theorem 3.25 *Let $R = |\nabla_{11}^2 \boldsymbol{\mathcal{O}} - \nabla_{11}^2 \boldsymbol{\mathcal{S}}|$. Let T be an integral operator with the kernel R, $T : f \mapsto R * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a) $$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 \le \beta \le p-1, \;\; -p/2-3 < \alpha+\beta \le 5p/2-3, \;\; \varepsilon > 0$

b) $$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 \le \beta \le p-1, \;\; -p/2-3 < \alpha+\beta \le 5p/2-3, \;\; -p/2-3 < \alpha < 5p/2-3$, $0 < \varepsilon < p/2+3+\alpha$.

Moreover, we have for $\alpha, \; \beta$ *specified in* $a)$ *and* $b)$, *respectively*

ad a)
$$\begin{aligned} \| |\nabla_{11}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla_{11}^2 \boldsymbol{\mathcal{S}}(\cdot)| * f \|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le \\ \le C \, \|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \end{aligned} \tag{3.107}$$

ad b)
$$\begin{aligned} \| |\nabla_{11}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla_{11}^2 \boldsymbol{\mathcal{S}}(\cdot)| * f \|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le \\ \le C \, \lambda^{\frac{\varepsilon}{p}} \|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}. \end{aligned} \tag{3.108}$$

Corollary 3.5 *Let T be an integral operator in the principal value sense with the kernel $\nabla_{11}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda)$, $T : f \mapsto R*f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a) $$L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1, \;\; -p/2-3 < \alpha+\beta < 5p/2-3, \;\; \varepsilon > 0$

b) $$L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3, \mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot\,;\lambda))$$

$$L^p(\mathbb{R}^3 \setminus \Omega, \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^3; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1, \;\; -p/2-3 < \alpha+\beta < 5p/2-3, \;\; -p/2-3 < \alpha < 5p/2-3$, $0 < \varepsilon < p/2+3+\alpha, \;\; \Omega \subset \mathbb{R}^3$ *– an arbitrary domain,* $\mathbf{0} \in \Omega$.

Moreover, we have for $\alpha, \; \beta$ *specified in* $a)$ *and* $b)$, *respectively*
ad a)

$$\|v.p.(\nabla_{11}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le C \|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^3}, \tag{3.109}$$

ad b)

$$\|v.p.(\nabla_{11}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^3} \le C \, \|f\|_{p,(\mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot;\lambda)),\mathbb{R}^3}, \tag{3.110}$$

$$\|v.p.\,(\nabla_{11}^2 \boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot;\lambda)),\mathbb{R}^3\setminus\Omega} \le C \, \lambda^{\frac{\varepsilon}{p}} \|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot;\lambda)),\mathbb{R}^3}. \tag{3.111}$$

Next we formulate analogical results also in the twodimensional case.

Theorem 3.26 *Let T be an integral operator with the kernel $\mathcal{O}_{11}(\cdot\,;\lambda)$ $T : f \mapsto \mathcal{O}_{11}(\cdot\,;\lambda) * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^2; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \eta_{\beta-\varepsilon}^{\alpha-p/2}(\cdot\,;\lambda))$$

for $-1/2 < \beta \le (p-1)/2$, $p/2-2 < \alpha+\beta < 3p/2-2$, $-p/2 < \alpha-\beta < p/2$, $\varepsilon > 0$.

b)
$$L^p(\mathbb{R}^2; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \nu_{\beta-\varepsilon}^{\alpha-p/2}(\cdot\,;\lambda))$$

for $-1/2 < \beta \le (p-1)/2$, $p/2-2 < \alpha+\beta < 3p/2-2$, $p/2-2 < \alpha < 3p/2-2$, $-p/2 < \alpha-\beta < p/2$, $\varepsilon > 0$.

Moreover, we have for α, β specified in a) and b), respectively

ad a)
$$\|\mathcal{O}_{11}(\cdot\,;\lambda) * f\|_{p,(\eta_{\beta-\varepsilon}^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-2}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.112}$$

ad b)
$$\|\mathcal{O}_{11}(\cdot\,;\lambda) * f\|_{p,(\nu_{\beta-\varepsilon}^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-1}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.113}$$

Theorem 3.27 *Let T be an integral operator with the kernel $R = \mathcal{O}_{ij}$, $i,j = 1,2$, $i\cdot j \ne 1$ or $R = e_i$; $T : f \mapsto R * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^2; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \eta_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $-1/2 < \beta < (p-1)/2$, $p/2-2 < \alpha+\beta < 3p/2-2$

b)
$$L^p(\mathbb{R}^2; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \nu_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $-1/2 < \beta < (p-1)/2$, $p/2-2 < \alpha+\beta < 3p/2-2$, $p/2-2 < \alpha < 3p/2-2$.

Moreover, we have for α, β specified in a) and b), respectively

ad a)
$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{p,(\eta_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-2}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.114}$$

$$\|e_i * f\|_{p,(\eta_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.115}$$

ad b)
$$\|\mathcal{O}_{ij}(\cdot\,;\lambda) * f\|_{p,(\nu_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-1}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.116}$$

$$\|e_i * f\|_{p,(\nu_\beta^{\alpha-p/2}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.117}$$

Theorem 3.28 *Let T be an integral operator with the kernel $\nabla_2\mathcal{O}_{11}$, $T: f \mapsto \nabla_2\mathcal{O}_{11} * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^2; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \eta_\beta^{\alpha}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $-5/2 < \alpha+\beta < 2p-5/2$, $-p-1/2 < \alpha < p-1/2$, $-2p+1 < \alpha-\beta < p/2+1$

b)
$$L^p(\mathbb{R}^2; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \nu_\beta^{\alpha-p/2}(\cdot\,;\lambda))$$

for $-1 < \beta < p-1$, $-5/2 < \alpha+\beta < 2p-5/2$, $\max\{-p-1/2, -2\} < \alpha < \min\{p-1/2, 3p/2-2\}$, $-2p+1 < \alpha-\beta < p/2+1$.

Moreover, we have for α, β *specified in* $a)$ *and* $b)$, *respectively*

ad a)
$$\|\nabla_2\mathcal{O}_{11}(\cdot\,;\lambda) * f\|_{p,(\eta_\beta^{\alpha}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.118}$$

ad b)
$$\|\nabla_2\mathcal{O}_{11}(\cdot\,;\lambda) * f\|_{p,(\nu_\beta^{\alpha}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-\frac{1}{2}}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.119}$$

Theorem 3.29 *Let T be an integral operator with the kernel $R = |\nabla^2\boldsymbol{\mathcal{O}} - \nabla^2\boldsymbol{\mathcal{S}}|$, or $R = |\nabla_i\mathcal{O}_{jk}|$, $(i,j,k) \neq (2,1,1)$; $T: f \mapsto R * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^2; \eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 \le \beta \le (p-1)/2$, $-p/2-2 < \alpha+\beta \le 3p/2-2$, $-3p/2 < \alpha-\beta \le p/2$, $\varepsilon > 0$

b)
$$L^p(\mathbb{R}^2; \nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2; \nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 \le \beta \le (p-1)/2$, $-p/2-2 < \alpha+\beta \le 3p/2-2$, $-3p/2 < \alpha-\beta \le p/2$, $-p/2-2 < \alpha < 3p/2-2$, $0 < \varepsilon < p/2+2+\alpha$.

Moreover, we have for α, β *specified in* $a)$ *and* $b)$, *respectively*

ad a)
$$\begin{aligned}\||\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2\boldsymbol{\mathcal{S}}(\cdot)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} &\le \\ \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}&,\end{aligned} \tag{3.120}$$

$$\||\nabla_i\mathcal{O}_{jk}(\cdot\,;\lambda)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.121}$$

ad b)
$$\begin{aligned}\||\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) - \nabla^2\boldsymbol{\mathcal{S}}(\cdot)| * f\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} &\le \\ \le C\,\lambda^{\frac{\varepsilon}{p}}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}&,\end{aligned} \tag{3.122}$$

$$\||\nabla_i\mathcal{O}_{jk}(\cdot\,;\lambda)| * f\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{\frac{\varepsilon}{p}-1}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.123}$$

Corollary 3.6 *Let T be an integral operator in the principal value sense with the kernel $\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda)$, $T: f \mapsto R * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^2;\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2;\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 < \beta < (p-1)/2$, $-p/2-2 < \alpha+\beta < 3p/2-2$, $-3p/2 < \alpha-\beta \le p/2$, $\varepsilon > 0$

b)
$$L^p(\mathbb{R}^2;\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2,\mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot\,;\lambda))$$

$$L^p(\mathbb{R}^2\setminus\Omega,\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2;\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 \le \beta \le (p-1)/2$, $-p/2-2 < \alpha+\beta < 3p/2-2$, $-p/2-2 < \alpha < 3p/2-2$, $-3p/2 < \alpha-\beta \le p/2$, $0 < \varepsilon < p/2+2+\alpha$, $\Omega \subset \mathbb{R}^2$ *– an arbitrary domain,* $\mathbf{0} \in \Omega$.

Moreover, we have for α, β *specified in* a) *and* b), *respectively*
ad a)

$$\|v.p.\,(\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.124}$$

ad b)

$$\|v.p.\,(\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot;\lambda)),\mathbb{R}^2} \le C\,\|f\|_{p,(\mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot;\lambda)),\mathbb{R}^2}, \tag{3.125}$$

$$\|v.p.\,(\nabla^2\boldsymbol{\mathcal{O}}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot;\lambda)),\mathbb{R}^2\setminus\Omega} \le C\,\lambda^{\frac{\varepsilon}{p}}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.126}$$

Theorem 3.30 *Let T be an integral operator with the kernel $R = |\nabla^2_{ij}\mathcal{O}_{kl} - \nabla^2_{ij}\mathcal{S}_{kl}|$, $(i,j,k,l) \ne (2,2,1,1)$ or $R = |\nabla\mathcal{O}_{22}|$, $R = |\nabla_1\mathcal{O}_{12}|$; $T: f \mapsto R * f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^2;\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2;\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 \le \beta \le (p-1)/2$, $-p/2-2 < \alpha+\beta \le 3p/2-2$, $\varepsilon > 0$

b)
$$L^p(\mathbb{R}^2;\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2;\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 \le \beta \le (p-1)/2$, $-p/2-2 < \alpha+\beta \le 3p/2-2$, $-p/2-2 < \alpha < 3p/2-2$, $0 < \varepsilon < p/2+2+\alpha$.

Moreover, we have for α, β, i, j, k, l *specified in* a) *and* b), *respectively*

ad a)

$$\begin{aligned} \||\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda) - \nabla^2_{ij}\mathcal{S}_{kl}(\cdot)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le \\ \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \end{aligned} \tag{3.127}$$

$$\||\nabla_i\mathcal{O}_{jk}(\cdot\,;\lambda)| * f\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{-1}\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.128}$$

ad b)

$$\||\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda) - \nabla^2_{ij}\mathcal{S}_{ij}(\cdot)| * f\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le \le C\,\lambda^{\frac{\varepsilon}{p}}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.129}$$

$$\||\nabla_i\mathcal{O}_{jk}(\cdot\,;\lambda)| * f\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\lambda^{\frac{\varepsilon}{p}-1}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.130}$$

Corollary 3.7 *Let T be an integral operator in the principal value sense with the kernel $\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda)$, $(i,j,k,l) \neq (2,2,1,1)$ $T : f \mapsto R*f$, and let $1 < p < \infty$. Then T is well defined continuous operator:*

a)
$$L^p(\mathbb{R}^2;\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2;\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 < \beta < (p-1)/2,\ \ -p/2-2 < \alpha+\beta < 3p/2-2,\ \ \varepsilon > 0$

b)
$$L^p(\mathbb{R}^2;\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2,\mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot\,;\lambda))$$

$$L^p(\mathbb{R}^2\setminus\Omega,\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)) \longmapsto L^p(\mathbb{R}^2;\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda))$$

for $-1/2 \le \beta \le (p-1)/2,\ \ -p/2-2 < \alpha+\beta < 3p/2-2,\ \ -p/2-2 < \alpha < 3p/2-2,\ \ 0 < \varepsilon < p/2+2+\alpha,\ \ \Omega \subset \mathbb{R}^2$ *– an arbitrary domain,* $\mathbf{0} \in \Omega$.

Moreover, we have for α, β *specified in* a) *and* b), *respectively*
ad a)

$$\|v.p.\,(\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda) * f)\|_{p,(\eta_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\|f\|_{p,(\eta_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}, \tag{3.131}$$

ad b)

$$\|v.p.\,(\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda)*f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot\,;\lambda)),\mathbb{R}^2} \le C\,\|f\|_{p,(\mu_\beta^{\alpha+p/2-\varepsilon,\alpha+p/2}(\cdot;\lambda)),\mathbb{R}^2} \tag{3.132}$$

$$\|v.p.\,(\nabla^2_{ij}\mathcal{O}_{kl}(\cdot\,;\lambda) * f)\|_{p,(\nu_\beta^{\alpha+p/2-\varepsilon}(\cdot;\lambda)),\mathbb{R}^2\setminus\Omega} \le C\,\lambda^{\frac{\varepsilon}{p}}\|f\|_{p,(\nu_\beta^{\alpha+p/2}(\cdot\,;\lambda)),\mathbb{R}^2}. \tag{3.133}$$

II.3.4 Stationary flow of the incompressible fluid in the whole space with non–zero velocity prescribed at infinity

The last subsection is devoted to the study of a very simple problem — the stationary flow of a viscous fluid in the whole $\mathbb{R}^N$. We assume that the prescribed velocity at infinity is non–zero and we study a small perturbation of the velocity from the steady state $\mathbf{v} = \mathbf{v}_\infty$; this perturbation is caused by a small external force which has certain asymptotic behaviour at infinity.

We apply the results from the previous subsections and show that the solution has also certain asymptotic properties. Let us emphasize that the anisotropy in the asymptotic behaviour is due to the anisotropy of the right hand side and unlike the exterior domain problem does not come from the problem itself. Nevertheless, the problem in the whole space easily demonstrate the estimates of the Oseen kernels and in two space dimensions we get the expected result

— different asymptotic properties of the first and second components of the velocity.

We study the following system in $\mathbb{R}^N$

$$\begin{aligned} \mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f} \\ \nabla\cdot\mathbf{v} &= 0 \\ \mathbf{v}\to\mathbf{v}_\infty \quad \text{as} &\quad |\mathbf{x}|\to\infty\,. \end{aligned}$$

We may, without loss of generality, assume that $\mathbf{v}_\infty = \lambda\mathbf{e}_1$. Moreover, we do not study the precise condition under which the solution exists. We put, without loss of generality, $\nu = 1$. Denoting $\mathbf{u} = \mathbf{v} - \mathbf{v}_\infty$ and assuming $\mathbf{f} = -\nabla\cdot\mathbf{G}$ we finally get

$$\begin{aligned} -\Delta\mathbf{u} + \lambda\tfrac{\partial\mathbf{u}}{\partial x_1} + \nabla p &= -\nabla\cdot(\mathbf{G} + \mathbf{u}\otimes\mathbf{u}) \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}\to\mathbf{0} \quad \text{as} &\quad |\mathbf{x}|\to\infty\,. \end{aligned} \tag{3.134}$$

We first show the existence of solution to (3.134) via successive approximation in Sobolev spaces. Then, using the integral representation of solution, we shall study asymptotic properties of the solution. Let us emphasize that we assume only small perturbation i.e. certain norms of $\mathbf{G}$ will be assumed sufficiently small.

Theorem 3.31 *Let $\|\mathbf{G}\|_q$, $q\in[\frac{N}{2};\frac{N+1}{2}]$, be sufficiently small. Then there exists weak solution of (3.134) such that the norms $\|\nabla\mathbf{u}\|_q$ and $\|\mathbf{u}\|_{\frac{(N+1)q}{N+1-q}}$ are finite.*

Proof: Let us denote $s = \frac{(N+1)q}{N+1-q}$. We shall apply the Banach fixed point theorem on the system (3.134) in the Banach space B,

$$B = \Big\{\mathbf{u};\mathbf{u}\in L^s(\mathbb{R}^N)\,,\nabla\mathbf{u}\in L^q(\mathbb{R}^N)\Big\}$$

equipped with the norm[7]

$$\|\mathbf{u}\|_B = \|\nabla\mathbf{u}\|_q + \lambda^{\frac{1}{N+1}}\|\mathbf{u}\|_s\,.$$

We denote by T the operator from B to B such that

$$T\mathbf{w} = \mathbf{u}\,,$$

where

$$\begin{aligned} -\Delta\mathbf{u} + \lambda\tfrac{\partial\mathbf{u}}{\partial x_1} + \nabla p &= -\nabla\cdot(\mathbf{G} + \mathbf{w}\otimes\mathbf{w}) \\ \nabla\cdot\mathbf{u} &= 0 \\ \textstyle\int_{S_1}|\mathbf{u}(R,\omega)|^q\mathrm{d}\omega \to 0 \quad \text{as} &\quad R\to\infty \end{aligned} \tag{3.135}$$

From [Ga1] (see also Section III.2) it follows that there exists unique solution to (3.135); moreover

$$\|\mathbf{u}\|_B \le C(\|\mathbf{G}\|_q + \|\mathbf{w}\otimes\mathbf{w}\|_q)\,.$$

[7]The size of λ does not play any role here. Nevertheless, we keep on writing λ in the norms.

Let us assume that $\|\mathbf{w}\|_B \leq \varepsilon$ with ε small. We show that for $\|\mathbf{G}\|_q$ sufficiently small also $\|\mathbf{u}\|_B \leq \varepsilon$. We have

$$\|\mathbf{w} \otimes \mathbf{w}\|_q \leq \|\mathbf{w}\|_{2q}^2 \leq \|\mathbf{w}\|_s^{2\alpha} \|\mathbf{w}\|_{\frac{Nq}{N-q}}^{2(1-\alpha)}$$

where the fact that $q \in [\frac{N}{2}; \frac{N+1}{2}]$ was used. We easily calculate that $\alpha = \frac{(N+1)(2q-N)}{2q}$ and so

$$\|\mathbf{w} \otimes \mathbf{w}\|_q \leq C\lambda^{-\frac{2q-N}{q}} \|\mathbf{w}\|_B^2 .$$

Therefore for the right hand side and ε sufficiently small we have

$$\|\mathbf{u}\|_q \leq \varepsilon$$

and the operator T maps ball with diameter ε into itself.

Next, let $\mathbf{U} = \mathbf{u}^1 - \mathbf{u}^2$, $P = p^1 - p^2$. We have

$$\begin{aligned} -\Delta \mathbf{U} + \lambda \tfrac{\partial \mathbf{U}}{\partial x_1} + \nabla P &= \nabla \cdot [(\mathbf{w}^1 - \mathbf{w}^2) \otimes \mathbf{w}^1 + \mathbf{w}^2 \otimes (\mathbf{w}^1 - \mathbf{w}^2)] \\ \nabla \cdot \mathbf{U} &= 0 \\ \textstyle\int_{S_1} |\mathbf{U}(R,\omega)|^q \mathrm{d}\omega \to 0 \quad &\text{as} \quad R \to \infty \end{aligned}$$

and denoting $\mathbf{W} = \mathbf{w}^1 - \mathbf{w}^2$

$$\|\mathbf{U}\|_B \leq C\|(\mathbf{w}^1 + \mathbf{w}^2) \otimes \mathbf{W}\|_q . \tag{3.136}$$

We proceed as above and get

$$\|(\mathbf{w}^1 + \mathbf{w}^2) \otimes \mathbf{W}\|_q \leq C\lambda^{-\frac{2q-N}{q}} (\|\mathbf{w}^1\|_B + \|\mathbf{w}^2\|_B)\|\mathbf{W}\|_B . \tag{3.137}$$

Therefore (3.136) and (3.137) yield for ε sufficiently small

$$\|\mathbf{U}\|_B \leq \kappa \|\mathbf{W}\|_B$$

with $\kappa < 1$. The Banach fixed point theorem applied on the set $B_\varepsilon = \{\mathbf{u} \in B; \|\mathbf{u}\|_B \leq \varepsilon\}$ finishes the proof.

□

The next aim is to study the asymptotic properties of $\mathbf{u}$ i.e. the estimates of $\mathbf{u}$ in certain weighted spaces. Under the assumption on $\mathbf{G}$ stated above we easily get

$$\begin{aligned} u_j(\mathbf{x}) &= \int_{\mathbb{R}^N} \frac{\partial \mathcal{O}_{ij}}{\partial y_k}(\mathbf{x}-\mathbf{y};\lambda)(w_i w_k + G_{ik})(\mathbf{y})\mathrm{d}\mathbf{y} \\ \frac{\partial u_j}{\partial x_l}(\mathbf{x}) &= -\text{v.p.} \int_{\mathbb{R}^N} \frac{\partial^2 \mathcal{O}_{ij}}{\partial y_l \partial y_k}(\mathbf{x}-\mathbf{y};\lambda)(w_i w_k + G_{ik})(\mathbf{y})\mathrm{d}\mathbf{y} + \\ &\quad + c_{ijkl}(w_i w_k + G_{ik})(\mathbf{x}) . \end{aligned} \tag{3.138}$$

The proof of (3.138) follows easily from the fact that $\boldsymbol{\mathcal{O}}$ is the fundamental Oseen tensor and from its asymptotic properties (see Section II.1) by means of a standard density argument.

Recalling that we have constructed the solution to (3.134) via successive approximation it is enough to show that, for $\mathbf{w}$ and $\nabla\mathbf{w}$ bounded in certain weighted spaces, the solution $\mathbf{u}$ to (3.135) lies in the same ball. Then the same holds for the fixed point constructed in Theorem 3.31.

We now study separately the cases $N = 2$ and $N = 3$.

Theorem 3.32 *Let* $\mathbf{G} \in L^p(\mathbb{R}^2; \eta_{Bp}^{Ap}(\cdot)) \cap L^q(\mathbb{R}^2)$, $p > \frac{8}{5}$, $1 < q \leq \frac{3}{2}$ *with the norms sufficiently small. Let* $\frac{3}{4} < A \leq 1$, $0 \leq B < \frac{1}{2} - \frac{1}{2p}$, $A + B \leq 2 - \frac{2}{p}$. *Then the solution to (3.134) from Theorem 3.31 has the following asymptotic properties:*

$$u_1 \in L^p(\mathbb{R}^2; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot))$$
$$u_2, \nabla\mathbf{u} \in L^p(\mathbb{R}^2; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot))$$

for any $\delta > 0$.

Proof: Let us assume that

$$\|w_1; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p + \|w_2; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p +$$
$$+ \|\nabla\mathbf{w}; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda))\|_p \leq \varepsilon\,,$$

where ε is (sufficiently) small positive number. From $(3.138)_1$ we have

$$u_1(\mathbf{x}) =$$
$$= \int_{\mathbb{R}^2} \Big[\frac{\partial \mathcal{O}_{11}}{\partial y_2}(\mathbf{x}-\mathbf{y};\lambda)(w_1w_2 + G_{12})(\mathbf{y}) + \frac{\partial \mathcal{O}_{11}}{\partial y_1}(\mathbf{x}-\mathbf{y};\lambda)(w_1^2 + G_{11})(\mathbf{y}) +$$
$$+ \frac{\partial \mathcal{O}_{21}}{\partial y_1}(\mathbf{x}-\mathbf{y};\lambda)(w_1w_2 + G_{21})(\mathbf{y}) + \frac{\partial \mathcal{O}_{21}}{\partial y_2}(\mathbf{x}-\mathbf{y};\lambda)(w_2^2 + G_{22})(\mathbf{y})\Big]\mathrm{d}\mathbf{y}\,.$$

We apply Theorems 3.28, 3.29 and get

$$\begin{gathered} \|u_1; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p \leq C\lambda^{-1}\Big\{\|w_1w_2; \eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p + \\ \|G_{12}; \eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p + \||w_1|^2 + |w_1w_2| + |w_2|^2; \eta_{Bp}^{(A-\frac{1}{2}+\kappa)p}(\cdot\,;\lambda)\|_p + \\ \||G_{11}| + |G_{21}| + |G_{22}|; \eta_{Bp}^{(A-\frac{1}{2}+\kappa)p}(\cdot\,;\lambda)\|_p\Big\} \end{gathered} \tag{3.139}$$

($\kappa > 0$) together with the conditions

$$\begin{gathered} -\tfrac{1}{2} \leq Bp \leq \tfrac{p-1}{2} \\ -\tfrac{5}{2} + \tfrac{p}{2} < (A+B)p < 2p - 2 \\ \max\{-\tfrac{3}{2}p + 1, -p\} < (A-B)p < p \\ -\tfrac{p+1}{2} < Ap < \tfrac{3}{2}p - \tfrac{1}{2}\,. \end{gathered} \tag{3.140}$$

Next we shall estimate the quadratic terms. We have for

$$A \geq \frac{3}{4} + \frac{\delta}{2p}\,, \qquad B \geq 0 \tag{3.141}$$

that

$$\begin{aligned}&\|w_1w_2;\eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p \le C\|w_1w_2;\eta_{Bp}^{(3A-\frac{3}{2}-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le\\ &\le C\|w_1;\eta_{Bp}^{2(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_{2p}\|w_2;\eta_{Bp}^{2(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_{2p} \le\\ &\le C\|w_1;\eta_{2Bp}^{2(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_{2p}\|w_2;\eta_{2Bp}^{2(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_{2p}\,.\end{aligned}$$

We apply the Sobolev imbedding theorem and get under the assumption (3.141)

$$\begin{aligned}&\|w_1w_2;\eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p \le C\lambda^{-1}(\|w_1;\eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p + \|\nabla\mathbf{w};\eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p)\,\cdot\\ &\cdot(\|w_2;\eta_{Bp}^{2(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p + \|\nabla\mathbf{w};\eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p) \le C\lambda^{-1}\varepsilon^2\,.\end{aligned}$$

Analogously we proceed for the other terms. Evidently, the most restrictive is those with w_2^2. We take $\kappa = \frac{\delta}{p}$ and get for A, B satisfying (3.141) and $\delta < \frac{p}{2}$

$$\begin{aligned}&\||w_2|^2;\eta_{Bp}^{(A-\frac{1}{2}+\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le \||w_2|^2;\eta_{2Bp}^{2(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le\\ &\le C(\|w_2;\eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p^2 + \|\nabla\mathbf{w};\eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p^2)\,.\end{aligned}$$

The other quadratic terms in (3.139) can be estimated

$$\begin{aligned}&\||w_1|^2 + |w_1w_2| + |w_2|^2;\eta_{Bp}^{(A-\frac{1}{2}+\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le C\lambda^{-1}(\|w_1;\eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p^2 +\\ &+\|w_2;\eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda))\|_p^2 + \|\nabla\mathbf{w};\eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p^2) \le C\lambda^{-1}\varepsilon^2\,.\end{aligned}$$

Using the evident inequality we get for $\delta < \frac{p}{2}$

$$\|\mathbf{G};\eta_{Bp}^{(A-\frac{1}{2}+\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le C(\lambda)\|\mathbf{G};\eta_{Bp}^{Ap}(\cdot\,;1)\|_p$$

we get for $\|\mathbf{G};\eta_{Bp}^{Ap}(\cdot\,;1)\|_p$ and ε sufficiently small

$$\|u_1;\eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p \le \frac{1}{3}\varepsilon\,. \tag{3.142}$$

Next we estimate the second component of the velocity. We have from $(3.138)_1$

$$\begin{aligned}&u_2(\mathbf{x}) =\\ &= \int_{\mathbb{R}^2}\Big\{\frac{\partial\mathcal{O}_{12}}{\partial y_2}(\mathbf{x}-\mathbf{y};\lambda)(w_1w_2 + G_{12})(\mathbf{y}) + \frac{\partial\mathcal{O}_{12}}{\partial y_1}(\mathbf{x}-\mathbf{y};\lambda)(w_1^2 + G_{11})(\mathbf{y}) +\\ &+\frac{\partial\mathcal{O}_{22}}{\partial y_1}(\mathbf{x}-\mathbf{y};\lambda)(w_1w_2 + G_{21})(\mathbf{y}) + \frac{\partial\mathcal{O}_{22}}{\partial y_2}(\mathbf{x}-\mathbf{y};\lambda)(w_2^2 + G_{22})(\mathbf{y})\Big\}\mathrm{d}\mathbf{y}\,.\end{aligned}$$

Theorem 3.29 yields

$$\begin{aligned}&\|u_2;\eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le C\lambda^{-1}\Big\{\||w_1|^2 + |w_1w_2| +\\ &+|w_2|^2;\eta_{Bp}^{(2A-1)p}(\cdot\,;\lambda))\|_p + \||\mathbf{G};\eta_{Bp}^{(2A-1)p}(\cdot\,;\lambda))\|_p\Big\}\end{aligned} \tag{3.143}$$

and the conditions

$$\begin{gathered} -\tfrac{1}{2} \le Bp \le \tfrac{p-1}{2} \\ p-2 < (2A+B)p < 3p-2 \\ 0 \le (2A-B)p < 2p\,. \end{gathered} \tag{3.144}$$

We now proceed as above and get for A, B satisfying (3.141) and

$$A \le 1 \tag{3.145}$$

that

$$\begin{gathered} \|u_2; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le C\lambda^{-1}(\|w_1; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p^2 + \\ + \|w_2; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p^2 + \|\nabla \mathbf{w}; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p^2 + \\ + \|\mathbf{G}; \eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p)\,, \end{gathered}$$

i.e. for the right hand side $\mathbf{G}$ and ε sufficiently small

$$\|u_2; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p \le \frac{1}{3}\varepsilon\,. \tag{3.146}$$

Finally we use the integral representation for the gradient of $\mathbf{u}$. We apply Corollary 3.6 to get

$$\begin{gathered} \|\nabla \mathbf{u}; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le \\ \le C(\||w_1|^2 + |w_1 w_2| + |w_2|^2; \eta_{Bp}^{(2A-1)p}(\cdot\,;\lambda)\|_p + \|\mathbf{G}; \eta_{Bp}^{(2A-1)p}(\cdot\,;\lambda)\|_p) \end{gathered}$$

under the assumptions (3.144) together with

$$-\frac{1}{2} < Bp < \frac{p-1}{2}\,. \tag{3.147}$$

As above, we easily verify that

$$\|\nabla \mathbf{u}; \eta_{Bp}^{(2A-1-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \le \frac{1}{3}\varepsilon\,. \tag{3.148}$$

Collecting (3.142), (3.146) and (3.148) we get the desired estimate. The conditions (3.140), (3.141) (3.144), (3.145) and (3.147) furnish the restrictions on A and B. The condition $2 - \frac{2}{p} > \frac{3}{4}$ implies $p > \frac{8}{5}$. As $\eta_B^A(\mathbf{x}) \ge 1$ for A, B non–negative, we can take $\delta > 0$, arbitrary.

□

The situation in three space dimensions is somewhat easier as there is no difference between the first and the other components of $\mathbf{u}$.

Theorem 3.33 *Let $\mathbf{G} \in L^p(\mathbb{R}^3; \eta_{Bp}^{Ap}(\cdot)) \cap L^q(\mathbb{R}^3)$, $p > \frac{3}{2}$, $\frac{3}{2} \le q \le 2$ with the norms sufficiently small. Let $0 \le B < 1 - \frac{1}{p}$, $1 \le A < \min(3 - \frac{3}{p} - B, 1 + \frac{1}{p} + B)$. Then the solution to (3.134) from Theorem 3.31 has the following asymptotic properties:*

$$\begin{gathered} \mathbf{u} \in L^p(\mathbb{R}^3; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot)) \\ \nabla \mathbf{u} \in L^p(\mathbb{R}^3; \eta_{Bp}^{(A-\frac{\delta}{p})p}(\cdot)) \end{gathered}$$

for any $\delta > 0$.

Proof: From $(3.138)_1$ and from Theorem 3.21 we get

$$\|\mathbf{u}; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p \leq C\lambda^{-1}\||\mathbf{w}|^2; \eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p + C(\lambda)\|\mathbf{G}; \eta_{Bp}^{Ap}(\cdot\,;1)\|_p$$

for

$$\begin{gathered} -\tfrac{3}{2} < Bp < \tfrac{3}{2}(p-1) \\ -\tfrac{5}{2}p + 2 < (A-B)p < 1 + p \\ -\tfrac{7}{2} + \tfrac{p}{2} < (A+B)p < \tfrac{7}{2}p - \tfrac{7}{2} \\ -p - \tfrac{1}{2} < Ap < 2p - \tfrac{1}{2}\,. \end{gathered} \tag{3.149}$$

We proceed as in the twodimensional case and get

$$\||\mathbf{w}|^2; \eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p \leq \|\mathbf{w}; \eta_{2Bp}^{(2A-1)p}(\cdot\,;\lambda)\|_{2p}^2$$

provided

$$A \geq 1\,, \qquad B \geq 0\,. \tag{3.150}$$

Then for $\delta < \frac{p}{2}$

$$\begin{aligned} \||\mathbf{w}|^2; \eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p &\leq C(\|\mathbf{w}; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p^2 + \|\nabla\mathbf{w}; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p^2) \leq \\ &\leq C(\|\mathbf{w}; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p^2 + \|\nabla\mathbf{w}; \eta_{Bp}^{(A-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p^2)\,. \end{aligned}$$

Combining this with the assumptions on the smallness of the right hand side we get

$$\|\mathbf{u}; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p \leq \tfrac{1}{2}\varepsilon \tag{3.151}$$

provided $\varepsilon \geq \|\mathbf{w}; \eta_{Bp}^{(A-\frac{1}{2})p}(\cdot\,;\lambda)\|_p + \|\nabla\mathbf{w}; \eta_{Bp}^{(A-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p$ is small enough.

Next, for $(3.138)_2$ we get from Corollary 3.3

$$\|\nabla\mathbf{u}; \eta_{Bp}^{(A-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \leq C\lambda^{-1}\||\mathbf{w}|^2; \eta_{Bp}^{Ap}(\cdot\,;\lambda)\|_p + C(\lambda)\|\mathbf{G}; \eta_{Bp}^{Ap}(\cdot\,;1)\|_p$$

under the conditions

$$\begin{gathered} -1 < Bp < p - 1 \\ -2p + 1 < (A-B)p < p + 1 \\ -3 < (A+B)p < 3(p-1)\,. \end{gathered} \tag{3.152}$$

As above we can get

$$\|\nabla\mathbf{u}; \eta_{Bp}^{(A-\frac{\delta}{p})p}(\cdot\,;\lambda)\|_p \leq \tfrac{1}{2}\varepsilon \tag{3.153}$$

provided ε is sufficiently small. The estimates (3.151) and (3.153) prove the theorem. Collecting (3.149), (3.150) and (3.152) we get the restrictions on A, B. The condition $3 - \frac{3}{p} > 1$ implies $p > \frac{3}{2}$. Again, due to the properties of $\eta_B^A(\mathbf{x})$, δ can be taken arbitrarily large.

□

III

Modified Oseen problem

As was shown in Chapter I, the problem for the viscoelastic fluid (its elliptic part) can be rewritten as

$$
\begin{aligned}
-\Delta \mathbf{u} + \beta^2 \frac{\partial^2 \mathbf{u}}{\partial x_1^2} + \beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla \pi &= N(\mathbf{u}, \mathbf{T}(\mathbf{u}), p(\mathbf{u}, \pi), \mathbf{f}) \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
\tag{0.1}
$$

(see (I.4.18)), where the right hand side $N(\mathbf{u}, \mathbf{T}(\mathbf{u}), p(\mathbf{u}, \pi), \mathbf{f})$ contains terms which are either nonlinear or contain the external force. In Chapter II we gave a detailed theory to the Oseen problem i.e. to the problem (II.0.1). The problem (0.1) differs from the Oseen problem due to the presence of the term $\beta^2 \frac{\partial^2 \mathbf{u}}{\partial x_1^2}$. Evidently, in order to have the operator

$$
A(\mathbf{u}) = -\Delta \mathbf{u} + \beta^2 \frac{\partial^2 \mathbf{u}}{\partial x_1^2} \tag{0.2}
$$

elliptic, we need to assume $\beta < 1$. Evidently, it would be easier if we could put the term $\beta^2 \frac{\partial^2 \mathbf{u}}{\partial x_1}$ on the right hand side of (0.1) and use the fact that it is small (β will be assumed small). In such a way we could get existence of solution in Sobolev spaces (compare also with an analogical situation for the second–grade fluid, see (I.4.28)–(I.4.29)). Nevertheless, as follows from Corollaries II.3.3, II.3.6, in such a case we would not be able to show the asymptotic structure of the solution as we loose ε in the weight. Therefore this chapter is devoted to the detailed study of the following problem

$$
\begin{aligned}
A(\mathbf{u}) + \beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla p &= \mathbf{f} \\
\nabla \cdot \mathbf{u} &= 0
\end{aligned}
\tag{0.3}
$$

with

$$
A(\mathbf{u}) = -(1-\mu) \frac{\partial^2 \mathbf{u}}{\partial x_1^2} - \frac{\partial^2 \mathbf{u}}{\partial x_2^2} - \cdots - \frac{\partial^2 \mathbf{u}}{\partial x_N^2}, \qquad 0 \le \mu < 1
$$

which we shall call the modified Oseen problem. (For $\mu = 0$ we get the (classical) Oseen problem which can be therefore considered as a special case). We shall first study the fundamental solution to (0.3) and show that it can be divided into two parts — one which is the fundamental Oseen tensor and the other one, which has at least the same asymptotic properties. We shall be therefore, in particular, allowed to use the L^q–weighted theory developed in Section II.3.

Next we shall study L^q–estimates to (0.3). We proof several results which can be regarded as analogues to the results given for the classical Oseen problem in Section II.2. Moreover, as the classical Oseen problem can be treated as a special case of (0.3) with $\mu = 0$, we in fact proof also Lemmas II.2.1–II.2.6.

III.1 Fundamental solution

This section is devoted to the study of the fundamental solution to (0.3). Let us consider in $\mathbb{R}^N$

$$
\begin{gathered}
-\Big[\Delta - \mu\frac{\partial^2}{\partial y_1^2} + 2\lambda\frac{\partial}{\partial y_1}\Big]\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};2\lambda) - \frac{\partial}{\partial y_i}e_j(\mathbf{x}-\mathbf{y}) = \delta_{ij}\delta_{\mathbf{x}} \\
\frac{\partial \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};2\lambda)}{\partial y_i} = 0\,,
\end{gathered}
\tag{1.1}
$$

where the derivatives are taken in the sense of distributions and $\delta_{\mathbf{x}}$ denotes the Dirac δ–distribution supported at $\mathbf{x}$.

We search $\boldsymbol{\mathcal{O}}^\mu$ in the form

$$
\boldsymbol{\mathcal{O}}^\mu(\mathbf{x}-\mathbf{y};2\lambda) = \boldsymbol{\mathcal{O}}(\mathbf{x}-\mathbf{y};2\lambda) + \mathbf{E}^\mu(\mathbf{x}-\mathbf{y};2\lambda)\,, \tag{1.2}
$$

where $\mathcal{O}_{ij}(\mathbf{x}-\mathbf{y};2\lambda)$ is the classical fundamental Oseen tensor (see Chapter II) and

$$
\Big[\Delta - \mu\frac{\partial^2}{\partial y_1^2} + 2\lambda\frac{\partial}{\partial y_1}\Big]E^\mu_{ij} = \mu\frac{\partial^2}{\partial y_1^2}\mathcal{O}_{ij} \tag{1.3}
$$

in the sense of distributions. We also easily see that the "fundamental pressure" e_j is exactly what we obtained in Section II.1, i.e.

$$
e_j(\mathbf{x}-\mathbf{y}) = -\frac{\partial}{\partial y_j}\mathcal{E}(|\mathbf{x}-\mathbf{y}|) \tag{1.4}
$$

(see (II.1.8)).

Let us start to study the solution to the problem (1.3). Recall that the second derivatives of the fundamental Oseen tensor behaves like second derivatives of fundamental solution to the Laplace operator. We must add to (1.3) the assumption that

$$
\frac{\partial E^\mu_{ij}}{\partial y_i} = 0 \tag{1.5}
$$

in the sense of distributions. We cannot require it directly as such problem does not have, in general, solution. But we shall verify later on that (1.5) is satisfied. The advantage of the problem (1.3), which we shall call the Oseen problem without pressure, is that we may use a change of variables in such a way that $A(\cdot)$ becomes the laplacian and we may study (1.3) via the same lines as the classical Oseen problem. This was impossible directly in (0.3) because of the pressure.

We can easily verify that for

$$
\begin{aligned}
Y_1 &= \frac{y_1}{\sqrt{1-\mu}} & X_1 &= \frac{x_1}{\sqrt{1-\mu}} \\
Y_j &= y_j \quad j=2,\cdots,N & X_j &= x_j \qquad j=2,\cdots,N
\end{aligned}
\tag{1.6}
$$

we have from (1.3)

$$
\Big(\Delta_{\mathbf{Y}} + \frac{2\lambda}{\sqrt{1-\mu}}\frac{\partial}{\partial Y_1}\Big)E^\mu_{ij}(\mathbf{X}-\mathbf{Y};2\lambda) = \frac{\mu}{1-\mu}\frac{\partial^2}{\partial Y_1^2}\mathcal{O}_{ij}(\mathbf{X}-\mathbf{Y};2\lambda)\,.
$$

We denote $\overline{\lambda} = \frac{\lambda}{\sqrt{1-\mu}}$ and finally get

$$\Big(\Delta_{\mathbf{Y}} + 2\overline{\lambda}\frac{\partial}{\partial Y_1}\Big)E^{\mu}_{ij}(\mathbf{X}-\mathbf{Y};2\lambda) = \frac{\mu}{1-\mu}\frac{\partial^2}{\partial Y_1^2}O_{ij}(\mathbf{X}-\mathbf{Y};2\lambda)\,. \tag{1.7}$$

Before starting to study the problem (1.7) let us shortly mention the change of variables (1.6). We easily see that

$$\sqrt{1-\mu}|\mathbf{X}| \le |\mathbf{x}| \le |\mathbf{X}|\,. \tag{1.8}$$

Moreover, as a consequence of Lemma II.3.1, we get ($s(\mathbf{X}) = |\mathbf{X}| - X_1$)

Corollary 1.1 *Let* $X_1 \ge 0$. *Then*

$$\frac{\sqrt{1-\mu}}{2}s(\mathbf{x}) \le s(\mathbf{X}) \le 2s(\mathbf{x})\,. \tag{1.9}$$

Let $X_1 < 0$. *Then*

$$\frac{1}{2}s(\mathbf{x}) \le s(\mathbf{X}) \le \frac{2}{\sqrt{1-\mu}}s(\mathbf{x})\,. \tag{1.10}$$

Proof: We proceed as in Lemma II.3.1 and get for $x_1 \ge 0$

$$s(\mathbf{x}) = \frac{|\mathbf{x}'|^2}{|\mathbf{x}|}\frac{1}{1+\cos\theta}\,, \qquad \text{where } \mathbf{x}' = (x_2,\cdots,x_N)\,,\, \theta \in \Big[-\frac{\pi}{2};\frac{\pi}{2}\Big] \text{ for } x_1 \ge 0\,.$$

As $|\mathbf{x}'| = |\mathbf{X}'|$, we have

$$s(\mathbf{x}) \le \frac{|\mathbf{x}'|^2}{|\mathbf{x}|} \le \frac{|\mathbf{X}'|^2}{|\mathbf{X}|}\frac{1}{\sqrt{1-\mu}} \le \frac{|\mathbf{X}'|^2}{|\mathbf{X}|}\frac{1}{1+\cos\widetilde{\theta}}\frac{2}{\sqrt{1-\mu}} = \frac{2}{\sqrt{1-\mu}}s(\mathbf{X})\,.$$

Analogously we get the other inequality in (1.9). For $x_1 < 0$ we use the evident fact that

$$|\mathbf{x}| \le s(\mathbf{x}) \le 2|\mathbf{x}|$$

and apply (1.8).

□

From (1.8)–(1.10) it follows that whatever we get for the asymptotic properties in the variables $\mathbf{X}$, the same holds also for the variables $\mathbf{x}$ and therefore, although we shall calculate the behaviour in the new variables $\mathbf{X}$, $\mathbf{Y}$, we can use finally the result for the original variables $\mathbf{x}$, $\mathbf{y}$. We come back to the problem (1.7). We have

Lemma 1.1 *The fundamental solution to the Oseen problem without pressure (1.7) is*

$$E^*(\mathbf{X}-\mathbf{Y};2\overline{\lambda}) = -\frac{1}{N-1}\sum_{i=1}^{N}\mathcal{O}_{ii}(\mathbf{X}-\mathbf{Y};2\overline{\lambda})\,, \tag{1.11}$$

where $\boldsymbol{\mathcal{O}}(\mathbf{X}-\mathbf{Y};2\overline{\lambda})$ *is the fundamental Oseen tensor.*

Proof: We proceed as in Section II.1. We search singular solution to

$$\Big(\Delta + 2\overline{\lambda}\frac{\partial}{\partial Y_1}\Big)E^*(\mathbf{X}-\mathbf{Y}) = \delta_{\mathbf{X}} = \Delta\mathcal{E}(|\mathbf{X}-\mathbf{Y}|)\,.$$

We suppose E^* in the form

$$E^*(\mathbf{X}-\mathbf{Y}) = \Delta\Phi(\mathbf{X}-\mathbf{Y})$$

and so

$$\Big(\Delta + 2\overline{\lambda}\frac{\partial}{\partial Y_1}\Big)\Delta\Phi(\mathbf{X}-\mathbf{Y}) = \Delta\mathcal{E}(|\mathbf{X}-\mathbf{Y}|)\,. \tag{1.12}$$

We see that we get the same equation for Φ as (II.1.4). So we calculate Φ in the same way as we did in case of the Oseen problem. Finally from (II.1.1) we see that $\Delta\Phi = \frac{-1}{N-1}\sum_{i=1}^N \mathcal{O}_{ii}$. The proof is finished.

□

Before constructing the solution to (1.7) we first show several integrability properties of the fundamental Oseen tensor which are a straightforward consequence of its asymptotic structure.

Lemma 1.2
a) Let $N=2$. Then[1]

(i) $\mathcal{O}_{11} \in L^p(\mathbb{R}^2)$ *for* $p \in (3;\infty)$

(ii) $\mathcal{O}_{ij} \in L^p(\mathbb{R}^2)$ *for* $p \in (2;\infty)$, $i+j \geq 3$

(iii) $\frac{\partial \mathcal{O}_{11}}{\partial y_2} \in L^p(\mathbb{R}^2)$ *for* $p \in (\frac{3}{2};2)$

(iv) $\nabla\boldsymbol{\mathcal{O}}$ *except of* $\frac{\mathcal{O}_{11}}{\partial y_2}$, *and regular parts of the second gradients of* $\boldsymbol{\mathcal{O}} \in L^p(\mathbb{R}^2)$ *for* $p \in (1;2)$.

b) Let $N=3$. Then

(i) $\mathcal{O}_{ij} \in L^p(\mathbb{R}^3)$ *for* $p \in (2;3)$, $i,j = 1,2,3$

(ii) $\frac{\partial \mathcal{O}_{ij}}{\partial y_k} \in L^p(\mathbb{R}^3)$ *for* $p \in (\frac{4}{3};\frac{3}{2})$, $k = 2,3$, $i,j = 1,2,3$

(iii) $\frac{\partial \mathcal{O}_{ij}}{\partial y_1}$ *and regular parts of the second gradients of* $\boldsymbol{\mathcal{O}} \in L^p(\mathbb{R}^3)$ *for* $p \in (1;\frac{3}{2})$.

Proof: It is a direct consequence of the asymptotic properties established in Section II.1 and Lemma II.3.2.

□

Corollary 1.2 *We have for* $N=2$ *that* $E^* \in L^p(\mathbb{R}^2)$ *for* $p \in (3;\infty)$ *and* $\frac{\partial}{\partial Y_1}E^* \in L^p(\mathbb{R}^2)$ *for* $p \in (1;2)$. *For* $N=3$ $E^* \in L^p(\mathbb{R}^3)$ *for* $p \in (2;3)$ *and* $\frac{\partial}{\partial Y_1}E^* \in L^p(\mathbb{R}^2)$ *for* $p \in (1;\frac{3}{2})$.

[1] under the regular part of $\nabla^2\boldsymbol{\mathcal{O}}$ we understand $\nabla^2\boldsymbol{\mathcal{O}} - \nabla^2\boldsymbol{\mathcal{S}}$

Remark 1.1 It is possible to show that $E^* \sim \mathrm{e}^{-\lambda s} r^{-\frac{N-1}{2}}$ for r sufficiently large. This follows for $N=2$ easily from (II.1.25) and for $N=3$ either from (II.1.37) or, easier, from (II.1.34) recalling that

$$E^* = -\frac{1}{N-1}\Delta\Phi = -\frac{1}{N-1}\frac{1}{8\pi\lambda}\Delta\int_0^{\lambda(|\mathbf{x}-\mathbf{y}|+y_1-x_1)}\frac{1-\mathrm{e}^{-\tau}}{\tau}\mathrm{d}\tau\,.$$

Nevertheless, even with this asymptotic behaviour at infinity we can only show the integrability proved in Corollary 1.2. In what follows we shall only use that $E^* \sim |\boldsymbol{\mathcal{O}}|$ and we shall not use this better properties in s — it is sufficient for our purposes.

We can now proof the following

Theorem 1.1 *The solution to (1.7) can be expressed as*

$$E_{ij}^{\mu}(\mathbf{Z};2\overline{\lambda}) = \frac{\mu}{1-\mu}\int_{\mathbb{R}^N}\frac{\partial}{\partial Z_1}E^*(\mathbf{Z}-\mathbf{Y};2\overline{\lambda})\frac{\partial}{\partial Y_1}\mathcal{O}_{ij}(\mathbf{Y};2\lambda)\,d\mathbf{Y}\,, \tag{1.13}$$

where the convolution can be understood in the usual notion of the L^p–spaces.

Proof: First, let us check that the convolution in (1.13) is well defined. We have that both $\frac{\partial}{\partial Z_1}E^*$ and $\frac{\partial}{\partial Y_1}\boldsymbol{\mathcal{O}}$ belong to $L^p(\mathbb{R}^N)$ for $p\in(1,2)$, $N=2$ and $p\in(1;\frac{3}{2})$, $N=3$. Applying the Young inequality (see Theorem VIII.2.1) we get that the convolution belongs to the $L^p(\mathbb{R}^N)$ for $p\in(1,\infty)$ if $N=2$ and for $p\in(1,3)$ if $N=3$. We have therefore to verify that E^{μ}_{ij} defined by (1.13) satisfy (1.7) in the sense of distributions.

The convolution (1.13) is well defined in $\mathcal{S}'$. Using the definition of convolution in $\mathcal{S}'$ we have for $\varphi\in\mathcal{S}(\mathbb{R}^N)$ (the brackets denotes the duality between $\mathcal{S}$ and $\mathcal{S}'$, $\mathbf{E}\times\mathbf{F}$ the direct product in $\mathcal{S}'$, $\eta_k(\mathbf{X},\mathbf{Y})\to 1$ in $C(\mathbb{R}^{2N})$ — see Section VIII.4)

$$\begin{aligned}
\Big\langle E^{\mu}_{ij},\Big(\Delta+2\overline{\lambda}\frac{\partial}{\partial X_1}\Big)\varphi\Big\rangle &= \lim_{k\to\infty}\frac{\mu}{1-\mu}\Big\langle\frac{\partial}{\partial X_1}E^*(\mathbf{X};2\overline{\lambda})\times\frac{\partial}{\partial Y_1}\mathcal{O}_{ij}(\mathbf{Y};2\lambda),\\
&\qquad\Big(\Delta+2\overline{\lambda}\frac{\partial}{\partial X_1}\Big)\varphi(\mathbf{X}+\mathbf{Y})\eta_k(\mathbf{X},\mathbf{Y})\Big\rangle =\\
&= \lim_{k\to\infty}\frac{\mu}{1-\mu}\Big\langle\frac{\partial}{\partial X_1}E^*(\mathbf{X};2\overline{\lambda})\times\frac{\partial}{\partial Y_1}\mathcal{O}_{ij}(\mathbf{Y};2\lambda),\\
&-2\frac{\partial}{\partial X_i}\varphi(\mathbf{X}+\mathbf{Y})\frac{\partial}{\partial X_i}\eta_k(\mathbf{X},\mathbf{Y})-2\overline{\lambda}\varphi(\mathbf{X}+\mathbf{Y})\frac{\partial}{\partial X_1}\eta_k(\mathbf{X},\mathbf{Y})\Big\rangle+\\
&+\lim_{k\to\infty}\frac{\mu}{1-\mu}\Big\langle\mathcal{O}_{ij}(\mathbf{Y};2\lambda)\times E^*(\mathbf{X};2\overline{\lambda}),\\
&\qquad\frac{\partial^2}{\partial X_1^2}\Big(\Delta+2\overline{\lambda}\frac{\partial}{\partial X_1}\Big)(\varphi(\mathbf{X}+\mathbf{Y})\eta_k(\mathbf{X},\mathbf{Y}))\Big\rangle =\\
&= \lim_{k\to\infty}\Big\langle\mathcal{O}_{ij}(\mathbf{Y};2\lambda),\frac{\partial^2}{\partial X_1^2}(\varphi(\mathbf{X}+\mathbf{Y})\eta_k(\mathbf{X},\mathbf{Y}))\Big|_{X=\mathbf{0}}\Big\rangle =\\
&= \Big\langle\frac{\mu}{1-\mu}O_{ij}(\mathbf{Y};2\lambda),\frac{\partial^2\varphi(\mathbf{Y})}{\partial Y_1^2}\Big\rangle\,,
\end{aligned}$$

where we used the fact that the convolution $\frac{\partial}{\partial X_1}E^* * \frac{\partial}{\partial Y_1}\mathcal{O}_{ij}$ exists. The theorem is shown.

□

III.1.1 Asymptotic properties of the fundamental solution

Having the fundamental solution for the modified Oseen problem expressed by (1.2) and (1.13), we can start to study its asymptotic properties. Our aim is to show that the decay at infinity is at least the same as for the fundamental Oseen tensor. Secondly, we want to show that E^{μ}_{ij} and its gradient are locally integrable while the second gradient has one part which is regular and the other part which represents a singular integral operator and can be studied (in our case) by means of the Marcinkiewicz multiplier theorem (see Theorem II.3.2).

We start to study the decay at infinity and as usually, we proceed separately for $N = 2$ and $N = 3$. Moreover, we put $2\overline{\lambda} = 1$ and finally show an analogue to the homogeneity property of $\mathcal{O}_{ij}$.

Lemma 1.3 *Let $N = 2$. Then we have for $k \geq 0\,, \varepsilon > 0$ arbitrarily small, and $|\mathbf{X}| \geq R \gg 1$:*

$$\nabla^k E^{\mu}_{ij}(\mathbf{X}; 1) \leq C|\mathbf{X}|^{-\frac{3+k-\varepsilon}{2}}(1+s(\mathbf{X}))^{-\frac{1+k}{2}}\,. \tag{1.14}$$

Proof: Let $|\mathbf{X}| \gg 1$. We divide the convolution (1.13) into three parts:

$$\begin{aligned} E^{\mu}_{ij}(\mathbf{X},1) = \frac{\mu}{1-\mu}\Big(&\int_{B_1(\mathbf{0})} \frac{\partial}{\partial X_1}E^*(\mathbf{X}-\mathbf{Y};1)\frac{\partial}{\partial Y_1}O_{ij}(\mathbf{Y};\sqrt{1-\mu})\ \mathrm{d}\mathbf{Y} + \\ &+ \int_{B_1(\mathbf{X})} \frac{\partial}{\partial X_1}E^*(\mathbf{X}-\mathbf{Y};1)\frac{\partial}{\partial Y_1}O_{ij}(\mathbf{Y};\sqrt{1-\mu})\ \mathrm{d}\mathbf{Y} + \\ &+ \int_{\mathbb{R}^2\setminus B_1(\mathbf{0})\setminus B_1(\mathbf{X})} \frac{\partial}{\partial X_1}E^*(\mathbf{X}-\mathbf{Y};1)\frac{\partial}{\partial Y_1}O_{ij}(\mathbf{Y};\sqrt{1-\mu})\ \mathrm{d}\mathbf{Y}\Big) \\ &\equiv I_1 + I_2 + I_3\,. \end{aligned} \tag{1.15}$$

We have easily ($E^* \sim \mathcal{O}_{11}$ at infinity)[2]

$$|I_1| \leq C\int_{B_1(\mathbf{0})} \frac{1}{|\mathbf{X}-\mathbf{Y}|^{\frac{3}{2}}(1+s(\mathbf{X}-\mathbf{Y}))^{\frac{1}{2}}}\frac{1}{|\mathbf{Y}|}\ \mathrm{d}\mathbf{Y} \leq \frac{C}{|\mathbf{X}|^{\frac{3}{2}}(1+s(\mathbf{X}))^{\frac{1}{2}}}$$

for $|\mathbf{X}|$ sufficiently large.

$$\begin{aligned} |I_2| &\leq C\int_{B_1(\mathbf{X})} \frac{1}{|\mathbf{X}-\mathbf{Y}|}\frac{1}{|\mathbf{Y}|^{\frac{3}{2}}}\frac{1}{(1+s(\mathbf{Y}))^{\frac{1}{2}}}\ \mathrm{d}\mathbf{Y} = \\ &= C\int_{B_1(\mathbf{0})} \frac{1}{|\mathbf{Z}|}\frac{1}{|\mathbf{X}-\mathbf{Z}|^{\frac{3}{2}}}\frac{1}{(1+s(\mathbf{X}-\mathbf{Z}))^{\frac{1}{2}}}\ \mathrm{d}\mathbf{Z} \leq \frac{C}{|\mathbf{X}|^{\frac{3}{2}}(1+s(\mathbf{X}))^{\frac{1}{2}}} \end{aligned}$$

for $|\mathbf{X}|$ sufficiently large.

Finally, the third part can be estimated as

$$|I_3| \leq C\int_{\mathbb{R}^2} \eta^{-\frac{3}{2}}_{-\frac{1}{2}}(\mathbf{X}-\mathbf{Y})\eta^{-\frac{3}{2}}_{-\frac{1}{2}}(\mathbf{Y})\ \mathrm{d}\mathbf{Y}$$

and applying Theorem II.3.18 (or, equivalently, Tab.3,4 from Chapter II)

$$|I_3| \leq C\eta^{-\frac{3}{2}+\varepsilon}_{-\frac{1}{2}}(\mathbf{X})\,, \varepsilon > 0,\ \text{arbitrarily small.}$$

[2] see also Remark 1.1

Summarizing the estimates for I_1, I_2 and I_3 we have

$$|E^{\mu}_{ij}(\mathbf{X};1)| \le \frac{C}{|\mathbf{X}|^{\frac{3}{2}-\varepsilon}(1+s(\mathbf{X}))^{\frac{1}{2}}}$$

for $|\mathbf{X}|$ sufficiently large, i.e (1.14) for $k = 0$. Next we start to estimate the derivatives of $\mathbf{E}^{\mu}$. We use again (1.15) and take the derivative with respect to X_i, $i = 1,2$. We get

$$\Big|\frac{\partial I_1}{\partial X_i}\Big| \le C\int_{B_1(\mathbf{0})}\Big|\frac{\partial}{\partial X_i}\frac{\partial}{\partial X_1}\mathbf{E}^*(\mathbf{X}-\mathbf{Y};1)\Big|\Big|\frac{\partial}{\partial Y_1}O_{ij}(\mathbf{Y};\sqrt{1-\mu})\Big|\,\mathrm{d}\mathbf{Y} \le$$
$$\le C\int_{B_1(\mathbf{0})}\frac{1}{|\mathbf{X}-\mathbf{Y}|^2}\frac{1}{(1+s(\mathbf{X}-\mathbf{Y}))}\frac{1}{|\mathbf{Y}|}\,\mathrm{d}\mathbf{Y} \le \frac{C}{|\mathbf{X}|^2(1+s(\mathbf{X}))}$$

for $|\mathbf{X}|$ sufficiently large.

Next, in I_2 we first change variables and get

$$\Big|\frac{\partial I_2}{\partial X_i}\Big| \le C\Big|\frac{\partial}{\partial X_i}\int_{B_1(\mathbf{0})}\frac{\partial}{\partial Z_1}\mathbf{E}^*(\mathbf{Z};1)\frac{\partial}{\partial Z_1}O_{ij}(\mathbf{X}-\mathbf{Z};\sqrt{1-\mu})\,\mathrm{d}\mathbf{Z}\Big| \le$$
$$\le C\int_{B_1(\mathbf{0})}\Big|\frac{\partial}{\partial Z_1}\mathbf{E}^*(\mathbf{Z};1)\Big|\Big|\frac{\partial^2}{\partial X_i\partial X_1}O_{ij}(\mathbf{X}-\mathbf{Z};\sqrt{1-\mu})\Big|\,\mathrm{d}\mathbf{Z} \le$$
$$\le \frac{C}{|\mathbf{X}|^2(1+s(\mathbf{X}))}$$

for $|\mathbf{X}|$ sufficiently large.

Before starting to estimate $|\frac{\partial}{\partial X_i}I_3|$, let us have a look on Tab.3,4 in Section II.3. It is clear that in order to get $e\ \frac{1}{2}$ greater, it is not sufficient to take either a or $c\ \frac{1}{2}$ greater. Nevertheless, we can proceed as follows. In domains $\Omega_0, \Omega_2, \Omega_6, \Omega_8$ and Ω_{11} (and the corresponding situations in $\Omega_4, \Omega_9, \Omega_{10}$ and Ω_{14-15}) we simply take the derivative of I_3 — we therefore increase c and d. Otherwise, we first change the variables as in the case of I_2 and therefore we increase a and b. So we get exactly what we want, i.e.

$$\Big|\frac{\partial I_3}{\partial X_i}\Big| \le \frac{C}{|\mathbf{X}|^{2-\varepsilon}(1+s(\mathbf{X}))^1},$$

where the ε–loss comes from some logarithmic terms. For $k > 1$ we can proceed similarly. The theorem is proved.

□

In the threedimensional case we have

Lemma 1.4 *Let $N = 3$. Then we have for $k \ge 0, \varepsilon > 0$ arbitrarily small, and $|\mathbf{X}| \ge R \gg 1$*

$$\nabla^k E^{\mu}_{ij}(\mathbf{X};1) \le C|\mathbf{X}|^{-\frac{4+k-\varepsilon}{2}}(1+s(\mathbf{X}))^{-\frac{2+k}{2}}\,. \tag{1.16}$$

Proof: It is more or less the same as in the twodimensional case. We only use the fact that $\frac{\partial}{\partial X_1}E^*(\mathbf{X};1) \sim \frac{\partial}{\partial X_1}O_{ij}(\mathbf{X};1) \sim |\mathbf{X}|^{-2}(1+s(\mathbf{X}))^{-1}$ for $|\mathbf{X}| \gg 1$, instead of Theorem II.3.18 we apply Theorem II.3.11 and use Tab.1,2 instead of Tab.3,4.

□

If we compare (1.15) and (1.16) with the asymptotic behaviour of $\mathcal{O}_{ij}$, we see that up to the first derivatives we have better behaviour of E^{μ}_{ij} than those of the fundamental Oseen tensor. For the second and higher derivatives we have still better behaviour than the worst terms from the higher gradients of the fundamental Oseen tensor.

The technique used in Lemmas 1.3 and 1.4 is not able to capture the different structure for the derivatives with respect to X_1 and with respect to X_j, $j \geq 2$. Using the fact that E^* behaves exponentially in $s(\mathbf{X})$, wo could establish also this faster decay for derivatives with respect to X_1. Nevertheless, we do not need it.

The next question is the behaviour for $|\mathbf{X}|$ small. The crucial problem is whether we can again divide the second gradient of E^{μ}_{ij} into two parts — one which is regular and the other one, which can be treated by theorems from Subsection II.3.1.

We shall take the advantage of the very easy structure of $\mathbf{E}^*(\mathbf{X})$ near zero. We have namely for $N = 2$

$$(\mathcal{S}_{11} + \mathcal{S}_{22})(\mathbf{X}) = \frac{1}{2\pi}\Big(1 + \log\frac{1}{|\mathbf{X}|}\Big)$$

and for $N = 3$

$$(\mathcal{S}_{11} + \mathcal{S}_{22} + \mathcal{S}_{33})(\mathbf{X}) = \frac{1}{2\pi}\frac{1}{|\mathbf{X}|}\,.$$

Therefore we have (see II.1.20 and II.1.37)

a) $N = 2$

$$\frac{\partial \mathbf{E}^*}{\partial X_1} = \frac{1}{2\pi}\frac{X_1}{|\mathbf{X}|^2} + \nu^*(|\mathbf{X}|) \tag{1.17}$$

with

$$\left.\begin{aligned} \nu(R) &= O(\ln R) \\ \frac{\partial}{\partial X_i}\nu(R) &= O\Big(\frac{1}{R}\Big) \end{aligned}\right\}\ \text{for } R = |\mathbf{X}| \to 0^+$$

b) $N = 3$

$$\frac{\partial E^*}{\partial X_1} = \frac{1}{4\pi}\frac{X_1}{|\mathbf{X}|^3} + \nu^*(|\mathbf{X}|) \tag{1.18}$$

with

$$\left.\begin{aligned} \nu(R) &= O\Big(\frac{1}{R}\Big) \\ \nu(R) &= O\Big(\frac{1}{R^2}\Big) \end{aligned}\right\}\ \text{for } R \to 0^+\,.$$

As we cannot treat directly the convolution $\frac{X_i}{|\mathbf{X}|^N} * \frac{\partial}{\partial y_1} S_{ij}(\mathbf{X}-\mathbf{Y})$ (more precisely, the derivative of this convolution), we proceed as follows. We denote by $g_N(\mathbf{X})$ function which is equal to $\frac{1}{2(N-1)\pi}\frac{X_i}{|\mathbf{X}|^N}$ on $B_3(\mathbf{0})$ and zero on $B^4(\mathbf{0})$ and is continuous and continuously differentiable up to certain order which will be specified later. This we adjust e.g. by taking

$$g_N(\mathbf{X}) = \frac{1}{2(N-1)\pi} \frac{X_i}{|\mathbf{X}|^N} P(|\mathbf{X}|)\,, \tag{1.19}$$

where $P(|\mathbf{X}|)$ is equal to 1 on $B_3(\mathbf{0})$, equal zero on $B^4(\mathbf{0})$ and is a polynomial for $3 \leq |\mathbf{X}| \leq 4$, satisfying certain continuity assumptions for $|\mathbf{X}| = 3$ and 4.

Now let $|\mathbf{X}| \leq 2$. We have

$$\begin{aligned} E^{\mu}_{ij}(\mathbf{X};1) &= \frac{\mu}{1-\mu}\Big(\int_{B_4(\mathbf{X})} g_N(\mathbf{X}-\mathbf{Y}) \frac{\partial \mathcal{S}_{ij}}{\partial Y_1}(\mathbf{Y})\mathrm{d}\mathbf{Y} + \\ &+ \int_{B_4(\mathbf{X})} g_N(\mathbf{X}-\mathbf{Y})\nu_{ij}(\mathbf{Y})\mathrm{d}\mathbf{Y} + \int_{B_3(\mathbf{X})} \nu^*(\mathbf{X}-\mathbf{Y})\frac{\partial \mathcal{O}_{ij}}{\partial Y_1}(\mathbf{Y})\mathrm{d}\mathbf{Y} + \\ &+ \int_{B^3(\mathbf{X})} \Big(\frac{\partial E^*}{\partial X_1} - g_N\Big)(\mathbf{X}-\mathbf{Y})\frac{\partial \mathcal{O}_{ij}}{\partial Y_1}(\mathbf{Y})\ \mathrm{d}\mathbf{Y}\Big)\,. \end{aligned} \tag{1.20}$$

We denote the integrals in (1.20) by $I_1 - I_4$ and estimate each of them separately. We need estimates up to the second order derivatives. The most crucial term is I_1; we leave it for a moment and start with the easier ones. The integral I_4 is very easy; as $\mathbf{0} \in B_2(\mathbf{X})$ we have no singularities and assuming $g_N \in C^2(\mathbb{R}^N \setminus \{\mathbf{0}\})$ (we shall require much more later, actually) we have easily

$$|D^\alpha I_4(\mathbf{X})| \leq C \qquad \text{for } |\mathbf{X}| \leq 2, \quad |\alpha| \leq 2. \tag{1.21}$$

Next we study the convolution I_3. Using (1.17) we have easily for $N = 2$

$$|I_3(\mathbf{X})| \leq C \int_{B_3(\mathbf{X})} \ln|\mathbf{X}-\mathbf{Y}| \frac{1}{|\mathbf{Y}|}\ \mathrm{d}\mathbf{Y} \leq C \tag{1.22}$$

and for $N = 3$, using (1.18) and Lemma II.3.11

$$|I_3(\mathbf{X})| \leq C \int_{B_3(\mathbf{X})} \frac{1}{|\mathbf{X}-\mathbf{Y}|} \frac{1}{|\mathbf{Y}|^2}\ \mathrm{d}\mathbf{Y} \leq C \ln|\mathbf{X}|\,. \tag{1.23}$$

We can easily verify that we may interchange integral and derivative. We therefore have for $N = 2$

$$\Big|\frac{\partial I_3(\mathbf{X})}{\partial X_i}\Big| \leq C \int_{B_3(\mathbf{X})} \frac{1}{|\mathbf{X}-\mathbf{Y}|} \frac{1}{|\mathbf{Y}|}\ \mathrm{d}\mathbf{Y}\,.$$

Applying Lemma II.3.11 we have

$$\Big|\frac{\partial I_3(\mathbf{X})}{\partial X_i}\Big| \leq C \ln|\mathbf{X}| \qquad \text{for } |\mathbf{X}| \neq 0\,. \tag{1.24}$$

Analogously, using Lemma II.3.11 we get also for $N = 3$ that

$$\Big|\frac{\partial I_3(\mathbf{X})}{\partial X_i}\Big| \leq \frac{C}{|\mathbf{X}|}\,. \tag{1.25}$$

But we cannot calculate the second derivative as we cannot interchange the integral and the derivative in this case. Nevertheless, we can look at the second derivative of $I_3(\mathbf{X})$ as on the first derivative of I_1 and therefore use the result for $\frac{\partial I_1}{\partial X_i}$ which will be obtained later.

In order estimate I_2, we can use the change of variables

$$|I_2(\mathbf{X})| = \int_{B_4(\mathbf{0})} g(\mathbf{Z})\nu_{ij}(\mathbf{X}-\mathbf{Z})\mathrm{d}\mathbf{Z}$$

and proceed analogously as in the estimate of $I_3(\mathbf{X})$.

We are left with the most difficult term $I_1(\mathbf{X})$. First we may easily verify that analogously as in the case of $\frac{\partial I_2}{\partial X_i}$, we have for $0 < |\mathbf{X}| \leq 2$

$$\begin{aligned} |I_1(\mathbf{X})| &\leq C \ln |\mathbf{X}|\,, \quad N=2 \\ |I_1(\mathbf{X})| &\leq \frac{C}{|\mathbf{X}|}\,, \quad N=3\,. \end{aligned} \tag{1.26}$$

Nevertheless, similarly as for the second derivatives of $I_3(\mathbf{X})$, we cannot interchange the derivative and the integral. Let us extend $g_N(\mathbf{X})$ by 0 outside $B_4(\mathbf{0})$. We may rewrite $I_1(\mathbf{X})$ as

$$I_1(\mathbf{X}) = C \int_{\mathbb{R}^N} g_N(\mathbf{X}-\mathbf{Y}) \frac{\partial S_{ij}}{\partial Y_1}(\mathbf{Y})\mathrm{d}\mathbf{Y}\,. \tag{1.27}$$

As $I_1(\mathbf{X}) \in L^1_{loc}(\mathbb{R}^N)$, we have $I_1(\mathbf{X}) \in \mathcal{S}'(\mathbb{R}^N)$. We can therefore calculate the Fourier transform of $I_1(\mathbf{X})$ in the sense of $\mathcal{S}'(\mathbb{R}^N)$. Moreover, as g has compact support, we have in $\mathcal{S}'$ (see Lemma VIII.4.12)

$$\mathcal{F}(I_1)(\xi) = (2\pi)^{\frac{N}{2}} \mathcal{F}(g_N)(\xi) \mathcal{F}\Big(\frac{\partial \mathcal{S}_{ij}}{\partial Y_1}\Big)(\xi)\,. \tag{1.28}$$

Using Lemma VIII.4.14

$$\mathcal{F}\Big(\frac{\partial \mathcal{S}_{ij}}{\partial Y_1}\Big)(\xi) = -(2\pi)^{-\frac{N}{2}} \frac{\delta_{ij}|\xi|^2 - \xi_i\xi_j}{|\xi|^4} i\xi_1\,,$$

i.e. $\mathcal{F}\Big(\frac{\partial \mathcal{S}_{ij}}{\partial Y_1}\Big) \in L^1_{loc}(\mathbb{R}^N)$. Moreover,

$$\mathcal{F}\Big(\frac{\partial \mathcal{S}_{ij}}{\partial Y_1}\Big) \in C^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$$

and

$$\Big|D^\alpha \mathcal{F}\Big(\frac{\partial \mathcal{S}_{ij}}{\partial Y_1}\Big)\Big| \leq \frac{C}{|\xi|^{|\alpha|+1}}$$

on $\mathbb{R}^N \setminus \{\mathbf{0}\}$.

As $g_N(\mathbf{X})$ has compact support and is bounded, we can calculate its Fourier transform directly.

We choose the polynomial $P(|\mathbf{Z}|)$ in such a way that

$$\begin{aligned} &P(3) = 1 \\ &P(4) = 0 \\ &D^k P(3) = D^k P(4) = 0 \qquad k = 1, 2 \ldots n\,,\, n \in \mathbb{N}\,. \end{aligned} \tag{1.29}$$

We have therefore $2n+2$ conditions — there exists exactly one polynomial with $\deg(2n+1)$ satisfying (1.29). In fact, for $N=3$ we shall require one more condition which will be precised later on. We first calculate the Fourier transform of $g_N(\mathbf{X})$ in two space dimensions. We have

$$\mathcal{F}(g_2)(\xi) = C\Big(\int_{B_3(\mathbf{0})} \frac{Z_1}{|\mathbf{Z}|^2} e^{i(\mathbf{Z},\xi)}\,\mathrm{d}\mathbf{Z} + \int_{B_4(\mathbf{0})\setminus B_3(\mathbf{0})} \frac{Z_1}{|\mathbf{Z}|^2} P(|\mathbf{Z}|) e^{i(\mathbf{Z},\xi)}\,\mathrm{d}\mathbf{Z}\Big)\,.$$

It is more convenient to work in the polar coordinates; we need in fact some estimates of $\mathcal{F}(g_2)(\xi)$ in terms of $|\xi|$. Denoting $|\xi|=s$, $|\mathbf{Z}|=R$, φ the angle in Z–coordinates and θ the angle in ξ–coordinates we get

$$\begin{aligned}\mathcal{F}(g_2)(s,\theta) = C\int_0^3\int_0^{2\pi} \cos\varphi e^{iRs(\cos\varphi\cos\theta+\sin\varphi\sin\theta)}\,\mathrm{d}\varphi\mathrm{d}R+\\ +\int_3^4\int_0^{2\pi}\cos\varphi P(R) e^{iRs(\cos\varphi\cos\theta+\sin\varphi\sin\theta)}\,\mathrm{d}\varphi\mathrm{d}R=\\ = C\Big(\int_0^3\int_0^{2\pi}\cos\varphi e^{iRs\cos(\varphi-\theta)}\,\mathrm{d}\varphi\mathrm{d}R + \int_3^4\int_0^{2\pi}\cos\varphi P(R)e^{iRs\cos(\varphi-\theta)}\,\mathrm{d}\varphi\mathrm{d}R\Big)\,.\end{aligned}$$

Moreover, as $\cos\varphi = \cos(\varphi-\theta)\cos\theta - \sin(\varphi-\theta)\sin\theta$ and

$$\int_0^{2\pi}\sin(\varphi-\theta)e^{iRs\cos(\varphi-\theta)}\,\mathrm{d}\varphi = 0\,,$$

we get (the integrals are evidently independent of θ)

$$\begin{aligned}\mathcal{F}(g_2)(s,\theta) = C\cos\theta\Big(\int_0^3\int_0^{2\pi}\cos\varphi e^{iRs\cos\varphi}\,\mathrm{d}\varphi\mathrm{d}R+\\ +\int_3^4\int_0^{2\pi}\cos\varphi P(R)e^{iRs\cos\varphi}\,d\varphi\,\mathrm{d}R\Big)\,.\end{aligned} \tag{1.30}$$

We shall need some estimates of $\mathcal{F}(g_2)(s,\theta)$ and of its derivatives in terms of s. We easily see from (1.30) that

$$|\mathcal{F}(g_2)(s,\theta)| \le C$$

and ($\frac{\partial}{\partial\xi_1}\cos\varphi = \frac{1}{s}\sin^2\theta$, $\frac{\partial}{\partial\xi_2}\cos\varphi = -\frac{1}{s}\sin\theta\cos\theta$)

$$|\nabla^k_\xi\mathcal{F}(g_2)(s,\theta)| \le C\Big(\frac{1}{s^k}+1\Big)\,,\quad k\in\mathbb{N}\,. \tag{1.31}$$

We shall prove a bit sharper estimates, namely for $s\in\mathbb{R}^+$ we show that[3]

$$|\nabla^k_\xi\mathcal{F}(g_2)| \le \frac{C}{(1+s)s^k} = \frac{C}{(1+|\xi|)|\xi|^k}\,,\qquad k\ \le n \tag{1.32}$$

[3] We have from (1.30) that

$$\mathcal{F}(g_2)(\xi) = \frac{\xi_1}{|\xi|}G(|\xi|)\,.$$

Therefore (1.32) follows if we show that

$$\Big|\frac{d^kG(s)}{ds^k}\Big| \le \frac{C}{(1+s)s^k}\,.$$

(see (1.29) for the definition of n). Using Fubini's theorem in the first integral of (1.30) and integrating by parts in the second one we have

$$\mathcal{F}(g_2)(s,\theta) = C\cos\varphi\Big(\int_0^{2\pi}\frac{1}{is}(e^{3is\cos\varphi}-1)\,\mathrm{d}\varphi+ \\ +\frac{1}{is}\int_0^{2\pi}\Big[P(R)e^{iRs\cos\varphi}\Big]_3^4\mathrm{d}\varphi-\frac{1}{is}\int_3^4\int_0^{2\pi}P'(R)e^{iRs\cos\varphi}\,\mathrm{d}\varphi\mathrm{d}R\Big)\,.$$

Recalling that $P(4)=0$ and $P(3)=1$ we finally get

$$\mathcal{F}(g_2)(s,\theta) = i\frac{C\cos\varphi}{s}\Big[2\pi+\int_3^4\int_0^{2\pi}P'(R)e^{iRs\cos\varphi}\,d\varphi\mathrm{d}R\Big]\,.$$

Combining this with the fact that $|g_1(s,\theta)|\le C$ for $s\to 0^+$ we have

$$|\mathcal{F}(g_2)(s,\theta)| \le \frac{C}{1+s}\,. \tag{1.33}$$

To show (1.32) for $k\le n$, we combine (1.31) with the result of the following lemma.

Lemma 1.5 *We have for* $0\le k\le n$

$$\frac{\partial^k\mathcal{F}(g_2)}{\partial s^k}(s,\theta) = \cos\theta\Big(\frac{C_k^1}{s^{k+1}}+ \\ +\sum_{j=1}^{k+1}\frac{C_k^j}{s^{k+1}}\int_3^4\int_0^{2\pi}\overbrace{(\dots(P'(R)R)'\dots R)'}^{j\text{ derivatives}}e^{iRs\cos\varphi}\,d\varphi dR\Big)\,, \tag{1.34}$$

where C_k^j *are constants depending only on* j *and* k.

Proof: We have shown (1.34) for $k=0$. We proceed by induction. Let (1.34) hold for some $k\in\mathbb{N}_0^+, k\ \le n$. Then

$$\frac{\partial}{\partial s}\Big(\frac{\partial^k\mathcal{F}(g_2)}{\partial s^k}\Big) = \cos\theta\Big[\frac{\partial}{\partial s}\Big(\frac{C_k^1}{s^{k+1}}\Big)+\sum_{j=1}^{k+1}\frac{\partial}{\partial s}\Big(\frac{C_k^j}{s^{k+1}}\Big)\cdot \\ \cdot\int_3^4\int_0^{2\pi}(\dots(P'(R)\cdot R)'\dots R)'e^{iRs\cos\varphi}\,d\varphi\mathrm{d}R+ \\ +\sum_{j=1}^{k+1}\frac{C_k^j}{s^{k+1}}\int_3^4\int_0^{2\pi}iR\cos\varphi(\dots(P'(R)\cdot R)'\dots R)'e^{iRs\cos\varphi}\,d\varphi\mathrm{d}R\Big]\,.$$

We have to calculate the last integrals. We get

$$\int_3^4\int_0^{2\pi}iR\cos\varphi\,\overbrace{(\dots(P'(R)\cdot R\dots R)'}^{j\text{ derivatives}}e^{iRs\cos\varphi}\,\mathrm{d}\varphi\mathrm{d}R = \\ =\frac{1}{s}\int_0^{2\pi}\Big[(\dots(P'(R)\cdot R)'\dots R)'Re^{iRs\cos\varphi}\,d\varphi\Big]_3^4\mathrm{d}\varphi- \\ -\frac{1}{s}\int_0^{2\pi}\int_3^4\underbrace{(\dots(P'(R)\cdot R)'\dots R)'}_{j+1\text{ derivatives}}e^{iRs\cos\varphi}\,\mathrm{d}\varphi\mathrm{d}R\,.$$

We therefore only need to check that the boundary terms disappear. As $k+1 \le n$ and $P^{(l)}(3) = P^{(l)}(4) = 0\ l = 1, 2 \dots, n$, the proof is complete.

□

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Remark 1.2 The conditions (1.29) say that our extension of $\frac{X_1}{|\mathbf{X}|^N}$ is of the class $C^n(\mathbb{R}^N \setminus \{\mathbf{0}\})$.

We next start to calculate the Fourier transform of $g_3(\mathbf{X})$. Before doing this let us observe that $\frac{X_1}{|\mathbf{X}|^3} = -\frac{\partial}{\partial X_1}(\frac{1}{|\mathbf{X}|})$. Therefore

$$\begin{aligned}\mathcal{F}(g_3)(\xi) = C\Big(\int_{B_3(\mathbf{0})} \frac{Z_1}{|\mathbf{Z}|^3} e^{i(\mathbf{Z},\xi)}\,\mathrm{d}\mathbf{Z} + \int_{B_4(\mathbf{0})\setminus B_3(\mathbf{0})} \frac{Z_1}{|\mathbf{Z}|^3} P(|\mathbf{Z}|) e^{i(\mathbf{Z},\xi)}\,\mathrm{d}\mathbf{Z}\Big) = \\ = C\Big(\int_{B_3(\mathbf{0})} \frac{1}{|\mathbf{Z}|} i\xi_1 e^{i(\mathbf{Z}\cdot\xi)}\,\mathrm{d}\mathbf{Z} - \int_{\partial B_3(\mathbf{0})} \frac{1}{|\mathbf{Z}|} e^{i(\mathbf{Z},\xi)} n_1(\mathbf{Z})\,\mathrm{d}S + \\ + \int_{B_4(\mathbf{0})\setminus B_3(\mathbf{0})} \frac{1}{|\mathbf{Z}|}\Big(P'(|\mathbf{Z}|)\frac{Z_1}{|\mathbf{Z}|} + P(|\mathbf{Z}|) i\xi_1\Big) e^{i(\mathbf{Z}\cdot\xi)}\,\mathrm{d}\mathbf{Z} + \\ + \int_{\partial B_3(\mathbf{0})} P(|\mathbf{Z}|) e^{i(\mathbf{Z},\xi)} n_1(\mathbf{Z})\mathrm{d}S - \int_{\partial B_4(\mathbf{0})} \frac{1}{|\mathbf{Z}|} P(|\mathbf{Z}|) e^{i(\mathbf{Z},\xi)} n_1(\mathbf{Z})\,\mathrm{d}S\Big)\end{aligned}$$

Using the assumptions on $P(|\mathbf{Z}|)$ we have

$$\begin{aligned}\mathcal{F}(g_3)(\xi) = iC\xi_1\left(\int_{B_3(\mathbf{0})} \frac{\mathrm{e}^{i(\mathbf{Z},\xi)}}{|\mathbf{Z}|}\mathrm{d}\mathbf{Z} + \int_{B_4(\mathbf{0})\setminus B_3(\mathbf{0})} \frac{\mathrm{e}^{i(\mathbf{Z},\xi)}}{|\mathbf{Z}|} P(|\mathbf{Z}|)\mathrm{d}\mathbf{Z}\right) \\ + C\int_{B_4(\mathbf{0})\setminus B_3(\mathbf{0})} \frac{\mathrm{e}^{i(\mathbf{Z},\xi)}}{|\mathbf{Z}|} P'(|\mathbf{Z}|)\frac{Z_1}{|\mathbf{Z}|}\mathrm{d}\mathbf{Z}\,.\end{aligned}$$

It is clear that there exists $Q(R)$ such that $Q'(R) = \frac{P'(R)}{R}$. We require also $Q(4) = Q(3) = 0$. This gives us one more condition on $P(R)$ (the other one can be justified by a proper choice of a constant). We can always find a polynomial $P(R)$ with deg $2n+2$ such that the above mentioned conditions will be satisfied.

Integrating by parts in the last integral we get

$$\mathcal{F}(g_3)(\xi) = iC\xi_1\Big(\int_{B_4(\mathbf{0})} \frac{\mathrm{e}^{i(\mathbf{Z},\xi)}}{|\mathbf{Z}|} P(|\mathbf{Z}|)\mathrm{d}\mathbf{Z} + \int_{B_4(\mathbf{0})\setminus B_3(\mathbf{0})} \mathrm{e}^{i(\mathbf{Z},\xi)} Q(|\mathbf{Z}|)\mathrm{d}\mathbf{Z}\Big) \quad (1.35)$$

The two integrals in (1.35) represent (up to a multiplicative constant) the Fourier transform of a radially symmetric function. It is well known (see Lemma VIII.4.7) that the Fourier transform of such a function is again radially symmetric and we have in spherical coordinates

$$\mathcal{F}(g_3)(s,\varphi,\theta) = C\cos\varphi\sin\theta F(s) \quad (1.36)$$

We come back to the calculation of $\mathcal{F}(g_3)$. Similarly as in two space dimensions, we change variables and use the spherical ones. We have

$$\begin{aligned}\mathcal{F}(g_3)(s,\varphi,\theta) = C\Big(\int_0^3\int_0^\pi\int_0^{2\pi} \cos\psi\sin^2\kappa\cdot \\ \cdot\mathrm{e}^{iRs(\cos\kappa\cos\theta + \sin\kappa\sin\theta(\cos\varphi\cos\psi + \sin\varphi\sin\psi))}\mathrm{d}\psi\mathrm{d}\kappa\mathrm{d}R + \\ + \int_3^4\int_0^\pi\int_0^{2\pi} \cos\psi\sin^2\kappa P(R)\cdot \\ \cdot\mathrm{e}^{iRs(\cos\kappa\cos\theta + \sin\kappa\sin\theta(\cos\varphi\cos\psi + \sin\varphi\sin\psi))}\mathrm{d}\psi\mathrm{d}\kappa\mathrm{d}R\Big)\,.\end{aligned}$$

We now apply (1.36); without loss of generality we may choose $\theta = \frac{\pi}{2}$, $\varphi = 0$ and get

$$\mathcal{F}(g_3)(s,0,\frac{\pi}{2}) = \mathcal{F}(g)(s) = C\Big(\int_0^3\int_0^\pi\int_0^{2\pi}\cos\psi\sin^2\kappa \mathrm{e}^{iRs\sin\kappa\cos\psi}\mathrm{d}\psi\mathrm{d}\kappa\mathrm{d}R+ \\ +\int_3^4\int_0^\pi\int_0^{2\pi}\cos\psi\sin^2\kappa P(R)\mathrm{e}^{iRs\sin\kappa\cos\psi}\mathrm{d}\psi\mathrm{d}\kappa\mathrm{d}R\Big)\,.$$

As in the twodimensional case we have

$$\frac{d^k\mathcal{F}(g)(s)}{ds^k} \le C\,, \quad s\ \in [0,\infty)\,, \quad k\ = 0,1,\ldots$$

and therefore, as $\mathcal{F}(g_3)(\xi) = \frac{\xi_1}{|\xi|}F(|\xi|)$, F bounded including all derivatives,

$$\frac{\partial^\alpha\mathcal{F}(g_3)(\xi)}{\partial\xi^\alpha} \le C\Big(1+\frac{1}{|\xi|^{|\alpha|}}\Big)\,, \quad |\alpha|\ge 0\,. \tag{1.37}$$

We shall prove a stronger result. We have

$$\mathcal{F}(g)(s) = \frac{C}{is}\Big[\int_0^\pi\int_0^{2\pi}\sin\kappa(\mathrm{e}^{3iR\sin\kappa\cos\psi}-1)\mathrm{d}\psi\mathrm{d}\kappa+ \\ +\int_0^\pi\int_0^{2\pi}\sin\kappa[P(R)\mathrm{e}^{iRs\sin\kappa\cos\psi}]_3^4\mathrm{d}\psi\mathrm{d}\kappa- \\ -\int_3^4\int_0^\pi\int_0^{2\pi}\sin\kappa P'(R)\mathrm{e}^{iRs\sin\kappa\cos\psi}\mathrm{d}\psi\mathrm{d}\kappa\mathrm{d}R\Big]\,.$$

Using (1.29) we get finally

$$\mathcal{F}(g)(s) = \frac{iC}{s}\left[4\pi+\int_3^4\int_0^\pi\int_0^{2\pi}\sin\kappa P'(R)\mathrm{e}^{iRs\sin\kappa\cos\psi}\mathrm{d}\psi\mathrm{d}\kappa\mathrm{d}R\right]$$

and therefore

$$\mathcal{F}(g_3)(s,\varphi,\theta) \le \frac{C}{1+s}\,. \tag{1.38}$$

Now, repeating mutatis mutandis all steps of the proof of Lemma 1.5 we finally get

$$|D^\alpha\mathcal{F}(g_3)(\xi)| \le \frac{C}{(1+|\xi|)|\xi|^{|\alpha|}}\,, \quad |\alpha|\le n\,. \tag{1.39}$$

Now we can come back to the study of (1.28). We know that

$$\mathcal{F}\Big(\frac{\partial\mathcal{S}_{ij}}{\partial x_1}\Big)(\xi) \in L^1_{loc}(\mathbb{R}^N)$$

and

$$\mathcal{F}(g_N)(\xi) \in L^\infty(\mathbb{R}^N)\,, \quad N = 2,3\,.$$

Therefore we easily verify that (1.28), which was written in the sense of $\mathcal{S}'$, is in fact a regular tempered distribution. We have namely for $\varphi\in\mathcal{S}$

$$\Big\langle\mathcal{F}\Big(\frac{\partial\mathcal{S}_{ij}}{\partial x_1}\Big)\mathcal{F}(g_N),\varphi\Big\rangle = \Big\langle\mathcal{F}\Big(\frac{\partial\mathcal{S}_{ij}}{\partial x_1}\Big),\mathcal{F}(g_N)\varphi\Big\rangle = \\ = \int_{\mathbb{R}^N}\mathcal{F}\Big(\frac{\partial\mathcal{S}_{ij}}{\partial x_1}\Big)\mathcal{F}(g_N)\varphi\mathrm{d}\xi \tag{1.40}$$

as $\mathcal{F}\Big(\frac{\partial \mathcal{S}_{ij}}{\partial x_1}\Big) \in L^1_{loc}(\mathbb{R}^N)$, $\mathcal{F}(g_N)\varphi \in L^\infty(\mathbb{R}^N)$ and decays sufficiently fast at infinity. We can therefore define a new functional on $\mathcal{S}$ by (1.40). Moreover, it is an easy matter to see that

$$\mathcal{F}\Big(\frac{\partial \mathcal{S}_{ij}}{\partial x_1}\Big)\mathcal{F}(g_N) \in L^q(\mathbb{R}^N) \quad \text{for } q \in \Big(\frac{N}{2};N\Big)\,.$$

We shall now reconstruct the decay properties of $I_1(\mathbf{x})$ due to its Fourier transform. In two dimensions we have the following statement:[4]

Lemma 1.6 *Let $\mathcal{F}(G) \in C^{m+1}(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ be such that*

$$A \;=\; \sup_{\substack{|\alpha|\le m+1\\ \xi\in\mathbb{R}^2}} |\xi|^{|\alpha|+1}(1+|\xi|)|D^\alpha \mathcal{F}(G)(\xi)| < \infty\,.$$

Then $G \in C^m(\mathbb{R}^2 \setminus \{\mathbf{0}\})$ and

$$\sup_{\substack{0<|\beta|\le m\\ \mathbf{x}\in\mathbb{R}^2}} |\mathbf{x}|^\beta |D^\beta G(\mathbf{x})| \le C(m)A$$
$$\sup_{\mathbf{x}\in\mathbb{R}^2} \Big|\ln\Big(\frac{1}{|\mathbf{x}|}+1\Big)\Big|^{-1} G(\mathbf{x}) \le CA\,.$$

Proof: We fix $\Phi \in C_0^\infty(\mathbb{R}^2)$ such that $\Phi(\xi) = 1$ for $|\xi| \le \frac{1}{2}$ and $\Phi(\xi) = 0$ for $|\xi| \ge 1$. For $\beta \in \mathbb{N}^2$ we put for $\lambda > 0$

$$\mathcal{F}(v_{\lambda,\beta}) = (i\xi)^\beta \mathcal{F}(G)(\xi)\Phi(\lambda\xi)$$
$$\mathcal{F}(w_{\lambda,\beta}) = (i\xi)^\beta \mathcal{F}(G)(\xi)(1-\Phi(\lambda\xi))$$

and we easily see that $\mathcal{F}(v_{\lambda,\beta}+w_{\lambda,\beta}) = (i\xi)^\beta\mathcal{F}(G)(\xi) = \mathcal{F}(D^\beta G)(\xi)$.[5] Moreover, we have $\mathcal{F}(v_{\lambda,\beta}) \in L^1(\mathbb{R}^2)$ and therefore

$$|v_{\lambda,\beta}| \le C\int_{|\xi|\le\lambda^{-1}} |\xi|^\beta|\xi|^{-1}(1+|\xi|)^{-1}\mathrm{d}\xi \le C\int_0^{\lambda^{-1}} \frac{r^{|\beta|}}{1+r}\mathrm{d}r\,.$$

If $|\beta| = 0$ then $|v_{\lambda,0}| \le C\ln(1+\frac{1}{\lambda})$, otherwise $v_{\lambda,\beta} \le C\lambda^{-|\beta|}$. Now let us take $p \in \mathbb{N}^2$ such that $|p| = |\beta|+1$. We shall show that $\mathbf{x}^p w_{\lambda,\beta}$ is a continuous function which tends to 0 as $|\mathbf{x}| \to \infty$, satisfying

$$|\mathbf{x}^p w_{\lambda,\beta}(\mathbf{x})| \le CA\;\lambda^{-|\beta|+|p|}\,.$$

The function $\mathcal{F}(w_{\lambda,\beta})$ has its support outside a ball with diameter $\frac{1}{\lambda}$. We have easily that $D^p\mathcal{F}(w_{\lambda,\beta}) \in L^1(\mathbb{R}^2)$ as it is a combination of

$$\xi^{\beta-q}\frac{\partial^{|\alpha|}\hat{G}(\xi)}{\partial\xi^\alpha}\frac{\partial^{|r|}}{\partial\xi^r}(1-\Phi(\lambda\xi))\,,$$

[4] see e.g. [Bou] for a similar result, but under slightly different assumptions
[5] the derivative is taken in the sense of $\mathcal{S}'$

where $\alpha + q + r = p$, $q \le \beta$. If $r \neq \mathbf{0}$ then its support lies between two balls with diameters $\frac{1}{2\lambda}$ and $\frac{1}{\lambda}$. Therefore

$$\begin{aligned} C\lambda^{|r|} \int_{(2\lambda)^{-1} \le |\xi| \le \lambda^{-1}} |\xi|^{|\beta|-|q|} |D^\alpha \mathcal{F}(G)(\xi)| \mathrm{d}\xi \le \\ \le CA\ \lambda^{|r|} \int_{(2\lambda)^{-1} \le |\xi| \le \lambda^{-1}} |\xi|^{|\beta|-|q|-|\alpha|-2} \mathrm{d}\xi \le \\ \le CA\ \lambda^{|r|+|q|+|\alpha|-|\beta|} \le CA\ \lambda^{|p|-|\beta|} . \end{aligned}$$

If $r = \mathbf{0}$ then

$$\begin{aligned} \int_{(2\lambda)^{-1} \le |\xi|} |\xi|^{|\beta|-|q|-|\alpha|-2} \mathrm{d}\xi \le CA \int_{(2\lambda)^{-1} \le |\xi|} |\xi|^{-3} \mathrm{d}\xi \le \\ \le CA\ |\lambda|^{|p|-|\beta|} . \end{aligned}$$

Now, setting $\lambda = |\mathbf{x}|$ we get for $|\mathbf{x}| \neq 0$

$$G(\mathbf{x}) = v_{\lambda,0} + w_{\lambda,0} \le C \ln\Big(\frac{1}{|\mathbf{x}|} + 1\Big) ,$$

while for $|\beta| > 0$

$$D^\beta G(\mathbf{x}) = v_{\lambda,\beta} + w_{\lambda,\beta} \le C|\mathbf{x}|^{-|\beta|} .$$

As $|p| \le m + 1$, we get $|\beta| \le m$.

□

In particular we have shown that $I_1(\mathbf{X}) \sim \ln(\mathbf{X})$ for $|\mathbf{X}|$ small; that we have already known. Moreover, we also obtained that for $0 < |\beta| \le 2$ we have outside $\mathbf{X} = \mathbf{0}$

$$D^\beta I_1(\mathbf{X}) \sim \frac{1}{|\mathbf{X}|^\beta} .$$

For $|\beta| = 1$ we have easily that the derivative in the sense of distribution and the classical derivative coincides[6]. We have therefore that the integral operator

$$\int_{\mathbb{R}^2} \frac{\partial I_1}{\partial X_i} (\mathbf{X} - \mathbf{Y}) f(\mathbf{Y}) \mathrm{d}\mathbf{Y}$$

is well defined on $C_0^\infty(\mathbb{R}^2)$ and can be (eventually) extended due to the density argument onto some $L^q(\mathbb{R}^2)$.

Finally let us consider the integral operator

$$Tf(\mathbf{X}) = \int_{\mathbb{R}^2} I_1(\mathbf{X} - \mathbf{Y}) \frac{\partial^2 f(\mathbf{Y})}{\partial Y_i \partial Y_j} \mathrm{d}\mathbf{Y}$$

for $f \in C_0^\infty(\mathbb{R}^2)$. Evidently, we have that

$$\mathcal{F}(Tf)(\xi) = (2\pi)^{\frac{N}{2}} \mathcal{F}(I_1)(\xi)(-\xi_i \xi_j) \mathcal{F}(f) ,$$

[6] their difference is supported at $\mathbf{0}$ and (see Lemma VIII.4.1) they may differ only up to the δ–distribution or its derivative; but the Fourier transform of $D^\beta I_1$ tends to zero as $|\xi| \to \infty$

where the multiplication is to be understood in the sense of $\mathcal{S}'$. Nevertheless, thanks to the properties of $\mathcal{F}(I_1)(\xi)$ it is an easy matter to see that

$$\mathcal{F}(Tf)(\xi) = -(2\pi)^{\frac{N}{2}}\xi_i\xi_j\mathcal{F}(I_1)(\xi)\mathcal{F}(f)(\xi) = m(\xi)\mathcal{F}(f)(\xi)\,,$$

where $m(\xi) \in L^\infty(\mathbb{R}^2)$. Moreover, we have for $|\alpha| \le n$ (see (1.29) for the definition of n) that

$$D^\alpha m(\xi) \le \frac{C}{|\xi|^{|\alpha|}} \text{ for } |\xi| \neq 0\,.$$

Therefore, assuming $n = 2$ for $N = 2$, we have in fact shown that $m(\xi)$ is a L^p–multiplier, $1 < p < \infty$ (see Theorem II.3.2) and $m \in M(a,2)$ for $a \in (1,\infty)$ (see Definition II.3.2).

We can therefore summarize the decay properties of $E^\mu_{ij}(\mathbf{X};1)$ for $|\mathbf{X}|$ small in two space dimensions. (In order to get the required information about the second derivative of $I_1(\mathbf{X})$, we have to take $n = 3$ in (1.29).)

Lemma 1.7 *Let $N = 2$. Then for $0 < |\alpha| \le 2$ we have for $|\mathbf{X}| \le 2$*

$$\begin{aligned} E^\mu_{ij}(\mathbf{X};1) &\le C\ln|\mathbf{X}| \\ D^\alpha E^\mu_{ij}(\mathbf{X};1) &\le C|\mathbf{X}|^{-|\alpha|}\,. \end{aligned}$$

Moreover,

$$E^\mu_{ij}(\mathbf{X};1) = I_1(\mathbf{X}) + I_2(\mathbf{X})\,,$$

where $D^\alpha I_2(\mathbf{X}) \le \frac{C}{|\mathbf{X}|}$ for $|\mathbf{X}| \le 2$, $|\alpha| = 2$ and $I_1(\mathbf{X})$, representing the singular part of the second gradient of $E^\mu_{ij}(\mathbf{X};1)$, has the following property:

$$\mathcal{F}\Big(\int_{\mathbb{R}^2} I_1(\cdot - \mathbf{Y})D^\alpha f(\mathbf{Y})\,d\mathbf{Y}\Big)(\xi) = m(\xi)\mathcal{F}(f)(\xi)\,, \quad |\alpha| = 2\,,$$

where $m(\xi)$ represents the L^p–Fourier multiplier, $1 < p < \infty$. Therefore the integral operator T,

$$Tf(\mathbf{X}) = \int_{\mathbb{R}^2} I_1(\mathbf{X} - \mathbf{Y})D^\alpha f(\mathbf{Y})\,d\mathbf{Y}$$

maps $C^\infty_0(\mathbb{R}^2)$ onto $L^p(\mathbb{R}^2)$, $1 < p < \infty$ and

$$\begin{aligned} \|Tf\|_{p,\mathbb{R}^2} &\le C\|f\|_{p,\mathbb{R}^2} \\ \|Tf\|_{p,(g),\mathbb{R}^2} &\le C\|f\|_{p,(g),\mathbb{R}^2} \end{aligned}$$

for all g, weights from the Muckenhoupt class A_p.

Next we continue with the threedimensional case. We have the following lemma which forms an analogue to Lemma 1.6; we prove it in N space dimensions, $N \ge 3$.

Lemma 1.8 *Let $\mathcal{F}(G) \in C^{m+N-1}(\mathbb{R}^N \setminus \{\mathbf{0}\})$, $N \geq 3$ be such that*

$$A = \sup_{\substack{|\alpha| \leq m+N-1 \\ \xi \in \mathbb{R}^N}} |\xi|^{|\alpha|+N-1}(1+|\xi|)|D^\alpha \mathcal{F}(G)(\xi)| < \infty .$$

Then $G \in C^m(\mathbb{R}^N \setminus \{\mathbf{0}\})$ and

$$\sup_{\substack{0 \leq |\beta| \leq m \\ \mathbf{x} \in \mathbb{R}^2}} |\mathbf{x}|^{|\beta|+N-2}|D^\beta G(\mathbf{x})| \leq C(m,N)A .$$

Proof: It is essentially the same as for $N = 2$. For $v_{\lambda,\beta}$ we get

$$|v_{\lambda,\beta}| \leq C \int_0^{\lambda^{-1}} \frac{r^{|\beta|+N-2}}{1+r} \mathrm{d}r$$

and therefore, unlike the twodimensional case, no logarithmic factor appears. In the estimates for $w_{\lambda,\beta}$, we have to take $|p| = |\beta| + N - 1$ in order to justify the integrability of

$$\int_{(2\lambda)^{-1} \leq |\xi|} |\xi|^{|\beta|-p-2} \mathrm{d}\xi .$$

□

Now we can proceed exactly as before Lemma 1.7 in order to verify the assumptions of the Marcinkiewicz multiplier theorem. Let us only remark two things. Unlike the twodimensional case we have different assumptions in Theorem II.3.2 (the Marcinkiewicz multiplier theorem) and Theorem II.3.5 (the weighted estimates via Kurtz and Wheeden). We therefore have to take $n = 3$ (see (1.29)) in order to have $m(\xi) \in M(a, 3)$ while for the multiplier theorem it is enough to have $n = 2$. The other remark concerns the behaviour of the type $\frac{1}{(1+|\xi|)|\xi|}$ for $\mathcal{F}(I_1)(\xi)$. This type of behaviour was essential only in two dimensions in order to have $\mathcal{F}(I_1)(\xi)$ locally integrable. For $N = 3$ (and eventually $N > 3$) it is enough to have behaviour of the type $\frac{1}{|\xi|^2}$.

We can now summarize the properties of $E^\mu_{ij}(\mathbf{X};1)$ in three space dimensions. Let us also mention that in order to get the required information about the second gradient of $I_1(\mathbf{X})$, we need to take $n = 4$ in (1.29).

Lemma 1.9 *Let $N = 3$. Then for $0 \leq |\alpha| \leq 2$ we have for $|\mathbf{X}| \leq 2$*

$$D^\alpha E^\mu_{ij}(\mathbf{X};1) \leq C|\mathbf{X}|^{-|\alpha|} .$$

Moreover,

$$E^\mu_{ij}(\mathbf{X};1) = I_1(\mathbf{X}) + I_2(\mathbf{X}) ,$$

where $D^\alpha I_2(\mathbf{X}) \leq \frac{C}{|\mathbf{X}|^2}$ for $|\mathbf{X}| \leq 2$, $|\alpha| = 2$ and $I_1(\mathbf{X})$, representing the singular part of the second gradient of $E^\mu_{ij}(\mathbf{X};1)$, has the following property:

$$\mathcal{F}\Big(\int_{\mathbb{R}^3} I_1(\cdot - \mathbf{Y}) D^\alpha f(\mathbf{Y}) d\mathbf{Y}\Big)(\xi) = m(\xi)\mathcal{F}(f)(\xi) , \quad (|\alpha| = 2) ,$$

where $m(\xi)$ represents the L^p–Fourier multiplier, $1 < p < \infty$. Therefore the integral operator T,

$$Tf(\mathbf{X}) = \int_{\mathbb{R}^3} I_1(\mathbf{X} - \mathbf{Y}) D^\alpha f(\mathbf{Y}) d\mathbf{Y}$$

maps $C_0^\infty(\mathbb{R}^3)$ onto $L^p(\mathbb{R}^3)$, $1 < p < \infty$ and

$$\|Tf\|_{p,\mathbb{R}^3} \le C\|f\|_{p,\mathbb{R}^3}$$
$$\|Tf\|_{p,(g),\mathbb{R}^3} \le C\|f\|_{p,(g),\mathbb{R}^3}$$

for all g, weights from the Muckenhoupt class A_p.

In order to show that $\boldsymbol{\mathcal{O}}^\mu = \boldsymbol{\mathcal{O}} + \mathbf{E}^\mu$ represents the fundamental solution to (1.1) we have to verify that

$$\frac{\partial E^\mu_{ij}}{\partial y_i} = 0$$

in the sense of distributions. Before doing this we have to proof the following lemma; we know (see Lemma VIII.4.12) that

$$\mathcal{F}(g * h) = (2\pi)^{\frac{N}{2}} \mathcal{F}(g)\mathcal{F}(h)$$

is well defined (in $\mathcal{S}'$) when e.g. $g \in \mathcal{S}'$ and $h \in \mathcal{D}'$ with compact support. We shall extend the result for g and h belonging to certain Lebesgue spaces. See also e.g. [St] for a similar kind of result result, nevertheless, not applicable in our situation.

Lemma 1.10 *Let $g \in L^p(\mathbb{R}^N)$, $h \in L^q(\mathbb{R}^N)$, p, $q \in [1; 2]$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \ge \frac{3}{2}$. Then $g*h \in L^r(\mathbb{R}^N)$ and $\mathcal{F}(g*h) = (2\pi)^{\frac{N}{2}}\mathcal{F}(g)\mathcal{F}(h)$, where the multiplication can be understood either in $\mathcal{S}'$ or in the a.e. sense.*

Proof: We have clearly $g * h \in L^r(\mathbb{R}^N)$ from the Young inequality (see Theorem VIII.2.1). Moreover, as $r \le 2$, we have from the Hausdorff–Young inequality (see Lemma VIII.4.10), $r' = \frac{r}{r-1}$

$$\|\mathcal{F}(g * h)\|_{r'} \le C\|g * h\|_r \le C\|g\|_p\|h\|_q \,.$$

Now, let $h_n \in C_0^\infty(\mathbb{R}^N)$, $h_n \to h$ in $L^q(\mathbb{R}^N)$. Then

$$\mathcal{F}(g * h_n) = (2\pi)^{\frac{N}{2}} \mathcal{F}(g)\mathcal{F}(h_n) \tag{1.41}$$

as $g \in \mathcal{S}'$ and $h_n \in C_0^\infty(\mathbb{R}^N) \subset \mathcal{D}'(\mathbb{R}^N)$ has a compact support. The product in (1.41) is well defined in both $\mathcal{S}'$ and the a.e. sense. Moreover, we have

$$\|\mathcal{F}(g * h_n) - \mathcal{F}(g * h)\|_{r'} = \|\mathcal{F}(g * (h_n - h))\|_{r'} \le C\|g\|_p\|h_n - h\|_q \to 0$$

as $n \to \infty$. It means that $\mathcal{F}(g * h_n) \to \mathcal{F}(g * h)$ in $L^{r'}(\mathbb{R}^N)$; $\frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}$. Moreover

$$\|\mathcal{F}(g)(\mathcal{F}(h_n) - \mathcal{F}(h))\|_{r'} \le \|\mathcal{F}(g)\|_{p'}\|\mathcal{F}(h_n) - \mathcal{F}(h)\|_{q'} \le C\|g\|_p\|h_n - h\|_q \to 0$$

i.e. $\mathcal{F}(g)\mathcal{F}(h_n) \to \mathcal{F}(g)\mathcal{F}(h)$ in $L^{r'}(\mathbb{R}^N)$. The lemma is shown.

□

We shall now show that $\frac{\partial}{\partial y_i} E^\mu_{ij} = 0$ in $\mathcal{S}'$. We start to calculate its Fourier transform. We have in $\mathcal{S}'$

$$\mathcal{F}\Big(\frac{\partial}{\partial y_i} E^\mu_{ij}\Big)(\xi) = -i\xi_i \mathcal{F}(E^\mu_{ij})(\xi)\,. \tag{1.42}$$

Moreover, as $\mathbf{E}^\mu \in L^q(\mathbb{R}^N)$ for $q \in (1,\infty)$ if $N = 2$ and $q \in (1,3)$ if $N = 3$, we have that the product can be understood in the a.e. sense ($\mathcal{F}(\mathbf{E}^\mu) \in L^p(\mathbb{R}^N)$, $p \in (2;\infty)$ from the Hausdorff–Young inequality). Recalling that

$$\frac{\partial}{\partial y_1} E^*,\ \frac{\partial}{\partial y_1}\boldsymbol{\mathcal{O}} \in L^q(\mathbb{R}^N)\,, \quad q \in \Big(1; \frac{N}{N-1}\Big)\,,$$

we have due to Lemma 1.10

$$\mathcal{F}(E^\mu_{ij}) = C\mathcal{F}\Big(\frac{\partial}{\partial y_1} E^*\Big)\mathcal{F}\Big(\frac{\partial}{\partial y_1}\mathcal{O}_{ij}\Big)\,, \tag{1.43}$$

where the product can be understood either in $\mathcal{S}'$ or in the a.e. sense. The first distributional and classical derivative[7] of $\mathcal{O}_{ij}$ coincide in $\mathcal{S}'$ and we have

$$\mathcal{F}\Big(\frac{\partial}{\partial y_1}\mathcal{O}_{ij}\Big) = -i\xi_1 \mathcal{F}(\mathcal{O}_{ij})\,, \tag{1.44}$$

where, in general, $\mathcal{F}(\mathcal{O}_{ij}) \in \mathcal{S}'$ is not a regular function. Nevertheless, we have for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$

$$\begin{aligned}\Big\langle \mathcal{F}\Big(\frac{\partial}{\partial y_i} E^\mu_{ij}\Big), \varphi\Big\rangle &= C\Big\langle \mathcal{F}\Big(\frac{\partial}{\partial y_1}\mathcal{O}_{ij}\Big), \xi_i \mathcal{F}\Big(\frac{\partial}{\partial y_1} E^*\Big)\varphi\Big\rangle = \\ &= C\Big\langle \mathcal{F}(\mathcal{O}_{ij}), \xi_i\xi_1 \mathcal{F}\Big(\frac{\partial}{\partial y_1} E^*\Big)\varphi\Big\rangle\,.\end{aligned} \tag{1.45}$$

Let us recall that $\frac{\partial}{\partial y_i}\mathcal{O}_{ij} = 0$ in $\mathcal{S}'$. Therefore

$$\Big\langle \mathcal{F}\Big(\frac{\partial}{\partial y_i}\mathcal{O}_{ij}\Big), \varphi\Big\rangle = \Big\langle \mathcal{F}(\mathcal{O}_{ij}), -i\xi_i\varphi\Big\rangle = 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^N)\,. \tag{1.46}$$

Moreover, as $\frac{\partial}{\partial y_i}\mathcal{O}_{ij} \in L^p(\mathbb{R}^N)$ for some $p \in (1;2)$, we have that $\mathcal{F}(\frac{\partial}{\partial y_i}\mathcal{O}_{ij}) \in L^q(\mathbb{R}^N)$ for some $q \in (2;\infty)$ and the first duality on the left hand side of (1.46) can be extended for $\varphi \in L^\infty(\mathbb{R}^N)$ with sufficiently fast decay at infinity. The same holds also for the duality on the right hand side. Coming back to (1.45) and recalling that $\xi_1\mathcal{F}(\frac{\partial}{\partial y_i} E^*) \in L^\infty(\mathbb{R}^N)$ (see Lemma VIII.4.16) we finally get

$$\Big\langle \mathcal{F}\Big(\frac{\partial}{\partial y_i} E^\mu_{ij}\Big), \varphi\Big\rangle = 0$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, i.e. $\mathcal{F}(\frac{\partial}{\partial y_i} E^\mu_{ij}) = 0 \in \mathcal{S}'$. But the Fourier transform is isomorphism on $\mathcal{S}'$ and therefore also $\frac{\partial}{\partial y_i} E^\mu_{ij} = 0$ in $\mathcal{S}'$. We have shown that

$$\mathcal{O}^\mu_{ij} = E^\mu_{ij} + \mathcal{O}_{ij}$$

is the fundamental solution to (1.1).

[7] taken outside of the origin

Remark 1.3 The reason why we used such a complicated method to verify that the fundamental solution has zero divergence is that we could not verify the assumptions in order to interchange the derivative and the integral.

Finally we shall study how E^{μ}_{ij} behaves for $2\lambda = \beta \neq 1$. Recall that we shall assume $\beta \ll 1$. We have

$$E^{\mu}_{ij}(\mathbf{X}-\mathbf{Y};\beta) = \frac{\mu}{1-\mu}\int_{\mathbb{R}^N} \frac{\partial}{\partial X_1} E^*\Big(\mathbf{X}-\mathbf{Y}-\mathbf{Z}; \frac{\beta}{\sqrt{1-\mu}}\Big)\frac{\partial}{\partial Z_1}\mathcal{O}_{ij}(\mathbf{Z};\beta)\mathrm{d}\mathbf{Z}\,.$$

Recalling that

$$\mathcal{O}_{ij}(\mathbf{Z};\beta) = \beta^{N-2}\mathcal{O}_{ij}(\beta\mathbf{Z};1)$$

we easily get

$$\begin{aligned} E^{\mu}_{ij}(\mathbf{X}-\mathbf{Y};\beta) &= \\ &= -\frac{\mu\beta^{N-2}}{1-\mu}\int_{\mathbb{R}^N} \frac{\partial}{\partial Z_1} E^*\Big(\beta(\mathbf{X}-\mathbf{Y})-\mathbf{Z}; \frac{1}{\sqrt{1-\mu}}\Big)\frac{\partial}{\partial Z_1}\mathcal{O}_{ij}(\mathbf{Z};1)\mathrm{d}\mathbf{Z} \end{aligned} \tag{1.47}$$

and therefore, for $\mu < 1$,

$$E^{\mu}_{ij}(\mathbf{X}-\mathbf{Y};\beta) = \beta^{N-2}E^{\mu}_{ij}(\beta(\mathbf{X}-\mathbf{Y});1)\,. \tag{1.48}$$

We can now summarize the results form this section. We have

Theorem 1.2 *Let $N = 2,3$, $\mu < 1$. Then the fundamental solution to (1.1) can be written in the form*

$$\boldsymbol{\mathcal{O}}^{\mu}(\mathbf{x};\beta) = \boldsymbol{\mathcal{O}}(\mathbf{x};\beta) + \mathbf{E}^{\mu}(\mathbf{x};\beta)\,;$$

here $\boldsymbol{\mathcal{O}}(\mathbf{x};\beta)$ denotes the fundamental Oseen tensor and the remainder $\mathbf{E}^{\mu}(\mathbf{x};\beta)$ has the following properties:

a) for $|\mathbf{x}| \geq R \gg 1$, $|\alpha| \geq 0$

$$D^{\alpha}E^{\mu}_{ij}(\mathbf{x};1) \leq C|\mathbf{x}|^{-\frac{N+1+|\alpha|}{2}}(1+s(\mathbf{x}))^{-\frac{N-1+|\alpha|}{2}}$$

b) for $|\mathbf{x}| < 1$

$$\begin{aligned} E^{\mu}_{ij}(\mathbf{x};1) &\leq C\ln|\mathbf{x}| \quad N=2 \\ E^{\mu}_{ij}(\mathbf{x};1) &\leq C|\mathbf{x}|^{-1} \quad N=3 \\ D^{\alpha}E^{\mu}_{ij}(\mathbf{x};1) &\leq C|\mathbf{x}|^{2-N-|\alpha|} \quad |\alpha| = 1,2\,, N=2,3\,. \end{aligned}$$

Moreover, $E^{\mu}_{ij}(\mathbf{x};1) = I_1(\mathbf{x}) + I_2(\mathbf{x})$, where $D^{\alpha}I_2(\mathbf{x}) = C|\mathbf{x}|^{1-N}$, $|\alpha| = 2$ for $|\mathbf{x}| \leq 1$ and $D^{\alpha}I_1(\mathbf{x})$, $|\alpha| = 2$, representing the singular part of the second gradient of $\mathbf{E}^{\mu}(\mathbf{x})$, defines a singular integral operator

$$Tf(\mathbf{x}) = \int_{\mathbb{R}^N} I_1(\mathbf{x}-\mathbf{y})D^{\alpha}f(\mathbf{y})\,d\mathbf{y}\,, \quad |\alpha| = 2$$

which maps $L^p(\mathbb{R}^N)$ onto $L^p(\mathbb{R}^N)$ and $\mathcal{L}^p(\mathbb{R}^N;g)$ onto $L^p(\mathbb{R}^N;g)$ for $1 < p < \infty$, $g \in A_p$ (Muckenhoupt class). Moreover,

$$\mathbf{E}^{\mu}(\mathbf{x};\beta) = \beta^{N-2}\mathbf{E}^{\mu}(\beta\mathbf{x};1)\,.$$

Remark 1.4 We shall use our fundamental solution $\mathcal{O}^{\mu}$ for $\mu = \beta^2$ with β sufficiently small. Therefore we shall have

$$\mathcal{O}^{\beta^2}(\mathbf{x};\beta) \sim \beta^{N-2}\mathcal{O}(\beta\mathbf{x};1) + \frac{\beta^2}{1-\beta^2}\mathbf{E}^{\frac{1}{2}}(\mathbf{x};\beta)$$

with $\mathbf{E}^{\frac{1}{2}}(\mathbf{x};\beta) = \beta^{N-2}\mathbf{E}^{\frac{1}{2}}(\beta\mathbf{x};1)$. Moreover, the remainder $\mathbf{E}^{\frac{1}{2}}(\mathbf{x};1)$ has at least the same asymptotic properties as $\mathcal{O}(\mathbf{x};1)$ and, therefore, all theorems proved in Subsection II.3.2 and II.3.3 for the Oseen kernels can be used also for the modified Oseen kernels. Let us also recall that we used the fact that having asymptotic properties in $\mathbf{X}$ means the same asymptotic properties in $\mathbf{x}$; here the relation (1.6) is supposed.

III.2 Modified Oseen problem in $\mathbb{R}^N$

This section is devoted to the study of existence, uniqueness and L^q– estimates of the problem (0.3) in the whole space. For our purpose, we shall need more general problem; namely

$$\left.\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = g \end{aligned}\right\} \text{ in } \mathbb{R}^N \tag{2.1}$$

where $\mathbf{f}$ and g will be given functions from $C_0^\infty(\mathbb{R}^N)$, $A(\mathbf{u}) = -\Delta\mathbf{u} + \mu\frac{\partial^2\mathbf{u}}{\partial x_1^2}$. As we already know from Section II.1, one possible construction of solution to (2.1) consists in the use of the fundamental solution to (0.3). We may search the solution in the form

$$\mathbf{u} = \mathbf{u}^1 + \mathbf{u}^2\,, \tag{2.2}$$

where

$$\mathbf{u}^1 = \nabla(\mathcal{E} * g)\,, \tag{2.3}$$

$\mathcal{E}$ fundamental solution to the Laplace equation, and $\mathbf{u}^2$ solves

$$\begin{gathered} A(\mathbf{u}^2) + \beta\frac{\partial\mathbf{u}^2}{\partial x_1} + \nabla p = \mathbf{f} - A(\mathbf{u}^1) - \beta\frac{\partial\mathbf{u}^1}{\partial x_1} \\ \nabla\cdot\mathbf{u}^2 = 0\,. \end{gathered} \tag{2.4}$$

We can rewrite the right hand side of $(2.4)_1$

$$\mathbf{f} - \mu\nabla\Big(\frac{\partial^2}{\partial x_1^2}(\mathcal{E} * g)\Big) - \beta\nabla\Big(\frac{\partial}{\partial x_1}(\mathcal{E} * g)\Big) + \nabla g$$

and so the solution to (2.4) can be sought in the form

$$\begin{gathered} u_j^2(\mathbf{x}) = (\mathcal{O}_{ij}^{\mu} * f_i)(\mathbf{x}) \\ p = \frac{\partial}{\partial x_j}(\mathcal{E} * f_j)(\mathbf{x}) - \mu\frac{\partial^2}{\partial x_1^2}(\mathcal{E} * g)(\mathbf{x}) - \beta\frac{\partial}{\partial x_1}(\mathcal{E} * g)(\mathbf{x}) + g\,. \end{gathered} \tag{2.5}$$

This form is not convenient for the L^q–estimates. We therefore use another approach and later on identify our solution with those from (2.2), (2.3) and (2.5). For the sake of simplicity we take $\beta = 1$. Later on, using the replacements

$$\begin{aligned} \mathbf{f} &\mapsto \frac{\mathbf{f}}{\beta^2} \\ g &\mapsto \frac{g}{\beta} \\ p &\mapsto \frac{p}{\beta} \\ \mathbf{x} &\mapsto \beta\mathbf{x} \end{aligned} \tag{2.6}$$

we get estimates with constants independent of β.

We search solution to (2.1) in the form

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \mathbf{U}(\xi) \mathrm{e}^{-i(\mathbf{x},\xi)} \mathrm{d}\xi = \mathcal{F}^{-1}(\mathbf{U})(\mathbf{x}) \\ p(\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} P(\xi) \mathrm{e}^{-i(\mathbf{x},\xi)} \mathrm{d}\xi = \mathcal{F}^{-1}(P)(\mathbf{x}) \,. \end{aligned} \tag{2.7}$$

Inserting (2.7) into (2.1) we get ($\xi' = (\xi_2, \dots, \xi_N)$)

$$\begin{aligned} [(1-\mu)\xi_1^2 + |\xi'|^2 - i\xi_1] U_m - i\xi_m P &= \mathcal{F}(f_m) \\ -i\xi_m U_m &= \mathcal{F}(g) \,. \end{aligned} \tag{2.8}$$

Solving (2.8) we have

$$\begin{aligned} U_m(\xi) &= \frac{(\delta_{mk}|\xi|^2 - \xi_m\xi_k)\mathcal{F}(f_k)(\xi)}{h(\xi)|\xi|^2} + \frac{i\xi_m}{|\xi|^2}\mathcal{F}(g)(\xi) \\ P(\xi) &= \frac{i\xi_k}{|\xi|^2}\mathcal{F}(f_k)(\xi) + \frac{h(\xi)}{|\xi|^2}\mathcal{F}(g)(\xi) \end{aligned} \tag{2.9}$$

with $h(\xi) = (1-\mu)\xi_1^2 + |\xi'|^2 - i\xi_1$. Denoting

$$\begin{aligned} V_m(\xi) &= \frac{(\delta_{mk}|\xi|^2 - \xi_m\xi_k)}{h(\xi)|\xi|^2}\mathcal{F}(f_k)(\xi) \\ W_m(\xi) &= \frac{i\xi_m}{|\xi|^2}\mathcal{F}(g)(\xi) \\ \Pi(\xi) &= \frac{i\xi_k}{|\xi|^2}\mathcal{F}(f_k)(\xi) \\ T(\xi) &= \frac{h(\xi)}{|\xi|^2}\mathcal{F}(g)(\xi) \end{aligned} \tag{2.10}$$

we can rewrite (2.9) as

$$\begin{aligned} \mathbf{U}(\xi) &= \mathbf{V}(\xi) + \mathbf{W}(\xi) \\ P(\xi) &= \Pi(\xi) + T(\xi) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{v}(\mathbf{x}) + \mathbf{w}(\mathbf{x}) \\ p(\mathbf{x}) &= \pi(\mathbf{x}) + \tau(\mathbf{x}) \end{aligned} \tag{2.12}$$

with

$$\mathbf{v}(\mathbf{x}) = \mathcal{F}^{-1}(\mathbf{V})(\mathbf{x})$$
$$\mathbf{w}(\mathbf{x}) = \mathcal{F}^{-1}(\mathbf{W})(\mathbf{x})$$
$$\pi(\mathbf{x}) = \mathcal{F}^{-1}(\Pi)(\mathbf{x})$$
$$\tau(\mathbf{x}) = \mathcal{F}^{-1}(T)(\mathbf{x})\,.$$

The advantage of (2.11) and (2.12), respectively, is that $\mathbf{v}$ and π represent the solution for $g \equiv 0$ while $\mathbf{w}$ and τ the solution for $\mathbf{f} \equiv \mathbf{0}$. Moreover , it is an easy matter to see that $\mathbf{u}$ and p are $C^\infty(\mathbb{R}^N)$.[8]

In order to get L^q–estimates of $\mathbf{u}$ and p, we shall apply the Lizorkin multiplier theorem, see Theorem II.3.3. A principle role will be played by the term $\psi_{mk} = \frac{\delta_{mk}|\boldsymbol{\xi}|^2 - \xi_m\xi_k}{|\boldsymbol{\xi}|^2 h(\boldsymbol{\xi})}$. We start with some observations.

Lemma 2.1 *If $N \geq 2$, $m, k = 1, 2, \ldots, N$. Then the assumptions of the Lizorkin multiplier theorem are satisfied*

a) by ψ_{mk} with $\beta = \frac{2}{N+1}$

b) by $\xi_l\psi_{mk}$ with $\beta = \frac{1}{N+1}$ and $l \in \{1, 2, \ldots, N\}$

c) by $\xi_1\psi_{mk}$ with $\beta = 0$

d) by $\xi_l\xi_s\psi_{mk}$ with $\beta = 0$ and $l, s \in \{1, 2, \ldots, N\}$

For $N = 2$ and $l, k \in \{1, 2\}$ we have also the assumptions satisfied

e) by ψ_{2k} with $\beta = \frac{1}{2}$

f) by $\xi_l\psi_{2k}$ with $\beta = 0$.

Proof: Let us recall that we need to get

$$|\xi_1|^{\kappa_1+\beta} \cdot \ldots \cdot |\xi_N|^{\kappa_N+\beta} \Big| \frac{\partial^\kappa m(\xi)}{\partial \xi_1^{\kappa_1} \ldots \partial \xi_N^{\kappa_N}} \Big| \leq C \tag{2.13}$$

for some $0 \leq \beta < 1$, $\kappa_i \in \{0, 1\}$, $\kappa = \sum \kappa_i \leq N$. We easily get that the left hand side is bounded by

$$\frac{C(\mu)}{|\xi|^2 + |\xi_1|} \Big(\prod_{i=1}^{N} |\xi_i| \Big)^\beta$$

for any $0 \leq \kappa \leq N$, $k, m \in \{1, \ldots, N\}$ and any $\mu \in [0, 1)$. We can therefore apply Lemma VII.4.2 from [Ga1].

□

We can now start to estimate the solution to (2.1). Let us start with $\mathbf{v}$ and π. Observing that the function $\frac{\xi_l\xi_k}{|\boldsymbol{\xi}|^2}$ satisfies the assumptions of Lizorkin

[8] The crucial fact is that the function $\frac{1}{|h(\boldsymbol{\xi})|}$ is locally integrable; see the proof of Lemma VIII.4.15.

multiplier Theorem II.3.3 (or, equivalently, those of Marcinkiewicz multiplier Theorem II.3.2) we get with help of Lemma 2.1 c) and d) that

$$\Big\|\frac{\partial \mathbf{v}}{\partial x_1}\Big\|_q + |\mathbf{v}|_{2,q} + |\pi|_{1,q} \le C\|\mathbf{f}\|_q\,, \quad q \in (1;\infty)\,. \tag{2.14}$$

Moreover, for $N=2$ we also have from Lemma 2.1 f)

$$\Big\|\frac{\partial \mathbf{v}}{\partial x_1}\Big\|_q + |\mathbf{v}|_{2,q} + |v_2|_{1,q} + |\pi|_{1,q} \le C\|\mathbf{f}\|_q\,, \quad q \in (1;\infty)\,. \tag{2.15}$$

Now we must restrict the values of q. We have for $1<q<N+1$ with help of Lemma 2.1 b)

$$|\mathbf{v}|_{1,s_1} \le C\|\mathbf{f}\|_q\,, \quad s_1 = \frac{(N+1)q}{N+1-q}\,, \quad q \in (1;N+1) \tag{2.16}$$

and for $N=2$ also

$$|v_2|_{\frac{2q}{2-q}} \le C\|\mathbf{f}\|_q\,, \quad q \in (1;2)\,. \tag{2.17}$$

Moreover, observing that $\frac{\xi_m}{|\xi|^2}$ satisfies the assumptions of Theorem II.3.3 with $\beta = \frac{1}{N}$, we have also

$$|\pi|_{\frac{Nq}{N-q}} \le C\|\mathbf{f}\|_q\,, \quad q \in (1;N)\,. \tag{2.18}$$

Finally, assuming $1<q<\frac{N+1}{2}$ we get from Lemma 2.1 a)

$$\|\mathbf{v}\|_{s_2} \le C\|\mathbf{f}\|_q\,, \quad s_2 = \frac{(N+1)q}{N+1-2q}\,, \quad q \in \Big(1;\frac{N+1}{2}\Big)\,. \tag{2.19}$$

Let us now estimate the pair $\mathbf{w}$, τ. We have easily

$$\begin{aligned} |\mathbf{w}|_{1,r} &\le C\|g\|_r \\ |\mathbf{w}|_{2,r} &\le C|g|_{1,r} \end{aligned} \quad r \in (1;\infty) \tag{2.20}$$

and

$$|\tau|_{1,r} \le C\|g\|_{1,r}\,, \quad r \in (1;\infty)\,. \tag{2.21}$$

Moreover, using the fact that $\frac{\xi_k}{|\xi|^2}$ satisfies the assumptions of Theorem II.3.3 with $\beta = \frac{1}{N}$, we get

$$\begin{aligned} |\mathbf{w}|_{\frac{Nr}{N-r}} &\le C\|g\|_r \\ |\tau|_{\frac{Nr}{N-r}} &\le C(\|g\|_{\frac{Nr}{N-r}} + \|g\|_r) \end{aligned} \quad r \in (1;N)\,. \tag{2.22}$$

Next, as $\|g\|_{s_1} \le C\|g\|_{1,q}$, $s_1 = \frac{(N+1)q}{N+1-q}$, $q \in (1;N+1)$, we have from $(2.20)_1$

$$|\mathbf{w}|_{1,s_1} \le C\|g\|_{1,q} \quad q \in (1;N+1)\,. \tag{2.23}$$

Finally, if $1<q<\frac{N+1}{2}$, we choose in $(2.22)_1$ the exponent r such that $\frac{Nr}{N-r} = \frac{(N+1)q}{N+1-2q}$ and get

$$\|\mathbf{w}\|_{s_2} \le C\|g\|_{r_1}\,, \quad r_1 = \frac{N(N+1)q}{N(N+1-q)+q}\,, \quad q \in \Big(1;\frac{N+1}{2}\Big)$$

and using the imbedding $W^{1,q}(\mathbb{R}^N) \hookrightarrow L^{r_1}(\mathbb{R}^N)$ we have

$$|\mathbf{w}|_{s_2} \le C\|g\|_{1,q}\,, \quad s_2 = \frac{(N+1)q}{N+1-2q}\,, \quad q \in \Big(1; \frac{N+1}{2}\Big)\,. \tag{2.24}$$

Combining estimates (2.14)–(2.24), applying (2.6) we have

Theorem 2.1 *Let* $\mathbf{f} \in W^{m,q}(\mathbb{R}^N)$, $g \in W^{m+1,q}(\mathbb{R}^N)$, $m \ge 0$, $1 < q < \infty$ *be given. Then there exists a pair of functions* $(\mathbf{u}, p)$,

$$\mathbf{u} \in W^{m+2,q}_{loc}(\mathbb{R}^N)\,,\ p \in W^{m+1,q}_{loc}(\mathbb{R}^N)$$

satisfying a.e. the modified Oseen problem (2.1). Moreover, we have following estimates for $l \in \{0, 1, \dots, m\}$

$$\beta\Big|\frac{\partial \mathbf{u}}{\partial x_1}\Big|_{l,q} + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q} \le C(|\mathbf{f}|_{l,q} + |g|_{l+1,q} + \beta|g|_{l,q})\,. \tag{2.25}$$

If $N = 2$, *then*

$$\beta|u_2|_{l+1,q} + \beta\Big|\frac{\partial u_1}{\partial x_1}\Big|_{l,q} + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q} \le C(\|\mathbf{f}\|_q + |g|_{l+1,q} + \beta|g|_{l,q})\,. \tag{2.26}$$

If $1 < q < N+1$, $s_1 = \frac{(N+1)q}{N+1-q}$, *then*

$$\begin{aligned}\beta^{\frac{1}{N+1}}|\mathbf{u}|_{l+1,s_1} + \beta\Big|\frac{\partial \mathbf{u}}{\partial x_1}\Big|_{l,q} + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q} \le\\ \le C(|\mathbf{f}|_{l,q} + |g|_{l+1,q} + \beta|g|_{l,q})\,.\end{aligned} \tag{2.27}$$

If $1 < q < N$, *then*

$$\begin{aligned}\beta^{\frac{1}{N+1}}|\mathbf{u}|_{l+1,s_1} + |p|_{l,\frac{Nq}{N-q}} + \beta\Big|\frac{\partial \mathbf{u}}{\partial x_1}\Big|_{l,q} + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q} \le\\ \le C(|\mathbf{f}|_{l,q} + |g|_{l+1,q} + \beta|g|_{l,q})\,.\end{aligned} \tag{2.28}$$

Moreover, if $N = 2$, *then for* $1 < q < 2$

$$\begin{aligned}\beta|u_2|_{l,\frac{2q}{2-q}} + \beta|u_2|_{l+1,q} + \beta^{\frac{1}{3}}|\mathbf{u}|_{l+1,\frac{3q}{3-q}} + |p|_{l,\frac{2q}{2-q}} + \beta\Big|\frac{\partial u_1}{\partial x_1}\Big|_{l,q} +\\ + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q} \le C(|\mathbf{f}|_{l,q} + |g|_{l+1,q} + \beta|g|_{l,q})\,.\end{aligned} \tag{2.29}$$

Furthermore, if $1 < q < \frac{N+1}{2}$, *we get for* $s_2 = \frac{(N+1)q}{N+1-2q}$

$$\begin{aligned}\beta^{\frac{2}{N+1}}|\mathbf{u}|_{l,s_2} + \beta^{\frac{1}{N+1}}|\mathbf{u}|_{l+1,s_1} + |p|_{l,\frac{Nq}{N-q}} + \beta\Big|\frac{\partial \mathbf{u}}{\partial x_1}\Big|_{l,q} +\\ + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q} \le C(|\mathbf{f}|_{l,q} + |g|_{l+1,q} + \beta|g|_{l,q})\,.\end{aligned} \tag{2.30}$$

Finally, especially for $N = 2$ *and* $1 < q < \frac{3}{2}$ *we have*

$$\begin{aligned}\beta|u_2|_{l,\frac{2q}{2-q}} + \beta|u_2|_{l+1,q} + \beta^{\frac{2}{3}}|\mathbf{u}|_{l,\frac{3q}{3-2q}} + \beta^{\frac{1}{3}}|\mathbf{u}|_{l+1,\frac{3q}{3-q}} + |p|_{l,\frac{2q}{2-q}} +\\ + \beta\Big|\frac{\partial u_1}{\partial x_1}\Big|_{l,q} + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q} \le C(|\mathbf{f}|_{l,q} + |g|_{l+1,q} + \beta|g|_{l,q})\,.\end{aligned} \tag{2.31}$$

If $\mathbf{w},\tau$ *is another solution to the same data such that* $|\frac{\partial \mathbf{w}}{\partial x_1}|_{l,q}$, $|\mathbf{w}|_{l+2,q}$ *are finite for some* $l \in \{0, 1, \dots, m\}$ *(or, equivalently,* $|\mathbf{w}|_{l+2,q}$ *and* $|\tau|_{l+1,q}$ *are finite), then*

$$\Big|\frac{\partial}{\partial x_1}(\mathbf{w} - \mathbf{u})\Big|_{l,q} = |\mathbf{w} - \mathbf{u}|_{l+2,q} = |\tau - p|_{l+1,q} = 0\,.$$

Proof: The estimates for $\mathbf{f}$, g smooth and $\beta = 1$ were shown above. To get the estimates for $\mathbf{f} \in W^{m,q}(\mathbb{R}^N)$, $g \in W^{m+1,q}(\mathbb{R}^N)$ only, we use the standard density argument. Next, for $\beta \neq 1$ we use (2.6). Let us only sketch the proof. Let $\mathbf{U}$, P solves

$$\begin{aligned} A_{\mathbf{y}}(\mathbf{U}) + \frac{\partial \mathbf{U}}{\partial y_1} + \nabla_{\mathbf{y}} P &= \mathbf{F} \\ \nabla_{\mathbf{y}} \cdot \mathbf{U} &= G\,. \end{aligned}$$

Now, $\mathbf{U}$, P satisfy the estimates (2.25)–(2.31) with $\beta = 1$. Taking $\mathbf{F} = \frac{\mathbf{f}}{\beta^2}$, $G = \frac{g}{\beta}$, $P = \frac{p}{\beta}$ and $\mathbf{y} = \beta\mathbf{x}$ we get that $\mathbf{u}$, p with $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{y})$, $p(\mathbf{x}) = P(\mathbf{y})$ satisfy the original problem (2.1) and

$$\begin{aligned} |\mathbf{U}|_{l+2,q} &\to \beta^{\frac{N}{q}-l-2}|\mathbf{u}|_{l+2,q} \\ |\mathbf{U}|_{l+1,q} &\to \beta^{\frac{N}{q}-l-1}|\mathbf{u}|_{l+1,q} \\ |\mathbf{U}|_{l+1,s_1} &\to \beta^{\frac{N}{q}-l-2+\frac{1}{N+1}}|\mathbf{u}|_{l+1,s_1} \\ |\mathbf{U}|_{l,s_2} &\to \beta^{\frac{N}{q}-l-2+\frac{2}{N+1}}|\mathbf{u}|_{l+1,s_2} \\ |P|_{l+1,q} &\to \beta^{\frac{N}{q}-l-2}|p|_{l+1,q} \end{aligned}$$

and the estimates with $\beta \neq 1$ follows easily by multiplying $\beta^{l+2-\frac{N}{q}}$. We are therefore left with the proof of the uniqueness part. We denote $\mathbf{z} = D^\alpha(\mathbf{w} - \mathbf{v})$, $s = D^\alpha(\tau - p)$, $|\alpha| = l$. The pair $\mathbf{z}$, s solves

$$\begin{aligned} A(\mathbf{z}) + \beta\frac{\partial \mathbf{z}}{\partial x_1} + \nabla s \; &= \mathbf{0} \\ \nabla \cdot \mathbf{z} &= 0 \end{aligned} \tag{2.32}$$

a.e. in $\mathbb{R}^N$. We multiply (2.32) by the standard mollifier $\omega_\varepsilon(|\mathbf{x} - \mathbf{y}|)$, $\mathbf{x} \in \mathbb{R}^N$ and integrate the variable $\mathbf{y}$ over $\mathbb{R}^N$. We get

$$\begin{aligned} A(\mathbf{z}_\varepsilon) + \beta\frac{\partial \mathbf{z}_\varepsilon}{\partial x_1} + \nabla s_\varepsilon &= \mathbf{0} \\ \nabla \cdot \mathbf{z}_\varepsilon &= 0\,, \end{aligned} \tag{2.33}$$

where $\mathbf{z}_\varepsilon$, s_ε are infinitely times differentiable functions and, moreover, $\nabla^2\mathbf{z}_\varepsilon$, $\frac{\partial \mathbf{z}_\varepsilon}{\partial x_1} \in L^q(\mathbb{R}^N)$ what implies $\nabla s_\varepsilon \in L^q(\mathbb{R}^N)$ (or $\nabla^2\mathbf{z}_\varepsilon$, $\nabla s_\varepsilon \in L^q(\mathbb{R}^N)$ imply $\frac{\partial \mathbf{z}_\varepsilon}{\partial x_1} \in L^q(\mathbb{R}^N)$). We can apply the divergence to $(2.33)_1$ and get

$$\Delta s_\varepsilon = 0 \quad \text{ in } \mathbb{R}^N\,. \tag{2.34}$$

Moreover, as $\nabla s_\varepsilon \in L^q(\mathbb{R}^N)$ we have that $\nabla s_\varepsilon = 0$ $\forall \varepsilon > 0$ and therefore $\nabla s = 0$ a.e. in $\mathbb{R}^N$. To show this, let us apply ∇ on (2.34), multiply it by $\eta_R \nabla s_\varepsilon |\nabla s_\varepsilon|^{q-2}$, η_R the standard cut–off function (see Section VIII.2), and integrate over $\mathbb{R}^N$. We get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \Delta(\nabla s_\varepsilon)\eta_R \nabla s_\varepsilon |\nabla s_\varepsilon|^{q-2} \mathrm{d}\mathbf{x} = \\ &= -(q-1)\int_{\mathbb{R}^N} |\nabla^2 s_\varepsilon|^2 |\nabla s_\varepsilon|^{q-2}\eta_R \mathrm{d}\mathbf{x} - \frac{1}{q}\int_{\mathbb{R}^N} \nabla|\nabla s_\varepsilon|^q \nabla\eta_R \mathrm{d}\mathbf{x} \end{aligned}$$

and therefore finally

$$q(q-1)\int_{\mathbb{R}^N} |\nabla^2 s_\varepsilon|^2 |\nabla s_\varepsilon|^{q-2}\eta_R \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} |\nabla s_\varepsilon|^q \Delta\eta_R \mathrm{d}\mathbf{x}\,.$$

As supp $\Delta\eta_R = B_{2R}(\mathbf{0}) \setminus B_R(\mathbf{0})$ and $|\Delta\eta_R| \leq \frac{C}{R^2}$, we have

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} |\nabla^2 s_\varepsilon|^2 |\nabla s_\varepsilon|^{q-2}\eta_R \mathrm{d}\mathbf{x} = 0\,.$$

Therefore $\nabla^2 s_\varepsilon = \mathbf{0}$ in $\mathbb{R}^N$ and so $\nabla s_\varepsilon =$ const. But as $\nabla s_\varepsilon \in L^q(\mathbb{R}^N)$, we have also $\nabla s_\varepsilon = \mathbf{0}$.

We return to (2.33). Let w_ε be any component of $\nabla^2 \mathbf{z}_\varepsilon$. Then

$$A(w_\varepsilon) + \beta\frac{\partial w_\varepsilon}{\partial x_1} = 0 \quad \text{in } \mathbb{R}^N\,. \tag{2.35}$$

Denoting

$$|w_\varepsilon|_A = \sqrt{\sum_{i=1}^N \Big(\frac{\partial w_\varepsilon}{\partial x_i}\Big)^2 - \mu\Big(\frac{\partial w_\varepsilon}{\partial x_1}\Big)^2} \tag{2.36}$$

we get after multiplying (2.35) by $|w_\varepsilon|^{q-2}w_\varepsilon\eta_R$ and proceeding as above

$$\int_{\mathbb{R}^N} \eta_R |w_\varepsilon|^{q-2}|w_\varepsilon|_A^2 \mathrm{d}\mathbf{x} \leq C\int_{\mathbb{R}^N} (|\Delta\eta_R| + |\nabla\eta_R|)|w_\varepsilon|^q \mathrm{d}\mathbf{x} \tag{2.37}$$

i.e. passing with $R \to \infty$

$$\nabla w_\varepsilon = 0 \quad \text{in } \mathbb{R}^N \quad \Longrightarrow w_\varepsilon = \text{ const}\,.$$

Again, as $w_\varepsilon \in L^q(\mathbb{R}^N)$, we get $w_\varepsilon = 0$ in $\mathbb{R}^N$ $\forall\varepsilon > 0$ i.e. $w = 0$ a.e. in $\mathbb{R}^N$. Proceeding analogously for any $|\alpha| = l$ we finish the proof.

□

Corollary 2.1 *Let* $\mathbf{f}$, $g \in C_0^\infty(\mathbb{R}^N)$. *Then* $\mathbf{u}$, p, *the solution to (2.1), constructed by (2.7) and (2.9) has the following decay properties for* $|\mathbf{x}|$ *sufficiently large:*

$$\begin{aligned} |D^\alpha \mathbf{u}(\mathbf{x})| &\leq C|\mathbf{x}|^{-\frac{N-1+|\alpha|}{2}}(1+s(\mathbf{x}))^{-\frac{N-1+|\alpha|}{2}} \\ |D^\alpha p(\mathbf{x})| &\leq C|\mathbf{x}|^{-(N-1+|\alpha|)} \end{aligned} \qquad k \geq 0\,. \tag{2.38}$$

Proof: Let us denote by $\mathbf{U}$, P the solution constructed in (2.5). We know that $\mathbf{U}$, $P \in C^\infty(\mathbb{R}^N)$ and moreover, as the right hand side has a compact support, we have also that $\mathbf{U}$, P behaves exactly as $\boldsymbol{\mathcal{O}}^\mu$ and $\nabla\mathcal{E}$, respectively.[9] Therefore we have in particular that $\mathbf{U}$, P satisfy (2.38). So (see also Lemma 1.2) $D^2\mathbf{U}$, ∇P, $\frac{\partial \mathbf{U}}{\partial x_1} \in L^q(\mathbb{R}^N)$ for all $q > 1$. Applying Theorem 2.1 we therefore get that

$$\|\nabla(P-p)\|_q = \|D^2(\mathbf{U}-\mathbf{u})\|_q = 0\,,$$

[9] Note that $\nabla\mathcal{E}(\mathbf{x}) \sim |\mathbf{x}|^{1-N} \leq |\mathbf{x}|^{-\frac{N-1}{2}}(1+s(\mathbf{x}))^{-\frac{N-1}{2}}$ for $|\mathbf{x}|$ sufficiently large.

where $\mathbf{u}$, p is the solution constructed by (2.7) and (2.9). Moreover, as both P, p and $\mathbf{U}$, $\mathbf{u}$ are r–integrable function for some r's $\in (1,\infty)$, we get $P = p$, $\mathbf{U} = \mathbf{u}$ and therefore also (2.38).

□

Next we shall study weak solutions to (2.1). We say that the pair $(\mathbf{u}, p) \in W^{1,q}_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N)$ is a q–weak solution to (2.1) if[10]

$$\int_{\mathbb{R}^N} \Big(\nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - \mu \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_1} + \beta \frac{\partial \mathbf{u}}{\partial x_1} \cdot \boldsymbol{\varphi}\Big) \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^N} p \frac{\partial \varphi_i}{\partial x_i} \mathrm{d}\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^N) \qquad \nabla \cdot \mathbf{u} = g \quad \text{a.e. in } \mathbb{R}^N . \tag{2.39}$$

First, let us assume that $\mathbf{f}$, $g \in C_0^\infty(\mathbb{R}^N)$. We can therefore construct a solution by means of the Fourier transform as we did above. Moreover, without loss of generality, we can write $\mathbf{f}$ and g in the divergence form (see e.g. [Ga1])

$$\begin{aligned} f_j(\mathbf{x}) &= \frac{\partial F_{lj}}{\partial x_l}(\mathbf{x}) \\ g(\mathbf{x}) &= \frac{\partial G_l}{\partial x_l}(\mathbf{x}) \end{aligned} \tag{2.40}$$

in such a way that

$$\begin{aligned} |\mathbf{f}|_{-1,q} &\le \|\mathbf{F}\|_q \le c_1 |\mathbf{f}|_{-1,q} \\ |g|_{-1,q} &\le \|\mathbf{G}\|_q \le c_2 |g|_{-1,q} \\ |G|_{1,q} &\le c_3 \|g\|_q . \end{aligned} \tag{2.41}$$

We have therefore $\mathcal{F}(f_j) = -i\xi_l \mathcal{F}(F_{lj})$, $\mathcal{F}(g) = -i\xi_l \mathcal{F}(G_l)$ and we can proceed as before Theorem 2.1. Moreover, using

Lemma 2.2 *Let $g \in L^q(\mathbb{R}^N) \cap D_0^{-1,q}(\mathbb{R}^N)$, $1 < q < \infty$. Then for any $\varepsilon > 0$ there exists $g_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ such that*

$$|g_\varepsilon - g|_{-1,q} + \|g_\varepsilon - g\|_q < \varepsilon .$$

Proof: see [Ga1], Lemma VII.4.3.

□

we have

Theorem 2.2 *Let $N \ge 2$, $\mathbf{f} \in D_0^{-1,q}(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N) \cap D_0^{-1,q}(\mathbb{R}^N)$, $1 < q < \infty$. Then there exists at least one q–weak solution to (2.1) in the sense of (2.39). This solution satisfies*

$$\begin{aligned} |\mathbf{u}|_{1,q} + \|p\|_q &\le C(|\mathbf{f}|_{-1,q} + \beta |g|_{-1,q} + \|g\|_q) \\ \beta \|u_2\|_q + |\mathbf{u}|_{1,q} + \|p\|_q &\le C(|\mathbf{f}|_{-1,q} + \beta |g|_{-1,q} + \|g\|_q) \ \textit{if } N = 2 , \end{aligned} \tag{2.42}$$

[10] this is slightly more general definition than in the case of Ω an exterior domain; see Definition 3.1

and if $1 < q < N+1$, $s_1 = \frac{(N+1)q}{N+1-q}$

$$\begin{aligned} \beta^{\frac{1}{N+1}}\|\mathbf{u}\|_{s_1} + |\mathbf{u}|_{1,q} + \|p\|_q &\le C(|\mathbf{f}|_{-1,q} + \beta|g|_{-1,q} + \|g\|_q) \\ \beta\|u_2\|_q + \beta\|\mathbf{u}\|_{\frac{3q}{3-q}} + |\mathbf{u}|_{1,q} + \|p\|_q &\le C(|\mathbf{f}|_{-1,q} + \beta|g|_{-1,q} + \|g\|_q) \end{aligned} \tag{2.43}$$

if $N = 2$*. Finally, if* $(\mathbf{w}, \tau)$ *is another* q*–weak solution to (2.1) corresponding to the same data* $\mathbf{f}$, g *such that* $\nabla \mathbf{w} \in L^q(\mathbb{R}^N)$, $\tau \in L^q(\mathbb{R}^N)$*, then* $\mathbf{w} = \mathbf{u} + c$ *and* $\tau = p$*.*

Proof: The estimates (2.42) and (2.43) are for $\mathbf{f}$, $g \in C_0^\infty(\mathbb{R}^N)$ obtained in the same way as in Theorem 2.1. Then, using the standard density argument together with Lemma 2.2 we can pass to the limit with non–smooth data. The uniqueness part is proved in the same lines as in Theorem 2.1. Denoting $\mathbf{z} = \mathbf{w} - \mathbf{v}$, $s = \tau - p$ we have that $\nabla \mathbf{z}$, $s \in L^q(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \Big(\nabla \mathbf{z} : \nabla \boldsymbol{\varphi} - \mu \frac{\partial \mathbf{z}}{\partial x_1} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_1} + \beta \frac{\partial \mathbf{z}}{\partial x_1} \cdot \boldsymbol{\varphi}\Big) \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} s \frac{\partial \varphi_i}{\partial x_i} \mathrm{d}\mathbf{x}$$
$$\forall \boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^N)$$
$$\nabla \cdot \mathbf{z} = 0 \quad \text{a.e. in } \mathbb{R}^N .$$

Taking in particular $\varphi_i(\mathbf{y}) = \omega_\varepsilon(|\mathbf{x} - \mathbf{y}|)$, $i = 1, 2, \ldots, N$, $\varepsilon > 0$, the standard mollifier, we get for $\mathbf{z}_\varepsilon = \mathbf{z} * \omega_\varepsilon \in C^\infty(\mathbb{R}^N)$, $\nabla z_\varepsilon \in L^q(\mathbb{R}^N)$ and $s_\varepsilon = s * \omega_\varepsilon \in C^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ that

$$\left.\begin{aligned} A(\mathbf{z}_\varepsilon) + \beta \frac{\partial \mathbf{z}_\varepsilon}{\partial x_1} + \nabla s_\varepsilon = 0 \\ \nabla \cdot \mathbf{z}_\varepsilon = 0 \end{aligned}\right. \quad \text{in } \mathbb{R}^N . \tag{2.44}$$

Now, applying the divergence to $(2.44)_1$ we have

$$\Delta s_\varepsilon = 0 \quad \text{in } \mathbb{R}^N \tag{2.45}$$

and therefore, multiplying (2.45) by $s_\varepsilon |s_\varepsilon|^{q-2} \eta_R$ with η_R the standard cut–off function and arguing as in Theorem 2.1 we get thanks to the integrability of s_ε and s that $s_\varepsilon = s = 0$ a.e. in $\mathbb{R}^N$. We are therefore left with

$$A(\mathbf{z}_\varepsilon) + \beta \frac{\partial \mathbf{z}_\varepsilon}{\partial x_1} = 0$$

and again, arguing as in Theorem 2.1, we get $\nabla \mathbf{z}_\varepsilon = 0$ and consequently $\mathbf{z} =$ const a.e. in $\mathbb{R}^N$.

□

We finish this section by proving a more general version of the uniqueness lemma

Lemma 2.3 *Let* $1 < q, r < N+1$ *and let* $(\mathbf{u}^1, p^1) \in W^{1,q}_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N)$ *and* $(\mathbf{u}^2, p^2) \in W^{1,r}_{loc}(\mathbb{R}^N) \times L^r_{loc}(\mathbb{R}^N)$ *be* q*– and* r*–weak solutions, respectively, corresponding the same data* $\mathbf{f}$, g *such that* $\nabla \mathbf{u}^{(1)} \in L^q(\mathbb{R}^N)$, $\nabla \mathbf{u}^{(2)} \in L^r(\mathbb{R}^N)$*. Then*

$$\mathbf{u} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)} = \mathbf{c} \in \mathbb{R}^N$$
$$p = p^{(1)} - p^{(2)} = a \in \mathbb{R}^N .$$

Proof: The couple $(\mathbf{u}, p)$ satisfies

$$\int_{\mathbb{R}^N} \Big(\nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - \mu \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_1} + \beta \frac{\partial \mathbf{u}}{\partial x_1} \cdot \boldsymbol{\varphi}\Big) \mathrm{d}\mathbf{x} = 0 \tag{2.46}$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^N)$ with zero divergence. Therefore, due to the density (see Remark VIII.3.4) also for all

$$\boldsymbol{\varphi} \in \widehat{H}^1_{q' \wedge r'}(\mathbb{R}^N) = \{\boldsymbol{\varphi} \in W^{1,q'}(\mathbb{R}^N) \cap W^{1,r'}(\mathbb{R}^N); \nabla \cdot \boldsymbol{\varphi} = 0 \text{ in } \mathbb{R}^N\}\,.$$

Let us consider an auxiliary problem

$$\begin{aligned} A(\mathbf{z}) - \beta \frac{\partial \mathbf{z}}{\partial x_1} + \nabla \tau = \xi \\ \nabla \cdot \mathbf{z} = 0 \end{aligned} \quad \text{in } \mathbb{R}^N \tag{2.47}$$

with

$$\xi \in \overline{C_0^\infty}(\mathbb{R}^N) = \Big\{\mathbf{u} \in C_0^\infty(\mathbb{R}^N); \int_{\mathbb{R}^N} \mathbf{u} \mathrm{d}\mathbf{x} = \mathbf{0}\Big\} \subset D_0^{-1,q}(\mathbb{R}^N) \quad \forall q \geq 1\,.$$

Theorem 2.2 guarantees existence of a couple $(\mathbf{z}, \tau)$ such that $\mathbf{z} \in L^q(\mathbb{R}^N)$ $\forall q \in (\frac{N+1}{N}; \infty)$, $(\nabla \mathbf{z}, \tau) \in L^q(\mathbb{R}^N)$ $\forall q \in (1; \infty)$ — a weak solution to (2.47). Evidently, this solution is also strong, in particular $C^\infty(\mathbb{R}^N)$. We multiply $(2.47)_1$ by $\mathbf{u}\zeta_R$, ζ_R the Sobolev cut–off function with $R > e^2$ (see Section VIII.2) and integrate over $\mathbb{R}^N$

$$\begin{aligned} \int_{\mathbb{R}^N} \Big(\nabla \mathbf{z} : \nabla \mathbf{u} - \mu \frac{\partial \mathbf{z}}{\partial x_1} \cdot \frac{\partial \mathbf{u}}{\partial x_1} + \beta \frac{\partial \mathbf{u}}{\partial x_1} \cdot \mathbf{z}\Big) \zeta_R \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} \xi \cdot \mathbf{u} \zeta_R \mathrm{d}\mathbf{x} - \\ - \int_{\mathbb{R}^N} \Big(\nabla \mathbf{z} \cdot \mathbf{u} \nabla \zeta_R - \mu \frac{\partial \mathbf{z}}{\partial x_1} \cdot \mathbf{u} \frac{\partial \zeta_R}{\partial x_1} + \beta \mathbf{z} \cdot \mathbf{u} \frac{\partial \zeta_R}{\partial x_1} + \tau \mathbf{u} \cdot \nabla \zeta_R\Big) \mathrm{d}\mathbf{x}\,. \end{aligned}$$

The second term on the right hand side can be bounded by

$$C(\|\mathbf{u}^{(1)} \nabla \zeta_R\|_q + \|\mathbf{u}^{(2)} \nabla \zeta_R\|_r)(\|\nabla \mathbf{z}\|_{q' \wedge r'} + \|\mathbf{z}\|_{q' \wedge r'} + \|\tau\|_{q' \wedge r'})$$

and tends to 0 as $R \to \infty$ (see Lemma VIII.2.2). We may therefore apply the Lebesgue dominated theorem to get

$$\int_{\mathbb{R}^N} \Big(\nabla \mathbf{z} : \nabla \mathbf{u} - \mu \frac{\partial \mathbf{z}}{\partial x_1} \cdot \frac{\partial \mathbf{u}}{\partial x_1} + \beta \frac{\partial \mathbf{u}}{\partial x_1} \cdot \mathbf{z}\Big) \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} \xi \cdot \mathbf{u} \mathrm{d}\mathbf{x}\,. \tag{2.48}$$

The left hand side is thanks to (2.46) equal to zero and we are left with

$$\int_{\mathbb{R}^N} \xi \cdot \mathbf{u} \mathrm{d}\mathbf{x} = 0 \quad \forall \xi \in \overline{C_0^\infty}(\mathbb{R}^N)\,.$$

This implies $\mathbf{u} = \mathbf{c} \in \mathbb{R}^N$ and easily we get also $p = a \in \mathbb{R}$. Let us finally note that the condition $\min\{q', r'\} > \frac{N+1}{N}$ implies $\max\{q, r\} < N + 1$.

□

III.3 Modified Oseen problem in exterior domains

In this section we shall study

$$
\left.\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega
\qquad
\begin{aligned} \mathbf{u} = \mathbf{u}_* \quad \text{at } \partial\Omega \\ \mathbf{u} \to 0 \quad \text{as } |\mathbf{x}| \to \infty \end{aligned} \tag{3.1}
$$

with $A(\mathbf{u}) = -\Delta\mathbf{u} + \mu\frac{\partial^2\mathbf{u}}{\partial x_1^2}$, $0 < \mu < 1$, $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, an exterior domain to a compact body $\mathcal{B} = \Omega^c$. We can assume, without loss of generality, that $B_{\frac{1}{2}}(\mathbf{0}) \subset \mathcal{B} \subset B_1(\mathbf{0})$. Using the results from the two preceding sections we shall prove existence, uniqueness, L^q–estimates as well as asymptotic properties at large distances for solutions to (3.1).

Definition 3.1 *We say that the vector field* $\mathbf{u} : \Omega \to \mathbb{R}^N$ *is a q–weak solution to (3.1) if for some* $q \in (1; \infty)$

(i) $\mathbf{u} \in D^{1,q}(\Omega)$

(ii) $\mathbf{u}$ *is (weakly) divergence free in* Ω

(iii) $\mathbf{u}\big/_{\partial\Omega} = \mathbf{u}_*$ *in the trace sense*

(iv) $\lim_{R\to\infty} \int_{S_N} |\mathbf{u}(R,\omega)|d\omega = 0$

(v) for all $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^N)$ *with zero divergence in* $\mathbb{R}^N$ *we have*

$$
\int_\Omega \Big(\nabla\mathbf{u} : \nabla\boldsymbol{\varphi} - \mu\frac{\partial \mathbf{u}}{\partial x_1}\cdot\frac{\partial \boldsymbol{\varphi}}{\partial x_1} + \beta\frac{\partial \mathbf{u}}{\partial x_1}\cdot\boldsymbol{\varphi}\Big) d\mathbf{x} = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle\,. \tag{3.2}
$$

Remark 3.1 Let us note that here we use a bit different definition of the weak solution than in the whole space.

If $\mathbf{f}$ has some (mild) degree of regularity, we can associate to every q–weak solution the corresponding pressure field

Lemma 3.1 *Let* $\Omega \subset \mathbb{R}^N$ *be a locally lipschitzian exterior domain in* $\mathbb{R}^N$, $N \geq 2$. *Let* $\mathbf{f} \in W_0^{-1,q}(\Omega_R)$ $\forall R > diam\,\Omega^c$. *Then to every weak solution* $\mathbf{u}$ *to (3.1) we can associate a pressure field* $p \in L^q_{loc}(\overline{\Omega})$ *such that*

$$
\int_\Omega \Big(\nabla\mathbf{u} : \nabla\boldsymbol{\varphi} - \mu\frac{\partial \mathbf{u}}{\partial x_1}\cdot\frac{\partial \boldsymbol{\varphi}}{\partial x_1} + \beta\frac{\partial \mathbf{u}}{\partial x_1}\cdot\boldsymbol{\varphi}\big) d\mathbf{x} = \int_\Omega p\nabla\cdot\boldsymbol{\varphi}\, d\mathbf{x} + \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \tag{3.3}
$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^N)$.

Proof: see Appendix, Theorem VIII.5.3

□

We shall also use several times the following result on the local regularity of q–weak solutions.

Theorem 3.1 *Let* $\mathbf{f} \in W^{m,q}_{loc}(\Omega)$*,* $m \geq 0$*,* $1 < q < \infty$ *and let*

$$\mathbf{u} \in W^{1,q}_{loc}(\Omega), p \in L^q_{loc}(\Omega)$$

with $\mathbf{u}$ *(weakly) divergence free satisfying (3.3) for all* $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$*. Then*

$$\mathbf{u} \in W^{m+2,q}_{loc}(\Omega), p \in W^{m+1,q}_{loc}(\Omega)\,.$$

In particular, if $\mathbf{f} \in C^\infty(\Omega)$*, then* $\mathbf{u}$*,* $p \in C^\infty(\Omega)$*. Furthermore, if* Ω *is of class* C^{m+2} *and*

$$\mathbf{f} \in W^{m,q}_{loc}(\overline{\Omega}), \mathbf{u}_* \in W^{m+2-\frac{1}{q},q}(\partial\Omega), \mathbf{u} \in W^{1,q}_{loc}(\overline{\Omega})\,,$$

then

$$\mathbf{u} \in W^{m+2,q}_{loc}(\overline{\Omega}), p \in W^{m+1,q}_{loc}(\overline{\Omega})\,.$$

In particular, if Ω *is of class* C^∞ *and* $\mathbf{f} \in C^\infty(\overline{\Omega}')$ *for all bounded* $\Omega' \subset \Omega$ *and* $\mathbf{u}_* \in C^\infty(\partial\Omega)$*, then* $\mathbf{u}$*,* $p \in C^\infty(\overline{\Omega}')$ *for all bounded* $\Omega' \subset \Omega$*.*

Proof: It si an easy consequence of Theorem VIII.5.4. We can assume the system (3.1) as the modified Stokes problem with the right hand side $\mathbf{f} - \beta\frac{\partial \mathbf{u}}{\partial x_1}$.

□

We have the following uniqueness results

Theorem 3.2 *Let* $1 < q < N+1$ *and* $\mathbf{u}$ *be a* q*–weak solution to (3.1) with* $\mathbf{f} = \mathbf{u}_* = 0$ *such that* $\mathbf{u} \in L^r(\Omega)$ *for some* $1 < r < \infty$*. Let* $\Omega \in C^2$ *be an exterior domain in* $\mathbb{R}^N$*. Then*

$$\mathbf{u} = \mathbf{0}\,, \quad p = a \in \mathbb{R}\,,$$

where p *is the associated pressure to* $\mathbf{u}$ *due to Lemma 3.1.*

Proof: From Lemma 3.1 we get the existence of the pressure field $p \in L^q_{loc}(\overline{\Omega})$. Applying Theorem 3.1 we see that $\mathbf{u}$, $p \in C^\infty(\Omega)$ and $\mathbf{u} \in W^{2,q}_{loc}(\overline{\Omega})$, $p \in W^{1,q}_{loc}(\overline{\Omega})$. Let $\eta \in C_0^\infty(B_{2R}(\mathbf{0}))$ be the usual cut–off function, $\eta = 1$ in $B_R(\mathbf{0})$, $R \geq 1$. Then

$$\mathbf{v} = \mathbf{u}(1-\eta)$$
$$\pi = p(1-\eta)$$

is a q–weak solution to

$$\left.\begin{aligned} A(\mathbf{v}) + \beta\frac{\partial \mathbf{v}}{\partial x_1} + \nabla\pi = \mathbf{F} \\ \nabla\cdot\mathbf{v} = G \end{aligned}\right\} \quad \text{in } \mathbb{R}^N\,, \tag{3.4}$$

where

$$\mathbf{F} = 2\nabla\mathbf{u}\nabla\eta - 2\mu\frac{\partial \mathbf{u}}{\partial x_1}\frac{\partial \eta}{\partial x_1} + \mathbf{u}\Big(\Delta\eta - \mu\frac{\partial^2\eta}{\partial x_1^2}\Big) - \beta\mathbf{u}\frac{\partial \eta}{\partial x_1} - p\nabla\eta$$
$$G = -\mathbf{u}\cdot\nabla\eta\,,$$

i.e. evidently $\mathbf{F}$, $G \in C_0^\infty(\mathbb{R}^N)$. Moreover, we know that $\mathbf{v} \in L^r(\mathbb{R}^N)$, $\nabla\mathbf{v} \in L^q(\mathbb{R}^N)$, $\pi \in L^q_{loc}(\mathbb{R}^N)$. Now, let $(\mathbf{V}, P)$ be solution to (3.4) constructed by

(2.9). Then, by Theorem 2.1 $\mathbf{V} \in L^t(\mathbb{R}^N)$ $\forall t \in (\frac{N+1}{N-1};\infty)$, $\nabla \mathbf{V} \in L^s(\mathbb{R}^N)$ $\forall s \in (\frac{N+1}{N};\infty)$ and $P \in L^a(\mathbb{R}^N)$ $\forall a \in (\frac{N}{N-1};\infty)$.

Lemma 2.4 gives us therefore (as both $\mathbf{v}$ and $\mathbf{V}$ are integrable) $\mathbf{v} = \mathbf{V}$ and $\pi - P = a \in \mathbb{R}$. We put $p^{(1)} = p - a$. Evidently, $p^{(1)}$ is again a pressure field and $(\mathbf{u}, p^{(1)})$ satisfies (3.3) for all $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^N)$. Moreover, $p^{(1)} \in L^a(\Omega)$, $a \in (\frac{N}{N-1};\infty)$ since $p^{(1)}$ and P coincide on $B^{2R}(\mathbf{0})$ and $p \in W^{1,q}_{loc}(\overline{\Omega})$. The couple $(\mathbf{v}, \pi^{(1)})$ with $\pi^{(1)} = p^{(1)}(1-\eta)$ is a q–weak solution to

$$\left.\begin{aligned} A(\mathbf{v}) + \beta\frac{\partial \mathbf{v}}{\partial x_1} + \nabla \pi^{(1)} = \mathbf{F}^{(1)} \\ \nabla \cdot \mathbf{v} = G \end{aligned}\right\} \quad \text{in } \mathbb{R}^N\,,$$

where $\mathbf{F}^{(1)} = \mathbf{F} + a\nabla\eta \in C_0^\infty(\mathbb{R}^N)$. We easily see that[11]

$$\begin{aligned} \mathbf{u} \in L^t(\Omega)\,, &\quad t \in \Big(\frac{N+1}{N-1};\infty\Big) \\ \nabla\mathbf{u} \in L^s(\Omega)\,, &\quad s \in \Big(\frac{N+1}{N};\infty\Big) \\ p^{(1)} \in L^a(\Omega)\,, &\quad a \in \Big(\frac{N}{N-1};\infty\Big)\,. \end{aligned}$$

Now, let $N = 3$. As (3.2) evidently holds (from the density) also for functions $\boldsymbol{\varphi}$ from $W^{1,q'}(\Omega)$ with zero divergence and (3.3) makes sense if moreover $\nabla\mathbf{v} \in L^b(\Omega)$ for some $b < 3$, we can use as test function in (3.3) $\boldsymbol{\varphi} = \mathbf{u}\zeta_R$, ζ_R the Sobolev cut–off function with $R > \mathrm{e}^2$. We get

$$\begin{aligned} \int_\Omega \Big(\nabla\mathbf{u} : \nabla\mathbf{u} - \mu\frac{\partial\mathbf{u}}{\partial x_1}\cdot\frac{\partial\mathbf{u}}{\partial x_1}\Big)\zeta_R \mathrm{d}\mathbf{x} = -\int_\Omega \Big(\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial x_i}\frac{\partial\zeta_R}{\partial x_i} - \mu\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial x_1}\frac{\partial\zeta_R}{\partial x_1}\Big)\mathrm{d}\mathbf{x} + \\ + \int_\Omega p^{(1)}\mathbf{u}\cdot\nabla\zeta_R \mathrm{d}\mathbf{x} + \frac{\beta}{2}\int_\Omega |\mathbf{u}|^2\frac{\partial\zeta_R}{\partial x_1}\mathrm{d}\mathbf{x}\,. \end{aligned} \tag{3.5}$$

The right hand side of (3.5) can be estimated by

$$C(\|\nabla\mathbf{u}\|_2 + \|p^{(1)}\|_2)\|\mathbf{u}\nabla\zeta_R\|_2 + \|\mathbf{u}\|_{\frac{5}{2}}\|\mathbf{u}\nabla\zeta_R\|_{\frac{5}{3}}$$

and tends to zero as $R \to \infty$ (see Lemma VIII.2.2). Hence $\nabla\mathbf{u} = \mathbf{0}$ a.e. in $\mathbb{R}^3$ and since $\mathbf{u}$ is summable, $\mathbf{u} = \mathbf{0}$. Now easily $p^{(1)} = 0$ and therefore $p = a \in \mathbb{R}$.

In two space dimensions, we have

$$\begin{aligned} \mathbf{u} \in L^t(\Omega)\,, &\quad t \in (3;\infty) \\ \nabla\mathbf{u} \in L^s(\Omega)\,, &\quad s \in \Big(\frac{3}{2};\infty\Big) \\ p^{(1)} \in L^a(\Omega)\,, &\quad a \in (2;\infty)\,. \end{aligned}$$

Therefore, unless $r < 3$ or $q < \frac{3}{2}$, we cannot control the term $\int_\Omega |\mathbf{u}|^2\frac{\partial\zeta_R}{\partial x_1}\mathrm{d}\mathbf{x}$. In such situations, we use as a test function

$$\boldsymbol{\varphi} = \mathbf{u}|\mathbf{u}|^\delta\zeta_R\,,$$

[11] the integrability at large distances follows from the properties of $\mathbf{V}$ and the fact that $\mathbf{u} = \mathbf{V}$ in $B^{2R}(\mathbf{0})$. Concerning the integrability near $\partial\Omega$, since $\mathbf{u} \in W^{1,q}_{loc}(\overline{\Omega})$, we have due to Theorem 3.1 that $\mathbf{u} \in W^{2,q}_{loc}(\overline{\Omega})$, i.e. $\mathbf{u} \in W^{1,\frac{Nq}{N-q}}_{loc}(\overline{\Omega})$, $q < N$. After finite number of steps we have $\mathbf{u} \in W^{1,r}_{loc}(\overline{\Omega})$, $r \in [1;\infty]$.

where $\delta > 0$ is sufficiently small. Instead of (3.5) we have

$$\int_\Omega \Big(|\nabla \mathbf{u}|^2 - \mu \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \mathbf{u}}{\partial x_1}\Big)|\mathbf{u}|^\delta \zeta_R \mathrm{d}\mathbf{x} = \int_\Omega \Big(\mu \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial x_1}\frac{\partial \zeta_R}{\partial x_1} - \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial x_i}\frac{\partial \zeta_R}{\partial x_i}\Big)|\mathbf{u}|^\delta \mathrm{d}\mathbf{x} +$$
$$+ \int_\Omega p^{(1)} \mathbf{u} \cdot \nabla \zeta_R |\mathbf{u}|^\delta \mathrm{d}\mathbf{x} + \frac{\beta}{2+\delta}\int_\Omega |\mathbf{u}|^{2+\delta}\frac{\partial \zeta_R}{\partial x_1} + \delta \int_\Omega p^{(1)} u_j |\mathbf{u}|^{\delta-2} u_k \frac{\partial u_k}{\partial x_j}\zeta_R -$$
$$-\delta \int_\Omega \Big[u_j \frac{\partial u_j}{\partial x_i} u_k \frac{\partial u_k}{\partial x_i} - \mu u_k \frac{\partial u_k}{\partial x_1} u_k \frac{\partial u_k}{\partial x_1}\Big]\zeta_R |\mathbf{u}|^{\delta-2}\mathrm{d}\mathbf{x}\,.$$

Passing with $R \to \infty$ and using

$$\int_\Omega |\mathbf{u}|^{2+\delta}\Big|\frac{\partial \zeta_R}{\partial x_1}\Big|\mathrm{d}\mathbf{x} \le \|\mathbf{u}\|_{3+\delta}^{1+\delta}\|\mathbf{u}\nabla\zeta_R\|_{\frac{3+\delta}{2}}\,,$$

we get

$$\int_\Omega \Big(|\nabla \mathbf{u}|^2 - \mu \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \mathbf{u}}{\partial x_1}\Big)|\mathbf{u}|^\delta \mathrm{d}\mathbf{x} \le \delta \int_\Omega (|p^{(1)}\|\mathbf{u}|^\delta |\nabla \mathbf{u}| + |\mathbf{u}|^\delta |\nabla \mathbf{u}|^2)\mathrm{d}\mathbf{x}\,.$$

Finally we pass with $\delta \to 0$ and using the Lebesgue dominated convergence theorem we get, as in the threedimensional case, $\mathbf{u} = \mathbf{0}$ and $p = a \in \mathbb{R}$.

□

Remark 3.2 If $q \ge 2$, then Theorem 3.2 holds also for $\Omega \in C^{0,1}$, an exterior domain. Then we have namely $\nabla \mathbf{u} \in L^2_{loc}(\overline{\Omega})$ and the first term on the left hand side of (3.5) is finite.

We start to construct weak solutions to (3.1). As the methods in two and three space dimensions are significantly different, we study each case separately. As Ω is unbounded, we expect that the compatibility condition

$$\int_{\partial\Omega} \mathbf{u}_* \cdot \mathbf{n}\mathrm{d}S \tag{3.6}$$

might be omitted. This is evidently true if $N = 3$ but if $N = 2$, the method presented in [Ga1] seems not to work as well as some modifications of it. We shall mention the crucial problem later on. Therefore, for $N = 2$ we shall suppose the condition (3.6) to be satisfied. For our application, the condition is trivially satisfied in both two– and threedimensional cases.

III.3.1 Threedimensional modified Oseen problem

Theorem 3.3 *Let Ω be a threedimensional exterior, locally lipschitzian domain. Let*

$$\mathbf{f} \in D_0^{-1,2}(\Omega)\,, \quad \mathbf{u}_* \in W^{\frac{1}{2},2}(\partial\Omega)\,.$$

Then there exists (2–) weak solution to (3.1). This solution satisfies the estimates

$$\|\mathbf{u}\|_{2,\Omega_R} + |\mathbf{u}|_{1,2} \le C(|\mathbf{f}|_{-1,2} + \|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})$$
$$\int_{S_3} |\mathbf{u}(R,\omega)|d\omega = O\Big(\frac{1}{R}\Big) \quad \text{as } R = |\mathbf{x}| \to \infty \tag{3.7}$$
$$\|p\|_{2,\Omega_R/\mathbb{R}} \le C(|\mathbf{f}|_{-1,2} + |\mathbf{u}|_{1,2})$$

for $R > diam\,\Omega^c$; p is the pressure associated to $\mathbf{u}$ by Lemma 3.1.

Proof: We look for the solution in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{w} + \boldsymbol{\sigma}\,, \tag{3.8}$$

where

$$\boldsymbol{\sigma} = \frac{\Phi}{4\pi}\nabla\Big(\frac{1}{|\mathbf{x}|}\Big)$$
$$\Phi = \int_{\partial\Omega} \mathbf{u}_* \cdot \mathbf{n} \mathrm{d}S\,.$$

Further, $\mathbf{v} \in W^{1,2}(\Omega)$ denotes the divergence free extension of $\mathbf{u}_* - \boldsymbol{\sigma}$ of bounded support in Ω, see Theorem VIII.3.1.[12] We have

$$\boldsymbol{\sigma} = O\Big(\frac{1}{|\mathbf{x}|^2}\Big) \qquad \text{as } |\mathbf{x}| \to \infty\,.$$

Finally, $(\mathbf{w}, p)$ will be 2–weak solution of the problem

$$\left.\begin{aligned} A(\mathbf{w}) + \beta\frac{\partial \mathbf{w}}{\partial x_1} + \nabla p = \mathbf{F} \\ \nabla\cdot\mathbf{w} = 0 \end{aligned}\right\} \quad \text{in } \Omega \qquad \mathbf{w} = \mathbf{0} \quad \text{at } \partial\Omega \tag{3.9}$$

with $\mathbf{F} = \mathbf{f} - A(\mathbf{v} + \boldsymbol{\sigma}) - \beta\frac{\partial}{\partial x_1}(\mathbf{v} + \boldsymbol{\sigma})$. Then, easily, $\mathbf{u} \in D^{1,2}(\Omega)$ is divergence free, $\mathbf{u}\big/_{\partial\Omega} = \mathbf{u}_*$ in the sense of traces, $\mathbf{u}$ satisfies (3.2) and

$$\int_{S_3} |\mathbf{u}(R,\omega)|\mathrm{d}\omega \le \int_{S_3} |\mathbf{w}(R,\omega)|\mathrm{d}\omega + O\Big(\frac{1}{R^2}\Big)\,.$$

From the properties of $\mathbf{w}$ it follows that $\int_{S_3} |\mathbf{w}(R,\omega)|\mathrm{d}\omega \le \frac{C}{R}$ for $R = |\mathbf{x}|$ sufficiently large, see Lemma VIII.1.12. We are therefore left with the proof of existence of 2–weak solution to (3.9); moreover the condition (iv) from Definition 3.1 is trivially satisfied.

Let us introduce an auxiliary problem. For any $\varepsilon > 0$ we look for $\mathbf{w}^\varepsilon \in \widehat{H}^1_2(\Omega)$ (see (3.13)) solution to the problem

$$((\mathbf{w}^\varepsilon, \boldsymbol{\varphi})) = \langle \mathbf{F}, \boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi} \in \widehat{H}^1_2(\Omega) \tag{3.10}$$

with

$$((\mathbf{w}^\varepsilon, \boldsymbol{\varphi})) = \int_\Omega \Big(\nabla\mathbf{w}^\varepsilon : \nabla\boldsymbol{\varphi} - \mu\frac{\partial \mathbf{w}^\varepsilon}{\partial x_1}\cdot\frac{\partial \boldsymbol{\varphi}}{\partial x_1} + \beta\frac{\partial \mathbf{w}^\varepsilon}{\partial x_1}\cdot\boldsymbol{\varphi} + \varepsilon\mathbf{w}^\varepsilon\cdot\boldsymbol{\varphi})\mathrm{d}\mathbf{x}\,.$$

The space $\widehat{H}^1_2(\Omega) = H^1_2(\Omega)$ is a Hilbert space with the scalar product

$$(\mathbf{w}, \boldsymbol{\varphi}) = \int_\Omega (\nabla\mathbf{w} : \nabla\boldsymbol{\varphi} + \varepsilon\mathbf{w}\cdot\boldsymbol{\varphi})\mathrm{d}\mathbf{x}$$

and $((\mathbf{w}^\varepsilon, \boldsymbol{\varphi}))$ is for any $\mu \in [0,1)$ a continuous sesquilinear form on $\widehat{H}^1_2(\Omega)$. To see this let us recall that $\int_\Omega \frac{\partial\boldsymbol{\varphi}}{\partial x_1}\cdot\boldsymbol{\varphi}\mathrm{d}\mathbf{x} = 0$ due to the density of $C^\infty_0(\Omega)$ in

[12] Evidently, the compatibility condition $\int_{\partial\Omega}(\mathbf{u}_* - \boldsymbol{\sigma})\cdot\mathbf{n}\mathrm{d}S = 0$ is satisfied.

$W_0^{1,2}(\Omega)$. Lax–Milgram theorem (see Theorem VIII.1.1) thus yields existence of a unique $\mathbf{w}^\varepsilon \in \widehat{H}_2^1(\Omega)$, the solution to (3.10). Thus we have

$$\|\nabla \mathbf{w}^\varepsilon\|_2 \leq C|\mathbf{F}|_{-1,2} \tag{3.11}$$

with C independent of ε. Moreover, easily

$$|\mathbf{F}|_{-1,2} \leq C(|\mathbf{f}|_{-1,2} + \|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})\,. \tag{3.12}$$

Therefore, at least for a chosen subsequence, there exists $\mathbf{w} \in \widehat{H}_2^1(\Omega)$ such that

$$\begin{aligned} \mathbf{w}^\varepsilon &\rightharpoonup \mathbf{w} \quad \text{in } L^6(\Omega) \\ \nabla \mathbf{w}^\varepsilon &\rightharpoonup \nabla \mathbf{w} \text{ in } L^2(\Omega)\,. \end{aligned}$$

From (3.10) we get

$$\int_\Omega \Big(\nabla \mathbf{w} \cdot \nabla \boldsymbol{\varphi} - \mu \frac{\partial \mathbf{w}}{\partial x_1} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_1} + \beta \frac{\partial \mathbf{w}}{\partial x_1} \cdot \boldsymbol{\varphi}\Big) \mathrm{d}\mathbf{x} = \langle \mathbf{F}, \boldsymbol{\varphi} \rangle \quad \forall \boldsymbol{\varphi} \in {}_0\mathcal{D}(\Omega)\,.$$

Using the Hölder inequality we easily have

$$\|\mathbf{w}\|_{2,\Omega_R} \leq |\Omega_R|^{\frac{1}{3}} \|\mathbf{w}\|_6 \leq C|\Omega_R|^{\frac{1}{3}} |\mathbf{w}|_{1,2}\,.$$

Finally, we can come back to the weak formulation of (3.1) and using Theorem VIII.5.3 we get

$$\begin{aligned} \|p\|_{2,\Omega_R/\mathbb{R}} &\leq C(|\mathbf{f}|_{-1,2} + \|\mathbf{u}\|_{2,\Omega_R} + |\mathbf{u}|_{1,2}) \leq \\ &\leq C(R)(|\mathbf{f}|_{-1,2} + |\mathbf{u}|_{1,2}) \end{aligned}$$

which finishes the proof.

□

Corollary 3.1 *Let Ω be an exterior threedimensional locally lipschitzian domain, $\mathbf{u}_* \in W^{\frac{1}{2},2}(\partial\Omega)$ and $\mathbf{f} \in C_0^\infty(\overline{\Omega})$. Then the solution, constructed in Theorem 3.3 has the following decay properties for $|\mathbf{x}|$ sufficiently large:*

$$\begin{aligned} |D^\alpha \mathbf{u}(\mathbf{x})| &\leq C|\mathbf{x}|^{-1-\frac{|\alpha|}{2}} (1 + s(\mathbf{x}))^{-1-\frac{|\alpha|}{2}} \quad |\alpha| \geq 0 \\ |D^\alpha p(\mathbf{x})| &\leq C|\mathbf{x}|^{-2-|\alpha|} \quad |\alpha| > 0\,. \end{aligned} \tag{3.13}$$

Remark 3.3 We can add to p such a constant that (3.13) holds also for $|\alpha| = 0$.

Proof: Theorem 3.1 implies that solution, constructed in Theorem 3.3, is of class $C^\infty(\Omega)$. Let us recall that we also have $\mathbf{u} \in L^6(\Omega)$, $\nabla \mathbf{u} \in L^2(\Omega)$ and $p \in L^2_{loc}(\overline{\Omega})$. The couple $(\mathbf{U}, P) \in C^\infty(\Omega)$, $\mathbf{U} = \mathbf{u}(1-\eta)$, $P = p(1-\eta)$, η as in Theorem 3.2, is a 2–weak solution to

$$\begin{aligned} A(\mathbf{U}) + \beta \frac{\partial \mathbf{U}}{\partial x_1} + \nabla P &= \mathbf{F} \\ \nabla \cdot \mathbf{U} &= G \end{aligned} \tag{3.14}$$

with $\mathbf{F}$, $G \in C_0^\infty(\mathbb{R}^3)$,

$$
\begin{gathered}
\mathbf{F} = (1-\eta)\mathbf{f} + 2\nabla\mathbf{u}\nabla\eta - 2\mu\frac{\partial \mathbf{u}}{\partial x_1}\frac{\partial \eta}{\partial x_1} + \mathbf{u}A(\eta) - \beta\mathbf{u}\frac{\partial \eta}{\partial x_1} - p\nabla\eta \\
G = -\mathbf{u}\cdot\nabla\eta\,.
\end{gathered} \tag{3.15}
$$

Clearly $\mathbf{U} \in L^6(\mathbb{R}^3)$, $\nabla\mathbf{U} \in L^2(\mathbb{R}^3)$ and $P \in L^2_{loc}(\mathbb{R}^3)$. Theorem 2.2 guarantees existence of another solution $(\mathbf{W}, \Pi)$ such that $\mathbf{W}$ and $\nabla\mathbf{W}$ have the same integrability as $\mathbf{U}$ and $\nabla\mathbf{U}$, respectively. So, due to Lemma 2.3 $\mathbf{U} = \mathbf{W}$ and $P = \Pi - a$.

We take $p^{(1)} = p + a$, denote $P^{(1)} = (1-\eta)p^{(1)}$. Then the pair $(\mathbf{U}, P^{(1)})$ solves the system

$$
\begin{gathered}
A(\mathbf{U}) + \beta\frac{\partial \mathbf{U}}{\partial x_1} + \nabla P^{(1)} = \mathbf{F}^{(1)} \\
\nabla\cdot\mathbf{U} = G
\end{gathered} \tag{3.16}
$$

with $\mathbf{F}^{(1)} = \mathbf{F} - a\nabla\eta$. Due to Lemma 2.3, $(\mathbf{U}, P^{(1)})$ coincides with the solution of (3.16) constructed in Theorem 2.1 and therefore, by Corollary 2.1 we get the asymptotic properties of $(\mathbf{U}, P^{(1)})$. As $(\mathbf{u}, p^{(1)})$, solution constructed in Theorem 3.3, coincides with it outside $B_{2R}(\mathbf{0})$, we get (3.13).

□

Remark 3.4 Using the same procedure as above for the twodimensional case we are not able to control the behaviour of $\mathbf{u}$ at infinity and therefore we cannot apply Lemma 2.3.

III.3.2 Twodimensional modified Oseen problem

As announced in Remark 3.4, we cannot easily use the technique from the threedimensional situation. We therefore follow the ideas of Finn and Smith (see [FiSm]; see also [Ga1] Section VII.5). Let us consider the following modification of (3.1)

$$
\left.\begin{aligned}
A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \delta\mathbf{u} + \nabla p = \mathbf{f} \\
\nabla\cdot\mathbf{u} = 0
\end{aligned}\right\} \quad \text{in } \Omega
\\
\mathbf{u} = \mathbf{u}_* \quad \text{at } \partial\Omega
\\
\mathbf{u} \to 0 \quad \text{as } |\mathbf{x}| \to \infty \tag{3.17}
$$

with $\delta > 0$. We first prove the existence of a solution to (3.17) in certain L^q–spaces and get some δ–independent estimates which will allow us to pass with δ to zero and therefore get solution to the original problem (3.1). The proof will be similar to the threedimensional situation. We therefore first consider the following non–homogeneous version of $(3.17)_{1,2}$ in $\mathbb{R}^2$

$$
\left.\begin{aligned}
A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \delta\mathbf{u} + \nabla\pi = \mathbf{f} \\
\nabla\cdot\mathbf{u} = g
\end{aligned}\right\} \quad \text{in } \mathbb{R}^2\,. \tag{3.18}
$$

We denote

$$\begin{aligned}(\mathbf{u},\mathbf{v}) &= \int_{\mathbb{R}^2} \mathbf{u}\cdot\mathbf{v}\mathrm{d}\mathbf{x}\\ a_\mu(\mathbf{u},\mathbf{v}) &= (\nabla\mathbf{u},\nabla\mathbf{v}) - \mu\Big(\frac{\partial \mathbf{u}}{\partial x_1},\frac{\partial \mathbf{v}}{\partial x_1}\Big)\,.\end{aligned} \tag{3.19}$$

Lemma 3.2 *Let* $\mathbf{f}\in L^q(\mathbb{R}^2)$, $g\in W^{1,q}(\mathbb{R}^2)\cap D_0^{-1,q}(\mathbb{R}^2)$, $1<q<\frac{3}{2}$. *Then for all* $\delta\in(0,1]$ *there exists a solution* $(\mathbf{u}^\delta,\pi^\delta)$ *to (3.18) such that*

$$\begin{aligned}\mathbf{u}^\delta &\in W^{2,q}(\mathbb{R}^2)\bigcap D_0^{1,\frac{3q}{3-q}}(\mathbb{R}^2)\bigcap L^{\frac{3q}{3-2q}}(\mathbb{R}^2)\\ \nabla\pi^\delta &\in L^q(\mathbb{R}^2)\\ u_2^\delta &\in D_0^{1,q}(\mathbb{R}^2)\bigcap L^{\frac{2q}{2-q}}(\mathbb{R}^2)\\ \frac{\partial u_1}{\partial x_1} &\in L^q(\mathbb{R}^2)\end{aligned}$$

and

$$\begin{aligned}\beta\|u_2^\delta\|_{\frac{2q}{2-q}} + \beta|u_2^\delta|_{1,q} + \beta^{\frac{2}{3}}|\mathbf{u}^\delta|_{1,\frac{3q}{3-2q}} + \beta^{\frac{1}{3}}|\mathbf{u}^\delta|_{1,\frac{3q}{3-q}} + \beta\Big\|\frac{\partial u_1^\delta}{\partial x_1}\Big\|_q +\\ +|\mathbf{u}^\delta|_{2,q} \le C(\|\mathbf{f}\|_q + |g|_{1,q} + \beta\|g\|_q)\\ |\pi^\delta|_{1,q} \le C(\|\mathbf{f}\|_q + |g|_{1,q} + \beta\|g\|_q + \delta|g|_{-1,q})\end{aligned} \tag{3.20}$$

with C *independent of* δ *and* β. *Moreover, if* $\mathbf{w}$, τ *are such that*

a) $\mathbf{w}\in W^{1,2}(\mathbb{R}^2)$, $\tau\in L^1_{loc}(\mathbb{R}^2)$

b) for all $\boldsymbol{\psi}\in C_0^\infty(\mathbb{R}^2)$

$$a_\mu(\mathbf{w},\boldsymbol{\psi}) + \beta\Big(\frac{\partial\mathbf{w}}{\partial x_1},\boldsymbol{\psi}\Big) + \delta(\mathbf{w},\boldsymbol{\psi}) = (\tau,\nabla\cdot\boldsymbol{\psi}) - \langle\mathbf{f},\boldsymbol{\psi}\rangle$$

and $\nabla\cdot\mathbf{w} = g$ *a.e. in* $\mathbb{R}^2$,

then necessarily $\mathbf{w}=\mathbf{u}^\delta$ *and* $\tau = \pi^\delta +$ *const a.e. in* $\mathbb{R}^2$.

Proof: The proof is very analogous to the proof of Theorem 2.1. We again assume first $\beta=1$ and use finally rescalling in order to obtain the constant independent on β; for this purpose we must take $\frac{\delta}{\beta^2}$ instead of δ. We can again calculate the Fourier transform of $\mathbf{u}^\delta$ and π^δ. Denoting $\mathbf{U}=\mathcal{F}(\mathbf{u}^\delta)$, $P=\mathcal{F}(\pi^\delta)$ we have

$$\begin{aligned}U_m(\xi) &= \frac{(\delta_{mk}|\xi|^2-\xi_m\xi_k)\mathcal{F}(f_k)(\xi)}{h(\xi,\delta)|\xi|^2} + \frac{i\xi_m}{|\xi|^2}\mathcal{F}(g)(\xi)\\ P(\xi) &= \frac{i\xi_k}{|\xi|^2}\mathcal{F}(f_k)(\xi) + \frac{h(\xi,\delta)}{|\xi|^2}\mathcal{F}(g)(\xi)\,,\end{aligned} \tag{3.21}$$

where $h(\xi,\delta) = (1-\mu)\xi_1^2 + \xi_2^2 - i\xi_1 + \frac{\delta}{\beta^2}$. It is an easy matter to see that $\frac{(\delta_{mk}|\xi|^2-\xi_m\xi_k)}{h(\xi,\delta)|\xi|^2}$ satisfies under certain conditions the assumptions of Lizorkin multiplier theorem; moreover Lemma 2.1 can be applied to show that the constants do not depend on $\frac{\delta}{\beta^2}$. We can therefore easily get the estimates for $\mathbf{u}^\delta$ from Theorem 2.1. Moreover we can show that $\mathbf{u}^\delta \in W^{1,q}(\mathbb{R}^2)$. We have namely that

$$\frac{1}{h(\xi,\delta)} \leq \frac{C}{\delta}\beta^2$$
$$\frac{|\xi|}{h(\xi,\delta)} \leq \frac{C}{\sqrt{\delta}}\beta$$

and therefore

$$\|\mathbf{u}^\delta\|_{1,q} \leq C(\delta)(\|\mathbf{f}\|_q + \|g\|_{1,q} + |g|_{-1,q})\,.$$

Without having $g \in D^{-1,q}(\mathbb{R}^2)$ we do not have any estimate of the gradient of the pressure; this is caused by the term $\frac{\delta}{\beta^2|\xi|^2}\mathcal{F}(g)$. But for $g \in D^{-1,q}(\mathbb{R}^2)$ we can similarly as in Theorem 2.2 get the estimate $(3.20)_2$.

It remains to prove the uniqueness part. Let us set $\mathbf{v} = \mathbf{w} - \mathbf{u}^\delta$, $p = \tau - \pi^\delta$. Then $\mathbf{v}$ obeys the identity

$$a_\mu(\mathbf{v},\boldsymbol{\varphi}) + \beta\Big(\frac{\partial \mathbf{v}}{\partial x_1},\boldsymbol{\varphi}\Big) + \delta(\mathbf{v},\boldsymbol{\varphi}) = 0 \quad \forall \boldsymbol{\varphi} \in {}_0\mathcal{D}(\mathbb{R}^2)\,. \tag{3.22}$$

We know that $\mathbf{u}^\delta \in W^{2,q}(\mathbb{R}^2)$ and therefore, by imbedding, $\mathbf{u}^\delta \in W^{1,2}(\mathbb{R}^2)$. Moreover, $\nabla\cdot\mathbf{u}^\delta = 0$ and we can easily extend the validity of (3.22) for $\boldsymbol{\varphi} \in \widehat{H}^1_2(\mathbb{R}^2)$, by density (see Remark VIII.3.4). Therefore we may use $\mathbf{v}$ as the test function in 3.22 and get (recall that $(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial x_1}) = 0$ by the density argument)

$$C\|\nabla \mathbf{v}\|_2^2 + \delta\|\mathbf{v}\|_2^2 \leq 0$$

i.e. $\mathbf{v} = \mathbf{0}$ in $\widehat{H}^1_2(\mathbb{R}^2)$. Therefore

$$(p, \nabla\cdot\boldsymbol{\psi}) = 0 \quad \forall \boldsymbol{\psi} \in C_0^\infty(\mathbb{R}^2)\,,$$

which implies $p =$ const and $\tau = \pi^\delta +$ const in $\mathbb{R}^2$.

□

Remark 3.5 Assuming more regularity about $\mathbf{f}$ and g we easily get

$$\|\nabla\pi^\delta\|_{\frac{2q}{2-q}} \leq C(\|\mathbf{f}\|_{\frac{2q}{2-q}} + \|\nabla g\|_{\frac{2q}{2-q}} + \beta\|g\|_{\frac{2q}{2-q}} + \delta\|g\|_q)\,. \tag{3.23}$$

This follows from the fact that $\frac{\xi_k}{|\xi|^2}$ satisfies the assumptions of Theorem II.3.3 with $\beta = \frac{1}{2}$.

The next step consists in proving the existence of a generalized solution to (3.17). We call $\mathbf{u}^\delta$ 2–generalized solution to (3.17) if the conditions (i)–(iv) from Definition 3.1 are satisfied with $\mathbf{u}^\delta$, $q = 2$ and

$$a_\mu(\mathbf{u}^\delta,\boldsymbol{\varphi}) + \beta\Big(\frac{\partial \mathbf{u}^\delta}{\partial x_1},\boldsymbol{\varphi}\Big) + \delta(\mathbf{u}^\delta,\boldsymbol{\varphi}) = \langle \mathbf{f},\boldsymbol{\varphi}\rangle \tag{3.24}$$

holds for all $\boldsymbol{\varphi} \in {}_0\mathcal{D}(\Omega)$.

Lemma 3.3 *Let Ω be a locally lipschitzian, exterior domain of $\mathbb{R}^2$ and let*

$$\mathbf{f} \in D_0^{-1,2}(\Omega)\,, \mathbf{u}_* \in W^{\frac{1}{2},2}(\partial\Omega)\,.$$

Let moreover

$$\int_{\partial\Omega} \mathbf{u}_* \cdot \mathbf{n} dS \ = 0\,.$$

Then $\forall \delta \in (0,1]$ there exists a generalized solution $\mathbf{u}^\delta$ to the problem (3.17). We have

$$\delta\|\mathbf{u}^\delta\|_2 + \|\mathbf{u}\|_{2,\Omega_R} + |\mathbf{u}|_{1,2} \le c_1(|\mathbf{f}|_{-1,2} + (1+\beta)\|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)}) \tag{3.25}$$

with $c_1 = c_1(R,\Omega)$. Moreover, denoting p^δ the associated pressure field, then

$$\|p^\delta\|_{2,\Omega_R/\mathbb{R}} \le c_2(|\mathbf{f}|_{-1,2} + (1+\beta)|\mathbf{u}^\delta|_{1,2}) \tag{3.26}$$

for all $R > diam\,\Omega^c$, $c_2 = c_2(R,\Omega)$.

Proof: The proof is analogous to the proof of Theorem 3.3; we only cannot extend $\mathbf{u}_*$ with nonzero flux over the boundary to a square integrable function; the extension would behave like $\frac{1}{|\mathbf{x}|}$ for $|\mathbf{x}|$ sufficiently large, see also Remark VIII.3.3. Therefore we must assume the condition (3.6) to be satisfied.

We search the solution in the form

$$\mathbf{u}^\delta = \mathbf{v}^\delta + \mathbf{w}^\delta$$

with $\mathbf{v}^\delta$ a divergence free extension of $\mathbf{u}_*$ with bounded support in Ω and $\mathbf{w}^\delta \in \widehat{H}_2^1(\Omega)$, solution to

$$a_\mu(\mathbf{w}^\delta, \boldsymbol{\varphi}) + \beta\Big(\frac{\partial \mathbf{w}^\delta}{\partial x_1}, \boldsymbol{\varphi}\Big) + \delta(\mathbf{w}^\delta, \boldsymbol{\varphi}) = \langle \mathbf{F}, \boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi} \in {}_0\mathcal{D}(\Omega) \tag{3.27}$$

with $\mathbf{F} = \mathbf{f} - A(\mathbf{v}^\delta) - \beta\frac{\partial}{\partial x_1}\mathbf{v}^\delta - \delta\mathbf{v}^\delta$. Applying the Lax–Milgram theorem in the same manner as in the proof of Theorem 3.3 we get the existence of a unique solution to (3.27). Moreover, using the properties of $\mathbf{v}^\delta$ we also have

$$\delta\|\mathbf{w}^\delta\|_2 + |\mathbf{w}^\delta|_{1,2} \le C(|\mathbf{f}|_{-1,2} + (1+\beta)\|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})\,.$$

Again, as in Theorem 3.3, we get estimates (3.25) and (3.26). We finish the proof by verifying the condition (iv) from Definition 3.1. Actually we have for $\delta > 0$ that $\mathbf{u}^\delta \in W^{1,2}(\Omega)$ and so, putting $|\mathbf{x}| = r$ and

$$\mathcal{J}(r) = \int_0^{2\pi} |\mathbf{u}^\delta(r,\theta)|^2 \mathrm{d}\theta\,, \quad r > \mathrm{diam}\,\Omega^c\,,$$

we recover

$$\mathcal{J}(r) \in L^1(1,\infty)\,, \frac{d\mathcal{J}}{dr} \in L^1(1,\infty)$$

which implies $\lim_{r\to\infty} \mathcal{J}(r) = 0$.

□

Lemma 3.4 *Let Ω be locally lipschitzian, $\mathbf{f}$, $\mathbf{u}_*$ satisfy the hypothesis of Lemma 3.3 and further, let $\mathbf{f} \in L^q(\Omega)$, $1 < q < \frac{3}{2}$. Then the solution $\mathbf{u}^\delta$ from the preceding lemma satisfy in addition for all $R > diam\,\Omega^c$*

$$\begin{gathered}\mathbf{u}^\delta \in D^{2,q}(\Omega^R) \cap D^{1,\frac{3q}{3-q}}(\Omega^R) \cap L^{\frac{3q}{3-2q}}(\Omega)\\ \pi^\delta \in D^{1,q}(\Omega^R)\\ u_2^\delta \in D^{1,q}(\Omega) \cap L^{\frac{2q}{2-q}}(\Omega)\\ \frac{\partial u_1^\delta}{\partial x_1} \in L^q(\Omega)\end{gathered}$$

along with the estimate

$$\begin{aligned}\beta(\|u_2^\delta\|_{\frac{2q}{2-q}} + |u_2^\delta|_{1,q} + \Big\|\frac{\partial u_1^\delta}{\partial x_1}\Big\|_q) + \min\{1,\beta^{\frac{2}{3}}\}|\mathbf{u}|_{1,\frac{3q}{3-2q}} +\\ +\beta^{\frac{1}{3}}|\mathbf{u}^\delta|_{1,\frac{3q}{3-q},\Omega^R} + |\mathbf{u}^\delta|_{2,q,\Omega^R} + |p^\delta|_{1,q,\Omega^R} \leq\\ \leq C(\|\mathbf{f}\|_q + |\mathbf{f}|_{-1,2} + (1+\beta)^2\|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})\end{aligned} \tag{3.28}$$

with $C = C(q,\Omega,\beta)$.

Proof: We take $\eta = \eta_{\frac{R}{2}}$ the usual cut–off function with $R > 2\,\mathrm{diam}\,\Omega^c$ and setting

$$\begin{gathered}\mathbf{U} = (1-\eta)\mathbf{u}^\delta\\ P = (1-\eta)p^\delta\end{gathered}$$

we have that

$$\left.\begin{gathered}A(\mathbf{U}) + \beta\frac{\partial \mathbf{U}}{\partial x_1} + \delta\mathbf{U} + \nabla P = \mathbf{F}\\ \nabla\cdot\mathbf{U} = G\end{gathered}\right\}\quad \text{in } \mathbb{R}^2 \tag{3.29}$$

with

$$\begin{gathered}\mathbf{F} = (1-\eta)\mathbf{f} - A(\eta)\mathbf{u}^\delta + 2\nabla\eta\nabla\mathbf{u}^\delta - 2\mu\frac{\partial\eta}{\partial x_1}\frac{\partial\mathbf{u}^\delta}{\partial x_1} - \beta\mathbf{u}^\delta\frac{\partial\eta}{\partial x_1} - p^\delta\nabla\eta\\ G = -\mathbf{u}^\delta\cdot\nabla\eta\,.\end{gathered}$$

As $1 < q < \frac{3}{2}$, we easily see

$$\begin{gathered}\|\mathbf{F}\|_q \leq C[\|\mathbf{f}\|_q + (1+\beta)\|\mathbf{u}^\delta\|_{1,2,\Omega_R} + \|p^\delta\|_{2,\Omega_R}]\\ \|G\|_q \leq C\|\mathbf{u}^\delta\|_{2,\Omega_R}\\ |G|_{1,q} \leq C\|\mathbf{u}\|_{1,2,\Omega_R}\\ |G|_{-1,q} \leq C[|\mathbf{u}^\delta\|_{2,\Omega_R} + \|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)}]\,.\end{gathered}$$

The last estimate follows from the fact that $\int_{\partial\Omega}\mathbf{u}_*\cdot\mathbf{n}\mathrm{d}S = 0$ and therefore there exists $\mathbf{w} \in W^{1,2}(\Omega^c)$ such that $\nabla\cdot\mathbf{w} = 0$ in Ω^c, $\mathbf{w} = \mathbf{u}_*$ at $\partial\Omega^c = \partial\Omega$ (see Lemma VIII.3.1) and denoting

$$\overline{\mathbf{u}} = \begin{cases}\mathbf{u} \text{ in } \Omega\\ \mathbf{w} \text{ in } \Omega^c\end{cases}$$

we see that $\mathbf{w} \in W^{1,2}(B_R(\mathbf{0}))$ for any $R > 0$ and

$$|\langle G, \varphi\rangle| = \Big| \int_{\mathbb{R}^N} \nabla \cdot (\overline{\mathbf{u}}\eta)\varphi \mathrm{d}\mathbf{x}\Big| \leq$$
$$\leq C\|\overline{\mathbf{u}}\|_{2,B_R(\mathbf{0})}\|\nabla\boldsymbol{\varphi}\|_{q'} \leq C(\|\mathbf{u}\|_{2,\Omega_R} + \|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})\|\nabla\varphi\|_{q'}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$. Therefore we get

$$\mathbf{U} \in D_0^{1,q}(\mathbb{R}^2) \cap D_0^{2,q}(\mathbb{R}^2) \cap D_0^{1,\frac{3q}{3-q}}(\mathbb{R}^2) \cap L^{\frac{3q}{3-2q}}(\mathbb{R}^2)$$
$$U_2 \in D_0^{1,q}(\mathbb{R}^2) \cap L^{\frac{2q}{2-q}}(\mathbb{R}^2)$$
$$\frac{\partial U_1}{\partial x_1} \in L^q(\mathbb{R}^2)$$
$$P \in D^{1,q}(\mathbb{R}^2)\,,$$

$\mathbf{U}$, P satisfy the estimate corresponding to (3.20). We finish the proof by recalling that

$$\|\mathbf{u}^\delta\|_{\frac{2q}{2-q},\Omega_R} + |u_2^\delta|_{1,q,\Omega_R} + \Big\|\frac{\partial u_1^\delta}{\partial x_1}\Big\|_{q,\Omega_R} + \|\mathbf{u}^\delta\|_{\frac{3q}{3-2q},\Omega_R} \leq C\|\mathbf{u}^\delta\|_{1,2,\Omega_R}\,.$$

□

Combining Lemmas 3.3 and 3.4 we are in a position to prove

Theorem 3.4 *Let Ω be a twodimensional exterior locally lipschitzian domain. Then for*

$$\mathbf{f} \in D_0^{-1,2}(\Omega) \cap L^q(\Omega)\,, 1 < q < \frac{3}{2}$$
$$\mathbf{u}_* \in W^{\frac{1}{2},2}(\partial\Omega)$$

there exists a generalized solution $\mathbf{u}$ to (3.1). Moreover, for all $R > \operatorname{diam}\Omega^c$ we have

$$\mathbf{u} \in D^{2,q}(\Omega^R) \cap D^{1,\frac{3q}{3-q}}(\Omega^R) \cap L^{\frac{3q}{3-2q}}(\Omega)$$
$$p \in D^{1,q}(\Omega^R) \cap L^2(\Omega_R)$$
$$u_2 \in D^{1,q}(\Omega) \cap L^{\frac{2q}{2-q}}(\Omega)$$
$$\frac{\partial u_1}{\partial x_1} \in L^q(\Omega)$$

with p, the pressure associated to $\mathbf{u}$ by Lemma 3.1. Finally we have

$$\|\mathbf{u}\|_{2,\Omega_R} + |\mathbf{u}|_{1,2} + \beta\Big(\|u_2\|_{\frac{2q}{2-q}} + |u_2|_{1,q} + \Big\|\frac{\partial u_1}{\partial x_1}\Big\|_q\Big) + \min\{1,\beta^{\frac{2}{3}}\}|\mathbf{u}|_{1,\frac{3q}{3-2q}} +$$
$$+\beta^{\frac{1}{3}}|\mathbf{u}|_{1,\frac{3q}{3-q},\Omega^R} + |\mathbf{u}|_{2,q,\Omega^R} + |p|_{1,q,\Omega^R} + |p|_{2,\Omega_R/\mathbb{R}} \leq \tag{3.30}$$
$$\leq C(\|\mathbf{f}\|_q + (1+\beta)|\mathbf{f}|_{-1,2} + (1+\beta)^2\|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})$$

with $C = C(q, \Omega, R)$.

Proof: We have that for all $\delta > 0$ the functions $\mathbf{u}^\delta$ satisfy

$$a_\mu(\mathbf{u}^\delta, \boldsymbol{\varphi}) + \beta\Big(\frac{\partial \mathbf{u}^\delta}{\partial x_1}, \boldsymbol{\varphi}\Big) + \delta(\mathbf{u}_\delta, \boldsymbol{\varphi}) = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi} \in {}_0\mathcal{D}(\Omega)\,.$$

From (3.25) and (3.28) we see that there exists at least subsequence such that

$$\begin{aligned} \nabla \mathbf{u}^\delta &\rightharpoonup \nabla \mathbf{u} \quad \text{in } L^2(\Omega) \\ \mathbf{u}^\delta &\rightharpoonup \mathbf{u} \quad \text{in } L^{\frac{3q}{3-2q}}(\Omega) \\ \delta \mathbf{u}^\delta &\to \mathbf{0} \quad \text{in } L^2(\Omega) \end{aligned}$$

and

$$a_\mu(\mathbf{u}, \boldsymbol{\varphi}) + \beta\Big(\frac{\partial \mathbf{u}}{\partial x_1}, \boldsymbol{\varphi}\Big) = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi} \in {}_0\mathcal{D}(\Omega)\,,$$

i.e. the condition (v) from Definition 3.1 is satisfied. The estimates (3.25) and (3.28), thanks to the weak compactness of $L^r(\Omega)$, remain satisfied and we get (3.30). We easily observe that $\mathbf{u}$ is divergence free and assumes the value $\mathbf{u}^*$ at $\partial\Omega$ in the sense of traces. It remains to verify the property (iv) from Definition 3.1. We shall prove even something more; namely that

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}) = \mathbf{0}\,. \tag{3.31}$$

We have that $\mathbf{u} \in D^{2,q}(\Omega^R)$, $1 < q < \frac{3}{2}$ and therefore $\mathbf{u} \in D^{1,\frac{2q}{2-q}}(\Omega^R)$. Thus

$$\mathbf{u} \in D^{1,\frac{2q}{2-q}}(\Omega^R) \cap L^{\frac{3q}{3-2q}}(\Omega^R)$$

and (3.31) follows from Theorem VIII.1.17 and Remark VIII.1.11.

□

Corollary 3.2 *Let $\mathbf{u}_*$ and Ω satisfy the hypothesis of Theorem 3.4. Moreover, let $\mathbf{f} \in C_0^\infty(\overline{\Omega})$. Then the solution, constructed in Theorem 3.4, has the following decay properties for $|\mathbf{x}|$ sufficiently large:*

$$\begin{aligned} |D^\alpha \mathbf{u}(\mathbf{x})| &\le C|\mathbf{x}|^{-\frac{1+|\alpha|}{2}}(1+s(\mathbf{x}))^{-\frac{1+|\alpha|}{2}} \quad |\alpha| \ge 0 \\ |D^\alpha p(\mathbf{x})| &\le C|\mathbf{x}|^{-1-|\alpha|} \quad |\alpha| > 0\,. \end{aligned} \tag{3.32}$$

Remark 3.6 We can add to p such a constant that $(3.32)_2$ holds also for $|\alpha| = 0$.

Proof: From Theorem 3.1 we know that our solution, constructed in Theorem 3.4, is infinitely times continuously differentiable in Ω. Recall that we have constructed the solution in such a way that $\mathbf{u} \in L^{\frac{3q}{3-2q}}(\Omega)$, $\nabla\mathbf{u} \in L^2(\Omega)$ and $p \in L^2_{loc}(\overline{\Omega})$. The couple $(\mathbf{U}, P) \in C^\infty(\Omega)$, $\mathbf{U} = \mathbf{u}(1-\eta)$, $P = p(1-\eta)$, η as in Theorem 3.2, is a 2–weak solution to

$$\begin{aligned} A(\mathbf{U}) + \beta\frac{\partial \mathbf{U}}{\partial x_1} + \nabla P &= \mathbf{F} \\ \nabla\cdot\mathbf{U} &= G \end{aligned}$$

with $\mathbf{F}$, G as in (3.15). Clearly $\mathbf{U} \in L^{\frac{3q}{3-2q}}(\mathbb{R}^2)$, $\nabla \mathbf{U} \in L^2(\mathbb{R}^2)$ and $P \in L^2_{loc}(\mathbb{R}^2)$. Theorem 2.1 guarantees existence of another solution $(\mathbf{W}, \Pi)$ such that $\mathbf{W}$ and $\nabla \mathbf{W}$ have the same integrability as $\mathbf{U}$ and $\nabla \mathbf{U}$, respectively. So, due to Lemma 2.3 $\mathbf{U} = \mathbf{W}$ and $P = \Pi - a$.

We take $p^{(1)} = p + a$ and denote $P^{(1)} = (1-\eta)p^{(1)}$. Then the couple $(\mathbf{U}, P^{(1)})$ solves the system

$$\begin{aligned} A(\mathbf{U}) + \beta \frac{\partial \mathbf{U}}{\partial x_1} + \nabla P^{(1)} &= \mathbf{F}^{(1)} \\ \nabla \cdot \mathbf{U} &= G \end{aligned} \tag{3.33}$$

with $\mathbf{F}^{(1)} = \mathbf{F} - a\nabla \eta$. Evidently, $(\mathbf{U}, P^{(1)})$ coincides with the solution to (3.33) constructed in Theorem 2.1 and therefore, by Corollary 2.1 we get the asymptotic properties of $(\mathbf{U}, P^{(1)})$. As $(\mathbf{u}, p^{(1)})$, solution constructed in Theorem 3.4, coincides with it outside $B_{2R}(\mathbf{0})$, we get (3.32).

□

We finish this subsection by proving the following uniqueness result.

Theorem 3.5 *Let Ω be an exterior domain*[13] *of class C^2 in $\mathbb{R}^N$, $N = 2, 3$. Let $1 < q < N + 1$. Then there exists at most one q–generalized solution to (3.1) in the sense of Definition 3.1.*

Proof: Let $(\mathbf{u}^1, p^1)$ and $(\mathbf{u}^2, p^2)$ be two different q–generalized solutions to (3.1) with the same data. Denoting $\mathbf{v} = \mathbf{u}^1 - \mathbf{u}^2$, $\pi = p^1 - p^2$ we have that $(\mathbf{v}, p)$ is a generalized solution to

$$\begin{aligned} A(\mathbf{v}) + \beta \frac{\partial \mathbf{v}}{\partial x_1} + \nabla p &= \mathbf{0} \\ \nabla \cdot \mathbf{v} &= 0 \\ \mathbf{v} &= \mathbf{0} \text{ at } \partial\Omega\,. \end{aligned}$$

Applying Theorem 3.1 we have that $\mathbf{v}$, $p \in C^\infty(\Omega)$. Proceeding as in the proof of Corollaries 3.1 and 3.2 and using that $\nabla \mathbf{v} \in L^q(\Omega)$ and $\mathbf{v}$ tends to 0 in the sense of Definition 3.1 (iv) we have that $\mathbf{v}$ satisfies the decay properties (3.13) and (3.32), respectively. So we have that $\mathbf{u} \in L^p(\Omega^R)$ $\forall p > 2$ $(N = 3)$ or $\forall p > 3$ $(N = 2)$. Due to Theorem 3.2 we have $\mathbf{v} = \mathbf{0}$ and $p =$ const.

□

III.3.3 Estimates in L^q–spaces

In this subsection, we shall study once more the solution obtained in the last two subsections and we shall extend the existence theorems to more general situations.

Theorem 3.6 *Let Ω be an exterior domain of class C^{m+2}, $m \geq 0$. Let $N = 2, 3$ and*

$$\mathbf{f} \in W^{m,q}(\Omega)\,, \mathbf{u}_* \in W^{m+2-\frac{1}{q},q}(\partial\Omega)\,, 1 < q < \frac{N+1}{2}\,.$$

[13] if $q \geq 2$, then it is sufficient to take $\Omega \in C^{0,1}$, see Remark 3.2

Then there exists one and only one solution $(\mathbf{u}, p)$ *to the Oseen problem (3.1) such that*

$$\mathbf{u} \in W^{m,s_2}(\Omega) \bigcap \Big\{ \bigcap_{l=0}^{m} D^{l+1,s_1}(\Omega) \bigcap D^{l+2,q}(\Omega) \Big\}$$
$$p \in \bigcap_{l=0}^{m} D^{l+1,q}(\Omega)$$

with $s_1 = \frac{(N+1)q}{N+1-q}$, $s_2 = \frac{(N+1)q}{N+1-2q}$. *If* $N = 2$, *then also*

$$u_2 \in W^{m,\frac{2q}{2-q}}(\Omega) \bigcap \Big\{ \bigcap_{l+0}^{m} D^{l+1,q}(\Omega) \Big\} .$$

Moreover

$$\begin{aligned} a_1\|\mathbf{u}\|_{m,s_2} + \beta\Big\|\frac{\partial \mathbf{u}}{\partial x_1}\Big\|_{m,q} + \sum_{l=0}^{m} [a_2|\mathbf{u}|_{l+1,s_1} + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q}] \leq \\ \leq C(\|\mathbf{f}\|_{m,q} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)}) \end{aligned} \tag{3.34}$$

and if $N = 2$,

$$\begin{aligned} \beta(\|u_2\|_{m,\frac{2q}{2-q}} + \|\nabla u_2\|_{m+1,q}) + a_1\|\mathbf{u}\|_{m,s_2} + \beta\Big\|\frac{\partial u_1}{\partial x_1}\Big\|_{m,q} + \\ + \sum_{l=0}^{m} [a_2|\mathbf{u}|_{l+1,s_1} + |\mathbf{u}|_{l+2,q} + |p|_{l+1,q}] \leq \\ \leq C(\|\mathbf{f}\|_{m,q} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)}) \end{aligned} \tag{3.35}$$

with $a_1 = \min\{1, \beta^{\frac{2}{N+1}}\}$, $a_2 = \min\{1, \beta^{\frac{1}{N+1}}\}$. *The constant* c *depends on* m, q, N, Ω *and* β. *However, for* $q \in (1, \frac{N}{2})$ *and* $\beta \in (0; B]$ *for some* $B > 0$, c *depends only on* m, q, N, Ω *and* B.

Proof: The uniqueness part follows easily from Theorem 3.5 as $q < \frac{N+1}{2}$ implies $s_1 < N + 1$. In order to prove the existence part, let us start to consider the problem (3.1) with data

$$\mathbf{f} \in C_0^\infty(\overline{\Omega}) \,, \mathbf{u}_* \in W^{m+2-\frac{1}{q},q}(\partial\Omega) \,, 1 < q < \infty \,.$$

Combining Theorems 3.1, 3.3 and 3.4 with Corollaries 3.1 and 3.2 we may construct a solution $(\mathbf{u}, p)$ such that

$$\mathbf{u} \in W^{m+2,q}_{loc}(\overline{\Omega}) \cap C^\infty(\Omega) \,, \quad p \in W^{m+1,q}_{loc}(\overline{\Omega}) \cap C^\infty(\Omega)$$

and $(\mathbf{u}, p)$ have at large distances the asymptotic properties as the fundamental Oseen tensor and $\nabla\mathcal{E}$, respectively.

Let η be the usual cut–off function equal one in $\Omega_{\frac{R}{2}}$ and zero in Ω^R, $R > 2 \operatorname{diam} \Omega^c$. Denoting $\mathbf{U} = \mathbf{u}(1 - \eta)$, $P = p(1 - \eta)$ we get that

$$\begin{aligned} A(\mathbf{U}) + \beta\frac{\partial \mathbf{U}}{\partial x_1} + \nabla P = \mathbf{F} \\ \nabla \cdot \mathbf{U} = G \end{aligned} \tag{3.36}$$

with

$$\begin{aligned}\mathbf{F} = (1-\eta)\mathbf{f} + 2\nabla\mathbf{u}\nabla\eta - 2\mu\frac{\partial\mathbf{u}}{\partial x_1}\frac{\partial\eta}{\partial x_1} - \mathbf{u}A(\eta) - \beta\mathbf{u}\frac{\partial\eta}{\partial x_1} - p\nabla\eta \\ G = -\mathbf{u}\cdot\nabla\eta\,.\end{aligned} \tag{3.37}$$

As $\mathbf{F}$, $G \in C_0^\infty(\mathbb{R}^N)$, there exists $(\mathbf{w},\tau)$ such that (see Theorem 2.1)

$$\begin{aligned}&\mathbf{w} \in \bigcap_{l=0}^{m} D^{l+2,q}(\mathbb{R}^N)\,, &&\tau \in \bigcap_{l=0}^{m} D^{l+1,q}(\mathbb{R}^N)\\ &\mathbf{w} \in \bigcap_{l=0}^{m} D^{l+1,s_1}(\mathbb{R}^N)\,, &&s_1 = \frac{(N+1)q}{N+1-q}\,, 1<q<N+1\\ &\mathbf{w} \in \bigcap_{l=0}^{q} W^{m,s_2}(\mathbb{R}^N)\,, &&s_2 = \frac{(N+1)q}{N+1-2q}\,, 1<q<\frac{N+1}{2}\,.\end{aligned}$$

Moreover, due Corollary 2.1, $(\mathbf{w},\tau)$ have the asymptotic structure of the fundamental Oseen solution (see (2.38)). As $\mathbf{U}$ and P coincide with $\mathbf{u}$ and p outside $B_R(\mathbf{0})$, respectively, we have the same structure for $\mathbf{U}$ and P and therefore, applying Lemma 2.3 together with the integrability properties of $\mathbf{U}$ and $\mathbf{w}$, we get $\mathbf{w} = \mathbf{U}$ and $\tau = P + a$, $a \in \mathbb{R}$. Theorem 2.1 yields

$$\begin{aligned}&\beta^{\frac{2}{N+1}}|\mathbf{u}|_{l,s_2,\Omega^R} + \beta^{\frac{1}{N+1}}|\mathbf{u}|_{l+1,s_1,\Omega^R} + \beta\Big|\frac{\partial\mathbf{u}}{\partial x_1}\Big|_{l,q,\Omega^R} + |\mathbf{u}|_{l+2,q,\Omega^R} + \\ &+|p|_{l+1,q,\Omega^R} \le C(|\mathbf{f}|_{l,q} + (1+\beta)|\mathbf{u}|_{l,q,\Omega_R} + |\mathbf{u}|_{l+1,q,\Omega_R} + |p|_{l,q,\Omega_R})\end{aligned} \tag{3.38}$$

with s_1, s_2 defined above, and if $N=2$

$$\begin{aligned}&\beta^{\frac{2}{3}}|\mathbf{u}|_{l,\frac{3q}{3-2q},\Omega^R} + \beta^{\frac{1}{3}}|\mathbf{u}|_{l+1,\frac{3q}{3-q}} + |\mathbf{u}|_{l+2,q,\Omega^R} + |p|_{l+1,q,\Omega^R} + \\ &\quad +\beta\Big(\Big|\frac{\partial u_1}{\partial x_1}\Big|_{l,q,\Omega^R} + |u_2|_{l+1,q,\Omega^R} + |u_2|_{l,\frac{2q}{2-q},\Omega^R}\Big) \le \\ &\quad \le C(|\mathbf{f}|_{l,q} + (1+\beta)|\mathbf{u}|_{l,q,\Omega_R} + |\mathbf{u}|_{l+1,q,\Omega_R} + |p|_{l,q,\Omega_R})\,.\end{aligned} \tag{3.39}$$

We shall now estimate $\mathbf{u}$, p in Ω_R. From Theorem VIII.5.4 (the term $\beta\frac{\partial\mathbf{u}}{\partial x_1}$ is put on the right hand side) we can get applying Corollary VIII.1.2

$$\begin{aligned}&\|\mathbf{u}\|_{m+2,q,\Omega_R} + \|p\|_{m+1,q,\Omega_R} \le C\Big[\|\mathbf{f}\|_{m,q,\Omega_{2R}} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)} + \\ &+\beta\Big\|\frac{\partial\mathbf{u}}{\partial x_1}\Big\|_{m,q,\Omega_{2R}} + \|\mathbf{u}\|_{m+2-\frac{1}{q},q(\partial B_{2R})} + \|p\|_{q,\Omega_{2R}} + \|\mathbf{v}\|_{q,\Omega_{2R}}\Big]\,.\end{aligned} \tag{3.40}$$

From the trace theorem (see Theorem VIII.1.6) we easily get

$$\|\mathbf{u}\|_{m+2-\frac{1}{q},q(\partial B_{2R})} \le C(|\mathbf{u}|_{m+2,q,\Omega^R} + \|\mathbf{u}\|_{m+1,q,\Omega_{2R}})\,. \tag{3.41}$$

As $\frac{(N+1)q}{N+1-kq} < \frac{Nq}{N-kq}$, $k=1,2$, $N \ge 2$, we have from the imbedding theorem (see Theorem VIII.1.2)

$$\|\mathbf{u}\|_{m,s_2,\Omega_R} + \sum_{l=0}^{m}|\mathbf{u}|_{l+1,s_1,\Omega_R} \le C\|\mathbf{u}\|_{m+2,q,\Omega_R} \tag{3.42}$$

and for $N = 2$, also

$$\|\mathbf{u}\|_{m,\frac{2q}{2-q},\Omega_R} \le C\|\mathbf{u}\|_{m+1,q,\Omega_R}\,. \tag{3.43}$$

Collecting (3.38)–(3.43) yields

$$\begin{aligned} a_1\|\mathbf{u}\|_{m,s_2,\Omega} + \beta\Big\|\frac{\partial\mathbf{u}}{\partial x_1}\Big\|_{m,q,\Omega} + \sum_{l=0}^{m}[a_2|\mathbf{u}|_{l+1,s_1,\Omega} + \\ +|\mathbf{u}|_{l+2,q,\Omega} + |p|_{l+1,q,\Omega}] \le C(\|\mathbf{f}\|_{m,q,\Omega} + \\ +\|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)} + (1+\beta)\|\mathbf{u}\|_{m+1,q,\Omega_{2R}} + \|p\|_{m,q,\Omega_{2R}}) \end{aligned} \tag{3.44}$$

and for $N = 2$

$$\begin{aligned} \beta(\|u_2\|_{m,\frac{2q}{2-q},\Omega} + \|\nabla u_2\|_{m+1,q,\Omega} + a_1\|\mathbf{u}\|_{m,\frac{3q}{3-2q},\Omega} + \beta\Big\|\frac{\partial\mathbf{u}}{\partial x_1}\Big\|_{m,q,\Omega} + \\ +\sum_{l=0}^{m}[a_2|\mathbf{u}|_{l+1,\frac{3q}{3-q},\Omega} + |\mathbf{u}|_{l+2,q,\Omega} + |p|_{l+1,q,\Omega}] \le \\ \le C(\|\mathbf{f}\|_{m,q,\Omega} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)} + (1+\beta)\|\mathbf{u}\|_{m+1,q,\Omega_{2R}} + \|p\|_{m,q,\Omega_{2R}}) \end{aligned} \tag{3.45}$$

with $a_1 = \min\{1, \beta^{\frac{2}{N+1}}\}$, $a_2 = \min\{1, \beta^{\frac{1}{N+1}}\}$, $s_1 = \frac{(N+1)q}{N+1-q}$, $s_2 = \frac{(N+1)q}{N+1-2q}$. Applying several times the interpolation inequality (see Theorem VIII.1.11) we have for ε sufficiently small

$$\|p\|_{m,q,\Omega_{2R}} \le C\|p\|_{q,\Omega_{2R}}^{\frac{1}{m+1}}\|p\|_{m+1,q,\Omega_{2R}}^{\frac{m}{m+1}} \le \varepsilon\|p\|_{m+1,q,\Omega_{2R}} + c(\varepsilon)\|p\|_{q,\Omega_{2R}}\,. \tag{3.46}$$

After modifying p by a suitable constant, we have also from Theorem VIII.5.3 (recall that $L^q(\Omega_{2R}) \hookrightarrow D^{-1,q}(\Omega_{2R})$)

$$\|p\|_{q,\Omega_{2R}} \le C((1+\beta)\|\mathbf{u}\|_{1,q,\Omega_{2R}} + \|\mathbf{f}\|_q)\,. \tag{3.47}$$

and so (3.44)–(3.47) yield

$$\begin{aligned} a_1\|\mathbf{u}\|_{m,s_2,\Omega} + \beta\Big\|\frac{\partial\mathbf{u}}{\partial x_1}\Big\|_{m,q,\Omega} + \\ +\sum_{l=0}^{m}[a_2|\mathbf{u}|_{l+1,s_1,\Omega} + |\mathbf{u}|_{l+2,q,\Omega} + |p|_{l+1,q,\Omega}] \le \\ \le C(\|\mathbf{f}\|_{m,q} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)} + (1+\beta)\|\mathbf{u}\|_{m+1,q,\Omega_{2R}}) \end{aligned} \tag{3.48}$$

and for $N = 2$

$$\begin{aligned} \beta(\|u_2\|_{m,\frac{2q}{2-q},\Omega} + \|\nabla u_2\|_{m+1,q,\Omega} + a_1\|\mathbf{u}\|_{m,\frac{3q}{3-2q},\Omega} + \beta\Big\|\frac{\partial\mathbf{u}}{\partial x_1}\Big\|_{m,q,\Omega} + \\ +\sum_{l=0}^{m}[a_2|\mathbf{u}|_{l+1,\frac{3q}{3-q},\Omega} + |\mathbf{u}|_{l+2,q,\Omega} + |p|_{l+1,q,\Omega}] \le \\ \le C(\|\mathbf{f}\|_{m,q} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)} + (1+\beta)\|\mathbf{u}\|_{m+1,q,\Omega_{2R}}) \end{aligned} \tag{3.49}$$

with a_1, a_2, s_1 and s_2 defined above.

We are now looking for an inequality of the type

$$\|\mathbf{u}\|_{m+1,q,\Omega_{2R}} \le C(\|\mathbf{f}\|_{m,q,\Omega} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)})$$

for a constant independent of $\mathbf{u}$, $\mathbf{u}_*$ and $\mathbf{f}$. We show this by a contradiction argument. Let $\{\mathbf{f}_k\} \subset C_0^\infty(\overline{\Omega})$, $\{\mathbf{u}_*\}_k \subset W^{m+2-\frac{1}{q},q}(\partial\Omega)$ be such that, denoting by $(\mathbf{u}_k, p_k)$ the corresponding solution to the Oseen problem with data $\mathbf{f}_k$, $(\mathbf{u}_*)_k$, we have

$$\|\mathbf{f}_k\|_{m,q,\Omega} + \|(\mathbf{u}_*)_k\|_{m+2-\frac{1}{q},q,(\partial\Omega)} \le \frac{1}{k}\,, \quad \|\mathbf{u}_k\|_{m+1,q,\Omega_{2R}} = 1\,. \tag{3.50}$$

Now, let $k \to \infty$. From (3.48) we have that $\mathbf{u}_k$ is bounded in $L^{s_2}(\Omega) \cap D^{1,s_1}(\Omega)$, $\nabla^2\mathbf{u}_k$ in $W^{m,q}(\Omega)$ and p_k in $D^{1,q}(\Omega)$. Therefore, at least for a chosen subsequence, $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $L^{s_2}(\Omega) \cap D^{1,s_1}(\Omega) \cap D^{2,q}(\Omega)$, $p_k \rightharpoonup p \in D^{1,q}(\Omega)$ and $\mathbf{u}$ is a s_1–weak solution to the modified Oseen problem (3.1) with $\mathbf{f} = \mathbf{0}$. Moreover, $\mathbf{u}_k \to \mathbf{u}$ in $W^{m+1,q}(\Omega_{2R})$ and therefore $\mathbf{u}$ assumes the zero trace at $\partial\Omega$, i.e. $\mathbf{u}$ is a s_1–weak solution to (3.1) with zero data. Applying Theorem 3.2 we see that $\mathbf{u} = \mathbf{0}$ and p is a constant. But $\mathbf{u}_k \to \mathbf{u} \in W^{m+1,q}(\Omega_{2R})$ and we get a contradiction to $\|\mathbf{u}_k\|_{m+1,q,\Omega_{2R}} = 1$. Therefore, (3.34) and (3.35) are shown.

Unfortunately, the constants in (3.34) and (3.35) depend apriori on β. We shall show that if $N = 3$ and $q \in (1, \frac{3}{2})$, the constant can be taken independent of β. Let $\beta \in (0, B]$, $B > 0$. Let the constant be dependent on β. Then there exists sequence $\{\mathbf{f}_k\} \subset C_0^\infty(\overline{\Omega})$, $\{\mathbf{u}_*\}_k \subset W^{m+2-\frac{1}{q},q}(\partial\Omega)$ and $\beta_k \in (0; B]$ such that

$$\begin{gathered} A(\mathbf{u}_k) + \beta_k \frac{\partial \mathbf{u}_k}{\partial x_1} + \nabla p_k = \mathbf{f}_k \\ \nabla \cdot \mathbf{u}_k = 0 \\ \mathbf{u}_k = (\mathbf{u}_*)_k \quad \text{at } \partial\Omega \end{gathered}$$

and

$$\|\mathbf{f}_k\|_{m,q,\Omega} + \|(\mathbf{u}_*)_k\|_{m+2-\frac{1}{q},q,(\partial\Omega)} \le \frac{1}{k}\,, \quad \|\mathbf{u}_k\|_{m+1,q,\Omega_{2R}} = 1\,.$$

So, there exists $\beta \ge 0$ such that $\beta_k \to \beta$, at least for a chosen subsequence.

If $\beta > 0$, we can deduce as above $\mathbf{u}_k \rightharpoonup \mathbf{u}$, $p_k \rightharpoonup p$ and $\mathbf{u} = \mathbf{0}$, $p = \text{const}$, yielding a contradiction.

If $\beta = 0$, we can no longer proceed as above as $a_1, a_2 \to 0$ and we cannot control the norms of $\mathbf{u}$ in $L^{s_2}(\Omega) \cap D^{1,s_1}(\Omega)$. Nevertheless, as for each fixed k $\mathbf{u}_k$ and $\nabla\mathbf{u}_k$ tend to zero uniformly (see Theorem VIII.1.17, $q < \frac{N}{2}$), we have for $1 < q < \frac{3}{2}$

$$\|\mathbf{u}_k\|_{\frac{3q}{3-2q}} \le C\|\nabla^2\mathbf{u}_k\|_q \le M\,.$$

We proceed as above and get that there exists $\mathbf{u} \in L^{\frac{3q}{3-2q}}(\Omega)$ and $\nabla\mathbf{u} \in L^{\frac{3q}{3-q}}(\Omega)$, $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $L^{\frac{3q}{3-2q}}(\Omega) \cap D^{1,\frac{3q}{3-q}}(\Omega) \cap D^{2,q}(\Omega)$, $p_k \rightharpoonup p$ in $D^{1,q}(\Omega)$ and $(\mathbf{u}, p)$ solves the modified Stokes problem

$$\begin{gathered} A(\mathbf{u}) + \nabla p = \mathbf{0} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} = \mathbf{0} \quad \text{at } \partial\Omega \end{gathered}$$

in the weak sense. Recalling that $q < \frac{N}{2}$, we get from Corollary VIII.5.4 $\mathbf{u} = \mathbf{0}$, $p = \text{const}$ and we get as above a contradiction. In order to finish the proof for general $\mathbf{f} \in W^{m,q}(\Omega)$ only, we use the standard density argument.

$\square$

Next we consider weak solutions to (3.1) in the sense of Definition 3.1. We have the following

Theorem 3.7 *Let $\Omega \subset \mathbb{R}^N$ be an exterior domain of class C^2. Moreover, let $\mathbf{f} \in D_0^{-1,q}(\Omega)$, $\mathbf{u}_* \in W^{1-\frac{1}{q},q}(\partial\Omega)$, $\frac{N}{N-1} < q < N+1$. Then there exists exactly one q–generalized solution to (3.1). Furthermore*

$$\mathbf{u} \in L^{s_2}(\Omega)\,, \quad s_2 = \frac{(N+1)q}{N+1-q}$$
$$p \in L^q(\Omega)$$

and if $N = 2$

$$u_2 \in L^q(\Omega)\,.$$

Finally

$$a_2\|\mathbf{u}\|_{s_2} + |\mathbf{u}|_{1,q} + \|p\|_q \le C(|\mathbf{f}|_{-1,q} + \|\mathbf{u}_*\|_{1-\frac{1}{q},q,(\partial\Omega)}) \tag{3.51}$$

and if $N = 2$

$$\beta\|u_2\|_q + a_2\|\mathbf{u}\|_{\frac{3q}{3-q}} + |\mathbf{u}|_{1,q} + \|p\|_q \le C(|\mathbf{f}|_{-1,q} + \|\mathbf{u}_*\|_{1-\frac{1}{q},q,(\partial\Omega)}) \tag{3.52}$$

with $c = c(N, q, \Omega, \beta)$ and $a_2 = \min\{1, \beta^{\frac{1}{N+1}}\}$. If $N = 3$ and $q \in (\frac{3}{2}; 3)$, then for $\beta \in (0; B]$ the constant c depends only on N, q, Ω and B.

Proof: We start with $\mathbf{f} \in C_0^\infty(\overline{\Omega})$, $\mathbf{u}_* \in W^{1-\frac{1}{q},q}(\partial\Omega) \cap W^{\frac{1}{2},2}(\partial\Omega)$. Due to Theorems 3.4 and 3.3 there exists $(\mathbf{u}, p)$, the unique solution to (3.1). Moreover, due to Corollaries 3.1 and 3.2 and Theorem VIII.5.4

$$\mathbf{u} \in W^{1,q}_{loc}(\overline{\Omega}) \bigcap C^\infty(\Omega)$$
$$p \in L^q_{loc}(\overline{\Omega}) \bigcap C^\infty(\Omega)\,.$$

We proceed as in the proof of Theorem 3.6. Let ψ be a cut–off function with support in $B^{\frac{R}{2}}(\mathbf{0})$, $\psi \equiv 1 \in B^R(\mathbf{0})$ with $R > 2\,\mathrm{diam}\,\Omega^c$. Then

$$\mathbf{w} = \psi\mathbf{u}$$
$$\pi = \psi p$$

solves in $\mathbb{R}^N$ (3.36)–(3.37) and satisfies (due to the uniqueness, see Lemma 2.3)

$$\beta^{\frac{1}{N+1}}\|\mathbf{w}\|_s + |\mathbf{w}|_{1,q} + \|\pi\|_q \le C(|\mathbf{F}|_{-1,q} + \beta|g|_{-1,q} + \|g\|_q)\,,$$

$N \ge 2$, $s = \frac{(N+1)q}{N+1-q}$

$$\beta\|w_2\|_q + \beta^{\frac{1}{3}}\|\mathbf{w}\|_{\frac{3q}{3-q}} + |\mathbf{w}|_{1,q} + \|\pi\|_q \le C(|\mathbf{F}|_{-1,q} + \beta|g|_{-1,q} + \|g\|_q)\,,$$

$N = 2$ (see Theorem 2.2). As $q > \frac{N}{N-1}$, we have[14]

$$|\mathbf{F}|_{-1,q} + \beta|g|_{-1,q} + \|g\|_q \le C(|\mathbf{f}|_{-1,q} + (1+\beta)\|\mathbf{u}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R})$$

and so

$$\begin{gathered}\beta^{\frac{1}{N+1}}\|\mathbf{u}\|_{s,\Omega^R} + |\mathbf{u}|_{1,q,\Omega^R} + \|\pi\|_{q,\Omega^R} \le \\ \le C(|\mathbf{f}|_{-1,q} + \beta\|\mathbf{u}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}), \quad (N \ge 2) \\ \beta\|u_2\|_{q,\Omega^R} + \beta^{\frac{1}{3}}\|\mathbf{u}\|_{\frac{3q}{3-q},\Omega^R} + |\mathbf{u}|_{1,q,\Omega^R} + \|p\|_{q,\Omega^R} \le \\ \le C(|\mathbf{f}|_{-1,q} + (1+\beta)\|\mathbf{u}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}), \quad (N = 2).\end{gathered}$$

Next, we have "near the boundary" (see Theorem VIII.5.5)

$$\begin{gathered}\|\mathbf{u}\|_{1,q,\Omega_R} + \|p\|_{q,\Omega_R} \le C(|\mathbf{f}|_{-1,q} + \|\mathbf{u}_*\|_{1-\frac{1}{q},q,(\partial\Omega)} + \\ +(1+\beta)\|\mathbf{u}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} + \|\mathbf{u}\|_{1-\frac{1}{q},q(\partial B_R)}),\end{gathered}$$

where the obvious inequality

$$\|\mathbf{f}\|_{-1,q,\Omega_R} \le |\mathbf{f}|_{-1,q}$$

was used. As $\|\mathbf{u}\|_{1-\frac{1}{q},q,(\partial B_R)} \le C\|\mathbf{u}\|_{1,q,\Omega^R_{2R}}$, we easily get due to the fact that $\|\mathbf{u}\|_{s,\Omega_R} \le C\|\mathbf{u}\|_{1,q,\Omega_R}$

$$\begin{gathered}a_2\|\mathbf{u}\|_{s,\Omega} + |\mathbf{u}|_{1,q,\Omega} + \|\pi\|_{q,\Omega} \le \\ \le C(|\mathbf{f}|_{-1,q} + \|\mathbf{u}_*\|_{1-\frac{1}{q},q,(\partial\Omega)} + (1+\beta)\|\mathbf{u}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}), \quad (N \ge 2) \\ \beta\|u_2\|_{q,\Omega} + a_2\|\mathbf{u}\|_{\frac{3q}{3-q},\Omega} + |\mathbf{u}|_{1,q,\Omega} + \|p\|_{q,\Omega} \le \\ \le C(|\mathbf{f}|_{-1,q} + \|\mathbf{u}_*\|_{1-\frac{1}{q},q,(\partial\Omega)} + (1+\beta)\|\mathbf{u}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R}), \quad (N = 2).\end{gathered}$$

We finish the proof by arguing as in Theorem 3.6; thanks to the uniqueness of solution to the Oseen problem

$$\|\mathbf{u}\|_{q,\Omega_R} + \|p\|_{-1,q,\Omega_R} \le C(|\mathbf{f}|_{-1,q} + \|\mathbf{u}_*\|_{1-\frac{1}{q},q,(\partial\Omega)}).$$

Finally, if $q \in (\frac{3}{2}, 3)$, the constant c can be taken independent of β as

$$\|\mathbf{u}\|_{\frac{Nq}{N-q}} \le C|\mathbf{u}|_{1,q}$$

and in this case and we may apply Theorem VIII.5.7.

□

[14] e.g. (recall $q' < N$)

$$\begin{gathered}\|\nabla\mathbf{u}\nabla\eta\|_{-1,q} = \sup_{\varphi\in D_0^{1,q'}(\mathbb{R}^N)} \left|\int_{\mathbb{R}^N} \nabla\mathbf{u}\nabla\eta\varphi \mathrm{d}\mathbf{x}\right| \le \\ \le \sup_{\varphi\in D_0^{1,q'}(\mathbb{R}^N)} C\|\mathbf{u}\|_q\|\varphi\|_{1,q',B_R(\mathbf{0})} \le C\|\mathbf{u}\|_q.\end{gathered}$$

III.4 Integral representation of solutions

The most important tool in the weighted estimates of solutions to the modified Oseen problem will be their integral representation. For our purpose, we can directly assume the right hand side in the divergence form, i.e. we study the following problem

$$\left.\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla p = \nabla\cdot\boldsymbol{\mathcal{G}} \\ \nabla\cdot\mathbf{u} = 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\mathbf{u} = \mathbf{u}_* \quad \text{at } \partial\Omega \tag{4.1}$$
$$\mathbf{u} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty\,.$$

Let us assume for a moment that $\boldsymbol{\mathcal{G}} \in C_0^\infty(\overline{\Omega})$, the domain $\Omega \in C^{0,1}$ and $\mathbf{u}_*$ satisfy such conditions that there exists a unique solution to (4.1), e.g. $\mathbf{u}_* \in W^{\frac{1}{2},2}(\partial\Omega)$. We first show the integral representation for such right hand sides. The standard density argument enables us later on to weaken the assumptions on $\boldsymbol{\mathcal{G}}$.

Let us denote

$$T_{ij}(\mathbf{u},p) = \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) - p\delta_{ij} - \mu\delta_{1j}\frac{\partial u_i}{\partial x_1}\,. \tag{4.2}$$

We easily see that $(4.1)_1$ can be rewritten as

$$-\nabla\cdot\mathbf{T}(\mathbf{u},p) + \beta\frac{\partial \mathbf{u}}{\partial x_1} = \nabla\cdot\boldsymbol{\mathcal{G}}\,, \tag{4.3}$$

where we used the fact that $\mathbf{u}$ is divergence free.

Let D be a bounded domain in $\mathbb{R}^N$. We easily get for $\mathbf{u}$, $\mathbf{v}$ smooth divergence free vector fields and p, π smooth scalar fields

$$\begin{aligned} \int_D \Big(\nabla\cdot\mathbf{T}(\mathbf{u},p) - \beta\frac{\partial\mathbf{u}}{\partial y_1}\Big)\cdot\mathbf{v}\mathrm{d}\mathbf{y} &= -\int_D \Big(\mathbf{T}(\mathbf{u},p) : \nabla\mathbf{v} - \beta\mathbf{u}\cdot\frac{\partial\mathbf{v}}{\partial y_1}\Big)\mathrm{d}\mathbf{y} + \\ &\quad + \int_{\partial D}(\mathbf{v}\cdot\mathbf{T}(\mathbf{u},p)\cdot\mathbf{n} - \beta\mathbf{u}\cdot\mathbf{v}n_1)\mathrm{d}_{\mathbf{y}}S \\ \int_D \Big(\nabla\cdot\mathbf{T}(\mathbf{v},-\pi) + \beta\frac{\partial\mathbf{v}}{\partial y_1}\Big)\cdot\mathbf{u}\mathrm{d}\mathbf{y} &= -\int_D \Big(\mathbf{T}(\mathbf{v},-\pi) : \nabla\mathbf{u} - \beta\mathbf{u}\cdot\frac{\partial\mathbf{v}}{\partial y_1}\Big)\mathrm{d}\mathbf{y} + \\ &\quad + \int_{\partial D}\mathbf{u}\cdot\mathbf{T}(\mathbf{v},-\pi)\cdot\mathbf{n}\mathrm{d}_{\mathbf{y}}S\,, \end{aligned} \tag{4.4}$$

where $n_1 = \mathbf{n}\cdot\mathbf{e}_1$ is the first component of the outer normal to D.

Recalling that for divergence free smooth vector fields

$$\int_D (\mathbf{T}(\mathbf{u},p) : \nabla\mathbf{v} - \mathbf{T}(\mathbf{v},-\pi) : \nabla\mathbf{u})\mathrm{d}\mathbf{y} = 0\,,$$

we get from (4.4)

$$\begin{aligned} \int_D \Big[\Big(\nabla\cdot\mathbf{T}(\mathbf{u},p) - \beta\frac{\partial\mathbf{u}}{\partial y_1}\Big)\cdot\mathbf{v} - \Big(\nabla\cdot\mathbf{T}(\mathbf{v},-\pi) + \beta\frac{\partial\mathbf{v}}{\partial y_1}\Big)\cdot\mathbf{u}\Big]\mathrm{d}\mathbf{y} &= \\ = \int_{\partial D}(\mathbf{v}\cdot\mathbf{T}(\mathbf{u},p) - \mathbf{u}\cdot\mathbf{T}(\mathbf{v},-\pi) - \beta\mathbf{u}\cdot\mathbf{v}\mathbf{e}_1)\cdot\mathbf{n}\mathrm{d}_{\mathbf{y}}S\,. \end{aligned} \tag{4.5}$$

We take in particular

$$D = \Omega_R^{(\varepsilon,\mathbf{x})} = \Omega_R \setminus \{\mathbf{y}; |\mathbf{x}-\mathbf{y}| < \varepsilon\}\,, \varepsilon < \operatorname{dist}(\mathbf{x}, \partial\Omega)$$
$$\mathbf{v}(\mathbf{y}) = \mathbf{w}_j(\mathbf{x}-\mathbf{y}) = (\mathcal{O}^\mu_{1j}(\mathbf{x}-\mathbf{y};\beta), \dots, \mathcal{O}^\mu_{Nj}(\mathbf{x}-\mathbf{y};\beta))$$
$$\pi(\mathbf{y}) = e_j(\mathbf{x}-\mathbf{y}) = \frac{\partial \mathcal{E}}{\partial x_j}(\mathbf{x}-\mathbf{y})\,,$$

$j = 1, 2, \dots, N$. Then $\mathbf{T}(\mathbf{v}, -\pi)(\mathbf{y}) = -\mathbf{T}(\mathbf{w}_j, e_j)(\mathbf{x}-\mathbf{y})$, $\partial D = \partial\Omega \cup \partial B_R(\mathbf{0}) \cup \partial B^\varepsilon(\mathbf{x})$. Moreover $\frac{\partial}{\partial y_k} T_{ik}(\mathbf{w}_j, e_j)(\mathbf{x}-\mathbf{y}) - \beta \frac{\partial \mathbf{w}_j}{\partial y_1} = \delta_{ij}\delta_{\mathbf{x}}$, where the derivatives are assumed in the sense of distribution, $\delta_{\mathbf{x}}$ is the Dirac distribution supported at the point $\mathbf{x}$. Moreover, let $(\mathbf{u}, p)$ satisfy (4.3). Then

$$\begin{gathered}\int_{\Omega_R^{(\varepsilon,\mathbf{x})}} -(\nabla \cdot \boldsymbol{\mathcal{G}}(\mathbf{y})) \cdot \mathbf{w}_j(\mathbf{x}-\mathbf{y}) \mathrm{d}\mathbf{y} = \\ = \int_{\partial\Omega \cup \partial B_R(\mathbf{0}) \cup \partial B^\varepsilon(\mathbf{x})} \Big(\mathbf{w}_j(\mathbf{x}-\mathbf{y}) \cdot \mathbf{T}(\mathbf{u}, p)(\mathbf{y}) + \\ + \mathbf{u}(\mathbf{y}) \cdot \mathbf{T}(\mathbf{w}_j, e_j)(\mathbf{x}-\mathbf{y}) - \beta \mathbf{u}(\mathbf{y}) \cdot \mathbf{w}_j(\mathbf{x}-\mathbf{y})\mathbf{e}_1\Big) \cdot \mathbf{n}(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S\,.\end{gathered} \tag{4.6}$$

We shall first pass with $\varepsilon \to 0^+$. Due to the properties of $\mathbf{w}_j = \{\mathcal{O}^\mu_{ij}\}_{i=1}^N$ (see Theorem 1.2), the integral on the left hand side converges to the integral over Ω_R. Next, let us regard the surface integral over $\partial B^\varepsilon(\mathbf{x})$.

Due to Theorem 1.2 we easily check that for any smooth functions $(\mathbf{u}, p)$

$$\lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} \Big(\mathbf{w}_j(\mathbf{x}-\mathbf{y}) \cdot \mathbf{T}(\mathbf{u}, p)(\mathbf{y}) - \beta \mathbf{u}(\mathbf{y}) \cdot \mathbf{w}_j(\mathbf{x}-\mathbf{y})\mathbf{e}_1\Big) \cdot \mathbf{n}(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S = 0\,. \tag{4.7}$$

Due to the definition of the fundamental solution we have for any smooth vector $\mathbf{u}$ (see also Subsection II.1.1 for $N = 2$)

$$\begin{gathered}u_j(\mathbf{x}) = \lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} \Big(\frac{\partial \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y})}{\partial n_{\mathbf{y}}} - \\ - \mu \frac{\partial \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y})}{\partial y_1} n_1(\mathbf{y}) + e_j(\mathbf{x}-\mathbf{y}) n_i(\mathbf{y})\Big) u_i(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S\,.\end{gathered} \tag{4.8}$$

Therefore

$$\begin{gathered}\lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} \mathbf{u}(\mathbf{y}) \cdot \mathbf{T}(\mathbf{w}_j, e_j)(\mathbf{x}-\mathbf{y}) \mathbf{n}(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S = \\ = -u_j(\mathbf{x}) - \lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} \frac{\partial \mathcal{O}^\mu_{kj}(\mathbf{x}-\mathbf{y})}{\partial y_i} u_i(\mathbf{y}) n_k(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S\,.\end{gathered} \tag{4.9}$$

Let us check that the limit on the right hand side of (4.9) is equal to zero. Let $\mathbf{u}$ be smooth, divergence free vector field with compact support in $\mathbb{R}^N$. Then due to the asymptotic properties of $\boldsymbol{\mathcal{O}}^\mu$

$$\begin{aligned}0 = \int_{\mathbb{R}^N} \frac{\partial^2 u_i(\mathbf{y})}{\partial y_i \partial y_k} \mathcal{O}^\mu_{kj}(\mathbf{x}-\mathbf{y}) \mathrm{d}\mathbf{y} &= -\int_{\mathbb{R}^N} \frac{\partial u_i(\mathbf{y})}{\partial y_k} \frac{\partial \mathcal{O}^\mu_{kj}(\mathbf{x}-\mathbf{y})}{\partial y_i} \mathrm{d}\mathbf{y} = \\ &= -\lim_{\varepsilon \to 0^+} \bigg[\int_{\partial B^\varepsilon(\mathbf{x})} u_i(\mathbf{y}) \frac{\partial \mathcal{O}^\mu_{kj}(\mathbf{x}-\mathbf{y})}{\partial y_i} n_k(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S - \\ &\quad - \int_{B^\varepsilon(\mathbf{x})} u_i(\mathbf{y}) \frac{\partial^2 \mathcal{O}^\mu_{kj}(\mathbf{x}-\mathbf{y})}{\partial y_i \partial y_k} (\mathbf{x}-\mathbf{y}) \mathrm{d}\mathbf{y}\bigg] = \\ &= -\lim_{\varepsilon \to 0^+} \int_{\partial B^\varepsilon(\mathbf{x})} u_i(\mathbf{y}) \frac{\partial \mathcal{O}^\mu_{kj}(\mathbf{x}-\mathbf{y})}{\partial y_i} n_k(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S\,.\end{aligned}$$

Therefore (4.9) implies

$$u_j(\mathbf{x}) = -\lim_{\varepsilon\to 0^+}\int_{\partial B^\varepsilon(\mathbf{x})} \mathbf{u}(\mathbf{y})\cdot \mathbf{T}(\mathbf{w}_j, e_j)(\mathbf{x}-\mathbf{y})\mathbf{n}(\mathbf{y})\mathrm{d}_{\mathbf{y}}S$$

and combining this with (4.7)

$$\begin{aligned} u_j(\mathbf{x}) = &\int_{\Omega_R} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\frac{\partial}{\partial y_k}\mathcal{G}_{ik}(\mathbf{y})d\mathbf{y} + \\ +\int_{\partial\Omega}&\Big[u_i(\mathbf{y})T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) - \beta\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k} + \\ &+\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},p)(\mathbf{y})\Big]n_k(\mathbf{y})\mathrm{d}S \; + \\ +\int_{\partial B_R(\mathbf{0})}&\Big[u_i(\mathbf{y})T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) - \beta\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k} + \\ &+\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},p)(\mathbf{y})\Big]n_k(\mathbf{y})\mathrm{d}S\,. \end{aligned} \tag{4.10}$$

Now let $\mathbf{x}\in\mathbb{R}^N$, $N=2,3$ be fixed. We want to pass with $R\to\infty$ and show that the boundary terms over $\partial B_R(\mathbf{0})$ tend to zero. As $\boldsymbol{\mathcal{G}}\in C_0^\infty(\overline{\Omega})$, we may use Corollaries 3.1 and 3.2, together with the properties of $\mathcal{O}^\mu_{ij}$. We have

$$\begin{aligned} \mathbf{u}(\mathbf{y}) &\sim \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};1) \sim R^{-\frac{N-1}{2}}(1+s(\mathbf{y}))^{-\frac{N-1}{2}} \\ \nabla\mathbf{u}(\mathbf{y}) &\sim \nabla\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};1) \sim R^{-\frac{N}{2}}(1+s(\mathbf{y}))^{-\frac{N}{2}} \\ p(\mathbf{y}) &\sim e_j(\mathbf{x}-\mathbf{y}) \sim R^{-N+1} \end{aligned}$$

for $R=|\mathbf{y}|$ sufficiently large. We get

$$\int_{\partial B_R} |\beta\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y})u_i(\mathbf{y})n_j(\mathbf{y})|\mathrm{d}S \;\sim \int\frac{R^{N-1}}{R^{N-1}(1+s(R\omega))^{N-1}}\mathrm{d}\omega \sim R^{-\frac{N-1}{2}} \to 0$$

for $R\to\infty$, see Lemma II.3.2. Analogously, even easier, we may show that also the other terms vanish for $R\to\infty$. Therefore

Theorem 4.1 *Let $\Omega\in C^{0,1}$, an exterior domain, $\boldsymbol{\mathcal{G}}\in C_0^\infty(\overline{\Omega})$ and $(\mathbf{u},p)$ be the unique solution to the Oseen problem (4.1). Let $\mathbf{T}$ be defined in (4.2). Then*

$$\begin{aligned} u_j(\mathbf{x}) = &\int_{\Omega} \frac{\partial}{\partial x_k}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}_{ik}(\mathbf{y})d\mathbf{y} + \\ +\int_{\partial\Omega}&\Big[u_i(\mathbf{y})T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) - \beta\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k} + \\ &+\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},p)(\mathbf{y}) + \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}_{ik}(\mathbf{y})\Big]n_k(\mathbf{y})\,dS \end{aligned} \tag{4.11}$$

$$\begin{aligned} D^\alpha u_j(\mathbf{x}) = &\int_{\Omega} D^\alpha_{\mathbf{x}}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\frac{\partial}{\partial y_k}\mathcal{G}_{ik}(\mathbf{y})d\mathbf{y} + \\ +\int_{\partial\Omega}&\Big[u_i(\mathbf{y})D^\alpha_{\mathbf{x}}T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) - \beta D^\alpha_{\mathbf{x}}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k} + \\ &+D^\alpha_{\mathbf{x}}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},p)(\mathbf{y})\Big]n_k(\mathbf{y})\,dS \end{aligned} \tag{4.12}$$

if $|\alpha| = 1$,

$$D^\alpha u_j(\mathbf{x}) = \mathcal{A}_j^{(1),\alpha}(\boldsymbol{\mathcal{G}}) - \int_\Omega D^\alpha_{\mathbf{x}} \frac{\partial \mathcal{N}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k} \mathcal{G}_{ik}(\mathbf{y}) d\mathbf{y} + \\ + \int_{\partial\Omega} \Big[u_i(\mathbf{y}) D^\alpha_{\mathbf{x}} T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) - \beta D^\alpha_{\mathbf{x}} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) u_i(\mathbf{y}) \delta_{1k} + \\ + D^\alpha_{\mathbf{x}} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) T_{ik}(\mathbf{u},p)(\mathbf{y}) + D^\alpha_{\mathbf{x}} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) \mathcal{G}_{ik}(\mathbf{y}) \Big] n_k(\mathbf{y})\, dS \tag{4.13}$$

if $|\alpha| = 1$,

$$D^\alpha u_j(\mathbf{x}) = \mathcal{A}_j^{(2),\alpha}(\nabla\cdot\boldsymbol{\mathcal{G}}) + \int_\Omega D^\alpha_{\mathbf{x}} \mathcal{N}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) \frac{\partial \mathcal{G}_{ik}(\mathbf{y})}{\partial y_k} d\mathbf{y} + \\ + \int_{\partial\Omega} \Big[u_i(\mathbf{y}) D^\alpha_{\mathbf{x}} T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) - \beta D^\alpha_{\mathbf{x}} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) u_i(\mathbf{y}) \delta_{1k} + \\ + D^\alpha_{\mathbf{x}} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) T_{ik}(\mathbf{u},p)(\mathbf{y}) \Big] n_k(\mathbf{y})\, dS \tag{4.14}$$

if $|\alpha| = 2$, *where* $\mathcal{A}_j^{(i),\alpha}$ *are operators which map* $L^q(\Omega)$ *into* $L^q(\Omega)$, $\mathcal{L}^q_{(g)}(\Omega)$ *into* $L^q_{(g)}(\Omega)$ *for* $1 < q < \infty$ *and* $g \geq 0$ *weights from the Muckenhoupt class* A_q.

Proof: The formula (4.11) follows from (4.10), using the Green theorem (see Theorem VIII.1.15) and passing with $R \to \infty$. In order to get (4.12), we first pass with R to infinity in (4.10) and then take the first order derivative with respect to x_l. Next, let us continue with (4.13). We have the volume term

$$\int_\Omega \frac{\partial \mathcal{G}_{ik}}{\partial y_k} D^\alpha \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y}) \mathrm{d}\mathbf{y}\,.$$

Let us recall that the first derivative of $\boldsymbol{\mathcal{O}}^\mu$ is locally integrable while the second derivative (in the sense of distributions) can be written in the form

$$D^\alpha \boldsymbol{\mathcal{O}}^\mu = D^\alpha \boldsymbol{\mathcal{S}}^\mu + D^\alpha \boldsymbol{\mathcal{N}}^\mu\,, \quad |\alpha| = 2\,,$$

where the part $D^\alpha \boldsymbol{\mathcal{S}}^\mu$ represents the singular and $D^\alpha \boldsymbol{\mathcal{N}}^\mu$ the weakly singular part. As $D^\alpha \boldsymbol{\mathcal{N}}^\mu$ is locally integrable (see Section II.1) and $D^\alpha \boldsymbol{\mathcal{S}}^\mu$ satisfies the hypothesis of the Marcinkiewicz multiplier theorem and the Kurtz–Wheeden theorem (see Theorems II.3.2 and II.3.5), the formula (4.13) is shown. Finally, to show (4.14), we proceed analogously.

□

Remark 4.1 The representation formulas (4.11)–(4.14) are not the only ones which may be proved. Evidently, instead of (4.13) we could get for example

$$D^\alpha u_j(\mathbf{x}) = \int_\Omega D^\alpha_{\mathbf{x}} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) \frac{\partial \mathcal{G}_{ik}}{\partial y_k}(\mathbf{y}) d\mathbf{y} + \\ + \int_{\partial\Omega} \Big[D^\alpha_{\mathbf{y}} u_i(\mathbf{y}) T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) - \beta \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) D^\alpha_{\mathbf{y}} u_i(\mathbf{y}) \delta_{1k} + \\ + \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) T_{ik}(D^\alpha_{\mathbf{y}}\mathbf{u}, D^\alpha_{\mathbf{y}} p)(\mathbf{y}) \Big] n_k(\mathbf{y}) \mathrm{d}S\,,$$

$|\alpha| = 1$ etc. As such formulas are not convenient for our purpose, we shall not write them explicitly out.

The formulas (4.11)–(4.14) are valid also for less smooth functions $\boldsymbol{\mathcal{G}}$. We can apply the density argument to get

Corollary 4.1 *The integral representation formulas holds for a.a.* $\mathbf{x} \in \Omega$ *if* $\mathbf{v} \in W^{2,q}_{loc}(\overline{\Omega})$, $p \in W^{1,q}_{loc}(\overline{\Omega})$ *for some* $1 < q < \infty$ *and*

a) (4.11) if $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$, $1 < q < N+1$, $1 < r < \infty$

b) (4.12) if $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $1 < q < N+1$.

c) (4.13) if $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$, $1 < q, r < \infty$

d) (4.14) if $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $1 < q < \infty$.

Proof: We have that $\nabla \boldsymbol{\mathcal{O}}^\mu \in L^r(\mathbb{R}^N)$ for $r \in (\frac{N+1}{N}; \frac{N}{N-1})$ and therefore the convolution is well defined whenever $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$ with $\frac{1}{q} + \frac{1}{r} \geq 1$ i.e. $q < N+1$. In order to have well defined the trace $\boldsymbol{\mathcal{G}} \cdot \mathbf{n}$ on $\partial\Omega$, it is enough to have $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$ for some $1 < r < \infty$, see Remark VIII.3.6. Analogously for (4.13) and (4.14) we use the fact that

$$\boldsymbol{\mathcal{N}}^\mu \in L^r(\Omega) \qquad \text{for } q \in \Big(1, \frac{N}{N-1}\Big)\,.$$

□

We now start to study the integral representation of pressure. Let $\mathbf{f} \in C_0^\infty(\overline{\Omega})$. We denote

$$W_j(\mathbf{x}) = \int_\Omega \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) f_i(\mathbf{y}) \mathrm{d}\mathbf{y}$$
$$S(\mathbf{x}) = \int_\Omega e_i(\mathbf{x}-\mathbf{y}) f_i(\mathbf{y}) \mathrm{d}\mathbf{y}\,.$$

We have

Lemma 4.1 *Let* $\Omega \in C^{0,1}$ *be an exterior domain,* $f \in C_0^\infty(\overline{\Omega})$*. The functions* $\mathbf{W}$ *and* S *are infinitely differentiable functions in* Ω *and for all* $\mathbf{x} \in \Omega$ *we have*

$$A(\mathbf{W})(\mathbf{x}) + \beta \frac{\partial \mathbf{W}}{\partial x_1}(\mathbf{x}) + \nabla S(\mathbf{x}) = \mathbf{f}(\mathbf{x})\,.$$

Proof: As $e_i(\mathbf{x}-\mathbf{y}) = \frac{\partial \mathcal{E}}{\partial x_i}(\mathbf{x}-\mathbf{y})$ and $\nabla \boldsymbol{\mathcal{O}}^\mu$ is locally integrable, we have

$$A(W_j) + \beta \frac{\partial W_j}{\partial x_1} + \frac{\partial S}{\partial x_j} = -(1-\delta_{1k}\mu)\frac{\partial}{\partial x_k}\int_\Omega \frac{\partial}{\partial x_k}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) f_i(\mathbf{y})\mathrm{d}\mathbf{y} +$$
$$+\beta \int_\Omega \frac{\partial}{\partial x_1}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) f_i(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{\partial}{\partial x_i}\int_\Omega \frac{\partial}{\partial x_j}\mathcal{E}(\mathbf{x}-\mathbf{y}) f_i(\mathbf{y})\mathrm{d}\mathbf{y} =$$
$$= -(1-\mu\delta_{1k})\int_\Omega \frac{\partial}{\partial x_k}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\frac{\partial f_i(\mathbf{y})}{\partial y_k}\mathrm{d}\mathbf{y} + \beta\int_\Omega \frac{\partial \mathcal{O}^\mu_{ij}}{\partial x_1}(\mathbf{x}-\mathbf{y};\beta) f_i(\mathbf{y})\mathrm{d}\mathbf{y} +$$
$$+\int_\Omega \frac{\partial \mathcal{E}}{\partial x_j}(\mathbf{x}-\mathbf{y})\frac{\partial f_i}{\partial y_i}\mathrm{d}\mathbf{y} + (1-\mu\delta_{1k})\int_{\partial\Omega}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) f_i(\mathbf{y}) n_k(\mathbf{y})\mathrm{d}_{\mathbf{y}}S \, -$$
$$-\int_{\partial\Omega}\frac{\partial \mathcal{E}}{\partial x_j}(\mathbf{x}-\mathbf{y}) f_i(\mathbf{y}) n_i(\mathbf{y})\mathrm{d}_{\mathbf{y}}S \;=\lim_{\varepsilon\to 0^+}\int_{\partial B^\varepsilon(\mathbf{x})}\Big[(1-\mu\delta_{1k})\frac{\partial \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial n_{\mathbf{y}}} +$$
$$+e_j(\mathbf{x}-\mathbf{y}) n_i(\mathbf{y})\Big]\mathrm{d}_{\mathbf{y}}S \; + \text{v.p.}\int_\Omega \Big[-(1-\mu\delta_{1k})\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_k}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) -$$
$$-\beta\frac{\partial \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial y_1} - \frac{\partial e_j}{\partial y_i}(\mathbf{x}-\mathbf{y})\Big] f_i(\mathbf{y})\mathrm{d}\mathbf{y} = f_j(\mathbf{x})\,.$$

We used the fact that the pair $(\boldsymbol{\mathcal{O}}^\mu, \mathbf{e})$ is the fundamental solution to the modified Oseen problem, see (4.8). The differentiability follows easily.

□

Moreover, we have that the pair $(\mathbf{u}, p)$ solves

$$A(\mathbf{u}) + \beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla p = \mathbf{f}\,.$$

Using the integral representation of $\mathbf{u}$ (see (4.11)) and the fact that $\mathbf{f} = \nabla \cdot \boldsymbol{\mathcal{G}}$, we have

$$\begin{aligned}
-\frac{\partial p}{\partial x_j} + f_j = A(W_j) + \beta\frac{\partial W_j}{\partial x_1} + \int_{\partial\Omega} \Big\{ -\beta\Big(A + \beta\frac{\partial}{\partial x_1}\Big)\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1l} + \\
+ u_i(\mathbf{y})\Big(A + \beta\frac{\partial}{\partial x_1}\Big)T_{il}(\boldsymbol{\mathcal{O}}^\mu_{\cdot,j}, e_j)(\mathbf{x}-\mathbf{y}) + \\
+ \Big(A + \beta\frac{\partial}{\partial x_1}\Big)\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{il}(\mathbf{u},p)(\mathbf{y})\Big\} n_l(\mathbf{y})\mathrm{d}_{\mathbf{y}}S\,.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
-\frac{\partial p}{\partial x_j} = -\frac{\partial S}{\partial x_j} + \int_{\partial\Omega} \Big\{ -\beta\Big(A + \beta\frac{\partial}{\partial x_1}\Big)\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)]u_i(\mathbf{y})\delta_{1l} + \\
+ u_i(\mathbf{y})\Big(A + \beta\frac{\partial}{\partial x_1}\Big)T_{il}(\boldsymbol{\mathcal{O}}^\mu_{\cdot,j}, e_j)(\mathbf{x}-\mathbf{y}) + \\
+ \Big(A + \beta\frac{\partial}{\partial x_1}\Big)\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y})T_{il}(\mathbf{u},p)(\mathbf{y})\Big\} n_l(\mathbf{y})\mathrm{d}_{\mathbf{y}}S\,.
\end{aligned} \tag{4.15}$$

We shall calculate the boundary terms. We easily check

$$\Big(A + \beta\frac{\partial}{\partial x_1}\Big)\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y}) = \frac{\partial e_j}{\partial y_i}(\mathbf{x}-\mathbf{y}) = -\frac{\partial e_i}{\partial x_j}(\mathbf{x}-\mathbf{y})$$

$$\Big(A + \beta\frac{\partial}{\partial x_1}\Big)T_{il}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j) = \Big(A + \beta\frac{\partial}{\partial x_1}\Big)\Big[\frac{\partial \mathcal{O}^\mu_{ij}}{\partial x_l} + \frac{\partial \mathcal{O}^\mu_{lj}}{\partial x_i} - e_j\delta_{il} - \mu\delta_{1l}\frac{\partial \mathcal{O}^\mu_{ij}}{\partial x_1}\Big]\,.$$

Denoting

$$\mathcal{T}_{il}(\mathbf{e}) = \frac{\partial e_i}{\partial x_l} + \frac{\partial e_l}{\partial x_i} + \beta e_1\delta_{il} + \mu\frac{\partial e_1}{\partial x_1}\delta_{il} - \mu\delta_{1l}\frac{\partial e_i}{\partial x_1} \tag{4.16}$$

we get

$$\Big(A + \beta\frac{\partial}{\partial x_1}\Big)T_{il}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j) = -\frac{\partial}{\partial x_j}\mathcal{T}_{il}(\mathbf{e})\,.$$

We have from (4.14)

$$\begin{aligned}
\frac{\partial p}{\partial x_j} = \frac{\partial S}{\partial x_j} + \frac{\partial}{\partial x_j}\int_{\partial\Omega}\Big[-\beta\delta_{1l}e_i(\mathbf{x}-\mathbf{y})u_i(\mathbf{y}) + \\
+ u_i(\mathbf{y})\mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + e_i(\mathbf{x}-\mathbf{y})T_{il}(\mathbf{u},p)(\mathbf{y})\Big] n_l(\mathbf{y})\mathrm{d}_{\mathbf{y}}S\,.
\end{aligned} \tag{4.17}$$

We can always add to p such a constant that $p \to 0$ as $|\mathbf{x}| \to \infty$. Therefore for such p we have

Theorem 4.2 *Let $\Omega \in C^{0,1}$, an exterior domain, $\boldsymbol{\mathcal{G}} \in C_0^\infty(\overline{\Omega})$ and $(\mathbf{u}, p)$ be the unique solution to the Oseen problem (4.1). Let $\mathcal{T}_{il}(\mathbf{e})$ be defined in (4.16) and $T_{ij}(\mathbf{u}, p)$ in (4.2). Then*

$$\begin{aligned} p(\mathbf{x}) = v.p. \int_\Omega \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_k} \mathcal{G}_{ik}(\mathbf{y}) d\mathbf{y} + c_{ik}\mathcal{G}_{ik}(\mathbf{x}) + \\ + \int_{\partial\Omega} \Big[u_i(\mathbf{y}) \mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + e_i(\mathbf{x}-\mathbf{y}) T_{il}(\mathbf{u}, p)(\mathbf{y}) - \\ -\beta e_i(\mathbf{x}-\mathbf{y}) u_i(\mathbf{y}) \delta_{1l} + e_i(\mathbf{x}-\mathbf{y}) \mathcal{G}_{il}(\mathbf{y}) \Big] n_l(\mathbf{y})\, d_{\mathbf{y}} S \end{aligned} \tag{4.18}$$

$$\begin{aligned} \frac{\partial p(\mathbf{x})}{\partial x_j} = v.p. \int_\Omega \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} \frac{\partial}{\partial y_k} \mathcal{G}_{ik}(\mathbf{y}) d\mathbf{y} + c_{ij} \frac{\partial}{\partial x_k} \mathcal{G}_{ik}(\mathbf{x}) + \\ + \int_{\partial\Omega} \Big[u_i(\mathbf{y}) \frac{\partial}{\partial x_j} \mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} T_{il}(\mathbf{u}, p)(\mathbf{y}) - \\ -\beta \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} u_i(\mathbf{y}) \delta_{1l} \Big] n_l(\mathbf{y})\, d_{\mathbf{y}} S\,. \end{aligned} \tag{4.19}$$

Proof: To get (4.18) it is enough to apply the Green theorem (see Theorem VIII.1.15) on (4.17) and recall that due to the choice of an appropriate constant we have $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. The formula (4.19) can be obtained from (4.18) differentiating with respect to x_j and using the same procedure as in the proof of Theorem 4.1 and Lemma 4.1.

□

Similarly as in Theorem 4.1, using the density argument one can extend Theorem 4.2 for less regular right hand sides.

Corollary 4.2 *The integral representation holds a.e. in Ω if $\mathbf{v} \in W^{2,q}_{loc}(\overline{\Omega})$, $p \in W^{1,q}_{loc}(\overline{\Omega})$ for some $1 < q < \infty$ and*

a) (4.18) if $\boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^r_{loc}(\overline{\Omega})$, $1 < q, r < \infty$

b) (4.19) if $\nabla \cdot \boldsymbol{\mathcal{G}} \in L^q(\Omega)$, $1 < q < \infty$.

Proof: It is completely analogous to the proof of Corollary 4.1.

□

III.5 Modified Oseen problem — L^q–estimates independent of β

The last section of this chapter is devoted to the study of the modified Oseen problem in exterior domains. We shall mostly study the case $\Omega \subset \mathbb{R}^2$, but the last theorem is devoted to the threedimensional flow. The aim is to develop an analogue to Theorem 3.5 for $\mathbb{R}^3$, where we got for $1 < q < \frac{3}{2}$ estimates independent of β. The study of such estimates in two space dimensions is more delicate; the essential tool will be the integral representation of solutions (see Theorems 4.1 and 4.2).

Lemma 5.1 *Let Ω be a twodimensional exterior domain of class C^2 and let for some $q \in (1;2]$*

$$\mathbf{u}_* \in W^{2-\frac{1}{q},q}(\partial\Omega)\,.$$

Let $(\mathbf{u}, p)$ be the corresponding solution to

$$\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla p &= \mathbf{0} \\ \nabla \cdot \mathbf{u} &= 0 \\ \mathbf{u} = \mathbf{u}_* \quad & at\ \partial\Omega \\ \lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}) &= \mathbf{0} \end{aligned} \tag{5.1}$$

and let $\mathbf{T}$ be defined in (4.2), $\mu = O(|\ln\beta|^{-1})$ as $\beta \to 0^+$. Then there exists $\overline{B} > 0$ and $C = C(\Omega, \overline{B})$ such that for all $0 < \beta \leq \overline{B}$

$$\Big|\int_{\partial\Omega} \mathbf{T}(\mathbf{u},p)\cdot \mathbf{n}dS\Big| \leq C|\ln\beta|^{-1}\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\,. \tag{5.2}$$

Proof: Putting the term $\beta\frac{\partial \mathbf{u}}{\partial x_1}$ on the right hand side, multiplying $(5.1)_1$ by a cut–off function which is equal to one in $B_{R_2}(\mathbf{0})$ and vanish outside of $B_{R_1}(\mathbf{0})$, $R_2 > R_1 > \operatorname{diam}\Omega^c$ we get from Theorem VIII.5.4

$$\|\mathbf{u}\|_{2,q,\Omega_{R_2}} + \|p\|_{1,q,\Omega_{R_2}} \leq C(\|\mathbf{u}\|_{1,q,\Omega_{R_1}} + \|p\|_{q,\Omega_{R_1}} + \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)})\,. \tag{5.3}$$

Applying the trace imbedding theorem (see Corollary VIII.1.1) we get for all $1 < q \leq 2$

$$\|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)} \leq C\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\,.$$

Moreover, in Theorem 3.4 we have shown the existence of a constant C, independent of β, such that

$$|\mathbf{u}|_{1,2} \leq C(1+\beta)\|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)} \leq C_1(1+\beta)\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\,. \tag{5.4}$$

Applying the Friedrichs inequality (see Theorem VIII.1.10)

$$\|\mathbf{u}\|_{q,\Omega_{R_1}} \leq C(\|\nabla\mathbf{u}\|_{q,\Omega_{R_1}} + \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}) \leq C(\Omega,B)\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} \tag{5.5}$$

for all $\beta \in (0, B]$. Moreover, from Theorem VIII.5.3 together with (5.4) and (5.5) after modifying the pressure by a suitable constant we have

$$\|p\|_{q,\Omega_{R_1}} \leq C\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} \tag{5.6}$$

and (5.3) together with (5.4)–(5.6) imply

$$\|\mathbf{u}\|_{2,q,\Omega_{R_2}} + \|p\|_{1,q,\Omega_{R_2}} \leq C\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\,. \tag{5.7}$$

Let us recall that

$$\mathcal{O}^{\mu}_{ij} = \mathcal{O}_{ij} + E^{\mu}_{ij}\,,$$

where $|E^{\mu}_{ij}(\mathbf{x};\beta)| \le C\mu|\mathcal{O}_{ij}(\mathbf{x};\beta)|$ for $|\mathbf{x}| \le 1$ and $\boldsymbol{\mathcal{O}}(\cdot\,;\beta)$ is the fundamental Oseen tensor. We have for $\mathbf{x} \in \Omega^{R_2}_{R_1}$, $\mathbf{y} \in \partial\Omega$

$$\begin{aligned} \mathcal{O}^{\mu}_{ij}(\mathbf{x}-\mathbf{y};\beta) &= \mathcal{R}(|\mathbf{x}-\mathbf{y}|) + C\ln\beta \\ |D^k_{\mathbf{x}}\mathcal{O}^{\mu}_{ij}(\mathbf{x}-\mathbf{y};\beta)| &= D^k_{\mathbf{x}}\mathcal{R}(|\mathbf{x}-\mathbf{y}|) \end{aligned} \tag{5.8}$$

with $\mathcal{R}(|\mathbf{x}-\mathbf{y}|)+D^k_{\mathbf{x}}\mathcal{R}(|\mathbf{x}-\mathbf{y}|) \le C(B)$; we used here that $\mu\ln\beta \le C(B)$. Since $|e_j(\mathbf{x}-\mathbf{y})| \le C$ in $\Omega^{R_2}_{R_1}$, denoting $\mathcal{J}(\beta) = |\ln\beta||\int_{\partial\Omega}\mathbf{T}(\mathbf{u},p)\cdot\mathbf{n}\mathrm{d}S|$ we have from (4.11)

$$\mathcal{J}(\beta) \le C\Big[|\mathbf{u}(\mathbf{x})| + \int_{\partial\Omega}|\mathbf{u}_*|\mathrm{d}S \;+ C\int_{\partial\Omega}|\mathbf{T}(\mathbf{u},p)|\mathrm{d}S\Big]\,, \tag{5.9}$$

where the constant C depends only on B, but not on β. Using the trace theorem (see Theorem VIII.1.6) we find out

$$\int_{\partial\Omega}|\mathbf{T}(\mathbf{u},p)|\mathrm{d}S \;\le C\int_{\partial\Omega}(|\nabla\mathbf{u}|+|p|)\mathrm{d}S \;\le C(\|\nabla\mathbf{u}\|_{1,q,\Omega_{R_2}}+\|p\|_{1,q,\Omega_{R_2}})\,. \tag{5.10}$$

Combining (5.9) with (5.10) and (5.7) we have

$$\mathcal{J}(\beta) \le |\mathbf{u}(\mathbf{x})| + C\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\,.$$

Integrating the inequality over $\Omega^{R_2}_{R_1}$ and using the Hölder inequality we get

$$\mathcal{J}(\beta) \le C(\|\mathbf{u}\|_{q,\Omega_{R_1}} + \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}) \le C\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\,.$$

The lemma is proved.

□

Let us denote for $1 < q \le \frac{6}{5}$

$$\begin{aligned} C_q = \Big\{\mathbf{u} \in L^{\frac{3q}{3-2q}}(\Omega) \cap D^{2,q}(\Omega) \cap D^{1,\frac{3q}{3-q}}(\Omega);\\ u_2 \in L^{\frac{2q}{2-q}}(\Omega); \nabla u_2 \in L^q(\Omega)\Big\}\,. \end{aligned} \tag{5.11}$$

Remark 5.1 If $\mathbf{u} \in C_q$ then $\mathbf{u} \in D^{1,\frac{2q}{2-q}}(\Omega) \cap L^{\frac{3q}{3-2q}}(\Omega)$ (see Lemma VIII.1.12) and therefore from Theorem VIII.1.17 and Remark VIII.1.11

$$\lim_{|\mathbf{x}|\to\infty}\mathbf{u}(\mathbf{x}) = \mathbf{0}$$

uniformly.

We shall investigate the L^q–estimates of solutions to the modified Oseen problem for $1 < q < \frac{6}{5}$. Let us first assume the right hand side equal to zero. We have

Lemma 5.2 *Let $\Omega \subset \mathbb{R}^2$ be exterior domain of class C^2 and let*

$$\mathbf{u}_* \in W^{2-\frac{1}{q},q}(\partial\Omega)\,, 1 < q < \frac{6}{5}\,.$$

Then for any $\beta > 0$ there exists a unique solution to the modified Oseen problem

$$\left.\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla p = \mathbf{0} \\ \nabla\cdot\mathbf{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega$$
$$\mathbf{u} = \mathbf{u}_* \quad \text{at } \partial\Omega$$
$$\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}) = \mathbf{0}$$

such that $\mathbf{u} \in C_q$, $p \in D^{1,q}(\Omega)$. Let $\mu = O(|\ln\beta|^{-1})$ as $\beta \to 0^+$. Then there exists $\beta_0 > 0$ such that for all $\beta \in (0, \beta_0]$

$$\begin{aligned} \langle \mathbf{u}\rangle_{\beta,q} \equiv \beta(\|u_2\|_{\frac{2q}{2-q}} + |u_2|_{1,q}) + \beta^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{3q}{3-2q}} + \beta^{\frac{1}{3}}|\mathbf{u}|_{1,\frac{3q}{3-q}} \leq \\ \leq C\beta^{2(1-\frac{1}{q})}|\ln\beta|^{-1}\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} \end{aligned} \tag{5.12}$$

with $C = C(\beta_0, q, \Omega)$.

Proof: As $\frac{3q}{3-q} < 3 \Longleftrightarrow q < \frac{3}{2}$, the uniqueness follows from Theorem 3.5. Moreover, from Theorems 3.4 and 3.6 we know that there exists a pair $(\mathbf{u}, p)$ such that $\langle \mathbf{u}\rangle_{\beta,q}$ is finite, $\mathbf{u} \in D^{2,q}(\Omega)$ and $p \in D^{1,q}(\Omega)$. Analogously as in Lemma 5.1 we show that (see (5.7))

$$\|\mathbf{u}\|_{2,q,\Omega_2} + \|p\|_{1,q,\Omega_2} \leq C\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} \tag{5.13}$$

and therefore, by imbedding theorem (see Theorem VIII.1.2)

$$\|u_2\|_{\frac{2q}{2-q},\Omega_2} + |u_2|_{1,q,\Omega_2} + \|\mathbf{u}\|_{\frac{3q}{3-2q},\Omega_2} + |\mathbf{u}|_{1,\frac{3q}{3-q},\Omega_2} \leq C\|\mathbf{u}\|_{2,q,\Omega_2}$$

and as $2(1-\frac{1}{q}) < \frac{1}{3}$ for $q \in (1, \frac{6}{5})$

$$\langle \mathbf{u}\rangle_{\beta,q} \leq C\beta^{2(1-\frac{1}{q})+\varepsilon}\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} \quad (\varepsilon > 0 \text{ for } 1 < q < \frac{6}{5})\,.$$

Next we give estimates of $\langle \mathbf{u}\rangle_{\beta,q}$ in Ω^2. We have

$$\begin{aligned} u_j(\mathbf{x}) = \mathcal{I}(\mathbf{u})\cdot\mathbf{w}_j(\mathbf{x};\beta) + \int_{\partial\Omega}\Big[-\beta\mathbf{w}_j(\mathbf{x}-\mathbf{z};\beta)\cdot\mathbf{u}_*(\mathbf{z})\mathbf{e}_1+ \\ +\mathbf{u}_*(\mathbf{z})\cdot\mathbf{T}(\mathbf{w}_j, e_j)(\mathbf{x}-\mathbf{z};\beta) - [\mathbf{w}_j(\mathbf{x}-\mathbf{z};\beta) + \mathbf{w}_j(\mathbf{z};\beta)]\cdot\mathbf{T}(\mathbf{u},\pi)(\mathbf{z})\Big]\cdot\mathbf{n}(\mathbf{z})\mathrm{d}S \end{aligned}$$

with

$$\mathcal{I}(\mathbf{u}) = \int_{\partial\Omega}\mathbf{T}(\mathbf{u}, p)\cdot\mathbf{n}\mathrm{d}S$$
$$\mathbf{w}_j = (\mathcal{O}^{\mu}_{1j}, \mathcal{O}^{\mu}_{2j})\,, j = 1, 2$$

(see (4.13) and (4.2)). Applying the mean value theorem we have (recall that $\Omega^c \subset B_1(\mathbf{0})$)

$$|u_j(\mathbf{x})| \le |\mathcal{I}(\mathbf{u})||\mathbf{w}_j(\mathbf{x};\beta)| + A \sup_{\mathbf{z}\in\Omega_1} \Big\{\beta|\mathbf{w}_j(\mathbf{x}-\mathbf{z};\beta)| + \\ + |\mathbf{e}(\mathbf{x}-\mathbf{z})| + |\nabla_{\mathbf{x}}\mathbf{w}_j(\mathbf{x}-\mathbf{z};\beta)|\Big\} \tag{5.14}$$

$$|\nabla u_j(\mathbf{x})| \le |\mathcal{I}(\mathbf{u})||\nabla\mathbf{w}_j(\mathbf{x};\beta)| + A \sup_{\mathbf{z}\in\Omega_1} \Big\{\beta|\nabla_{\mathbf{x}}\mathbf{w}_j(\mathbf{x}-\mathbf{z};\beta)| + \\ |\nabla_{\mathbf{x}}\mathbf{e}(\mathbf{x}-\mathbf{z})| + |\nabla^2_{\mathbf{x}}\mathbf{w}_j(\mathbf{x}-\mathbf{z};\beta)|\Big\} \tag{5.15}$$

with

$$A = \|\nabla\mathbf{u}\|_{1,(\partial\Omega)} + \|p\|_{1,(\partial\Omega)} + \|\mathbf{u}_*\|_{1,(\partial\Omega)} .$$

Using the rescalling (see Theorem 1.2) we get from (5.14) ($\mathbf{y} = \beta\mathbf{x}$)

$$|u_j(\mathbf{x})| \le |\mathcal{I}(\mathbf{u})||\mathbf{w}_j(\mathbf{y};1)| + \\ + C\beta A \sup_{|\mathbf{z}|\le\beta} \Big\{|\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)| + |\mathbf{e}(\mathbf{y}-\mathbf{z})| + |\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|\Big\}$$

and

$$\|u_j\|^t_{t,\Omega^2} \le C\beta^{-2}\Big[|\mathcal{I}(\mathbf{u})|^t\|\mathbf{w}_j(\mathbf{y};1)\|^t_{t,\mathbb{R}^2} + \\ + \beta^t A^t \int_{|\mathbf{y}|\ge 2\beta} \sup_{|\mathbf{z}|\le\beta} \Big\{|\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)| + \\ + |\mathbf{e}(\mathbf{y}-\mathbf{z})| + |\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|\Big\}^t \mathrm{d}\mathbf{y}\Big] . \tag{5.16}$$

We have that $\mathbf{w}_j(\cdot\,;1) \in L^t(\mathbb{R}^2)$ for $t \in (3;\infty)$ if $j=1$ and $t \in (2;\infty)$ if $j=2$ (see Lemma 1.2 and Theorem 1.2). Therefore

$$\int_{|\mathbf{y}|\ge 2\beta} \sup_{|\mathbf{z}|\le\beta} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} \le \\ \le \int_{4\ge|\mathbf{y}|\ge 2\beta} \sup_{|\mathbf{z}|\le\beta} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} + \int_{|\mathbf{y}|\ge 4} \sup_{|\mathbf{z}|\le 1} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} .$$

The first integral can be estimated as follows

$$\int_{4\ge|\mathbf{y}|\ge 2\beta} \sup_{|\mathbf{z}|\le\beta} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} \le \\ \le C \int_{4\ge|\mathbf{y}|\ge 2\beta} \sup_{|\mathbf{z}|\le\beta} (|\ln|\mathbf{y}-\mathbf{z}|| + 1)^t \mathrm{d}\mathbf{y} \le \\ \le C \int_{4\ge|\mathbf{y}|\ge 2\beta} (|\ln|\mathbf{y}|| + 1)^t \mathrm{d}\mathbf{y} \le C_1 ,$$

where C_1 does not depend on β. Using the fact that for $|\mathbf{y}| \ge |\mathbf{y}_0|$ (large enough)

$$\sup_{|\mathbf{z}|\le 1} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)| \le |\mathbf{w}_j(\mathbf{y};1)| + \sup_{|\mathbf{z}|\le 1} |\nabla\mathbf{w}_j(\mathbf{y}-\gamma\mathbf{z};1)| , \quad \gamma \in (0;1) ,$$

we can estimate the second term

$$\int_{|\mathbf{y}|\ge 4} \sup_{|\mathbf{z}|\le 1} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} \le \int_{|\mathbf{y}|\ge\mathbf{y}_0} (|\mathbf{w}_j(\mathbf{y};1)|^t + |\mathbf{y}|^{-tr_j}) \mathrm{d}\mathbf{y} + \\ + \int_{4\le|\mathbf{y}|\le|\mathbf{y}_0|} \sup_{|\mathbf{z}|\le 1} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} ,$$

where $r_j = 1$ if $j = 1$ and $r_j = \frac{3}{2}$ if $j = 2$. As $\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)$ is locally regular, we have

$$\int_{|\mathbf{y}|\geq 2\beta} \sup_{|\mathbf{z}|\leq\beta} |\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} \leq C \tag{5.17}$$

with C independent of β and $t > 3$ if $j = 1$ and $t > 2$ if $j = 2$. We continue with the pressure term on the right hand side of (5.16). We have for $t > 2$

$$\begin{aligned}\int_{|\mathbf{y}|\geq 2\beta} \sup_{|\mathbf{z}|\leq\beta} |\mathbf{e}(\mathbf{y}-\mathbf{z})|^t \mathrm{d}\mathbf{y} \leq \\ \leq C\Big(\int_{4\geq|\mathbf{y}|\geq 2\beta} |\mathbf{y}|^{-t}\mathrm{d}\mathbf{y} + \int_{|\mathbf{y}|\geq 4} \sup_{|\mathbf{z}|\leq 1} |\mathbf{y}-\mathbf{z}|^t \mathrm{d}\mathbf{y}\Big) \leq C(\beta^{2-t}+1)\,.\end{aligned} \tag{5.18}$$

Analogously

$$\begin{aligned}\int_{|\mathbf{y}|\geq 2\beta} \sup_{|\mathbf{z}|\leq\beta} |\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} \leq \int_{4\geq|\mathbf{y}|\geq 2\beta} \sup_{|\mathbf{z}|\leq\beta} |\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} + \\ + \int_{|\mathbf{y}|\geq 4} \sup_{|\mathbf{z}|\leq 1} |\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} \leq C \int_{4\geq|\mathbf{y}|\geq 2\beta} |\mathbf{y}|^{-t}\mathrm{d}\mathbf{y} + \\ + \int_{4\leq|\mathbf{y}|\leq|\mathbf{y}_0|} |\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} + \int_{|\mathbf{y}|\geq|\mathbf{y}_0|} (|\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y};1)|^t + |\mathbf{y}|^{-ts_j})\mathrm{d}\mathbf{y}\end{aligned}$$

with $s_j = \frac{3}{2}$ if $j = 1$, $s_j = 2$ if $j = 2$. Therefore we have for $t > 2$ (if $j = 1$) or $t > 1$ (if $j = 2$)

$$\int_{|\mathbf{y}|\geq 2\beta} \sup_{|\mathbf{z}|\leq\beta} |\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|^t \mathrm{d}\mathbf{y} \leq C(1+\beta^{2-t})\,. \tag{5.19}$$

Combining (5.17)–(5.19) we finally have

$$\|u_j\|_{t,\Omega^2} \leq C(\beta^{-\frac{2}{t}}|\mathcal{I}(\mathbf{u})| + A(1+\beta^{\frac{t-2}{t}})) \tag{5.20}$$

for all $t > 2$ if $j = 2$ and $t > 3$ if $j = 1$. Therefore for all $\beta < \beta_0$

$$\|u_j\|_{t,\Omega^1} \leq C(\beta_0)(\beta^{-\frac{2}{t}}|\mathcal{I}(\mathbf{u})| + A)\,.$$

Setting $t = \frac{2q}{2-q}$ $(j = 2)$ and $t = \frac{3q}{3-2q}$ $(j = 1)$ we have

$$\beta\|u_2\|_{\frac{2q}{2-q},\Omega^2} + \beta^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{3q}{3-2q},\Omega^2} \leq C(\beta_0)(\beta^{-2(1-\frac{1}{q})}|\mathcal{I}(\mathbf{u})| + \beta^{\frac{2}{3}}A)\,. \tag{5.21}$$

For gradients we have from (5.15)

$$\begin{aligned}|u_j|^\tau_{1,\tau,\Omega^2} \leq C\beta^{\tau-2}\Big[|\mathcal{I}(\mathbf{u})|^\tau \|\nabla\mathbf{w}_j(\cdot\,;1)\|^\tau_{\tau,\mathbb{R}^2} + \\ +\beta^\tau A^\tau \int_{|\mathbf{y}|\geq 2\beta} \sup_{|\mathbf{z}|\leq\beta} \Big\{|\nabla_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)| + \\ +|\nabla_{\mathbf{y}}\mathbf{e}(\mathbf{y}-\mathbf{z})| + |\nabla^2_{\mathbf{y}}\mathbf{w}_j(\mathbf{y}-\mathbf{z};1)|\Big\}^\tau \mathrm{d}\mathbf{y}\Big]\,.\end{aligned} \tag{5.22}$$

Now, $\|\nabla\mathbf{w}_j(\cdot\,;1)\|_{\tau,\mathbb{R}^2} \leq C$ if $1 < \tau < 2$ $(j = 2)$, $\frac{3}{2} < \tau < 2$ $(j = 1)$. As $\nabla\mathbf{e}(\mathbf{x}-\mathbf{y}) \sim \nabla^2\mathbf{w}_j(\mathbf{x}-\mathbf{y};1) \sim \frac{1}{|\mathbf{x}-\mathbf{y}|^2}$ and $\nabla\mathbf{w}_j(\mathbf{x}-\mathbf{y};1) \sim \frac{1}{|\mathbf{x}-\mathbf{y}|}$ for $|\mathbf{x}-\mathbf{y}|$ small, we get as above

$$|u_j|_{1,\tau,\Omega^2} \leq C\Big(\beta^{1-\frac{2}{\tau}}|\mathcal{I}(\mathbf{u})| + A(1+\beta^{2-\frac{2}{\tau}}+\beta)\Big) \leq C(\beta_0)(\beta^{1-\frac{2}{\tau}}|\mathcal{I}(\mathbf{u})| + A)$$

and again, setting $\tau = q$ $(j = 2)$ and $\tau = \frac{3q}{3-q}$ $(j = 1, 2)$

$$\beta^{\frac{1}{3}} |\mathbf{u}|_{1,\frac{3q}{3-q},\Omega^2} + \beta |u_2|_{1,q,\Omega^2} \leq C(\beta_0)(\beta^{2(1-\frac{1}{q})} |\mathcal{I}(\mathbf{u})| + \beta^{\frac{1}{3}} A) \,. \tag{5.23}$$

Moreover

$$A \ \leq C\Big(\|\mathbf{u}\|_{2,q,\Omega_2} + \|p\|_{1,q,\Omega_2} + \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\Big) \leq C\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}$$

(see (5.13)) and ($\varepsilon > 0$ for $1 < q < \frac{6}{5}$)

$$\langle \mathbf{u} \rangle_{\beta,q} \leq C\beta^{2(1-\frac{1}{q})} (|\mathcal{I}(\mathbf{u})| + \beta^{\varepsilon} \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}) \,.$$

Applying Lemma 5.1 we finish the proof.

□

Remark 5.2 If $q = \frac{6}{5}$, we could show analogously as in Lemma 5.2 that

$$\langle \mathbf{u} \rangle_{\beta,\frac{6}{5}} \leq C\beta^{\frac{1}{3}} \|\mathbf{u}_*\|_{\frac{7}{6},\frac{6}{5},(\partial\Omega)} \,.$$

Next we investigate the situation when $\mathbf{u}_* \equiv \mathbf{0}$ and $\mathbf{f} \neq \mathbf{0}$.

Lemma 5.3 *Let $\Omega \in C^2$ be an exterior domain in $\mathbb{R}^2$. Let $\mathbf{f} \in L^q(\Omega)$, $1 < q < \frac{6}{5}$. Then there exists exactly one solution to the modified Oseen problem*[15]

$$\left.\begin{aligned} A(\mathbf{w}) + \beta\frac{\partial \mathbf{w}}{\partial x_1} + \nabla\tau = \mathbf{f} \\ \nabla\cdot\mathbf{w} = 0 \end{aligned}\right\} \quad \text{in } \Omega$$
$$\mathbf{u} = \mathbf{0} \qquad \text{at } \partial\Omega$$
$$\lim_{|\mathbf{x}|\to\infty} \mathbf{w}(\mathbf{x}) = \mathbf{0}$$

such that $\mathbf{w} \in C_q$, $\tau \in D^{1,q}(\Omega)$. This solution satisfies

$$\langle \mathbf{w} \rangle_{\beta,q} \leq C\|\mathbf{f}\|_q$$

for all $\beta \in (0, \beta_0]$ and $C = C(\Omega, q, \beta_0)$.

Proof: Extend $\mathbf{f}$ by zero outside of Ω. We put

$$\mathbf{w} = \mathbf{v} + \mathbf{z}$$
$$\tau = p + r \,,$$

where

$$\left.\begin{aligned} A(\mathbf{v}) + \beta\frac{\partial \mathbf{v}}{\partial x_1} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{v} = 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^2 \tag{5.24}$$

[15] As usually, if $\mathbf{w}_1$, τ_1 is another solution to the same data with $\langle \mathbf{w}_1 \rangle_{\beta,q}$ finite, then $\mathbf{w}_1 = \mathbf{w}$ and $\tau_1 = \tau + \text{const}$

and

$$\left.\begin{aligned} A(\mathbf{z}) + \beta\frac{\partial \mathbf{z}}{\partial x_1} + \nabla r = \mathbf{0} \\ \nabla\cdot\mathbf{z} = 0 \end{aligned}\right\} \quad \text{in } \Omega$$
$$\mathbf{u} = -\mathbf{v} \quad \text{at } \partial\Omega \tag{5.25}$$
$$\lim_{|\mathbf{x}|\to\infty} \mathbf{w}(\mathbf{x}) = \mathbf{0}\,.$$

From Theorem 2.1 we have

$$\langle \mathbf{v}\rangle_{\beta,q} + |\mathbf{v}|_{2,q} + |p|_{1,q} \le c_1\|\mathbf{f}\|_q \tag{5.26}$$

with c_1 independent on β. Moreover, from Lemma 2.3 we have that $\overline{\mathbf{v}}$, defined

$$\overline{v}_j(\mathbf{x}) = \int_{\mathbb{R}^2} \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) f_i(\mathbf{y})\mathrm{d}\mathbf{y} \tag{5.27}$$

is equal to $\mathbf{v}$ a.e. in $\mathbb{R}^2$. To show this, it is enough to verify that $\overline{\mathbf{v}} \in L^{q_1}(\mathbb{R}^2) \cap D^{1,q_2}(\mathbb{R}^2)$ for $1 < q_2 < 3$, $1 < q_1 < \infty$. But

$$\|\overline{\mathbf{v}}\|_{q_1} \le \|\boldsymbol{\mathcal{O}}^\mu\|_{p_1}\|\mathbf{f}\|_q\,, \quad \frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{q} - 1$$
$$\|\nabla\overline{\mathbf{v}}\|_{q_2} \le \|\boldsymbol{\mathcal{O}}^\mu\|_{p_2}\|\mathbf{f}\|_q\,, \quad \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q} - 1$$

and we may apply Lemma 2.3 as, evidently, $p_2 \in (\frac{3}{2};2)$ and so we can find for any $q \in (1;\frac{6}{5})$ such q_2 that $q_2 < 3$. Moreover, both functions are globally integrable and therefore $\overline{\mathbf{v}} = \mathbf{v}$ a.e. in $\mathbb{R}^2$.

From (5.27) we have

$$|\mathbf{v}(\mathbf{x})|^{q'} \le C\|\mathbf{f}\|_q^{q'} \int_{\mathbb{R}^2} |\boldsymbol{\mathcal{O}}^\mu(\beta(\mathbf{x}-\mathbf{y});1)|^{q'}\mathrm{d}\mathbf{y}$$

and therefore

$$\|\mathbf{v}\|^{q'}_{q',B_1(\mathbf{0})} \le C\beta^{-2}\|\mathbf{f}\|_q^{q'}$$

what implies

$$\|\mathbf{v}\|_{q',B_1(\mathbf{0})} \le C\beta^{-2(1-\frac{1}{q})}\|\mathbf{f}\|_q\,. \tag{5.28}$$

We pass to the problem (5.25). We apply Lemma 5.2 to get the existence of a unique couple $(\mathbf{z},r)$ such that for all $\beta \in (0;\beta_0]$

$$\langle\mathbf{z}\rangle_{\beta,q} \le C\beta^{2(1-\frac{1}{q})}\|\mathbf{v}\|_{2-\frac{1}{q},q,(\partial\Omega)}\,.$$

The trace and interpolation theorems (see Theorems VIII.1.6 and VIII.1.11) imply

$$\|\mathbf{v}\|_{2-\frac{1}{q},q,(\partial\Omega)} \le C\|\mathbf{v}\|_{2,q,\Omega_1} \le C(\|\mathbf{v}\|_{q,\Omega_1} + |\mathbf{v}|_{2,q,\Omega_1})\,.$$

Since $q' > q$, (5.28) and (5.26) yield

$$\langle\mathbf{z}\rangle_{\beta,q} \le C\beta^{2(1-\frac{1}{q})}\Big(\beta^{-2(1-\frac{1}{q})} + 1\Big)\|\mathbf{f}\|_q \le C(\beta_0)\|\mathbf{f}\|_q.$$

The lemma is shown.

□

Remark 5.3 The same result holds also for $q = \frac{6}{5}$.

Now, let $(\mathbf{u}, \pi)$ solves

$$\begin{gathered} A(\mathbf{u}) + \beta \frac{\partial \mathbf{u}}{\partial x_1} + \nabla \pi = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} = \mathbf{u}_* \text{ at } \partial\Omega \\ \mathbf{u} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty\,. \end{gathered} \tag{5.29}$$

Combining Lemmas 5.2 and 5.3 we get

Theorem 5.1 *Let Ω be a C^2 exterior domain in $\mathbb{R}^2$, $1 < q < \frac{6}{5}$. Let $\mu = O(|\ln \beta|^{-1})$ for $\beta \to 0^+$. For a given $\mathbf{f} \in L^q(\Omega)$, $\mathbf{u}_* \in W^{2-\frac{1}{q},q}(\partial\Omega)$ there exists a unique solution to the modified Oseen problem (5.29) such that $\mathbf{u} \in C_q$, $\pi \in D^{1,q}(\Omega)$. For $\beta_0 > 0$ sufficiently small there exists $C = C(q, \Omega, \beta_0)$ such that for all $\beta \in (0; \beta_0]$*

$$\langle \mathbf{u} \rangle_{\beta,q} \le C\Big[\beta^{2(1-\frac{1}{q})} |\ln \beta|^{-1} \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} + \|\mathbf{f}\|_q\Big]\,. \tag{5.30}$$

Remark 5.4 For $q = \frac{6}{5}$ we have

$$\langle \mathbf{u} \rangle_{\beta,q} \le C\Big[\beta^{2(1-\frac{1}{q})} \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} + \|\mathbf{f}\|_q\Big]\,.$$

Next we prove the following extension of Theorem 5.1.

Theorem 5.2 *Let $\Omega \in C^{l+2}$ be an exterior domain in $\mathbb{R}^2$, $\mathbf{f} \in W^{l,q}(\Omega)$, $\mathbf{u}_* \in W^{l+2-\frac{1}{q},q}(\partial\Omega)$, $q \in (1; \frac{3}{2})$, $l = 0, 1, \ldots$. Then the unique solution to the modified Oseen problem (5.29) satisfies for all $\beta \in (0; \beta_0]$, β_0 sufficiently small,*

$$\begin{gathered} \beta^{2(1-\frac{1}{q})}\Big[\|\nabla^2 \mathbf{u}\|_{l,q} + \|\nabla \pi\|_{l,q}\Big] \le \\ \le C_1\Big[\|\mathbf{f}\|_q + \beta^{2(1-\frac{1}{q})}(\|\mathbf{f}\|_{l,q} + \|\mathbf{u}_*\|_{l+2-\frac{1}{q},q,(\partial\Omega)})\Big]\,. \end{gathered} \tag{5.31}$$

Moreover, if the right hand side $\mathbf{f} \in W^{l,q}(\Omega) \cap W^{k,p}(\Omega)$, $p \in (1; \infty)$, $k = 0, 1, \ldots$, the boundary condition $\mathbf{u}_* \in W^{l+2-\frac{1}{q},q}(\partial\Omega) \cap W^{k+2-\frac{1}{p},p}(\partial\Omega)$ and $\Omega \in C^{2+\max\{k,l\}}$, then

$$\begin{gathered} \beta^{2(1-\frac{1}{q})}\Big[\|\nabla^2 \mathbf{u}\|_{l,q} + \|\nabla \pi\|_{l,q} + \|\nabla^2 \mathbf{u}\|_{k,p} + \|\nabla \pi\|_{k,p}\Big] \le \\ \le C_2\Big[\|\mathbf{f}\|_q + \beta^{2(1-\frac{1}{q})}(\|\mathbf{f}\|_{l,q} + \|\mathbf{f}\|_{k,p} + \\ + \|\mathbf{u}_*\|_{l+2-\frac{1}{q},q,(\partial\Omega)} + \|\mathbf{u}_*\|_{k+2-\frac{1}{p},p,(\partial\Omega)})\Big]\,. \end{gathered} \tag{5.32}$$

Especially, if

$$p \ge \frac{2q}{2-q} \text{ and } k \ge l\,, \tag{5.33}$$

then[16]

$$\beta^{2(1-\frac{1}{q})}\Big[\|\nabla^2\mathbf{u}\|_{l,q}+\|\nabla\pi\|_{l,q}+\|\nabla\mathbf{u}\|_{k+1,p}+\|\pi\|_{k+1,p}\Big]\le \\ \le C_3\Big[\|\mathbf{f}\|q+\beta^{2(1-\frac{1}{q})}(\|\mathbf{f}\|_{l,q}+\|\mathbf{f}\|_{k,p}+\|\mathbf{u}_*\|_{k+2-\frac{1}{p},p,(\partial\Omega)})\Big]\,, \tag{5.34}$$

where the constants $C_i = C_i(\beta_0, q, p, \Omega, N)$.

Proof: Let us first show that (5.32) implies, under the assumptions (5.33), the inequality (5.34). From Lemma VIII.1.12 we have that there exists $w_0 \in \mathbb{R}$ such that for all $p \in (1;\infty)$, $w \in D^{1,\frac{2p}{2+p}}(\Omega)$

$$\|w - w_0\|_p \le C|w|_{1,\frac{2p}{p+2}}\,.$$

Evidently, $p > \frac{2p}{p+2}$ and $q \le \frac{2p}{p+2} \iff p \ge \frac{2q}{2-q}$. From Theorem 3.7 we easily check that the solution to the modified Oseen problem is such that $\nabla\mathbf{u} \in L^{\frac{3q}{3-q}}(\Omega)$ whenever $1 < q \le \frac{3}{2}$. Therefore

$$\|\nabla\mathbf{u}\|_{1,p} \le C\|\nabla\mathbf{u}\|_{2,\frac{2p}{p+2}} \le C(\|\nabla\mathbf{u}\|_{2,p} + \|\nabla\mathbf{u}\|_{2,q})\,.$$

Analogously for π we have

$$\|\pi + \pi_0\|_p \le C(|\pi|_{1,p} + |\pi|_{1,q})\,.$$

As the pressure is determined up to a additive constant, we can take $\tilde\pi = \pi + \pi_0$ in such a way that $\tilde\pi \in L^p(\Omega)$ is a new pressure. Finally, applying the trace imbedding theorem (see Corollary VIII.1.1) we have

$$\|\mathbf{u}_*\|_{l+2-\frac{1}{q},q,(\partial\Omega)} \le C\|\mathbf{u}_*\|_{k+2-\frac{1}{p},p,(\partial\Omega)}$$

for $k \ge l$ and $q \le \frac{2p}{p+2} \le p$. We are therefore left with inequalities (5.31) and (5.32). We proceed as in the proof of Theorem 5.1. We search the solution $(\mathbf{u}, \pi)$ in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{w}$$
$$\pi = p + r\,,$$

where

$$\left.\begin{aligned} A(\mathbf{v}) + \beta\frac{\partial\mathbf{v}}{\partial x_1} + \nabla p = \mathbf{0} \\ \nabla\cdot\mathbf{v} = 0 \end{aligned}\right\} \text{ in } \Omega \\ \mathbf{v} = \mathbf{u}_* \text{ at } \partial\Omega \tag{5.35}$$

and

$$\left.\begin{aligned} A(\mathbf{w}) + \beta\frac{\partial\mathbf{w}}{\partial x_1} + \nabla r = \mathbf{f} \\ \nabla\cdot\mathbf{w} = 0 \end{aligned}\right\} \text{ in } \Omega \\ \mathbf{v} = \mathbf{0} \text{ at } \partial\Omega\,. \tag{5.36}$$

As in Lemma 5.1 (see (5.7)) we may show that

$$\|\mathbf{v}\|_{s+2,t,\Omega_R} + \|p\|_{s+1,t,\Omega_R} \le C\|\mathbf{u}_*\|_{s+2-\frac{1}{t},t,(\partial\Omega)}$$

[16] we add to π a suitable constant such that π is integrable

with $s \geq 0$, $1 < t < \infty$.

Next, let $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $\varphi(\mathbf{x}) = 1$ in $B^R(\mathbf{0})$, $\varphi(\mathbf{x}) = 0$ in $B_{\frac{R}{2}}(\mathbf{0})$, $\frac{R}{2} > \operatorname{diam} \Omega^c$. Let us put $\mathbf{U} = \mathbf{v}\varphi$, $P = p\varphi$. Then the couple $(\mathbf{U}, P)$ solves in the whole space

$$\begin{gathered} A(\mathbf{U}) + \beta \frac{\partial \mathbf{U}}{\partial x_1} + \nabla P = \mathbf{F} \\ \nabla \cdot \mathbf{U} = G \end{gathered}$$

with

$$\begin{gathered} \mathbf{F} = A(\varphi)\mathbf{v} - 2\nabla\varphi\nabla\mathbf{v} + 2\mu \frac{\partial \varphi}{\partial x_1}\frac{\partial \mathbf{v}}{\partial x_1} + \beta \frac{\partial \varphi}{\partial x_1}\mathbf{v} + p\nabla\varphi \\ G = \mathbf{v} \cdot \nabla \varphi \,. \end{gathered}$$

As $\operatorname{supp} \mathbf{F}$ and $\operatorname{supp} g \subset B_R(\mathbf{0})$, and

$$\|\mathbf{F}\|_{s,t,\mathbb{R}^2} + \|G\|_{s,t,\mathbb{R}^2} \leq C(\|\mathbf{v}\|_{s+1,t,\Omega_R} + \|p\|_{s,t,\Omega_R})\,,$$

we have due to Theorem 2.1 that

$$\|\nabla^2 \mathbf{U}\|_{s,t,\mathbb{R}^2} + \|\nabla P\|_{s,t,\mathbb{R}^2} \leq C(\|\mathbf{v}\|_{s+1,t,\Omega_R} + \|p\|_{s,t,\Omega_R})$$

and $\nabla^2 \mathbf{U}$, ∇P coincides with $\nabla^2 \mathbf{v}$, ∇p in Ω^R (see Lemma 2.3). Arguing as e.g. in Corollary 3.2 we have

$$\|\nabla^2 \mathbf{v}\|_{s,t,\Omega} + \|\nabla p\|_{s,t,\Omega} \leq C\|\mathbf{u}_*\|_{s+2-\frac{1}{t},t,(\partial\Omega)}\,, \tag{5.37}$$

$s \geq 0$, $1 < t < \infty$. Next we study the problem (5.36). We extend $\mathbf{f}$ onto $\mathbb{R}^2$ in such a way that

$$\|\mathbf{f}\|_{s,t,\mathbb{R}^2} + \|\mathbf{f}\|_{k,p,\mathbb{R}^2} \leq C(\|\mathbf{f}\|_{s,t,\Omega} + \|\mathbf{f}\|_{k,p,\Omega})$$

and similarly as in Lemma 5.3 we search $(\mathbf{w}, p)$ as a sum of a solution to the modified Oseen problem in $\mathbb{R}^2$ with the extension of $\mathbf{f}$ and of a solution to the modified Oseen problem in Ω, i.e.

$$(\mathbf{w}, p) = (\mathbf{w}^1, r^1) + (\mathbf{w}^2, r^2)\,,$$

$$\left.\begin{gathered} A(\mathbf{w}^1) + \beta \frac{\partial \mathbf{w}^1}{\partial x_1} + \nabla r^1 = \mathbf{f} \\ \nabla \cdot \mathbf{w} = 0 \end{gathered}\right\} \text{ in } \mathbb{R}^2 \tag{5.38}$$

and

$$\begin{gathered} \left.\begin{gathered} A(\mathbf{w}^2) + \beta \frac{\partial \mathbf{w}^2}{\partial x_1} + \nabla r^2 = \mathbf{0} \\ \nabla \cdot \mathbf{w}^2 = 0 \end{gathered}\right\} \text{ in } \Omega \\ \mathbf{w}^2 = -\mathbf{w}^1 \text{ at } \partial\Omega\,. \end{gathered} \tag{5.39}$$

We have from Theorem 2.1 that

$$\|\nabla^2 \mathbf{w}^1\|_{s,t,\mathbb{R}^2} + \|\nabla r^1\|_{s,t,\mathbb{R}^2} \leq C\|\mathbf{f}\|_{s,t,\Omega}\,. \tag{5.40}$$

Moreover, as in Lemma 5.3 we can show that for $1 < q < \frac{3}{2}$

$$\|\mathbf{w}^1\|_{q',B_1(\mathbf{0})} \leq C\beta^{-2(1-\frac{1}{q})}\|\mathbf{f}\|_q \tag{5.41}$$

and from (5.37)

$$\|\nabla^2\mathbf{w}^2\|_{s,t,\Omega} + \|\nabla r^2\|_{s,t,\Omega} \leq C\|\mathbf{w}^1\|_{s+2-\frac{1}{t},t,(\partial\Omega)} \,. \tag{5.42}$$

Let us put in (5.40) and (5.42) $s = l$, $t = q$ and $s = k$, $t = p$, respectively. We have to control the boundary terms $\|\mathbf{w}^1\|_{l+2-\frac{1}{q},q,(\partial\Omega)}$ and $\|\mathbf{w}^1\|_{k+2-\frac{1}{p},p,(\partial\Omega)}$. Applying the trace and the imbedding theorems (see Theorems VIII.1.6 and VIII.1.2) and recalling that $q < \frac{3}{2}$ i.e. $q < q'$

$$\begin{aligned} \|\mathbf{w}^1\|_{l+2-\frac{1}{q},q,(\partial\Omega)} &\leq C\|\mathbf{w}^1\|_{l+2,q,\Omega_1} \leq C_1(\|\mathbf{w}^1\|_{q',\Omega_1} + \|\nabla^2\mathbf{w}^1\|_{l,q,\Omega_1}) \\ \|\mathbf{w}^1\|_{k+2-\frac{1}{p},p,(\partial\Omega)} &\leq C(\|\mathbf{w}^1\|_{p,\Omega_1} + \|\nabla^2\mathbf{w}^1\|_{k,p,\Omega_1}) \leq \\ &\leq C(\|\mathbf{w}^1\|_{2,q,\Omega_1} + \|\nabla^2\mathbf{w}^1\|_{k,p,\Omega_1}) \leq \\ &\leq C(\|\mathbf{w}^1\|_{q',\Omega_1} + \|\nabla^2\mathbf{w}^1\|_{q,\Omega_1} + \|\nabla^2\mathbf{w}^1\|_{k,p,\Omega_1})\,. \end{aligned} \tag{5.43}$$

Combining (5.40)–(5.43) we finally get for $1 < q < \frac{3}{2}$, $1 < p < \infty$

$$\begin{aligned} \|\nabla^2\mathbf{w}\|_{l,q,\Omega} + \|\nabla r\|_{l,q,\Omega} &\leq C(\beta^{-2(1-\frac{1}{q})}\|\mathbf{f}\|_q + \|\mathbf{f}\|_{l,q}) \\ \|\nabla^2\mathbf{w}\|_{k,p,\Omega} + \|\nabla r\|_{k,p,\Omega} &\leq C\Big((1+\beta^{-2(1-\frac{1}{q})})\|\mathbf{f}\|_q + \|\mathbf{f}\|_{k,p}\Big)\,. \end{aligned} \tag{5.44}$$

The estimates (5.37) and (5.44) finish the proof.

□

Finally we get similar result also in three space dimensions. Here, the situation is much easier. We have namely

Theorem 5.3

(i) Let $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap W^{k,p}(\Omega)$, $1 < p < \infty$, $k \geq 0$, Ω *be a threedimensional exterior domain of class* C^{k+2}, $\mathbf{u}_* \in W^{k+2-\frac{1}{p},p}(\partial\Omega)$. *Then the unique solution to the modified Oseen problem (5.29)* $(\mathbf{u},\pi)$ *satisfies*

$$\begin{aligned} &\beta^{\frac{1}{4}}\|\mathbf{u}\|_4 + |\mathbf{u}|_{1,2} + \|\pi\|_2 + \|\nabla^2\mathbf{u}\|_{k,p} + \|\nabla\pi\|_{k,p} \leq \\ &\leq C\Big(|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_{k,p} + \|\mathbf{u}_*\|_{k+2-\frac{1}{p},p,(\partial\Omega)} + \|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)}\Big)\,. \end{aligned} \tag{5.45}$$

(ii) Let $\mathbf{f} \in L^q(\Omega) \cap W^{k,p}(\Omega)$, $1 < q < \frac{3}{2}$, $1 < p < \infty$, $k \geq 0$, Ω *be a threedimensional exterior domain of class* C^{k+2}, $\mathbf{u}_* \in W^{k+2-\frac{1}{p},p}(\partial\Omega) \cap W^{2-\frac{1}{q},q}(\partial\Omega)$. *Then the unique solution to the modified Oseen problem (5.29)* $(\mathbf{u},\pi)$ *satisfies*

$$\begin{aligned} &a_1\|\mathbf{u}\|_{\frac{2q}{2-q}} + a_2|\mathbf{u}|_{1,\frac{4q}{4-q}} + \|\nabla^2\mathbf{u}\|_q + \|\nabla^2\mathbf{u}\|_{k,p} + \|\nabla\pi\|_{k,p} \leq \\ &\leq C\Big(\|\mathbf{f}\|_q + \|\mathbf{f}\|_{k,p} + \|\mathbf{u}_*\|_{k+2-\frac{1}{p},p,(\partial\Omega)} + \|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)}\Big) \end{aligned} \tag{5.46}$$

with $a_1 = \min\{1, \beta^{\frac{1}{2}}\}$, $a_2 = \min\{1, \beta^{\frac{1}{4}}\}$.

Proof: We proceed as in Theorem 5.2 and combine the estimates with Theorems 3.7 and 3.6. We can assume that $p \geq \frac{3}{2}$ if $k = 0$ (otherwise, we can apply directly Theorems 3.6 and 3.7). We have to estimate $\mathbf{w}^1$, solving in $\mathbb{R}^3$

$$A(\mathbf{w}^1) + \beta \frac{\partial \mathbf{w}^1}{\partial x_1} + \nabla r^1 = \overline{\mathbf{f}}$$
$$\nabla \cdot \mathbf{w}^1 = 0$$

with $\overline{\mathbf{f}}$, an extension of $\mathbf{f}$ in some $W^{k,p}(\Omega)$ space.

In the case (i) we apply Theorem 3.7 to get

$$\|\mathbf{w}^1\|_{2,\Omega_R} \leq C\|\mathbf{w}^1\|_{6,\Omega_R} \leq C|\mathbf{w}^1|_{1,2,\mathbb{R}^3} \leq C|\overline{\mathbf{f}}|_{-1,2} \leq C(|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_{k,p}),$$

since, denoting $\boldsymbol{\varphi}^1$ and $\boldsymbol{\varphi}^2$ the parts of $\boldsymbol{\varphi}$ supported in $B^1(\mathbf{0})$ and $B_2(\mathbf{0})$, respectively, we have

$$\sup_{|\boldsymbol{\varphi}|_{1,2} \leq 1} \langle \overline{\mathbf{f}}, \boldsymbol{\varphi} \rangle \leq \sup_{|\boldsymbol{\varphi}|_{1,2} \leq 1} (\langle \overline{\mathbf{f}}, \boldsymbol{\varphi}^1 \rangle + \langle \overline{\mathbf{f}}, \boldsymbol{\varphi}^2 \rangle) \leq$$
$$\sup_{|\boldsymbol{\varphi}|_{1,2} \leq 1} (|\mathbf{f}|_{-1,2}|\boldsymbol{\varphi}^1|_{1,2} + \|\mathbf{f}\|_{\frac{6}{5},B_2(\mathbf{0})}\|\boldsymbol{\varphi}^2\|_{6,B_2(\mathbf{0})}) \leq C(|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_{\frac{6}{5}}),$$

where the evident inequality $\|\boldsymbol{\varphi}^2\|_{6,B_2(\mathbf{0})} \leq \|\boldsymbol{\varphi}\|_{6,\mathbb{R}^3} \leq |\boldsymbol{\varphi}|_{1,2}$ was used. We apply this estimate instead of (5.41).

In the case (ii) we use the integral representation to get

$$|\mathbf{w}^1(\mathbf{x})| \leq \beta \Big| \int_{\mathbb{R}^3} \boldsymbol{\mathcal{O}}^\mu(\beta(\mathbf{x}-\mathbf{y});1)\overline{\mathbf{f}}(\mathbf{y})\mathrm{d}\mathbf{y} \Big| \leq C \int_{\mathbb{R}^3} \frac{|\overline{\mathbf{f}}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\mathbf{y}$$

and due to the Sobolev theorem on weakly singular integrals (see Theorem VIII.2.4) we have

$$\|\mathbf{w}^1\|_s \leq C\|\overline{\mathbf{f}}\|_q$$

with $1 < q < \frac{3}{2}$ and $s = \frac{3q}{3-2q}$. Then we use this inequality instead of (5.41) and in the estimate of the solution to (5.39) we have

$$\|\nabla^2 \mathbf{w}^2\|_{k,p,\Omega} \leq C\|\mathbf{w}^1\|_{k+2-\frac{1}{p},p,(\partial\Omega)} \leq \|\mathbf{w}^1\|_{k+2,p,B_1(\mathbf{0})} \leq$$
$$C(\|\mathbf{w}^1\|_{s,B_1(\mathbf{0})} + \|\nabla^2\mathbf{w}^1\|_{k,p,B_1(\mathbf{0})}).$$

□

IV

Steady transport equation

IV.1 Definitions, basic properties

We shall now consider the other linear problem needed for the study of the stationary flow of viscoelastic fluids. The results presented in this chapter are mostly taken from [No1]; see also [No2], [No3] and [No4]. Nevertheless, some extensions are added; see Theorems 2.3 and 2.5.

Let $\Omega \subset \mathbb{R}^N$ be an exterior domain of class $C^{0,1}$ or let $\Omega = \mathbb{R}^N$. We study

$$\lambda z \; + \mathbf{w} \cdot \nabla z \; + az = f \quad \text{ in } \Omega \,, \tag{1.1}$$

where the unknown function is z; λ is a (without loss of generality positive) constant, $\mathbf{w} \in C^1_{loc}(\Omega)$, $a \in C_{loc}(\Omega)$ and f are given functions.

Remark 1.1 Although we study only scalar equation, all results presented in this chapter can be without changes applied also on the system

$$\lambda z_i + \sum_{j=1}^{N} w_{ij} \frac{\partial z_i}{\partial x_j} + \sum_{s=1}^{m} a_{is} z_s = f_i \,, \quad i = 1, 2, \ldots, m \,, \tag{1.2}$$

where $\mathbf{z} = (z_1, \ldots, z_m)$ are unknown functions, while $\{w_{ij}\}$, $i = 1, \ldots, m$, $j = 1, \ldots N$, $\{a_{is}\}$, $i, s = 1, \ldots, m$ and $\mathbf{f} = (f_1, \ldots, f_m)$ are given functions. The norms of vector–(eventually tensor–)valued functions are the maxima of their components.

Let us first take $f \in W^{1,q}_{loc}(\Omega)$.

Definition 1.1 *The function $z \in W^{1,q}_{loc}(\Omega)$ is called a strong solution to (1.1) if (1.1) is satisfied a.e. in Ω.*

In the case of lower regularity we have to modify the definition.

Remark 1.2 If $f \in D^{1,q}(\Omega)$, then $f \in W^{1,q}_{loc}(\Omega)$ and we can again use Definition 1.1.

Definition 1.2 *Let $f \in L^q_{loc}(\Omega)$. Then $z \in L^q_{loc}(\Omega)$ is called a q–weak solution if*

$$\int_\Omega z(\lambda\varphi - \mathbf{w} \cdot \nabla\varphi + (a - \nabla \cdot \mathbf{w})\varphi)\, d\mathbf{x} = \int_\Omega f \varphi d\mathbf{x} \quad \forall \varphi \in C_0^\infty(\Omega) \,. \tag{1.3}$$

Remark 1.3 We define the distribution $\mathbf{w}\cdot\nabla z \ \in \mathcal{D}'(\Omega)$

$$\langle \mathbf{w}\cdot\nabla z, \varphi\rangle = -\int_\Omega z(\mathbf{w}\cdot\nabla\varphi + \nabla\cdot\mathbf{w}\varphi)\mathrm{d}\mathbf{x} \quad \forall\varphi\in\mathcal{D}(\Omega)\,. \tag{1.4}$$

We can rewrite (1.3) as

$$\mathbf{w}\cdot\nabla z \ = f - \lambda z \ - az\,, \tag{1.5}$$

where the equality (1.5) is assumed in $\mathcal{D}'(\Omega)$. Having $z\in L^q_{loc}(\Omega)$ we see that $\mathbf{w}\cdot\nabla z \ \in L^q_{loc}(\Omega)$ and (1.1) is fulfilled a.e. in Ω. Therefore any weak solution in the sense of Definition 1.2 is also a strong solution in the sense of Definition 1.1. Analogously to (1.1) we can define

$$\mathbf{w}\cdot\nabla\nabla\xi = \nabla(\mathbf{w}\cdot\nabla\xi) - \nabla\mathbf{w}\cdot\nabla\xi \quad \text{for } \xi\in W^{1,q}_{loc}(\Omega) \tag{1.6}$$

i.e.

$$\langle \mathbf{w}\cdot\nabla\nabla\xi, \varphi\rangle = -\int_\Omega (\varphi\nabla\mathbf{w} + \nabla\varphi\mathbf{w})\cdot\nabla\xi\mathrm{d}\mathbf{x} \quad \forall\varphi\in\mathcal{D}(\Omega)\,. \tag{1.7}$$

In order to construct solutions to (1.1) we follow [No1]. Our method is essentially based on the following classical lemma, which is due to Friedrichs (see e.g. [Mis]). By $C^1_B(\mathbb{R}^N)$ we understand the set of functions continuously differentiable on $\mathbb{R}^N$ which are together with the first derivative bounded.

Lemma 1.1 (Friedrichs)
Let $1<q<\infty$, $\mathbf{w}\in C^1_B(\mathbb{R}^N)$, $z\in L^q(\mathbb{R}^N)$. If $\mathbf{w}\cdot\nabla z \ \in L^q(\mathbb{R}^N)$, then

$$\mathbf{w}\cdot\nabla z_h \to \mathbf{w}\cdot\nabla z \qquad in\ L^q(\mathbb{R}^N)\,,$$

where z_h stays for $z*\omega_h$, ω_h the classical mollifier.

Proof: Since $\mathbf{w}\cdot\nabla z \ \in L^q(\mathbb{R}^N)$, we have easily $(\mathbf{w}\cdot\nabla z)_h \to \mathbf{w}\cdot\nabla z$ in $L^q(\mathbb{R}^N)$ and therefore it is sufficient to verify that $\mathbf{w}\cdot\nabla z_h - (\mathbf{w}\cdot\nabla z)_h \to 0$ in $L^q(\mathbb{R}^N)$.

Applying (1.4) we have

$$(\mathbf{w}\cdot\nabla z)_h(\mathbf{x}) = \langle(\mathbf{w}\cdot\nabla z)(\cdot), \omega_h(\mathbf{x}-\cdot)\rangle = -\int_{\mathbb{R}^N} z(\mathbf{y})\nabla_{\mathbf{y}}\cdot[\mathbf{w}(\mathbf{y})\omega_h(\mathbf{x}-\mathbf{y})]\mathrm{d}\mathbf{y}\,.$$

So

$$\begin{aligned}I(\mathbf{x}) = \int_{\mathbb{R}^N}\Big[w_i(\mathbf{x})z(\mathbf{y})\frac{\partial\omega_h(\mathbf{x}-\mathbf{y})}{\partial x_i} + z(\mathbf{y})\frac{\partial}{\partial y_i}(w_i(\mathbf{y})\omega_h(\mathbf{x}-\mathbf{y}))\Big]\mathrm{d}\mathbf{y} = \\ = \int_{\mathbb{R}^N} z(\mathbf{y})\frac{\partial}{\partial y_i}\Big[(w_i(\mathbf{y})-w_i(\mathbf{x}))\omega_h(\mathbf{x}-\mathbf{y})\Big]\mathrm{d}\mathbf{y} = \\ = \int_{\mathbb{R}^N}(z(\mathbf{y})-z(\mathbf{x}))\frac{\partial}{\partial y_i}\Big[(w_i(\mathbf{y})-w_i(\mathbf{x}))\omega_h(\mathbf{x}-\mathbf{y})\Big]\mathrm{d}\mathbf{y}\,,\end{aligned}$$

where the last equality follows from the fact that ω_h has compact support in $\mathbb{R}^N$. Therefore

$$\begin{aligned}|I(\mathbf{x})| \le \int_{\mathbb{R}^N}|z(\mathbf{y})-z(\mathbf{x})|\omega_h(\mathbf{x}-\mathbf{y})|\nabla\mathbf{w}(\mathbf{y})|\mathrm{d}\mathbf{y} + \\ + \int_{\mathbb{R}^N}|z(\mathbf{y})-z(\mathbf{x})||\nabla\omega_h(\mathbf{x}-\mathbf{y})||\mathbf{w}(\mathbf{y})-\mathbf{w}(\mathbf{x})|\mathrm{d}\mathbf{y} = I_1(\mathbf{x}) + I_2(\mathbf{x})\,.\end{aligned}$$

We easily estimate

$$\int_{\mathbb{R}^N} |I_1(\mathbf{x})|^q \mathrm{d}\mathbf{x} \le C \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} |z(\mathbf{y}) - z(\mathbf{x})| \omega_h(\mathbf{x}-\mathbf{y}) \mathrm{d}\mathbf{y} \Big)^q \mathrm{d}\mathbf{x} \le$$
$$\le C \int_{\mathbb{R}^N} \int_{B_1(\mathbf{0})} |z(\mathbf{x} - h\xi) - z(\mathbf{x})|^q \mathrm{d}\xi \Big(\int_{B_1(\mathbf{0})} |\omega_1(\xi)|^{q'} \mathrm{d}\xi \Big)^{\frac{q}{q'}} \mathrm{d}\mathbf{x}\,.$$

Using the p–mean continuity and absolute continuity of the integral (see e.g. [KuFuJo]) we easily verify that for any $\varepsilon > 0$ we can assure that

$$\int_{\mathbb{R}^N} |I_1(\mathbf{x})|^q \mathrm{d}\mathbf{x} \le \frac{\varepsilon^q}{2} \tag{1.8}$$

for h sufficiently small. Similarly, as $|\mathbf{w}(\mathbf{x}) - \mathbf{w}(\mathbf{y})| \le C|\mathbf{x} - \mathbf{y}|$, and for any $r \in (1;\infty)$

$$\int_{B_h(\mathbf{x})} (|\mathbf{x}-\mathbf{y}||\nabla_{\mathbf{y}} \omega_h(\mathbf{x}-\mathbf{y})|)^r \mathrm{d}\mathbf{y} \le h^{r-N} \int_{B_h(\mathbf{x})} \Big|\nabla_{\mathbf{y}} \omega_1\Big(\frac{\mathbf{x}-\mathbf{y}}{h}\Big)\Big|^r \mathrm{d}\mathbf{y} \le$$
$$\le \int_{B_1(\mathbf{0})} |\nabla_{\mathbf{z}} \omega_1(\mathbf{z})|^r \mathrm{d}\mathbf{z} \le C\,,$$

we easily verify that

$$\int_{\mathbb{R}^N} |I_2(\mathbf{x})|^q \mathrm{d}\mathbf{x} \le \frac{\varepsilon^q}{2} \tag{1.9}$$

for h sufficiently small. Therefore (1.8) and (1.9) imply that $\|I(\cdot)\|_q \to 0$ as $h \to 0^+$ and the lemma is proved.

□

Corollary 1.1 *Let* $1 < q < \infty$, $\Omega \in C^{0,1}$, $\mathbf{w} \in C^1(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$. *Let* $z \in L^q(\Omega)$ *and* $\mathbf{w} \cdot \nabla z \in L^q(\Omega)$. *Denote*

$$\overline{z}(\mathbf{x}) = \begin{cases} z(\mathbf{x}) & \textit{if } \mathbf{x} \in \Omega \\ 0 & \textit{if } \mathbf{x} \in \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

Then

$$\mathbf{w} \cdot \nabla \overline{z}_h \to \mathbf{w} \cdot \nabla z \qquad \textit{in } L^q(\Omega)\,,$$

where f_h *denotes the standard mollification of* f.

Proof: Let $\overline{\mathbf{w}} \in C^1(\mathbb{R}^N)$ be any extension of $\mathbf{w}$ onto $\mathbb{R}^N$ (see Remark VIII.1.3). Then by (1.4) for $\Omega = \mathbb{R}^N$ we have for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$

$$\langle \overline{\mathbf{w}} \cdot \nabla \overline{z}, \varphi \rangle = - \int_{\mathbb{R}^N} \overline{z}(\overline{\mathbf{w}} \cdot \nabla\varphi + \varphi \nabla \cdot \overline{\mathbf{w}}) \mathrm{d}\mathbf{x} =$$
$$= - \int_\Omega z(\mathbf{w} \cdot \nabla\varphi + \varphi \nabla \cdot \mathbf{w}) \mathrm{d}\mathbf{x} = \int_\Omega (\nabla \cdot (z\mathbf{w}) - z\nabla \cdot \mathbf{w})\varphi \mathrm{d}\mathbf{x} +$$
$$+ \int_{\partial\Omega} \mathbf{w} \cdot \mathbf{n} z\ \varphi \mathrm{d}S \ = \int_\Omega [\nabla \cdot (z\mathbf{w}) - z\ \nabla \cdot \mathbf{w}]\varphi \mathrm{d}\mathbf{x}\,.$$

We used the fact that $\mathbf{w} \cdot \nabla z \in L^q(\Omega)$, $z \in L^q(\Omega)$, $\mathbf{w} \in C^1(\Omega)$ imply $\nabla \cdot (z\mathbf{w}) \in L^q(\Omega)$, we may apply the Green theorem and the trace $\mathbf{w} \cdot \mathbf{n}z$ is well defined (see Remark VIII.3.6 and Lemma VIII.3.3). So

$$|\langle \overline{\mathbf{w}} \cdot \nabla \overline{z}, \varphi \rangle| \le \|\varphi\|_{q',\mathbb{R}^N} (\|\nabla \cdot (z\mathbf{w})\|_{q,\Omega} + \|z\nabla \cdot \mathbf{w}\|_{q,\Omega})$$

and $\overline{\mathbf{w}} \cdot \nabla \overline{z} \in L^q(\mathbb{R}^N)$. Lemma 1.1 yields

$$\overline{\mathbf{w}} \cdot \nabla \overline{z}_h \to \overline{\mathbf{w}} \cdot \nabla \overline{z} \qquad \text{in } L^q(\mathbb{R}^N)$$

and therefore, particularly,

$$\mathbf{w} \cdot \nabla \overline{z}_h \to \mathbf{w} \cdot \nabla z \qquad \text{in } L^q(\Omega)\,.$$

□

Remark 1.4 Let the hypothesis of Corollary 1.1 be satisfied, $f \in L^q(\Omega)$, $1 < q < \infty$. If z is a weak solution to (1.1) in the sense of Definition 1.2, then (1.3) holds for all $\varphi \in L^{q'}(\Omega)$ such that $\mathbf{w} \cdot \nabla \varphi \in L^{q'}(\Omega)$. To show this, it is enough to observe that (1.3) holds for any $\varphi \in C_0^\infty(\Omega)$ and $\mathbf{w} \cdot \mathbf{n} = 0$ at $\partial\Omega$, i.e. (1.3) holds also for $\varphi \in C_0^\infty(\overline{\Omega})$. This can be shown by means of Corollary 1.1, Remark 1.3 and Lemma VIII.3.3.

Similarly, if

$$\int_\Omega z(\lambda\varphi - \mathbf{w} \cdot \nabla\nabla\varphi + (a - \nabla \cdot \mathbf{w})\nabla\varphi)\mathrm{d}\mathbf{x} = \int_\Omega f \nabla\varphi \mathrm{d}\mathbf{x} \tag{1.10}$$

holds for all $\varphi \in C_0^\infty(\Omega)$, then (1.10) holds for any $\varphi \in W_0^{1,q'}(\Omega)$ such that $\mathbf{w} \cdot \nabla\varphi \in W^{1,q'}(\Omega)$.

Corollary 1.2

(i) Let $1 < q < \infty$, $\Omega \in C^{0,1}$, $\mathbf{w} \in C^1(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$ and $z \in L^q(\Omega)$, $\mathbf{w} \cdot \nabla z \in L^q(\Omega)$. Then

$$\int_\Omega \mathbf{w} \cdot \nabla z \; |z|^{q-2} z \, d\mathbf{x} = -\frac{1}{q} \int_\Omega \nabla \cdot \mathbf{w} |z|^q \, d\mathbf{x}\,. \tag{1.11}$$

(ii) Let the assumptions of (i) are satisfied and let moreover $z \in W^{1,q}(\Omega)$, $\mathbf{w} \cdot \nabla z \in W^{1,q}(\Omega)$. Then $\mathbf{w} \cdot \nabla\nabla z \in L^q(\Omega)$ and

$$\int_\Omega \mathbf{w} \cdot \nabla\Big(\frac{\partial z}{\partial x_j}\Big) |\nabla z|^{q-2} \frac{\partial z}{\partial x_j} \, d\mathbf{x} = -\frac{1}{q} \int_\Omega \nabla \cdot \mathbf{w} |\nabla z|^q \, d\mathbf{x}\,. \tag{1.12}$$

Proof: Easily, the assertion (ii) follows from (i). We show therefore only (1.11). We have for $R > \operatorname{diam} \Omega^c$

$$\int_{\Omega_R} \overline{\mathbf{w}} \cdot \nabla \overline{z}_h |\overline{z}_h|^{q-2} \overline{z}_h \mathrm{d}\mathbf{x} = \frac{1}{q} \int_{\Omega_R} \overline{\mathbf{w}} \cdot \nabla |\overline{z}_h|^q \mathrm{d}\mathbf{x} =$$
$$= -\frac{1}{q} \int_{\Omega_R} \nabla \cdot \overline{\mathbf{w}} |\overline{z}_h|^q \mathrm{d}\mathbf{x} + \frac{1}{q} \int_{\partial B_R(\mathbf{0})} \overline{\mathbf{w}} \cdot \mathbf{n} |\overline{z}_h|^q \mathrm{d}S \; + \frac{1}{q} \int_{\partial\Omega} \overline{\mathbf{w}} \cdot \mathbf{n} |\overline{z}_h|^q \mathrm{d}S\,.$$

Since $\mathbf{w} \cdot \mathbf{n} = 0$ at $\partial\Omega$, $z_h \in L^q(\Omega)$ and $\overline{\mathbf{w}}$ is bounded, we have that

$$\lim_{R\to\infty} \Big(\int_{\partial B_R(\mathbf{0})} \overline{\mathbf{w}} \cdot \mathbf{n} |\overline{z}_h|^q \mathrm{d}S + \int_{\partial\Omega} \overline{\mathbf{w}} \cdot \mathbf{n} |\overline{z}_h|^q \mathrm{d}S \Big) = 0\,.$$

Then, Corollary 1.1 and the well–known properties of the mollifier yield (1.11).

□

We now apply the above obtained equalities in order to prove the apriori estimates of solutions to (1.1). Although our main interest is devoted to the study of Ω an exterior domain, due to the technique we must include also the case $\Omega = \mathbb{R}^N$.

Lemma 1.2 *Let $1 < q < \infty$, $k = 0, 1, \ldots$, $\Omega = \mathbb{R}^N$ or $\Omega \in C^{0,1}$. Suppose that*

(i) $k = 0$, $\mathbf{w} \in C^1(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$ ($\Omega \neq \mathbb{R}^N$), $a \in C^0(\overline{\Omega})$

(ii) $k \geq 1$, $\mathbf{w} \in C^k(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$ ($\Omega \neq \mathbb{R}^N$), $a \in C^k(\overline{\Omega})$

and $f \in W^{k,q}(\Omega)$. Then there exists $\alpha_0(k) \geq 1$ such that if $\mathbf{z} \in W^{k,q}(\Omega)$ is a solution to (1.1), then

$$\lambda \|z\|_{s,q} \leq \|f\|_{s,q} + \alpha_0 \theta_0 \|z\|_{s,q} \quad s = 0, 1, \ldots, k\,, \tag{1.13}$$

where

$$\theta_0 = \theta_0^{(k)}(a, \mathbf{w}) = \begin{cases} \|\nabla \cdot \mathbf{w}\|_{C^0} + \|a\|_{C^0} & k = 0 \\ \|\nabla \mathbf{w}\|_{C^{k-1}} + \|a\|_{C^k} & k \geq 1\,. \end{cases}$$

Lemma 1.3 *Let $1 < q < \infty$, $k = 1, 2, \ldots$, $\Omega = \mathbb{R}^N$ or $\Omega \in C^{0,1}$, an exterior domain, $f \in W^{k,q}(\Omega)$. Let $\mathbf{w} \in C^k(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$ ($\Omega \neq \mathbb{R}^N$), $a \in C^{k-1}(\overline{\Omega})$ and*

$$\begin{aligned} \nabla^k a \in L^q(\Omega) &\quad \text{for } kq > N \\ \nabla^k a \in L^N(\Omega) &\quad \text{for } 1 < q < N\,. \end{aligned} \tag{1.14}$$

Then there exists $\alpha_0(k, q) \geq 1$ such that if $\mathbf{z} \in W^{k,q}(\Omega)$ is a solution to (1.1), then

$$\lambda \|z\|_{s,q} \leq \|f\|_{s,q} + \alpha_0 \theta_i \|z\|_{s,q} \quad s = 0, 1, \ldots, k\,, \tag{1.15}$$

where $i = 1, 2$ corresponds to $(1.14)_{1,2}$ and

$$\theta_i = \theta_i^{(k)}(a, \mathbf{w}) = \begin{cases} \|\nabla \mathbf{w}, a\|_{C^{k-1}} + \|\nabla^k a\|_q & i = 1 \\ \|\nabla \mathbf{w}, a\|_{C^{k-1}} + \|\nabla^k a\|_N & i = 2\,. \end{cases}$$

Lemma 1.4 *Let $1 < q < \infty$, $k = 2, 3, \ldots$, $\Omega = \mathbb{R}^N$ or $\Omega \in C^{0,1}$, an exterior domain, $f \in W^{k,q}(\Omega)$. Let $\mathbf{w} \in C^{k-1}(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$ ($\Omega \neq \mathbb{R}^N$), $a \in C^{k-1}(\overline{\Omega})$ and*[1]

$$\begin{aligned} \nabla^k w, \nabla^k a \in L^q(\Omega) &\quad \text{for } (k-1)q > N \\ \nabla^k w \in L^N(\Omega), \nabla^k a \in L^q(\Omega) &\quad \text{for } 1 < q < N, kq > N \\ \nabla^k w, \nabla^k a \in L^N(\Omega) &\quad \text{for } 1 < q < N\,. \end{aligned} \tag{1.16}$$

Then there exists $\alpha_0(k, q) \geq 1$ such that if $\mathbf{z} \in W^{k,q}(\Omega)$ is a solution to (1.1), then

$$\lambda \|z\|_{s,q} \leq \|f\|_{s,q} + \alpha_0 \overline{\theta}_i \|z\|_{s,q} \quad s = 0, 1, \ldots, k\,, \tag{1.17}$$

[1]Some other combinations like $a \in C^k(\overline{\Omega})$, $\mathbf{w} \in C^{k-1}(\overline{\Omega})$, $\nabla^k \mathbf{w} \in L^N(\Omega)$ etc. are also possible. But we do not need them.

where $i = 1,2,3$ *corresponds to (1.16)*$_{1,2,3}$ *and*

$$\overline{\theta}_i = \overline{\theta}_i^{(k)}(a, \mathbf{w}) = \begin{cases} \|\nabla \mathbf{w}\|_{C^{k-2}} + \|a\|_{C^{k-1}} + \|\nabla^k a, \nabla^k \mathbf{w}\|_q & i = 1 \\ \|\nabla \mathbf{w}\|_{C^{k-2}} + \|a\|_{C^{k-1}} + \|\nabla^k a\|_q + \|\nabla^k \mathbf{w}\|_N & i = 2 \\ \|\nabla \mathbf{w}\|_{C^{k-2}} + \|a\|_{C^{k-1}} + \|\nabla^k a, \nabla^k \mathbf{w}\|_N & i = 3 \,. \end{cases}$$

Lemma 1.5 *Let* $1 < q < \infty$, $k = 1,2,\ldots$, $\Omega = \mathbb{R}^N$ *or* $\Omega \in C^{0,1}$, *an exterior domain,* $f \in W^{k,q}_{loc}(\Omega)$, $\nabla f \in W^{k-1,q}(\Omega)$. *Suppose that* $\mathbf{w} \in C^k(\overline{\Omega})$, $\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$ $(\Omega \neq \mathbb{R}^N)$, *and one of the following conditions be satisfied*

$$\begin{aligned} &a = 0 \\ &a \in C^{k-1}(\Omega), \nabla^k a \in L^N(\Omega) \quad \text{for } 1 < q < N \,. \end{aligned} \tag{1.18}$$

Then there exists $\alpha_0(k,q) \geq 1$ *such that*

$$\lambda \|\nabla z\|_{s-1,q} \leq \|\nabla f\|_{s-1,q} + \alpha_0 \theta'_i \|\nabla z\|_{s-1,q} \quad s = 1,2,\ldots,k \tag{1.19}$$

(i) *for any solution* $\mathbf{z} \in W^{k,q}_{loc}(\Omega)$ *such that* $\nabla z \in W^{k-1,q}(\Omega)$ *(case (1.18)*$_1$*)*

(ii) *for any* $z \in W^{k,q}_{loc}(\Omega)$ *such that* $\nabla z \in W^{k-1,q}(\Omega)$ *and such that the Sobolev–Poincaré inequality*

$$\|z\|_{\frac{Nq}{N-q}} \leq C(q,N) \|\nabla z\|_{0,q}$$

holds (case (1.18)$_2$*).*

Here $i = 1,2$ *corresponds to (1.18)*$_{1,2}$, α_0 *is from Lemma 1.3 and*

$$\theta'_i = \theta'^{(k)}_i(a, \mathbf{w}) = \begin{cases} \|\nabla \mathbf{w}\|_{C^{k-1}} & i = 1 \\ \|\nabla \mathbf{w}, a\|_{C^{k-1}} + \|\nabla^k a\|_N & i = 2 \,. \end{cases}$$

Lemma 1.6 *Let* $1 < q < \infty$, $k = 1,2,\ldots$, $\Omega = \mathbb{R}^N$ *or* $\Omega \in C^{0,1}$, *an exterior domain,* $f \in W^{k,q}_{loc}(\Omega)$, $\nabla f \in W^{k-1,q}(\Omega)$. *Suppose that* $\mathbf{w} \in C^{k-1}(\overline{\Omega})$,

$$\begin{aligned} &\nabla^k \mathbf{w} \in L^N(\Omega) \quad \text{for } 1 < q < N \\ &\nabla^k \mathbf{w} \in L^q(\Omega) \quad \text{for } (k-1)q > N \,, \end{aligned} \tag{1.20}$$

$\mathbf{w} \cdot \mathbf{n}/_{\partial\Omega} = 0$ $(\Omega \neq \mathbb{R}^N)$, *and one of the following conditions be satisfied*

$$\begin{aligned} &a = 0 \\ &a \in C^{k-1}(\Omega), \nabla^k a \in L^N(\Omega) \quad \text{for } 1 < q < N \,. \end{aligned} \tag{1.21}$$

Then there exists $\alpha_0(k,q) \geq 1$ *such that*

$$\lambda \|\nabla z\|_{s-1,q} \leq \|\nabla f\|_{s-1,q} + \alpha_0 \theta'_{i,j} \|\nabla z\|_{s-1,q} \quad s = 1,2,\ldots,k \tag{1.22}$$

(i) *for any solution* $\mathbf{z} \in W^{k,q}_{loc}(\Omega)$ *such that* $\nabla z \in W^{k-1,q}(\Omega)$ *(case (1.21)*$_1$*)*

(ii) for any $z \in W^{k,q}_{loc}(\Omega)$ such that $\nabla z \in W^{k-1,q}(\Omega)$ and such that the Sobolev–Poincaré inequality

$$\|z\|_{\frac{Nq}{N-q}} \leq C(q,N)\|\nabla z\|_{0,q}$$

holds (case $(1.21)_2$).

Here $i = 1,2$ corresponds to $(1.21)_{1,2}$, $j = 1,2$ corresponds to $(1.20)_{1,2}$, α_0 is from Lemma 1.4 and

$$\theta'^{(k)}_{i,j}(a,\mathbf{w}) = \begin{cases} \|\nabla \mathbf{w}\|_{C^{k-2}} + \|\nabla^k \mathbf{w}\|_N & i=1, j=1 \\ \|\nabla \mathbf{w}\|_{C^{k-2}} + \|\nabla^k \mathbf{w}\|_q & i=1, j=2 \\ \|\nabla \mathbf{w}\|_{C^{k-2}} + \|a\|_{C^{k-1}} + \|\nabla^k \mathbf{w}\|_N + \|\nabla^k a\|_N & i=2, j=1 \\ \|\nabla \mathbf{w}\|_{C^{k-2}} + \|a\|_{C^{k-1}} + \|\nabla^k \mathbf{w}\|_q + \|\nabla^k a\|_N & i=2, j=2\,. \end{cases}$$

Proof of Lemmas 1.2–1.6: Differentiating (1.1) we find

$$\lambda \nabla^r z = -\mathbf{w}\cdot\nabla(\nabla^r z) - \sum_{\substack{i+j=r\\ 1\leq j\leq r-1}} \nabla^i \mathbf{w}\cdot\nabla\nabla^j z - \nabla^r \mathbf{w}\cdot\nabla z - \nabla^r a z - \\ - \sum_{\substack{i+j=r\\ 0\leq i\leq r-1}} \nabla^i a \nabla^j z + \nabla^r f \tag{1.23}$$

$r = 0,1,\ldots,s$, a.e. in Ω. We multiply (1.23) scalarly by $|\nabla^r z|^{q-2}\nabla^r z$, integrate over Ω and get

$$\lambda\|\nabla^r z\|_q^q \leq \sum_{i=1}^{6} I_i\,.$$

Applying Corollary 1.2 and the Hölder inequality we can estimate each term as follows

$$I_1 = \Big|\int_\Omega \mathbf{w}\cdot\nabla(\nabla^r z) : (|\nabla^r z|^{q-2}\nabla^r z)\mathrm{d}\mathbf{x}\Big| \leq \frac{1}{q}\|\nabla\cdot\mathbf{w}\|_{C^0}\|\nabla^r z\|_q^q \tag{1.24}$$

$$I_2 = \Big|\int_\Omega \sum_{\substack{i+j=r\\ 0\leq j\leq r-1}} \nabla^i \mathbf{w}\cdot\nabla(\nabla^j z) : (|\nabla^r z|^{q-2}\nabla^r z)\mathrm{d}\mathbf{x}\Big| \leq \\ \leq C(r)\|\nabla \mathbf{w}\|_{C^{r-1}}\|\nabla z\|^q_{r-1,q} \tag{1.25}$$

$$I_3 = \Big|\int_\Omega (\nabla^r \mathbf{w}\cdot\nabla z) : (|\nabla^r z|^{q-2}\nabla^r z)\mathrm{d}\mathbf{x}\Big| \leq \\ \leq \begin{cases} \|\nabla^r \mathbf{w}\|_{C^{r-1}}\|\nabla z\|^q_{r-1,q} & 1<q<\infty \\ \|\nabla^r \mathbf{w}\|_N\|\nabla z\|_{\frac{Nq}{N-q}}\|\nabla^r z\|_q^{q-1} & 1<q<N \\ \|\nabla^r \mathbf{w}\|_q\|\nabla z\|_{C^0}\|\nabla^r z\|_q^{q-1} & 1<q<\infty \end{cases} \tag{1.26}$$

$$I_4 = \Big|\int_\Omega z\,\nabla^r a : (|\nabla^r z|^{q-2}\nabla^r z)\mathrm{d}\mathbf{x}\Big| \leq \\ \leq \begin{cases} \|\nabla^r a\|_{C^0}\|z\|^q_{r,q} & 1<q<\infty \\ \|\nabla^r a\|_N\|z\|_{\frac{Nq}{N-q}}\|\nabla^r z\|_q^{q-1} & 1<q<N \\ \|\nabla^r a\|_q\|z\|_{C^0}\|\nabla^r z\|_q^{q-1} & 1<q<\infty \end{cases} \tag{1.27}$$

$$I_5 = \Big| \int_\Omega \sum_{\substack{i+j=r \\ 0\le i\le r-1}} (\nabla^i a \nabla^j z) : (|\nabla^r z|^{q-2} \nabla^r z)\,\mathrm{d}\mathbf{x}| \le \\ \le C(r)\|a\|_{C^{r-1}} \|\nabla z\|^q_{r-1,q} \tag{1.28}$$

$$I_6 = \Big| \int_\Omega \nabla^r f : (|\nabla^r z|^{q-2}\nabla^r z)\mathrm{d}\mathbf{x}\Big| \le \|\nabla^r f\|_q \|\nabla^r z\|_q^{q-1} \,. \tag{1.29}$$

In order to prove Lemma 1.2, let $k = 0$ first. Then $r = 0$ and $I_2 = I_4 = 0$. We sum up (1.24), $(1.26)_1$, (1.28), (1.29) and get (1.13) with $s = 0$. Next, for $k \ge 1$ we use $(1.27)_1$, sum it up with (1.24), (1.25), $(1.26)_1$, (1.28) and (1.29) for $r = 0, 1, \ldots, s$ and get (1.13). Lemma 1.2 is proved.

To show Lemma 1.3 we use for $s = k$ the estimate $(1.27)_2$ if $1 < q < N$ and $(1.27)_3$ if $kq > N$ and apply the Sobolev imbedding theorem (see Theorems VIII.1.2, VIII.1.3). Summing up (1.24)–(1.29) we get (1.15).

Next, let $k \ge 2$. For $s = k$ we use the estimate $(1.26)_2$ (for $1 < q < N$) or $(1.26)_3$ (for $(k-1)q > N$) and apply the Sobolev imbedding theorems VIII.1.2, VIII.1.3. Summing up (1.24)–(1.29) we get (1.17) which proves Lemma 1.4.

Further, let us observe that if $a = 0$, then (1.19) easily follows by summing up (1.24), (1.26), (1.25) and (1.29). If $a \neq 0$, we use $(1.27)_2$ together with the Sobolev–Poincaré inequality. The estimate (1.19) follows by summing up (1.24)–(1.29) for $r = 1, 2, \ldots, s$.

Finally, combining the proofs of Lemmas 1.4 and 1.5 we get Lemma 1.6.

□

IV.2 Existence of solution

This section is devoted to the construction of solutions to (1.1). We first construct the solution in the whole $\mathbb{R}^N$ and then take its restriction onto Ω — this restriction evidently solves (1.1) in Ω. The following lemma enables to extend the data onto $\mathbb{R}^N$.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ (exterior or bounded) domain of class $C^{0,1}$. For any $s = 0, 1, \ldots, k$ and $p_1^{(s)}, \ldots, p_{k-1}^{(s)}$, $1 < p_i^{(s)} < \infty$, we define $X_s(\Omega)$ the Banach space of functions with finite norm*

$$\|v\|_{X_s(\Omega)} = \|v\|_{C^s(\overline{\Omega})} + \sum_{i=1}^{k-s} \|\nabla^{i+s} v\|_{p_i^{(s)}}$$

and for $1 < p_i < \infty$ we define $X(\Omega)$ as a space of functions with finite norm

$$\|v\|_{X(\Omega)} = \sum_{i=1}^{k} \|\nabla^i v\|_{p_i} \,.$$

Let $p_{k-s} \ge p_{k-s-1} \ge \ldots \ge p_1$ (or $p_k \ge p_{k-1} \ge \ldots \ge p_0$). Then there exists a common extension E from $X(\Omega)$ to $X(\mathbb{R}^N)$ and $X_s(\Omega)$ to $X_s(\mathbb{R}^N)$, $s = 0, 1, \ldots, k$, respectively. It means that

$$\|Eu\|_{X(\mathbb{R}^N)} \le C(k, \Omega, \{p_i\}) \|u\|_{X(\Omega)} \\ \|Eu\|_{X_s(\mathbb{R}^N)} \le C(k, s, \Omega, \{p_i\}) \|u\|_{X_s(\Omega)} \,. \tag{2.1}$$

In particular

$$\|Eu\|_{X_k(\mathbb{R}^N)} \le C(k,\Omega)\|u\|_{X_k(\Omega)} \,. \tag{2.2}$$

Proof: It is an easy consequence of Lemma VIII.1.7 and Remark VIII.1.3.

□

We start with the construction of solutions in $\mathbb{R}^N$.

Lemma 2.2 *Let $k = 0, 1, \ldots$, $a \equiv 0$, $\mathbf{w}$, $f \in C_0^\infty(\mathbb{R}^N)$. If $\alpha_0(k)\theta_0^{(k)}(0,\mathbf{w}) < \lambda$, then there exists a unique $z \in \cap_{1<q<\infty} W^{k,q}(\mathbb{R}^N)$, the solution to (1.1), which satisfies the estimate*

$$\|z\|_{k,q} \le \frac{\|f\|_{k,q}}{\lambda - \alpha_0\theta_0} \quad \forall\, 1 < q < \infty \tag{2.3}$$

with α_0, θ_0 from Lemma 1.2.

Proof: Let $\varepsilon > 0$. We consider the problem

$$((z_\varepsilon, \varphi)) \equiv \int_{\mathbb{R}^N} (\varepsilon \nabla z_\varepsilon \nabla \varphi + \lambda z_\varepsilon \varphi + \mathbf{w}\cdot\nabla z_\varepsilon \varphi)\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} f\varphi \mathrm{d}\mathbf{x} \tag{2.4}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Provided

$$\frac{1}{2}\|\nabla\cdot\mathbf{w}\|_{C^0} < \lambda \tag{2.5}$$

the bilinear form $((\cdot,\cdot))$ is continuous and coercive on $W^{1,2}(\mathbb{R}^N)$; namely

$$((z,z)) \ge \varepsilon\|\nabla z\|_2^2 + (\lambda - \frac{1}{2}\|\nabla\cdot\mathbf{w}\|_{C^0})\|z\|_2^2 \,.$$

Therefore, under the assumption (2.5) there exists a unique $z_\varepsilon \in W^{1,2}(\mathbb{R}^N)$, the solution to (2.4). This solution satisfies

$$-\varepsilon\Delta z_\varepsilon + \lambda z_\varepsilon = f - \mathbf{w}\cdot\nabla z_\varepsilon \tag{2.6}$$

in the sense of distributions. Moreover, as $f - \mathbf{w}\cdot\nabla z_\varepsilon \in L^2(\mathbb{R}^N)$, the local regularity of elliptic systems (see [AgDoNi]) gives $z_\varepsilon \in W^{2,2}_{loc}(\mathbb{R}^N)$ and, on bootstrapping, $z_\varepsilon \in C^\infty(\mathbb{R}^N)$.

Since $\mathbf{w}$, $f \in C_0^\infty(\mathbb{R}^N)$ and $z_\varepsilon \in C^\infty(\mathbb{R}^N)$, we see that

$$g \equiv f - \mathbf{w}\cdot\nabla z_\varepsilon \in W^{k,q}(\Omega) \quad \forall k \in \mathbb{N}_0^+ \,,\; q \in [1;\infty] \,.$$

Applying the Fourier transform in (2.6) we see that a (generally different) solution to (2.6), $\tilde{z}_\varepsilon$, can be written as

$$\tilde{z}_\varepsilon = \mathcal{F}^{-1}\Big(\frac{\mathcal{F}(g)}{\varepsilon|\xi|^2 + \lambda}\Big)$$

and therefore, by the Marcinkiewicz multiplier theorem (see Theorem II.3.2) we have $\tilde{z}_\varepsilon \in W^{k,q}(\mathbb{R}^N)$ for all $k \in \mathbb{N}_0^+$, $q \in (1,\infty)$. But $W^{1,2}(\mathbb{R}^N)$ is the uniqueness class and therefore $\tilde{z}_\varepsilon = z_\varepsilon$. Further we estimate z_ε. We have

$$-\varepsilon\Delta\nabla^r z_\varepsilon + \lambda\nabla^r z_\varepsilon = \nabla^r f - \mathbf{w}\cdot\nabla\nabla^r z_\varepsilon - \sum_{\substack{i+j=r\\0\le j\le r-1}} \nabla^i\mathbf{w}\cdot\nabla\nabla^j z_\varepsilon \tag{2.7}$$

for any $r = 0,1,\ldots,k$. Similarly as in Lemmas 1.2–1.6 we can get

$$\begin{aligned}\varepsilon(q-1)\sum_{r=0}^{k}\int_{\mathbb{R}^N}|\nabla^{r+1}z_\varepsilon|^2|\nabla^r z_\varepsilon|^{q-2}\mathrm{d}\mathbf{x} + \lambda\|z_\varepsilon\|_{k,q}^q \le\\ \le \|f\|_{k,q}\|z_\varepsilon\|_{k,q}^{q-1} + \alpha_0(k)\theta_0^{(k)}(0,\mathbf{w})\|z_\varepsilon\|_{k,q}^q\,.\end{aligned} \tag{2.8}$$

The hypothesis (2.5) is trivially satisfied if $\alpha_0\theta_0 < \lambda$. From (2.8) we deduce

$$\|z_\varepsilon\|_{k,q} \le \frac{\|f\|_{k,q}}{\lambda - \alpha_0(k)\theta_0^{(k)}(0,\mathbf{w})}\,,\quad 1<q<\infty\,, \tag{2.9}$$

i.e. an estimate independent of ε. From (2.9) we easily get that for any $1 < t_0 < t_1 < \infty$ there exists $z \in W^{k,t_0}(\mathbb{R}^N)\cap W^{k,t_1}(\mathbb{R}^N)$ such that

$$\begin{aligned} z_{\varepsilon_n} \rightharpoonup z \quad &\text{in } W^{k,q}(\mathbb{R}^N) \quad \forall q\in[t_0;t_1]\\ \varepsilon z_{\varepsilon_n} \to 0 \quad &\text{in } W^{k+1,2}(\mathbb{R}^N)\end{aligned}$$

at least for a chosen subsequence $\varepsilon_n \to 0^+$. The diagonalisation procedure yields

$$z_{\varepsilon_k} \rightharpoonup z \qquad \text{in } W^{k,q}(\mathbb{R}^N) \quad \forall q\in(1,\infty)\,,$$

at least for a chosen subsequence $\varepsilon_k \to 0^+$. Easily z solves (1.1) and from Lemma 1.2 we conclude (2.3). Since the solution is unique in the class where (2.3) holds, we even conclude that the whole sequence $z_\varepsilon \rightharpoonup z$ in $W^{k,q}(\mathbb{R}^N)$.

□

Next we weaken the assumptions on $\mathbf{w}$

Lemma 2.3 *Let $k = 0,1,\ldots$, $a \equiv 0$, $f \in C_0^\infty(\mathbb{R}^N)$, $\mathbf{w} \in C_B^1(\mathbb{R}^N)$ ($k = 0$) or $\mathbf{w} \in C_B^k(\mathbb{R}^N)$ ($k \ge 1$). If $\alpha_0(k)\theta_0^{(k)}(0,\mathbf{w}) < \lambda$, then there exists a unique $z \in \cap_{1<q<\infty}W^{k,q}(\mathbb{R}^N)$, the solution to (1.1), which satisfies (2.3).*

Proof: Let $\varepsilon > 0$, η_R the usual cut–off function (see Section VIII.2) and denote

$$\mathbf{w}_{R,\varepsilon} = (\mathbf{w}\eta_R)_\varepsilon\,,$$

i.e. the mollification of $\mathbf{w}\eta_R$. Then $\mathbf{w}_{R,\varepsilon} \in C_0^\infty(\mathbb{R}^N)$ and evidently $\mathbf{w}_{R,\varepsilon} \to \mathbf{w}\eta_R$ in $C_B^k(\mathbb{R}^N)$, $\theta_0^{(k)}(0,\mathbf{w}_{R,\varepsilon}) \to \theta_0^{(k)}(0,\mathbf{w}\eta_R)$ as $\varepsilon \to 0^+$. Finally $\theta_0^{(k)}(0,\mathbf{w}\eta_R) \to \theta_0^{(k)}(0,\mathbf{w})$ as $R \to \infty$. Therefore if $\alpha_0(k)\theta_0^{(k)}(0,\mathbf{w}) < \lambda$, there exist $R_0 > 0$ such that $\alpha_0(k)\theta_0^{(k)}(0,\mathbf{w}\eta_R) < \lambda$ for $R > R_0$ and $\varepsilon_0(R) > C(R_0) > 0$ such that $\alpha_0(k)\theta_0^{(k)}(0,\mathbf{w}_{R,\varepsilon}) < \lambda$ for $0 < \varepsilon < \varepsilon_0(R)$. We consider the problem

$$\lambda z + \mathbf{w}_{R,\varepsilon} \cdot \nabla z = f$$

with R, ε satisfying $\alpha_0(k)\theta_0^{(k)}(0,\mathbf{w}_{R,\varepsilon}) < \lambda$. From Lemma 2.2 we conclude the existence of a unique solution $z_{R,\varepsilon}$ which satisfies

$$\|z_{R,\varepsilon}\|_{k,q} \leq \frac{\|f\|_{k,q}}{\lambda - \alpha_0(k)\theta_0^{(k)}(0,\mathbf{w}_{\varepsilon,R})} \quad 1 < q < \infty\,. \tag{2.10}$$

We pass with $\varepsilon \to 0^+$ and $R \to \infty$ and get as in Lemma 2.2

$$\begin{aligned} z_{R,\varepsilon} &\rightharpoonup z_R \quad \text{in } W^{k,q}(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0^+ \\ z_R &\rightharpoonup z \quad \text{in } W^{k,q}(\mathbb{R}^N) \quad \text{as } R \to \infty\,, \end{aligned}$$

at least for chosen subsequences. The limit function z satisfies for all functions $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} [\lambda z\, \varphi - z(\mathbf{w}\cdot\nabla\varphi - \nabla\cdot\mathbf{w}\varphi)]\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} f\varphi \mathrm{d}\mathbf{x}\,,$$

i.e. z is a weak solution to $(1.1)_{a=0}$. As any weak solution is the strong solution (see Remark 1.3) and $z \in W^{k,q}(\mathbb{R}^N)$, we conclude (2.3) from Lemma 1.2. The proof is complete.

□

We keep for a moment $f \in C_0^\infty(\mathbb{R}^N)$ but we add the function a.

Lemma 2.4 *Let $f \in C_0^\infty(\mathbb{R}^N)$ and k, a, $\mathbf{w}$ and q satisfy the assumptions of Lemma 1.2 with $\Omega = \mathbb{R}^N$. Let $\alpha_0(k)\theta_0^{(k)}(a,\mathbf{w}) < \lambda$. Then there exists a unique $z \in \cap_{1<t<\infty} W^{k,t}(\mathbb{R}^N)$, the solution to (1.1), which satisfies the estimate*

$$\|z\|_{s,t} \leq \frac{\|f\|_{s,t}}{\lambda - \alpha_0\theta_0} \quad \forall s = 0,1,\ldots,k\,, \quad t \in (1,\infty)\,. \tag{2.11}$$

Proof: Let us consider the equation

$$\lambda z - \mathbf{w}\cdot\nabla z = f - a\tau \quad \text{in } \mathbb{R}^N \tag{2.12}$$

with $\tau \in W^{k,t_0}(\mathbb{R}^N) \cap W^{k,t_1}(\mathbb{R}^N)$, $1 < t_0 < t_1 < \infty$ and $\mathbf{w}$, a satisfying $\alpha_0\theta_0 < \lambda$. Using the standard density property together with Lemma 2.3 we show that there a exists unique $z_\tau \in W^{k,t_0}(\mathbb{R}^N) \cap W^{k,t}(\mathbb{R}^N)$, the solution to (2.12) satisfying the estimate

$$\lambda\|z\|_{k,q} \leq \|f\|_{k,q} + \alpha_0(k)\theta_0^{(k)}(0,\mathbf{w})\|z\|_{k,q} + \|a\|_{C^k}\|\tau\|_{k,q} \quad \forall q \in [t_0;t_1]\,.$$

Therefore the operator $T : W^{k,t_0}(\mathbb{R}^N) \cap W^{k,t_1}(\mathbb{R}^N) \mapsto W^{k,t_0}(\mathbb{R}^N) \cap W^{k,t_1}(\mathbb{R}^N)$ which assigns to τ z_τ, the solution to (2.12), is well defined. We shall show that T is a contraction operator. Let τ_1 and τ_2 be two different functions from $W^{k,t_0}(\mathbb{R}^N) \cap W^{k,t_1}(\mathbb{R}^N)$ and z_1, z_2 the corresponding solutions. Then

$$\lambda \|z_1 - z_2\|_{k,q} \le \alpha_0(k)\theta_0^{(k)}(0,\mathbf{w})\|z_1 - z_2\|_{k,q} + \|a\|_{C^k}\|\tau_1 - \tau\|_{k,q} \,.$$

As $\alpha_0(k) \ge 1$ and $\alpha_0(k)(\|a\|_{C^k} + \theta_0^{(k)}(0,\mathbf{w})) < \lambda$, we have

$$\|a\|_{C^k} < \frac{\lambda - \alpha_0(k)\theta_0^{(k)}(0,\mathbf{w})}{\alpha_0(k)} \le \lambda - \alpha_0(k)\theta_0^{(k)}(0,\mathbf{w})$$

and the operator T is contraction in $W^{k,q}(\mathbb{R}^N)$ for any $q \in [t_0; t_1]$. Denote by z its (unique) fixed point. Then $z \in W^{k,q}(\mathbb{R}^N)$, $t_0 \le q \le t_1$, and solves (1.1). Since t_0, t_1 can be taken arbitrarily and Lemma 1.2 guarantees (2.11), Lemma 2.4 is proved.

□

We are now in a position to prove the existence of solution to (1.1) for $f \in W^{k,q}(\mathbb{R}^N)$ and for Ω an exterior domain in $\mathbb{R}^N$.

Theorem 2.1

(i) *Let q, k, Ω a, $\mathbf{w}$ and f satisfy the assumptions of Lemma 1.2. Then there exists $\alpha(k) \ge \alpha_0(k)$ such that if*

$$\alpha(k)\theta_0^{(k)}(a,\mathbf{w}) < \lambda \,,$$

then there exists a unique solution $z \in W^{k,q}(\Omega)$ to the problem (1.1) which satisfies the estimate

$$\|z\|_{k,q} \le \frac{\|f\|_{k,q}}{\lambda - \alpha_0\theta_0} \tag{2.13}$$

(for the definition of α_0 see Lemma 1.2). If $\Omega = \mathbb{R}^N$, then $\alpha = \alpha_0$.

(ii) *If moreover $f \in W_0^{l,q}(\Omega)$ for some $l = 1, 2 \ldots, k$ and*

$$(\mathbf{w}, \nabla\mathbf{w}, \ldots \nabla^{l-1}\mathbf{w})/_{\partial\Omega} = \mathbf{0}$$

in the sense of traces, then also $z \in W_0^{l,q}(\Omega)$.

Proof: Let us first note that it is enough to prove the theorem for $\Omega = \mathbb{R}^N$. Suppose that the theorem holds true for $\Omega = \mathbb{R}^N$, i.e. there exists $\mathcal{A} > 0$ such that for any $\mathbf{W}$, A satisfying

$$\mathcal{A}\theta_0^{(k)}(A,\mathbf{W}) < \lambda$$

and $F \in W^{k,q}(\mathbb{R}^N)$ there exists $Z \in W^{k,q}(\mathbb{R}^N)$, solution to the problem

$$\lambda Z \ + \mathbf{W} \cdot \nabla Z \ + AZ = F \quad \text{ in } \mathbb{R}^N \,.$$

Let us extend $\mathbf{w}$, a and f to the whole $\mathbb{R}^N$ due to Lemma 2.1 and denote their extensions by $\mathbf{W}$, A and F, respectively. Moreover, from (2.1) and (2.2) it follows that

$$\theta_0^{(k)}(A, \mathbf{W}) \le M\theta_0^{(k)}(a, \mathbf{w}) .$$

Suppose that $\mathbf{w}$, a satisfy

$$\max\{\alpha_0, \mathcal{A}M\}\theta_0^{(k)}(a, \mathbf{w}) < \lambda .$$

Then certainly

$$\mathcal{A}\theta_0^{(k)}(A, \mathbf{W}) < \lambda$$

and according to our assumptions, there exists $Z \in W^{k,q}(\mathbb{R}^N)$, a unique solution to (1.1) in $\mathbb{R}^N$. Then $z = Z/_{\Omega}$ belongs to $W^{k,q}(\Omega)$ and solves the transport equation (1.1) in Ω. In particular $\alpha_0(k)\theta_i^{(k)}(a, \mathbf{w}) < \lambda$ and (2.13) follows from Lemma 1.2.

Now the statement (i) for $\Omega = \mathbb{R}^N$ follows easily from Lemma 2.4 by the standard density argument. To prove (ii), let us observe that we have at $\partial\Omega$

$$\mathbf{w} \cdot \nabla z \; = (\mathbf{w} \cdot \mathbf{t})(\mathbf{t} \cdot \nabla z) + (\mathbf{w} \cdot \mathbf{n})(\mathbf{n} \cdot \nabla z)$$

with $\mathbf{t}$ and $\mathbf{n}$ the tangent and normal vectors to $\partial\Omega$, respectively. If $l = 1$, then $(\mathbf{w} \cdot \mathbf{n}) = (\mathbf{w} \cdot \mathbf{t}) = 0$ at $\partial\Omega$ and from (1.1) we see that

$$(\lambda - a)z = 0$$

in the sense of traces at $\partial\Omega$. As $\lambda > \|a\|_{C^0}$, we have $z = 0$ at $\partial\Omega$. For higher derivatives we proceed analogously.

□

Theorem 2.2

(i) *Let q, k, Ω a, $\mathbf{w}$ and f satisfy the assumptions of Lemma 1.3. Moreover, for $(1.14)_1$, let $\nabla a \in W^{k-1,q}(\Omega)$. Then there exists $\alpha(k,q) \ge \alpha_0(k,q)$ such that if*

$$\alpha(k,q)\theta_i^{(k)}(a, \mathbf{w}) < \lambda$$

($i = 1, 2$, see Lemma 1.3), then there exists a unique solution $z \in W^{k,q}(\Omega)$ to the problem (1.1) satisfying the estimate

$$\|z\|_{k,q} \le \frac{\|f\|_{k,q}}{\lambda - \alpha_0\theta_i} . \tag{2.14}$$

If $\Omega = \mathbb{R}^N$, then $\alpha = \alpha_0$.

(ii) *If moreover $f \in W_0^{l,q}(\Omega)$ for some $l = 1, 2 \ldots, k$ and*

$$(\mathbf{w}, \nabla\mathbf{w}, \ldots \nabla^{l-1}\mathbf{w})/_{\partial\Omega} = \mathbf{0}$$

in the sense of traces, then also $z \in W_0^{l,q}(\Omega)$.

Proof: Analogously to Theorem 2.1 we can show that it is sufficient to prove the statement (i) for $\Omega = \mathbb{R}^N$ and use Lemma 2.1 in the case of $\Omega \neq \mathbb{R}^N$. Let us start with $\nabla a \in W^{k-1,q}(\mathbb{R}^N)$, $kq > N$.

Let ζ_R be the Sobolev cut–off function with $R > \mathrm{e}$, see Section VIII.2. Put $a_{R,\varepsilon} = (a\zeta_R)_\varepsilon$, $\varepsilon > 0$ (the mollification). Then $a_{R,\varepsilon} \in C_0^\infty(\mathbb{R}^N)$ and for fixed $R > \varepsilon$

$$\left.\begin{aligned} a_{R,\varepsilon} \to a\zeta_R \quad & \text{in } C_B^{k-1}(\mathbb{R}^N) \\ \nabla^k(a_{R,\varepsilon}) \to \nabla^k(a\zeta_R) \quad & \text{in } L^q(\mathbb{R}^N) \end{aligned}\right\} \text{ as } \varepsilon \to 0^+ .$$

Therefore $\theta_1^{(k)}(a_{R,\varepsilon}, \mathbf{w}) \to \theta_1^{(k)}(a\zeta_R, \mathbf{w})$ as $\varepsilon \to 0^+$. Since $\nabla a \in W^{k-1,q}(\mathbb{R}^N)$, we have also due to Lemma VIII.2.2

$$\left.\begin{aligned} \|\nabla^k u \zeta_R\|_q \to \|\nabla^k u\|_q \\ \|\nabla^{k-l} u \; \nabla^l \zeta_R\|_q \to 0 \quad l = 1, 2, \dots, k \end{aligned}\right\} \text{ as } R \to \infty \tag{2.15}$$

and therefore

$$\|\nabla^k(u\zeta_R)\|_q \to \|\nabla^k u\|_q \quad \text{as } R \to \infty .$$

Moreover

$$\|a\zeta_R\|_{C^{k-1}} \to \|a\|_{C^{k-1}} \quad \text{as } R \to \infty$$

and therefore

$$\theta_1^{(k)}(a\zeta_R, \mathbf{w}) \to \theta_1^{(k)}(a, \mathbf{w}) \quad \text{as } R \to \infty .$$

Let $\mathbf{w}$, a be such that $\alpha_0(k,q)\theta_1^{(k)}(a, \mathbf{w}) < \lambda$. Then there exists R_0 such that for any $R > R_0$

$$\alpha_0(k,q)\theta_1^{(k)}(a\zeta_R, \mathbf{w}) < \lambda .$$

Furthermore, for any $R > R_0$ there exists $\varepsilon_0(R) \geq C_0(R_0) > 0$ such that for any $0 < \varepsilon < \varepsilon_0(R)$

$$\alpha_0(k,q)\theta_1^{(k)}(a_{R,\varepsilon}, \mathbf{w}) < \lambda . \tag{2.16}$$

Let us consider the problem

$$z \; + \mathbf{w} \cdot \nabla z \; + a_{R,\varepsilon} z \; = f \tag{2.17}$$

for ε, R discussed above. Now two possibilities may happen

a) $\mathbf{w}$, $a_{R,\varepsilon}$ are such that $\alpha_0(k,q)\theta_1^{(k)}(a_{R,\varepsilon}, \mathbf{w}) + \alpha_0^{(0)}(k)\theta_0^{(k)}(a_{R,\varepsilon}, \mathbf{w}) < \lambda$, $\alpha_0^{(0)}$, $\theta_0^{(k)}$ are defined in Lemma 1.2. Then by Theorem 2.1 there exists a unique solution $z_{R,\varepsilon} \in W^{k,q}(\mathbb{R}^N)$ and according to Lemma 1.3

$$\|z_{R,\varepsilon}\|_{k,q} \leq \frac{\|f\|_{k,q}}{\lambda - \alpha_0(k,q)\theta_1^{(k)}(a_{R,\varepsilon}, \mathbf{w})} \tag{2.18}$$

b) if $\alpha_0(k,q)\theta_1^{(k)}(a_{R,\varepsilon}, \mathbf{w}) + \alpha_0^{(0)}(k)\theta_0^{(k)}(a_{R,\varepsilon}, \mathbf{w}) \geq \lambda$, then there exists $\lambda_0 > \lambda$ such that $\alpha_0(k,q)\theta_1^{(k)}(a_{R,\varepsilon}, \mathbf{w}) + \alpha_0^{(0)}(k)\theta_0^{(k)}(a_{R,\varepsilon}, \mathbf{w}) < \lambda_0$. Let us consider the equation

$$\lambda_0 z \; + \mathbf{w} \cdot \nabla z \; + a_{R,\varepsilon} z \; = f + (\lambda_0 - \lambda)\tau \tag{2.19}$$

for $\tau \in W^{k,q}(\mathbb{R}^N)$. Applying Theorem 2.1 to (2.19) we easily conclude that there exists a unique solution $z^\tau_{R,\varepsilon}$ to (2.19) together with the estimate

$$(\lambda_0 - \alpha_0(k,q)\theta_1^{(k)}(a_{R,\varepsilon},\mathbf{w}))\|z^\tau_{R,\varepsilon}\|_{k,q} \le \|f\|_{k,q} + (\lambda_0 - \lambda)\|\tau\|_{k,q}\,. \quad (2.20)$$

From (2.16) we have $\lambda_0 - \alpha_0(k,q)\theta_1^{(k)}(a_{R,\varepsilon},\mathbf{w}) > \lambda_0 - \lambda$ and therefore (2.19) defines a linear mapping $\tau \mapsto z^\tau_{R,\varepsilon}$ which is due to (2.20) contraction in $W^{k,q}(\mathbb{R}^N)$. We denote its unique fixed point $z_{R,\varepsilon}$ and from Lemma 1.3 we conclude that it satisfies the estimate (2.18).

We may therefore pass with $\varepsilon \to 0^+$ and then with $R \to \infty$ to get $z \in W^{k,q}(\mathbb{R}^N)$ such that

$$\begin{aligned} z_{R,\varepsilon} &\rightharpoonup z_R \quad \text{in } W^{k,q}(\mathbb{R}^N) \quad \text{as } \varepsilon \to 0^+ \\ z_R &\rightharpoonup z \quad \text{in } W^{k,q}(\mathbb{R}^N) \quad \text{as } R \to \infty\,, \end{aligned}$$

at least for chosen subsequences. Moreover, it is an easy matter to verify that the function z satisfies

$$\int_{\mathbb{R}^N} z(\lambda\varphi - \mathbf{w}\cdot\nabla\varphi + (a - \nabla\cdot\mathbf{w})\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^N} f\varphi\mathrm{d}\mathbf{x} \quad (2.21)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ together with the estimate

$$\|z\|_{k,q} \le \frac{\|f\|_{k,q}}{\lambda - \alpha_0(k,q)\theta_1^{(k)}(a,\mathbf{w})}\,, \quad (2.22)$$

i.e. z is a weak (and therefore also a strong) solution to (1.1).

The proof of the statement (i) for $\nabla^k a \in L^N(\mathbb{R}^N)$ can be done analogously. Only instead of (2.15) we have

$$\begin{aligned} \|\nabla^k a\zeta_R\|_N &\to \|\nabla^k a\|_N \quad \text{as } R \to \infty \\ \|\nabla^{k-j} a\nabla^j\zeta_R\|_N &\to 0 \quad \text{as } R \to \infty\,, j = 1,2,\dots,k \end{aligned}$$

due to the properties of the Sobolev cut–off function (see Section VIII.2). We have namely for $l \ge 2$ that $\nabla^l\zeta_R$ is in $L^N(\mathbb{R}^N)$ bounded by a constant which tends to 0 as $R \to \infty$. The uniqueness follows from the estimate (2.22). The statement (ii) can be shown as in Theorem 2.1.

□

Remark 2.1 Let us note that for $k \ge 2$, $kq > N$ we can weaken the assumptions on a. Let $mq \ge N$. Then it is enough to take $\nabla^{k-m+2}a \in W^{m-2,q}(\Omega)$ for $m \le k-2$ and to proceed as above. Indeed,

$$\|\nabla^{k-1}a\nabla\zeta_R\|_q + \ldots + \|\nabla^{k-m+1}a\nabla^{m-1}\zeta_R\|_q \to 0$$

due to Lemma VIII.2.2 and

$$\|\nabla^{k-m}a\nabla^m\zeta_R\|_q + \ldots + \|a\nabla^k\zeta_R\|_q \to 0$$

due to the fact that $\|\nabla^m\zeta_R\|_q \to 0$ as $R \to \infty$.

Theorem 2.3

(i) *Let q, k, Ω a, $\mathbf{w}$ and f satisfy the assumptions of Lemma 1.4. Moreover, let $\nabla a \in W^{k-1,q}(\Omega)$ if $kq > N$ and $\nabla \mathbf{w} \in W^{k-1,q}(\Omega)$ if $(k-1)q > N$. Then there exists $\alpha(k,q) \geq \alpha_0(k,q)$ such that if*

$$\alpha(k,q)\overline{\theta}_i^{(k)}(a,\mathbf{w}) < \lambda$$

($i = 1,2,3$, see Lemma 1.4), then there exists a unique solution $z \in W^{k,q}(\Omega)$ to the problem (1.1) satisfying the estimate

$$\|z\|_k \leq \frac{\|f\|_{k,q}}{\lambda - \alpha_0\overline{\theta}_i} \,. \tag{2.23}$$

If $\Omega = \mathbb{R}^N$, then $\alpha = \alpha_0$.

(ii) *If moreover $f \in W_0^{l,q}(\Omega)$ for some $l = 1, 2 \ldots, k$ and*

$$(\mathbf{w}, \nabla\mathbf{w}, \ldots \nabla^{l-1}\mathbf{w})/_{\partial\Omega} = \mathbf{0}$$

in the sense of traces, then also $z \in W_0^{l,q}(\Omega)$.

Proof: It is essentially the same as the proof of Theorem 2.2; instead of $a_{R,\varepsilon}$ we take $\mathbf{w}_{R,\varepsilon} \in C_0^\infty(\mathbb{R}^N)$, use Theorem 2.2 for the existence and pass with $\varepsilon \to 0^+$, $R \to \infty$. Instead of Lemma 1.3, we apply Lemma 1.4.

□

Remark 2.2 Similarly as in Remark 2.1, we can weaken the assumptions on $\mathbf{w}$ if $k \geq 2$ and $(k-1)q > N$.

Remark 2.3 We can prove even a bit stronger version of the uniqueness. We have:

Let $1 < p, q < \infty$. Let $z_1 \in L^q(\Omega)$ and $z_2 \in L^p(\Omega)$ be two (apriori different) weak solutions to (1.1) with $f \in L^q(\Omega) \cap L^p(\Omega)$. Let $\mathbf{w}$, a satisfy assumptions (i) of Lemma 1.2 and $\alpha_0\theta_0^{(0)}(a,\mathbf{w}) < \lambda$. Then $z_1 = z_2$ a.e. in Ω.

Proof: Again, it is enough to consider $\Omega = \mathbb{R}^N$. Let $f_n \in C_0^\infty(\Omega)$ be a sequence such that $f_n \to f$ in $L^q(\Omega) \cap L^p(\Omega)$ and z_n denotes the unique solution (by Lemma 2.4) to

$$\lambda z \ + \mathbf{w}\cdot\nabla z \ + az = f_n \,.$$

But $f_n \to f$ in $L^q(\Omega) \cap L^p(\Omega)$ and $L^q(\Omega)$ and $L^p(\Omega)$, respectively, are uniqueness classes, hence $z_n \to z_1$ in $L^p(\Omega)$ and $z_n \to z_2$ in $L^q(\Omega)$, i.e. $z_1 = z_2$ a.e. in $\mathbb{R}^N$.

□

Theorem 2.4 *Let a, k, Ω, a, $\mathbf{w}$ and f satisfy the assumptions of Lemma 1.5. Moreover, let $f \in L^p(\Omega)$, $1 < p < \infty$. Then there exists $\alpha(k,q) \geq \alpha_0(k,q)$ such that if*

$$\alpha(k,q)\theta_i'^{(k)}(k,q) < \lambda \tag{2.24}$$

($i=1,2$, see Lemma 1.5) then there exists unique solution $z \in L^p(\Omega)$ such that $\nabla z \in W^{k-1,q}(\Omega)$ satisfying the estimates

$$\begin{aligned} \|z\|_p &\le \frac{\|f\|_p}{\lambda - \alpha_0\theta_0} \\ \|\nabla z\|_{k-1,q} &\le \frac{\|\nabla f\|_{k-1,q}}{\lambda - \alpha_0\theta_i'}\,. \end{aligned} \tag{2.25}$$

If $\Omega = \mathbb{R}^N$, then $\alpha = \alpha_0$.

Proof: Let us note that there exists a sequence $f_n \in C_0^\infty(\overline{\Omega})$ such that

$$\begin{aligned} \nabla f_n \to \nabla f \quad &\text{in } W^{k-1,q}(\Omega) \\ f_n \to f \quad &\text{in } L^p(\Omega)\,. \end{aligned}$$

Moreover, (2.24) implies that $\alpha_0\theta_0^{(0)}(a,\mathbf{w}) < \lambda$, α_0, $\theta_0^{(0)}(a,\mathbf{w})$ from Lemma 1.2. Therefore for any $n \in \mathbb{N}$ there exists z_n, solution to

$$\lambda z + \mathbf{w}\cdot\nabla z + az = f_n \quad \text{in } \Omega$$

(see Theorems 2.1 and 2.2) and Lemmas 1.2 and 1.5 imply

$$\begin{aligned} \|z_n\|_p &\le \frac{\|f_n\|_p}{\lambda - \alpha_0(0)\theta_0^{(0)}(a,\mathbf{w})} \\ \|\nabla z_n\|_{k-1,q} &\le \frac{\|f_n\|_{k-1,q}}{\lambda - \alpha_0(k,q)\theta_i'^{(k)}(a,\mathbf{w})}\,, \end{aligned}$$

$i = 1,2$. Passing with $n \to \infty$ and recalling that $z_n \rightharpoonup z$ in $L^p(\Omega)$, $\nabla z_n \rightharpoonup \nabla z$ in $W^{k-1,q}(\Omega)$ (the whole sequences) we easily verify that z solves (1.1). The estimates (2.25) follow from Lemmas 1.2 and 1.5.[2]

□

Theorem 2.5 *Let a, k, Ω, a, $\mathbf{w}$ and f satisfy the assumptions of Lemma 1.6. Moreover, let $f \in L^p(\Omega)$, $1 < p < \infty$. Then there exists $\alpha(k,q) \ge \alpha_0(k,q)$ such that if*

$$\alpha(k,q)\theta_{i,j}'^{(k)} < \lambda \tag{2.26}$$

($i,j = 1,2$, see Lemma 1.6) then there exists unique solution $z \in L^p(\Omega)$ such that $\nabla z \in W^{k-1,q}(\Omega)$ satisfying the estimates

$$\begin{aligned} \|z\|_p &\le \frac{\|f\|_p}{\lambda - \alpha_0\theta_0} \\ \|\nabla z\|_{k-1,q} &\le \frac{\|\nabla f\|_{k-1,q}}{\lambda - \alpha_0\theta_{i,j}'}\,. \end{aligned} \tag{2.27}$$

If $\Omega = \mathbb{R}^N$, then $\alpha = \alpha_0$.

[2]Let us note that if $z \in L^p(\Omega)$, $1 < p < \infty$, and $\nabla z \in W^{k-1,q}(\Omega)$, $1 < q < N$, then the Sobolev–Poincaré inequality $\|z\|_{\frac{Nq}{N-q}} \le C(q,N)\|\nabla z\|_q$ holds.

Proof: It is essentially the same as the proof of Theorem 2.4.

□

We finish this section by studying the weighted estimates of solutions to (1.1). Let $g : \Omega \mapsto \mathbb{R}$ be a weight function such that

$$g \in C^k(\Omega)\,, \quad g > 0 \text{ in } \Omega \tag{2.28}$$

and denote

$$\theta^{(k)}_{(g),0}(a, \mathbf{w}) = \theta^{(k)}_0(a - \mathbf{w} \cdot \nabla \ln g, \mathbf{w})\,. \tag{2.29}$$

Let us suppose that $\theta^{(k)}_{(g),0}(a, \mathbf{w}) < \infty$. We define $\widetilde{W}^{k,q}_{(g)}(\Omega)$ as follows

$$u \in \widetilde{W}^{k,q}_{(g)}(\Omega) \Longleftrightarrow ug \in W^{k,q}(\Omega)\,. \tag{2.30}$$

Then $\widetilde{W}^{k,q}_{(g)}(\Omega)$ is a Banach space with

$$\|u\|_{k,q,(g)} = \|ug\|_{k,q}\,. \tag{2.31}$$

Moreover we suppose that g is such that

$$\widetilde{W}^{k,q}_{(g)}(\Omega) \subset W^{k,q}(\Omega)\,, \tag{2.32}$$

i.e. $g \geq C_1$ as $|\mathbf{x}| \to \infty$, $g \geq C_2$ as $|\mathbf{x}| \to \partial\Omega$ and the same assumptions on the derivatives of g. Then we have

Theorem 2.6 *Let k, q, Ω, a, $\mathbf{w}$, f satisfy the assumptions of Theorem 2.1. Let g be such that (2.28)–(2.30) hold. Let $f \in \widetilde{W}^{k,q}_{(g)}(\Omega)$ and*

$$\alpha_0(k)(\theta^{(k)}_{(g),0} + \theta^{(k)}_0)(a, \mathbf{w}) < \lambda\,. \tag{2.33}$$

Let $z \in W^{k,q}(\Omega)$ be the solution to (1.1) guaranteed by Theorem 2.1. Then $z \in \widetilde{W}^{k,q}_{(g)}(\Omega)$ and

$$\|z\|_{k,q,(g)} \leq \frac{\|f\|_{k,q,(g)}}{\lambda - \alpha_0\theta_{(g),0}}\,. \tag{2.34}$$

Proof: Let us solve

$$\lambda\xi + \mathbf{w} \cdot \nabla\xi + (a - \mathbf{w} \cdot \nabla \ln g)\xi = fg \quad \text{ in } \Omega. \tag{2.35}$$

Theorem 2.1 guarantees the existence of a unique solution to (2.34) in $W^{k,q}(\Omega)$ together with the estimate

$$\|\xi\|_{k,q} \leq \frac{\|fg\|_{k,q}}{\lambda - \alpha_0(k)\theta^{(k)}_{(g),0}(a, \mathbf{w})}\,. \tag{2.36}$$

The function $\eta = \frac{\xi}{g} \in \widetilde{W}^{k,q}_{(g)}(\Omega) \subset W^{k,q}(\Omega)$ and solves

$$\lambda\eta + \mathbf{w} \cdot \nabla\eta + a\eta = f\,. \tag{2.37}$$

But denoting by z the unique solution to (2.36) in $W^{k,q}(\Omega)$, we easily deduce that $\eta = z$ i.e. $\xi = zg$. The estimate (2.34) follows from (2.36).

□

V

Existence of solutions in Sobolev spaces

This chapter is devoted to the construction of solutions to the problem (I.4.14)–(I.4.15) in two and three space dimensions. We shall combine results from the last two chapters — the estimate on the modified Oseen problem and transport equation. In the following chapter we then prove some auxiliary estimates in order to study the asymptotic structure of solutions. The method of demonstration is a perturbativ one — we study only small perturbations with respect to the rest state $\mathbf{u} = \mathbf{0}$, $p =$ const caused by a small external force and by a small velocity prescribed at infinity. The method is based on the following version of the Banach fixed point theorem (see e.g. [Vi])

Theorem 0.1 *Let X, Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Let H be non–empty, closed, convex and bounded subset of X and let $\mathcal{M} : H \mapsto H$ be a mapping such that*

$$\|\mathcal{M}(u) - \mathcal{M}(v)\|_Y \leq \kappa \|u - v\|_Y \quad \forall u, v \in H\,,$$

$0 \leq \kappa < 1$. *Then $\mathcal{M}$ has a unique fixed point in H.*

Proof: Let $u_n \in H$ be a sequence strongly convergent to u in Y. As H is weakly closed and X is reflexive, there exists $\{u_{n_k}\}$, subsequence chosen from $\{u_n\}$, and $v \in H$ such that $u_{n_k} \rightharpoonup v$ in X. But $X \hookrightarrow Y$ and therefore $u_{n_k} \rightharpoonup v$ in Y. The uniqueness of the weak limit implies $u = v$ and therefore $u \in H$. We see that H is closed in Y and the result follows from the Banach fixed point theorem.

□

We shall study separately the three– and twodimensional flows.

V.1 Threedimensional flow

We shall prove several existence theorems under different assumptions on the right hand side $\mathbf{f}$. We start with some auxiliary lemmas.

Lemma 1.1 *Let Ω be an exterior domain in $\mathbb{R}^3$ of class $C^{0,1}$. Then we have*

$$\|u\|_\infty \leq C\|u\|_4^{\frac{2}{5}} |u|_{2,2}^{\frac{3}{5}} + C(\varepsilon)\|u\|_4^{\frac{2}{5}+\varepsilon} |u|_{2,2}^{\frac{3}{5}-\varepsilon} \tag{1.1}$$

$u \in L^4(\Omega) \cap D^{2,2}(\Omega)$, $0 < \varepsilon \leq \frac{3}{5}$

$$\|u\|_\infty \leq C\|u\|_q^{1-\alpha}|u|_{1,p}^{\alpha} + C(\varepsilon)\|u\|_q^{1-\alpha+\varepsilon}|u|_{1,p}^{\alpha-\varepsilon} \tag{1.2}$$

$u \in L^q(\Omega) \cap D^{1,p}(\Omega)$, $p > 3$, $1 < q < \infty$, $\alpha = \frac{3p}{3p+pq-3p}$, $0 < \varepsilon \leq \alpha$

$$\|u\|_p \leq C\|u\|_q^{\alpha}|u|_{1,p}^{1-\alpha} \tag{1.3}$$

$u \in L^q(\Omega) \cap D^{1,p}(\Omega)$, $p > 3$, $1 \leq q \leq p$, $\alpha = \frac{3(p-q)}{3p+pq-3q}$

$$\|u\|_4 \leq C\|u\|_2^{\frac{7p-12}{10p-12}}|u|_{1,p}^{\frac{3p}{10p-12}} \tag{1.4}$$

$u \in L^2(\Omega) \cap D^{1,p}(\Omega)$, $p > \frac{12}{7}$

$$\|u\|_{\frac{2p}{p-2}} \leq C\|u\|_4^{1-\alpha}|u|_{1,p}^{\alpha} \tag{1.5}$$

$u \in L^4(\Omega) \cap D^{1,p}(\Omega)$, $\alpha = \frac{12-3p}{7p-12}$, $\frac{12}{5} < p \leq 4$.

Proof: It is a consequence of Theorem VIII.1.13 and Remark VIII.1.10.

□

Let us recall that we study the following system

$$\left.\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla\pi = \mathbf{N}(\mathbf{f}, \mathbf{T}(\mathbf{u}), p(\pi, \mathbf{u}), \mathbf{u}) \\ \nabla \cdot \mathbf{u} = 0 \end{aligned}\right\} \text{ in } \Omega \\ \mathbf{u} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \\ \mathbf{u} = -\mathbf{v}_\infty \quad \text{at } \partial\Omega \tag{1.6}$$

$$p + ((\mathbf{u} + \mathbf{v}_\infty) \cdot \nabla)p = \pi \tag{1.7}$$

$$\mathbf{T} + ((\mathbf{u} + \mathbf{v}_\infty) \cdot \nabla)\mathbf{T} + \mathbf{G}(\nabla \mathbf{u}, \mathbf{T}) = 2\eta\mathbf{D}(\mathbf{u})\,, \tag{1.8}$$

i.e. we search a fixed point of the operator $\mathcal{M} : (\mathbf{w}, s) \to (\mathbf{u}, \pi)$, where

$$\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial \mathbf{u}}{\partial x_1} + \nabla\pi = \mathbf{F}(\mathbf{f}, \mathbf{T}(\mathbf{w}), p(s, \mathbf{w}), \mathbf{w}) \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \\ \mathbf{u} = -\mathbf{v}_\infty \quad \text{at } \partial\Omega \end{aligned} \tag{1.9}$$

and

$$p + ((\mathbf{w} + \mathbf{v}_\infty) \cdot \nabla)p = s \tag{1.10}$$

$$\mathbf{T} + ((\mathbf{w} + \mathbf{v}_\infty) \cdot \nabla)\mathbf{T} + \mathbf{G}(\nabla \mathbf{w}, \mathbf{T}) = 2\eta\mathbf{D}(\mathbf{w})\,. \tag{1.11}$$

Moreover

$$\mathbf{N}(\mathbf{f}, \mathbf{T}, p, \mathbf{w}) = \mathbf{f} + \nabla \cdot \Big[\mathbf{F}(\nabla \mathbf{w}, \mathbf{T}) + p(\nabla \mathbf{w})^T - ((\mathbf{w} \cdot \nabla)\mathbf{w}) \otimes \mathbf{w} - \\ -\mathbf{w} \otimes \mathbf{w} - \beta\Big(\frac{\partial \mathbf{w}}{\partial x_1} \otimes \mathbf{w} + ((\mathbf{w} \cdot \nabla)\mathbf{w}) \otimes \mathbf{e}_1\Big) + \mathbf{f} \otimes (\mathbf{w} + \beta\mathbf{e}_1)\Big] \equiv \mathbf{f} + \nabla \cdot \mathbf{g}\,, \tag{1.12}$$

where $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are bilinear functions, $\mathbf{v}_\infty = \text{const} = \beta\mathbf{e}_1$.

We first start with the right hand side $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap W^{k,2}(\Omega)$, $k \geq 2$. From Theorem IV.2.3 we have

Corollary 1.1 *Let $\Omega \in C^{0,1}$ be an exterior domain in $\mathbb{R}^3$. Let $\|\nabla\mathbf{w}\|_{C^{k-2}}$, $|\mathbf{w}|_{k,3}$ and $\mathbf{v}_\infty = \beta\mathbf{e}_1$ be sufficiently small. Then for $k \geq 2$ the solution to (1.10) and (1.11) satisfies*

$$\begin{aligned} \|\mathbf{T}\|_{k,2} &\leq C\|\nabla\mathbf{w}\|_{k,2} \\ \|p\|_{k,2} &\leq C\|s\|_{k,2}\,. \end{aligned} \tag{1.13}$$

Next we study the modified Oseen problem (1.9). We first need some estimates of the right hand side.

Lemma 1.2 *Let $\Omega \in C^{0,1}$ be exterior domain in $\mathbb{R}^3$, $l \geq 2$. Then for u, v sufficiently smooth we have*

$$\|uv\|_{l,2} \leq \|u\|_{l,2}\|v\|_{l,2} \tag{1.14}$$

$u, v \in W^{l,2}(\Omega)$

$$\|uv\|_{1,2} \leq \|u\|_{1,2}\|v\|_{2,2} \tag{1.15}$$

$u \in W^{1,2}(\Omega)$, $v \in W^{2,2}(\Omega)$

$$\|uv\|_{l,2} \leq (\|u\|_4 + \|\nabla u\|_{l-1,2})\|v\|_{l,2} \tag{1.16}$$

$u \in L^4(\Omega)$, $\nabla u \in W^{l-1,2}(\Omega)$, $v \in W^{l,2}(\Omega)$

$$\|uv\|_{1,2} \leq (\|u\|_4 + \|\nabla u\|_2)\|v\|_{2,2} \tag{1.17}$$

$u \in L^4(\Omega)$, $\nabla u \in L^2(\Omega)$, $v \in W^{l,2}(\Omega)$

$$\|uv\|_{l,2} \leq (\|u\|_4 + \|\nabla u\|_{l-1,2})(\|v\|_4 + \|\nabla v\|_{l-1,2}) \tag{1.18}$$

$u, v \in L^4(\Omega)$, $\nabla u, \nabla v \in W^{l-1,2}(\Omega)$

$$\|uv\|_{1,2} \leq (\|u\|_4 + \|\nabla u\|_2)(\|v\|_4 + \|\nabla v\|_{1,2}) \tag{1.19}$$

$u, v \in L^4(\Omega)$, $\nabla u \in L^2(\Omega)$, $\nabla v \in W^{1,2}(\Omega)$.

Proof: The inequality (1.14) follows directly by means of the imbeddings $\|\mathbf{u}\|_\infty \leq C\|\mathbf{u}\|_{2,2}$, $\|\mathbf{u}\|_4 \leq C\|\mathbf{u}\|_{1,2}$, while for (1.16), (1.18) we also apply Lemma 1.1, inequality (1.1). Similarly we show the other inequalities with $l = 1$.

□

We can now prove the first existence theorem:

Theorem 1.1 *Let $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap W^{k,2}(\Omega)$, $k \geq 2$, $\Omega \in C^{k+1}$ and let β_0 and $|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_{k,2}$ be sufficiently small. Then for any $0 < \beta \leq \beta_0$ there exists a solution to the system (1.6)–(1.8) such that*

$$\mathbf{u} \in L^4(\Omega)\,, \quad \nabla\mathbf{u}, \pi, p \in W^{k,2}(\Omega)\,.[1]$$

[1] Let us recall (see Chapter I) that π plays the role of the effective pressure; the real pressure is p.

Proof: We define the operator $\mathcal{M} : V_k \mapsto V_k$

$$V_k = \{(\mathbf{u}, p); \mathbf{u} \in L^4(\Omega), \nabla \mathbf{u}, \pi \in W^{k,2}(\Omega)\},$$

$\mathcal{M} : (\mathbf{w}, s) \to (\mathbf{u}, \pi)$, where the pair $(\mathbf{u}, \pi)$ solves (1.9) and $\mathbf{T}$, p solves the transport equations (1.10) and (1.11) with $\mathbf{G}(\cdot,\cdot)$ a bilinear tensor function. We denote for $(\mathbf{u}, \pi) \in V_k$

$$\|(\mathbf{u}, \pi)\|_{V^k} = \beta^{\frac{1}{4}} \|\mathbf{u}\|_4 + \|\nabla \mathbf{u}\|_{k,2} + \|\pi\|_{k,2}$$

and show first that for sufficiently small $\delta > 0$ the operator $\mathcal{M}$ maps

$$B_\delta = \{(\mathbf{u}, \pi) \in V_k; \|(\mathbf{u}, \pi)\|_{V^k} \le \delta\}$$

into itself.

Applying Theorem III.5.3 (i) we get (see (1.12))

$$\begin{aligned}\|(\mathbf{u}, \pi)\|_{V^k} &\le C(|\mathbf{N}|_{-1,2} + \|\mathbf{N}\|_{k-1,2} + \|\mathbf{v}_\infty\|_{k+\frac{3}{2},2,(\partial\Omega)}) \le \\ &\le C(|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_{k-1,2} + \|\mathbf{g}\|_{k,2} + \beta).\end{aligned}$$

We have to estimate $\mathbf{g}$ in $W^{k,2}(\Omega)$. Lemma 1.2 and Corollary 1.1 yield for δ small enough ($\|\nabla^k \mathbf{w}\|_3 + \|\nabla \mathbf{w}\|_{C^{k-2}} \le C\|\nabla \mathbf{w}\|_{k,2} \le C\delta$)

$$\begin{aligned}\|\mathbf{g}\|_{k,2} \le C\Big[&\|\mathbf{T}\|_{k,2}\|\nabla \mathbf{w}\|_{k,2} + \|p\|_{k,2}\|\nabla \mathbf{w}\|_{k,2} + \|\nabla \mathbf{w}\|_{k,2}(\|\mathbf{w}\|_4 + \|\nabla \mathbf{w}\|_{k-1,2})^2 \\ &+(\|\mathbf{w}\|_4 + \|\nabla \mathbf{w}\|_{k-1,2})^2 + \beta\|\nabla \mathbf{w}\|_{k,2}(\|\mathbf{w}\|_4 + \|\nabla \mathbf{w}\|_{k-1,2}) + \\ &+\|\mathbf{f}\|_{k,2}(\beta + \|\mathbf{w}\|_4 + \|\nabla \mathbf{w}\|_{k-1,2})\Big] \le \\ &\le C\Big[\|(\mathbf{w}, s)\|^2_{V_k}(\beta^{-\frac{1}{2}} + \beta) + \|(\mathbf{w}, s)\|^3_{V_k}(\beta^{-\frac{1}{2}} + 1) + \\ &+\|\mathbf{f}\|_{k,2}(\beta + (1 + \beta^{-\frac{1}{4}})\|(\mathbf{w}, s)\|_{V^k})\Big].\end{aligned}$$

We put $\delta = \varepsilon\beta^\alpha$ and assume $\|(\mathbf{w}, s)\|_{V_k} \le \delta$. The exponent α is positive and will be specified below. Then

$$\begin{aligned}\|(\mathbf{u}, \pi)\|_{V_k} \le C\Big[&|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_{k-1,2} + \beta + \varepsilon^2\beta^{2\alpha}(\beta^{-\frac{1}{2}} + \beta) + \\ &+\varepsilon^3\beta^{3\alpha}(\beta^{-\frac{1}{2}} + 1) + \|\mathbf{f}\|_{k,2}(\beta + \varepsilon\beta^\alpha(1 + \beta^{-\frac{1}{4}}))\Big].\end{aligned}$$

Taking ε, β, $|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_{k,2}$ sufficiently small and $\frac{1}{2} \le \alpha < 1$ we get

$$\|(\mathbf{u}, \pi)\|_{V_k} \le \varepsilon\beta^\alpha = \delta$$

and the operator $\mathcal{M}$ maps B_δ into itself. We have to verify that $\mathcal{M}$ is a contraction in B_δ in the topology of V_{k-1}. Let $(\mathbf{w}^1, s^1)$, $(\mathbf{w}^2, s^2) \in V_k$ and $(\mathbf{u}^1, \pi^1)$, $(\mathbf{u}^2, \pi^2)$ be the corresponding images of the operator $\mathcal{M}$. Denoting $\mathbf{U} = \mathbf{u}^1 - \mathbf{u}^2$,

$\Pi = \pi^1 - \pi^2$, $\mathbf{W} = \mathbf{w}^1 - \mathbf{w}^2$, $S = s^1 - s^2$ we have

$$\begin{gathered}
A(\mathbf{U}) + \beta \frac{\partial \mathbf{U}}{\partial x_1} + \nabla \Pi = \\
= \nabla \cdot \Big[\mathbf{F}(\nabla \mathbf{w}^1, \mathbf{T}^1 - \mathbf{T}^2) + \mathbf{F}(\nabla \mathbf{W}, \mathbf{T}^2) + (p^1 - p^2)(\nabla \mathbf{w}^1)^T + \\
+ p^2 (\nabla \mathbf{W})^T - ((\mathbf{W} \cdot \nabla)\mathbf{w}^1) \otimes \mathbf{w}^1 - ((\mathbf{w}^2 \cdot \nabla)\mathbf{W}) \otimes \mathbf{w}^1 - \\
- ((\mathbf{w}^2 \cdot \nabla)\mathbf{w}^2) \otimes \mathbf{W} - \mathbf{W} \otimes \mathbf{w}^1 - \mathbf{w}^2 \otimes \mathbf{W} - \beta \Big(\frac{\partial \mathbf{W}}{\partial x_1} \otimes \mathbf{w}^1 - \\
- \frac{\partial \mathbf{w}^2}{\partial x_1} \otimes \mathbf{W} + ((\mathbf{W} \cdot \nabla)\mathbf{w}^1) \otimes \mathbf{e}_1 + ((\mathbf{w}^2 \cdot \nabla)\mathbf{W}) \otimes \mathbf{e}_1 \Big) + \mathbf{f} \otimes \mathbf{W} \Big] \\
\nabla \cdot \mathbf{U} = 0 \\
\mathbf{U} = \mathbf{0} \text{ at } \partial \Omega \\
\mathbf{U} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty
\end{gathered} \tag{1.20}$$

$$p^1 - p^2 + ((\mathbf{w}^1 + \mathbf{v}_\infty) \cdot \nabla)(p^1 - p^2) = S - (\mathbf{W} \cdot \nabla) p^2 \tag{1.21}$$

$$\begin{gathered}
\mathbf{T}^1 - \mathbf{T}^2 + ((\mathbf{w}^1 + \mathbf{v}_\infty) \cdot \nabla)(\mathbf{T}^1 - \mathbf{T}^2) + \mathbf{G}(\nabla \mathbf{w}^1, \mathbf{T}^1 - \mathbf{T}^2) = \\
= 2\eta \mathbf{D}(\mathbf{W}) - (\mathbf{W} \cdot \nabla)\mathbf{T}^2 - \mathbf{G}(\nabla \mathbf{W}, \mathbf{T}^2) \,.
\end{gathered} \tag{1.22}$$

Corollary 1.1 applied once on (1.21), (1.22) and once on (1.7), (1.8) yield together with Lemma 1.1

$$\begin{gathered}
\|p^1 - p^2\|_{k-1,2} \le C(\|S\|_{k-1,2} + \|(\mathbf{W} \cdot \nabla)p^2\|_{k-1,2}) \le \\
\le C\|(\mathbf{W}, S)\|_{V_{k-1}} (\|\mathbf{w}^2, s^2\|_{V_k} \beta^{-\frac{1}{4}} + 1)
\end{gathered} \tag{1.23}$$

$$\begin{gathered}
\|\mathbf{T}^1 - \mathbf{T}^2\|_{k-1,2} \le C(\|\nabla \mathbf{W}\|_{k-1,2} + \|\mathbf{W} \cdot \nabla \mathbf{T}^2\|_{k-1,2} + \\
+ \|\mathbf{G}(\nabla \mathbf{W}, \mathbf{T}^2\|_{k-1,2}) \le C\|(\mathbf{W}, S)\|_{V_{k-1}} (\|\mathbf{w}^2, s^2\|_{V_k} \beta^{-\frac{1}{4}} + 1) \,.
\end{gathered} \tag{1.24}$$

We now estimate (1.20) applying Theorem III.5.3 together with (1.23) and (1.24) and Lemma 1.2. Let us consider $k = 2$ (the case $k > 2$ is much simpler; we can proceed as above for the space V_k).

$$\begin{aligned}
\|(\mathbf{U}, \Pi)\|_{V_1} \le\; & C\Big\{ \|\mathbf{g}(\mathbf{f}, \mathbf{T}^1, p^1, \mathbf{w}^1) - \mathbf{g}(\mathbf{f}, \mathbf{T}^2, p^2, \mathbf{w}^2)\|_{1,2} \Big\} \le \\
& \le C\Big\{ \|\nabla \mathbf{W}\|_2 + \|\mathbf{W}\|_4)\|\mathbf{f}\|_{2,2} + \|\mathbf{T}^1 - \mathbf{T}^2\|_{1,2} \|\nabla \mathbf{w}^1\|_{2,2} + \\
& + \|\mathbf{T}^2\|_{2,2} \|\nabla \mathbf{W}\|_{1,2} + \|p^1 - p^2\|_{1,2} \|\nabla \mathbf{w}^1\|_{2,2} + \|p^2\|_{2,2} \|\nabla \mathbf{W}\|_{1,2} + \\
& + (\|\mathbf{W}\|_4 + \|\nabla \mathbf{W}\|_2) \|\mathbf{w}^i \cdot \nabla \mathbf{w}^j\|_{2,2} + \|\nabla \mathbf{W}\|_{1,2} \|\mathbf{w}^i \mathbf{w}^j\|_{2,2} + \\
& + (\|\mathbf{W}\|_4 + \|\nabla \mathbf{W}\|_2)(\|\mathbf{w}^i\|_4 + \|\nabla \mathbf{w}^i\|_{1,2}) + \\
& + \beta \Big[\|\nabla \mathbf{W}\|_{1,2} \|\mathbf{w}^i\|_{2,2} + (\|\mathbf{W}\|_4 + \|\nabla \mathbf{W}\|_2) \|\nabla \mathbf{w}^j\|_{2,2}) \Big] \Big\} \le \\
& \le C(1 + \beta^{-\frac{1}{4}} + \beta^{-\frac{1}{2}}) \|(\mathbf{W}, S)\|_{V_1} \cdot \\
& \cdot [c(\mathbf{f}) + \|\mathbf{w}^1, s^1\|_{V_2} + \|\mathbf{w}^2, s^2\|_{V_2} + \|\mathbf{w}^i, s^i\|_{V_2} \|\mathbf{w}^j, s^j\|_{V_2})] \,.
\end{aligned}$$

$(i, j = 1, 2)$. Recalling that $\|\mathbf{w}^i, s^i\|_{V_2} \le \varepsilon \beta^\alpha$, $\alpha \in [\frac{1}{2}; 1)$ we can always choose ε, β sufficiently small such that

$$\|(\mathbf{U}, \Pi)\|_{V_1} \le \kappa \|(\mathbf{W}, S)\|_{V_1} \tag{1.25}$$

with $\kappa \in (0;1)$, $(\mathbf{w}^i, s^i) \in B_\delta$, $\delta = \varepsilon\beta^\alpha$. Analogously we get for $k \geq 3$ the same inequality in V_{k-1}. Therefore $\mathcal{M}$ is a contraction in V_{k-1} and Theorem 0.1 finishes the proof.

□

We have required quite high regularity on the right hand side — $\mathbf{f} \in W^{2,2}(\Omega) \cap D_0^{-1,2}(\Omega)$ at least. Applying Theorem III.5.3 (ii) with $p > 3$ we can a bit weaken the assumptions on the right hand side — $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap L^2(\Omega) \cap W^{1,p}(\Omega)$, $p \in (3;4]$. We shall show the existence of solution in this situation; analogously as in the proof we proceed for $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap L^2(\Omega) \cap W^{k,p}(\Omega)$, $k \geq 2$ and get smoother solution. The proof is similar to the case $k = 1$ and therefore, we shall not do it.

Theorem 1.2 *Let $\mathbf{f} \in D_0^{-1,2}(\Omega) \cap L^2(\Omega) \cap W^{k,p}(\Omega)$, $k \geq 1$, $3 < p \leq 4$, $\Omega \in C^{k+1}$. Let $|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_2 + \|\mathbf{f}\|_{k,p}$ and β_0 be sufficiently small. Then for any $0 < \beta \leq \beta_0$ there exists solution to the problem (1.6)–(1.8) such that*

$$\begin{gathered}\mathbf{u} \in L^4(\Omega) \cap D^{1,2}(\Omega) \\ p, \pi \in L^2(\Omega) \\ \nabla^2\mathbf{u}, \nabla p, \nabla\pi \in W^{k-1,p}(\Omega)\,.\end{gathered}$$

Proof: We show the theorem for $k = 1$. As in Theorem 1.1 we define the operator $\mathcal{M} : V_1 \mapsto V_1$, where now

$$V_1 = \{(\mathbf{u}, \pi); \mathbf{u} \in L^4(\Omega), \pi, \nabla\mathbf{u} \in L^2(\Omega), \nabla^2\mathbf{u}, \nabla\pi \in L^p(\Omega)\}$$

and

$$\|(\mathbf{u}, \pi)\|_{V_1} = \beta^{\frac{1}{4}}\|\mathbf{u}\|_4 + \|\nabla\mathbf{u}\|_2 + \|\pi\|_2 + \|\nabla^2\mathbf{u}\|_p + \|\nabla\pi\|_p\,.$$

We show that $\mathcal{M}$ maps sufficiently small balls in V_1 into itself and that $\mathcal{M}$ is a contraction in V_1 in the topology of V_0, where

$$\begin{gathered}V_0 = \{(\mathbf{u}, \pi); \mathbf{u} \in L^4(\Omega), \pi, \nabla\mathbf{u} \in L^2(\Omega)\} \\ \|(\mathbf{u}, \pi)\|_{V_0} = \beta^{\frac{1}{4}}\|\mathbf{u}\|_4 + \|\nabla\mathbf{u}\|_2 + \|\pi\|_2.\end{gathered}$$

Theorem III.5.3 yields (recall that $p > 3$, i.e. $W^{2-\frac{1}{p},p}(\partial\Omega) \hookrightarrow W^{\frac{1}{2},2}(\partial\Omega)$)

$$\begin{aligned}\|(\mathbf{u}, \pi)\|_{V_1} &\leq C(|\mathbf{N}|_{-1,2} + \|\mathbf{N}\|_p + \|\mathbf{v}_\infty\|_{2-\frac{1}{p},p,(\partial\Omega)}) \leq \\ &\leq C(|\mathbf{f}|_{-1,2} + \|\mathbf{f}\|_p + \|\mathbf{g}\|_2 + \|\nabla\cdot\mathbf{g}\|_p + \beta)\,.\end{aligned} \tag{1.26}$$

We have to estimate $\|\mathbf{g}\|_2 + \|\nabla\cdot\mathbf{g}\|_p$. We easily get due to (1.2) with $q = 2$ and 4 and (1.3) with $q = 2$

$$\begin{aligned}\|\mathbf{g}\|_2 \leq C\Big(&(\|\mathbf{T}\|_2 + \|p\|_2)(\|\nabla\mathbf{w}\|_2 + \|\nabla^2\mathbf{w}\|_p) + \\ &+\|\nabla\mathbf{w}\|_2(\|\mathbf{w}\|_4^2 + \|\nabla\mathbf{w}\|_2^2 + \|\nabla^2\mathbf{w}\|_p^2) + \\ &+\beta\|\nabla\mathbf{w}\|_2(\|\mathbf{w}\|_4 + \|\nabla\mathbf{w}\|_p) + \|\mathbf{w}\|_4^2 + \beta\|\mathbf{f}\|_2 + \|\mathbf{w}\|_4\|\mathbf{f}\|_{1,p}\Big)\,.\end{aligned} \tag{1.27}$$

Now for $\|\nabla^2\mathbf{w}\|_p + \|\nabla\mathbf{w}\|_2$ small enough ($\|\nabla\mathbf{w}\|_{C^0}$ is bounded by this) we have

$$\begin{gathered}\|\mathbf{T}\|_2 \le C\|\nabla\mathbf{w}\|_2 \\ \|\nabla\mathbf{T}\|_{1,p} \le C\|\nabla\mathbf{w}\|_{1,p} \le C(\|\nabla\mathbf{w}\|_2 + \|\nabla^2\mathbf{w}\|_p) \\ \|p\|_2 \le C\|s\|_2 \\ \|\nabla p\|_p \le C\|\nabla s\|_p\end{gathered} \tag{1.28}$$

(see Theorems IV.2.1 and IV.2.4) and therefore

$$\|\mathbf{g}\|_2 \le C(\beta^{-\frac{1}{2}} + \beta)\|(\mathbf{w},s)\|_{V_1}^2 + (\beta^{-\frac{1}{2}}+1)\|(\mathbf{w},s)\|_{V_1}^3 + \\ +\beta\|\mathbf{f}\|_2 + \beta^{-\frac{1}{4}}\|(\mathbf{w},s)\|_{V_1}\|\mathbf{f}\|_{1,p}\,.$$

Next we estimate

$$\|\nabla\cdot\mathbf{g}\|_p \le (\|\mathbf{T}\|_{1,p} + \|\nabla p\|_p)(\|\nabla^2\mathbf{w}\|_p + \|\nabla\mathbf{w}\|_\infty) + \|\nabla^2\mathbf{w}\|_p\|\mathbf{w}\|_\infty^2 + \\ +\|\nabla\mathbf{w}\|_{1,p}^2(\|\nabla\mathbf{w}\|_\infty + \beta) + \|\nabla\mathbf{w}\|_{1,p}\|\mathbf{w}\|_\infty + (\beta + \|\mathbf{w}\|_\infty)\|\mathbf{f}\|_{1,p}\,.$$

Again, applying (1.28), Lemma 1.1 and standard inequalities we end up with

$$\begin{aligned}\|\nabla\cdot\mathbf{g}\|_p \; &\le C(\beta^{-\frac{1}{2}} + \beta)\|(\mathbf{w},s)\|_{V_1}^2 + (\beta^{-\frac{1}{2}}+1)\|(\mathbf{w},s)\|_{V_1}^3 + \\ &+\beta\|\mathbf{f}\|_{1,p} + \beta^{-\frac{1}{4}}\|(\mathbf{w},s)\|_{V_1}\|\mathbf{f}\|_{1,p}\,.\end{aligned} \tag{1.29}$$

Assuming the norms of $\mathbf{f}$ and β sufficiently small, $\delta = \varepsilon\beta^\alpha$ for ε small and $\alpha \in [\frac{1}{2};1)$ we get as in Theorem 1.2

$$\|(\mathbf{u},\pi)\|_{V_1} \le \delta = \varepsilon\beta^\alpha\,,$$

whenever $\|(\mathbf{w},s)\|_{V_1} \le \delta$.

Now let $(\mathbf{w}^1,s^1)$ and $(\mathbf{w}^2,s^2)$ be two elements of V_1. Denoting $(\mathbf{u}^i,\pi^i)$ the corresponding images of the operator $\mathcal{M}$ and $\mathbf{U}$, Π, $\mathbf{W}$ and S as in Theorem 1.1 we have

$$\|(\mathbf{U},\Pi)\|_{V_0} \le C\|\mathbf{g}(\mathbf{f},\mathbf{T}^1,p^1,\mathbf{w}^1) - \mathbf{g}(\mathbf{f},\mathbf{T}^2,p^2,\mathbf{w}^2)\|_2\,. \tag{1.30}$$

As in (1.23) and (1.24) we have to estimate first $\mathbf{T}^1 - \mathbf{T}^2$ and $p^1 - p^2$, solutions to (1.21) and (1.22). We have for sufficiently small ε and β (i.e. sufficiently small norms of $\mathbf{w}$ and s)

$$\|p^1 - p^2\|_2 \le (\|S\|_2 + \|\mathbf{W}\cdot\nabla p^2\|_2) \le C(\|S\|_2 + \|\nabla p^2\|_p\|\mathbf{W}\|_{\frac{2p}{p-2}})\,.$$

As $\frac{2p}{p-2} \ge 4 \Longleftrightarrow p \le 4$, we have for $3 < p \le 4$

$$\begin{aligned}\|p^1 - p^2\|_2 &\le C(\|S\|_2 + \|\nabla p^2\|_p(\|\mathbf{W}\|_4 + \|\nabla\mathbf{W}\|_p)) \le \\ &\le C(1 + (\beta^{-\frac{1}{4}}+1)\|(\mathbf{w}^2,s^2)\|_{V_1})\|(\mathbf{W},S)\|_{V_0}\,.\end{aligned} \tag{1.31}$$

Analogously, for $3 < p \le 4$

$$\begin{aligned}\|\mathbf{T}^1 - \mathbf{T}^2\|_2 &\le C(\|\nabla\mathbf{W}\|_2 + \|(\mathbf{W}\cdot\nabla)\mathbf{T}^2\|_2 + \|\mathbf{G}(\nabla\mathbf{W},\mathbf{T}^2)\|_2) \le \\ &\le C(1 + (\beta^{-\frac{1}{4}}+1)\|(\mathbf{w}^2,s^2)\|_{V_1})\|(\mathbf{W},S)\|_{V_0}\,.\end{aligned} \tag{1.32}$$

We have

$$\begin{aligned}\|(\mathbf{U},\Pi)\|_{V_0} \le C\Big[&\|\mathbf{T}^1-\mathbf{T}^2\|_2\|\nabla\mathbf{w}^1\|_\infty+\|\nabla\mathbf{W}\|_2\|\mathbf{T}^2\|_\infty+\\ &+\|p^1-p^2\|_2\|\nabla\mathbf{w}^1\|_\infty+\|\nabla\mathbf{W}\|_2\|p^2\|_\infty+\\ &+\|\nabla\mathbf{W}\|_2\|\mathbf{w}^1\|_\infty\|\mathbf{w}^2\|_\infty+\|\mathbf{W}\|_4(\|\mathbf{w}^1\|_4+\|\mathbf{w}^2\|_4)+\\ &+\beta\Big(\|\nabla\mathbf{W}\|_2(\|\mathbf{w}^1\|_\infty+\|\mathbf{w}^2\|_\infty)+\|\mathbf{W}\|_4(\|\nabla\mathbf{w}^1\|_4+\|\nabla\mathbf{w}^2\|_4)\Big)+\\ &+\|\mathbf{W}\|_4(\|(\mathbf{w}^1\cdot\nabla)\mathbf{w}^1\|_4+\|(\mathbf{w}^2\cdot\nabla)\mathbf{w}^2\|_4)+\|\mathbf{f}\|_{1,p}\|\mathbf{W}\|_4\Big]\,.\end{aligned}$$

The estimates (1.31), (1.32) and definitions of the spaces finally yield

$$\begin{aligned}\|(\mathbf{U},\Pi)\|_{V_0} \le C\|(\mathbf{W},S)\|_{V_0}\Big[&\|(\mathbf{w}^i,s^i)\|_{V_1}(\beta^{-\frac12}+\beta)+\\ &+\|(\mathbf{w}^i,s^i)\|_{V_1}^2(\beta^{-\frac12}+1)\Big]\end{aligned} \tag{1.33}$$

and taking ε, β sufficiently small we end up with

$$\|(\mathbf{U},\Pi)\|_{V_0} \le \kappa\|(\mathbf{W},S)\|_{V_0}\,, \quad 0<\kappa<1\,,$$

i.e. the operator $\mathcal{M}$ is a contraction in V_0. Again, Theorem 0.1 finishes the proof.

□

The assumption $\mathbf{f}\in D^{-1,2}(\Omega)$ together with the restriction $p\le 4$ can be replaced by $\mathbf{f}\in W^{1,q}(\Omega)$ for some $1<q<\frac43$. We have in this case

Theorem 1.3 *Let $\Omega\in C^{k+1}$ be exterior domain in $\mathbb{R}^3$.*

(i) Let $\mathbf{f}\in W^{1,q}(\Omega)\cap W^{k,p}(\Omega)$, $k\ge1$, $p\in(3;\infty)$, $q=\frac43$ ($q=\frac65$ if $k=1$).[2] *Let β_0 and $\|\mathbf{f}\|_{1,q}+\|\mathbf{f}\|_{k,p}$ be sufficiently small. Then for all $0<\beta\le\beta_0$ there exists solution $(\mathbf{u},p)$ to the problem (1.6)–(1.8) such that*

$$\begin{gathered}\mathbf{u}\in L^{\frac{2q}{2-q}}(\Omega)\cap D^{1,\frac{4q}{4-q}}(\Omega)\\ \nabla^2\mathbf{u},\nabla p,\nabla\pi\in L^q(\Omega)\cap W^{k-1,p}(\Omega)\\ p,\pi\in L^{\frac{3q}{3-q}}(\Omega)\,.\end{gathered}$$

(ii) Let $\mathbf{f}\in W^{1,\frac43}(\Omega)\cap W^{k,2}(\Omega)$, $k\ge2$. Let β_0 and $\|\mathbf{f}\|_{1,\frac43}+\|\mathbf{f}\|_{k,2}$ be sufficiently small. Then for all $0<\beta\le\beta_0$ there exists solution $(\mathbf{u},p)$ to the problem (1.6)–(1.8) such that

$$\begin{gathered}\mathbf{u}\in L^4(\Omega)\cap D^{1,\frac43}(\Omega)\\ \nabla^2\mathbf{u},\nabla p,\nabla\pi\in L^{\frac43}(\Omega)\cap W^{k-1,2}(\Omega)\\ p,\pi\in L^{\frac{12}{5}}(\Omega)\,.\end{gathered}$$

[2]If $\mathbf{f}\in L^q(\Omega)\cap W^{k,p}(\Omega)$, $q\in(1;\frac65)$ ($q\in(1;\frac43)$), then $\mathbf{f}\in L^q(\Omega)\cap L^\infty(\Omega)$, i.e. $\mathbf{f}\in L^{\frac65}(\Omega)$ ($L^{\frac43}(\Omega)$).

Proof: We show only the most difficult case (i) with $k=1$. Let us denote

$$V_1 = \Big\{(\mathbf{u},\pi); \mathbf{u} \in L^{\frac{2q}{2-q}}(\Omega) \cap D^{1,\frac{4q}{4-q}}(\Omega), \\ \nabla^2\mathbf{u}, \nabla\pi \in L^q(\Omega) \cap L^p(\Omega); 1 < q < \frac{6}{5}, p > 3\Big\}$$

and

$$\|(\mathbf{u},\pi)\|_{V_1} = \beta^{\frac12}\|\mathbf{u}\|_{\frac{2q}{2-q}} + \beta^{\frac14}\|\nabla\mathbf{u}\|_{\frac{4q}{4-q}} + \\ + \|\nabla^2\mathbf{u}\|_p + \|\nabla^2\mathbf{u}\|_q + \|\nabla\pi\|_q + \|\nabla\pi\|_p\,.$$

Let us note that due to the fact that π is generally determined up to an additive constant we may assume due to the imbedding theorem (see Theorem VIII.1.2) that $\pi \in L^{\frac{3q}{3-q}}$ together with the inequality

$$\|\pi\|_{\frac{3q}{3-q}} \le C(q)\|\nabla\pi\|_q\,.$$

As in Theorems 1.1 and 1.2 we define operator $\mathcal{M} : V_1 \mapsto V_1$ and show that $\mathcal{M}$ maps sufficiently small balls into itself. Theorem III.5.3 (ii) yields (for $\beta \le 1$)

$$\|(\mathbf{u},\pi)\|_{V_1} \le C(\|\mathbf{f}\|_q + \|\mathbf{f}\|_p + \|\nabla\cdot\mathbf{g}\|_q + \|\nabla\cdot\mathbf{g}\|_p + \beta)\,.$$

We have for $1 \le q < \frac{12}{7}$, $p > 3$ due to Lemma 1.1 and standard inequalities

$$\|\nabla\cdot\mathbf{g}\|_p \le C\Big[\|\nabla\mathbf{T}\|_p\|\nabla\mathbf{w}\|_\infty + \|\mathbf{T}\|_\infty\|\nabla^2\mathbf{w}\|_p + \|\nabla p\|_p\|\nabla\mathbf{w}\|_\infty + \\ + \|\nabla^2\mathbf{w}\|_p\|\mathbf{w}\|_\infty^2 + (\|\nabla^2\mathbf{w}\|_p^2 + \|\nabla\mathbf{w}\|^2_{\frac{4q}{4-q}})\|\mathbf{w}\|_\infty + \|\mathbf{w}\|_\infty\|\nabla\mathbf{w}\|_p + \\ + \beta\Big((\|\nabla^2\mathbf{w}\|_p\|\nabla\mathbf{w}\|_\infty + \|\nabla^2\mathbf{w}\|_p^2 + \|\nabla\mathbf{w}\|^2_{\frac{4q}{4-q}}\Big) + \\ + \|\nabla\mathbf{f}\|_p(\|\mathbf{w}\|_\infty + \beta)\Big]\,.$$

Using the estimates of the type (see Theorems IV.2.1 and IV.2.4)

$$\|\mathbf{T}\|_{1,p} \le C\|\nabla\mathbf{w}\|_{1,p} \le C(\|\nabla\mathbf{w}\|_{\frac{4q}{4-q}} + \|\nabla^2\mathbf{w}\|_p) \\ \|\nabla p\|_p \le C\|\nabla s\|_p \\ \|\nabla p\|_q \le C\|\nabla s\|_q \\ \|p\|_{\frac{3q}{3-q}} \le C\|s\|_{\frac{3q}{3-q}}$$

we finally get

$$\|\nabla\cdot\mathbf{g}\|_p \le C\Big[\|(\mathbf{w},s)\|^2_{V_1}(\beta^{-\frac34} + \beta) + \|(\mathbf{w},s)\|^3_{V_1}(\beta^{-1} + 1) + \\ + \beta\|\nabla\mathbf{f}\|_p + C(\mathbf{f})(1 + \beta^{-\frac12})\|(\mathbf{w},s)\|_{V_1}\Big]\,.$$

Analogously for $1 < q \le \frac43$ (i.e. $\frac{4q}{4-q} \le 2$)

$$\|\nabla\cdot\mathbf{g}\|_q \le C\Big[\|\nabla\mathbf{T}\|_q\|\nabla\mathbf{w}\|_\infty + \|\mathbf{T}\|_\infty\|\nabla^2\mathbf{w}\|_q + \|\nabla p\|_q\|\nabla\mathbf{w}\|_\infty + \\ + \|\nabla^2\mathbf{w}\|_q\|\mathbf{w}\|_\infty^2 + \|\nabla\mathbf{w}\|_{2q}^2 + \|\mathbf{w}\|_{\frac{2q}{2-q}}\|\nabla\mathbf{w}\|_2 + \\ + \beta\Big(\|\nabla^2\mathbf{w}\|_q\|\mathbf{w}\|_\infty + \|\nabla\mathbf{w}\|_{2q}^2\Big) + \|\nabla\mathbf{f}\|_q(\|\mathbf{w}\|_\infty + \beta)\Big] \le \\ \le C\Big[\|(\mathbf{w},s)\|^2_{V_1}(\beta^{-\frac34} + \beta) + \|(\mathbf{w},s)\|^3_{V_1}(\beta^{-1} + 1) + \\ + \beta\|\nabla\mathbf{f}\|_p + C(\mathbf{f})\beta^{-\frac12}\|(\mathbf{w},s)\|_{V_1}\Big]$$

and therefore

$$\|(\mathbf{u},\pi)\|_{V_1}^2 \le C\Big[\|(\mathbf{w},s)\|_{V_1}^2(\beta^{-\frac{3}{4}}+\beta)+\|(\mathbf{w},s)\|_{V_1}^3(\beta^{-1}+1)+C(\mathbf{f})+\beta\Big]\,.$$

Again, assuming $\|(\mathbf{w},s)\|_{V_1} \le \delta = \varepsilon\beta^\alpha$, $\alpha \in [\frac{3}{4};1)$ we get for β, ε sufficiently small

$$\|(\mathbf{u},\pi)\|_{V_1} \le \varepsilon\beta^\alpha = \delta$$

and $\mathcal{M}$ maps again sufficiently small balls into itself.

The definition of the space on which $\mathcal{M}$ is a contraction, is a bit more complicated now. Let us note that it is not possible to take just a subset of V_1 such that we skip the L^p–norms. But we can take

$$V_0 = \{\mathbf{u} \in L^4(\Omega); \nabla\mathbf{u}, \pi \in L^2(\Omega)\}\,.$$

Taking $(\mathbf{w}^i, s^i) \subset V_0 \cap V_1$, we have

$$\|(\mathbf{U},\Pi)\|_{V_0} \le C\|\mathbf{g}(\mathbf{f},\mathbf{T}^1,p^1,\mathbf{w}^1)-\mathbf{g}(\mathbf{f},\mathbf{T}^2,p^2,\mathbf{w}^2)\|_2\,. \tag{1.34}$$

Moreover we can easily verify that $V_1 \hookrightarrow V_0$. We have[3]

$$\begin{aligned}\|(\mathbf{u},\pi)\|_{V_0} &\le C(\beta)\Big[\|\mathbf{u}\|_{\frac{2q}{2-q}}+\|\mathbf{u}\|_\infty+\|\nabla\mathbf{u}\|_{\frac{4q}{4-q}}+\|\nabla\mathbf{u}\|_\infty+ \\ &+\|\pi\|_{\frac{3q}{3-q}}+\|\pi\|_\infty\Big] \le C(\beta)\Big[\|\mathbf{u}\|_{\frac{2q}{2-q}}+\|\nabla^2\mathbf{u}\|_p+\|\nabla\mathbf{u}\|_{\frac{4q}{4-q}}+ \\ &+\|\nabla\pi\|_p+\|\nabla\pi\|_q\Big] \le C(\beta)\|(\mathbf{u},\pi)\|_{V_1}\,.\end{aligned}$$

We have to estimate $\|(\mathbf{U},\Pi)\|_{V_0}$ by means of (1.34). For $p \le 4$ we can proceed exactly as in the estimate (1.33) and get for ε, β sufficiently small

$$\|(\mathbf{U},\Pi)\|_{V_0} \le \kappa\|(\mathbf{W},S)\|_{V_0}$$

for $0<\kappa<1$ and the operator is contraction in V_0. If $p>4$ we use instead of (1.31)

$$\|\mathbf{W}\nabla p^2\|_2 \le \|\mathbf{W}\|_4\|\nabla p^2\|_4 \le \|\mathbf{W}\|_4\|\nabla p^2\|_q^a\|\nabla p^2\|_p^{1-a}$$

and again verify that $\mathcal{M}$ is a contraction. Analogously we proceed in (i) for $k \ge 2$ and in the case (ii); we shall show that $\mathcal{M}$ is a contraction in V_{k-1},

$$\begin{aligned}V_k = \Big\{&(\mathbf{u},\pi); \mathbf{u} \in L^{\frac{2q}{2-q}}(\Omega)\cap D^{1,\frac{4q}{4-q}}(\Omega), \\ &\nabla^2\mathbf{u}, \nabla\pi \in L^q(\Omega)\cap W^{k-1,p}(\Omega); 1<q<\frac{6}{5}\Big\}\end{aligned}$$

with $p>3$ in the case (i) and $p=2$ in the case (ii).

□

[3]Let us note that exactly here we have to restrict ourselves, because of pressure, on $q \le \frac{6}{5}$.

V.2 Plane flow

This section is devoted to the study of the twodimensional flow. Unlike the threedimensional flow, we do not have any β–independent estimates for $\mathbf{f} \in D^{-1,q}(\Omega)$; we can dispose only with Theorems III.5.1 and III.5.2. As in the previous section, we start with some auxiliary interpolation inequalities.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^2$ be exterior domain of class $C^{0,1}$. Then we have for $1 < q < \frac{6}{5}$, $2 < p < \infty$*

$$\|u\|_\infty \le C\|\nabla u\|_p^\alpha \|u\|_{\frac{3q}{3-2q}}^{1-\alpha} + C(\varepsilon)\|\nabla u\|_p^{\alpha-\varepsilon}\|u\|_{\frac{3q}{3-2q}}^{1-\alpha+\varepsilon}\,, \tag{2.1}$$

where $\alpha = \frac{2p(3-2q)}{6p-6q-pq}$, $0 < \varepsilon \le \alpha$.

Proof: See Theorem VIII.1.13 and Remark VIII.1.10.

□

Let $1 < q < \frac{6}{5}$, $p > 2$, $k \ge 0$. We denote by

$$\begin{aligned} S_{p,q}^k = \Big\{&(\mathbf{u},\pi); u_2 \in L^{\frac{2q}{2-q}}(\Omega), \nabla u_2 \in L^q(\Omega), \mathbf{u} \in L^{\frac{3q}{3-2q}}(\Omega), \\ &\nabla \mathbf{u} \in L^{\frac{3q}{3-q}}(\Omega) \cap L^p(\Omega), \nabla^2\mathbf{u}, \nabla\pi \in W^{1,q}(\Omega) \cap W^{k,p}(\Omega)\Big\}\,. \end{aligned} \tag{2.2}$$

and

$$\langle \mathbf{u}\rangle_{\beta,q} = \beta(\|u_2\|_{\frac{2q}{2-q}} + \|\nabla u_2\|_q) + \beta^{\frac{2}{3}}\|\mathbf{u}\|_{\frac{3q}{3-2q}} + \beta^{\frac{1}{3}}\|\nabla \mathbf{u}\|_{\frac{3q}{3-q}} \tag{2.3}$$

$$[|(\mathbf{u},\pi)|]_k = \beta^{2(1-\frac{1}{q})}(\|\nabla^2\mathbf{u}\|_{1,q} + \|\nabla\pi\|_{1,q} + \|\nabla\mathbf{u}\|_{k+1,p} + \|\nabla\pi\|_{k,p}) \tag{2.4}$$

$$|||(\mathbf{u},\pi)|||_0 = \langle \mathbf{u}\rangle_{\beta,q} + \beta^{2(1-\frac{1}{q})}(\|\nabla^2\mathbf{u}\|_q + \|\nabla\pi\|_q)\,. \tag{2.5}$$

We shall show the following

Theorem 2.1 *Let $\mathbf{f} \in W^{2,q}(\Omega) \cap W^{k,p}(\Omega)$, $1 < q < \frac{6}{5}$, $k \ge 2$ and let $\|\mathbf{f}\|_{2,q} + \|\mathbf{f}\|_{k,p}$ and β be sufficiently small. Let $\Omega \subset \mathbb{R}^2$ be an exterior domain of the class C^{k+1}. Then there exists solution to the problem (1.6)–(1.8) such that $(\mathbf{u},p) \in S_{p,q}^k$.*

Before proving the theorem we first show some more auxiliary estimates.

Lemma 2.2 *Let $\mathbf{v}$, $\mathbf{w}$ be divergence free*[4] *and sufficiently smooth fields. Let $1 < q < \frac{6}{5}$, $2 < p < \infty$, $k \ge 1$. Then the following inequalities hold with constants independent of $\mathbf{v}$, $\mathbf{w}$ and β.*

$$\|(\mathbf{v}\cdot\nabla)\mathbf{w}\|_q \le \beta^{-1-2(1-\frac{1}{q})}\langle \mathbf{v}\rangle_{\beta,q}\langle \mathbf{w}\rangle_{\beta,q} \tag{2.6}$$

$$\|(\mathbf{w}\cdot\nabla)\mathbf{w}\|_{k,p} \le C\Big[\beta^{-\frac{2}{3}-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle_{\beta,q}[|(\mathbf{w},\cdot)|]_0 + \beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_{k-1}^2\Big] \tag{2.7}$$

$$\|\nabla^2\mathbf{w}|\mathbf{w}|^2\|_{k,p} \le C\beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_k\Big[\beta^{-\frac{4}{3}}\langle \mathbf{w}\rangle_{\beta,q}^2 + \beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_k^2\Big] \tag{2.8}$$

[4]it is fundamental only for the inequality (2.6)

$$|\nabla^2\mathbf{w}|\mathbf{w}|^2|_{1,q} \le C\beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_1\Big[\beta^{-\frac{4}{3}}\langle\mathbf{w}\rangle^2_{\beta,q} + \beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0^2\Big] \tag{2.9}$$

$$\|\mathbf{w}\nabla^2\mathbf{w}\|_{k,p} \le C\beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_k\Big[\beta^{-\frac{2}{3}}\langle\mathbf{w}\rangle_{\beta,q} + \beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_{k-1}\Big] \tag{2.10}$$

$$|\mathbf{w}\nabla^2\mathbf{w}|_{1,q} \le C\beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_1\Big[\beta^{-\frac{2}{3}}\langle\mathbf{w}\rangle_{\beta,q} + \beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_1\Big] \tag{2.11}$$

$$\||\nabla\mathbf{w}|^2\|_{k-1,p} \le C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_k^2 \tag{2.12}$$

$$\|\nabla\mathbf{w}\nabla\mathbf{w}\|_q \le C\beta^{-6(1-\frac{1}{q})\frac{2-q}{q}}\langle\mathbf{w}\rangle_{\beta,q}^{6(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0^{\frac{6-4q}{q}} \tag{2.13}$$

$$\|\nabla^2\mathbf{v}|\mathbf{w}|^2\|_q \le C\beta^{-2(1-\frac{1}{q})}\||(\mathbf{v},\cdot)|\|_0\Big[\beta^{-\frac{4}{3}}\langle\mathbf{w}\rangle^2_{\beta,q} + \beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0^2\Big] \tag{2.14}$$

$$\begin{aligned}\|\nabla^2\mathbf{w}\,\mathbf{v}\,\mathbf{w}\|_q \le C\beta^{-\frac{2}{3}}\langle\mathbf{v}\rangle_{\beta,q}\Big[&\beta^{-\frac{2}{3}-2(1-\frac{1}{q})}\langle\mathbf{w}\rangle_{\beta,q}[|(\mathbf{w},\cdot)|]_0 + \\ &+\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0^2\Big]\end{aligned} \tag{2.15}$$

$$\|\mathbf{v}\nabla^2\mathbf{w}\|_q \le C\beta^{-\frac{2}{3}}\langle\mathbf{v}\rangle_{\beta,q}\Big[\beta^{-\frac{1}{3}}\langle\mathbf{w}\rangle_{\beta,q} + \beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0\Big] \tag{2.16}$$

$$\|\mathbf{w}\nabla^2\mathbf{v}\|_q \le C\beta^{-2(1-\frac{1}{q})}\||(\mathbf{v},\cdot)|\|_0\Big[\beta^{-\frac{2}{3}}\langle\mathbf{w}\rangle_{\beta,q} + \beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0\Big] \tag{2.17}$$

$$\|\nabla\mathbf{v}\nabla\mathbf{w}\|_q \le C\beta^{-\frac{1}{3}}\langle\mathbf{v}\rangle_{\beta,q}\Big[\beta^{-\frac{1}{3}}\langle\mathbf{w}\rangle_{\beta,q} + \beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_1\Big] \tag{2.18}$$

$$\|\nabla\mathbf{f}\,\mathbf{w}\|_q \le C\beta^{-\frac{2}{3}}\langle\mathbf{w}\rangle_{\beta,q}\Big[\|\nabla\mathbf{f}\|_q + \|\nabla\mathbf{f}\|_p\Big] \tag{2.19}$$

$$\|\nabla\mathbf{f}\,\mathbf{w}\|_{k,p} \le C\|\mathbf{f}\|_{k+1,p}\Big[\beta^{-\frac{2}{3}}\langle\mathbf{w}\rangle_{\beta,q} + \beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_{\max\{1,k-2\}}\Big] \tag{2.20}$$

$$|\nabla\mathbf{f}\,\mathbf{w}|_{1,q} \le C\|\mathbf{f}\|_{1,q}\Big[\beta^{-\frac{2}{3}}\langle\mathbf{w}\rangle_{\beta,q} + \beta^{-2(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0\Big]\,. \tag{2.21}$$

Proof: The first inequality is classical, nevertheless, we repeat the proof (see also [Ga2]). We have

$$(\mathbf{v}\cdot\nabla)\mathbf{w} = \Big(-v_1\frac{\partial w_2}{\partial x_2} + v_2\frac{\partial w_1}{\partial x_2}, v_1\frac{\partial w_2}{\partial x_1} + v_2\frac{\partial w_2}{\partial x_2}\Big)\,,$$

where we used that $\nabla\cdot\mathbf{w} = 0$. So we have

$$\begin{aligned}\|(\mathbf{v}\cdot\nabla)\mathbf{w}\|_q &\le \|v_1\|_{\frac{3q}{3-2q}}\|\nabla w_2\|_{\frac{3}{2}} + \|v_2\|_3\|\nabla\mathbf{w}\|_{\frac{3q}{3-q}} \le \\ &\le \|v_1\|_{\frac{3q}{3-2q}}\|\nabla w_2\|_q^{3(1-\frac{1}{q})}\|\nabla w_2\|_{\frac{3q}{3-q}}^{\frac{3}{q}-2} + \|v_2\|_{\frac{2q}{2-q}}^{6(1-\frac{1}{q})}\|v_2\|_{\frac{3q}{3-2q}}^{\frac{6}{q}-5}\|\nabla\mathbf{w}\|_{\frac{3q}{3-q}} \le \\ &\le \beta^{-1-2(1-\frac{1}{q})}\langle\mathbf{v}\rangle_{\beta,q}\langle\mathbf{w}\rangle_{\beta,q}\,.\end{aligned}$$

The other inequalities are easier and follow from Lemma 2.1, imbedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ and the following inequalities

$$\|\mathbf{w}\cdot\nabla\mathbf{w}\|_{k,p} \le \|\nabla\mathbf{w}\|_{k,p}\|\mathbf{w}\|_{k,\infty}$$
$$\|\nabla^2\mathbf{w}|\mathbf{w}|^2\|_{k,p} \le \|\nabla^2\mathbf{w}\|_{k,p}\|\mathbf{w}\|_{k,\infty}^2$$
$$|\nabla^2\mathbf{w}|\mathbf{w}|^2|_{1,q} \le \|\nabla^3\mathbf{w}\|_q\|\mathbf{w}\|_\infty^2 + \|\nabla^2\mathbf{w}\|_q\|\mathbf{w}\|_\infty\|\nabla\mathbf{w}\|_\infty$$
$$\|\mathbf{w}\nabla^2\mathbf{w}\|_{k,p} \le \|\nabla^2\mathbf{w}\|_{k,p}\|\mathbf{w}\|_{k,\infty}$$
$$|\mathbf{w}\nabla^2\mathbf{w}|_{1,q} \le \|\nabla^3\mathbf{w}\|_q\|\mathbf{w}\|_\infty + \|\nabla^2\mathbf{w}\|_q\|\nabla\mathbf{w}\|_\infty$$
$$\||\nabla\mathbf{w}|^2\|_{k,p} \le \|\nabla\mathbf{w}\|_{k,p}\|\nabla\mathbf{w}\|_{k-1,\infty}$$
$$\|\nabla^2\mathbf{v}|\mathbf{w}|^2\|_q \le \|\nabla^2\mathbf{v}\|_q\|\mathbf{w}\|_\infty^2$$
$$\|\nabla^2\mathbf{w}\,\mathbf{v}\,\mathbf{w}\|_q \le \|\mathbf{v}\|_{\frac{3q}{3-2q}}\|\nabla^2\mathbf{w}\|_{\frac{3}{2}}\|\mathbf{w}\|_\infty \le \|\mathbf{v}\|_{\frac{3q}{3-2q}}(\|\nabla^2\mathbf{w}\|_q + \|\nabla^2\mathbf{w}\|_p)\|\mathbf{w}\|_\infty$$
$$\|\mathbf{v}\cdot\nabla^2\mathbf{w}\|_q \le \|\mathbf{v}\|_{\frac{3q}{3-2q}}\|\nabla^2\mathbf{w}\|_{\frac{3}{2}} \le \|\mathbf{v}\|_{\frac{3q}{3-2q}}(\|\nabla^2\mathbf{w}\|_p + \|\nabla^2\mathbf{w}\|_q)$$
$$\|\mathbf{w}\cdot\nabla^2\mathbf{v}\|_q \le \|\nabla^2\mathbf{v}\|_q\|\mathbf{w}\|_\infty$$
$$\|\nabla\mathbf{v}\nabla\mathbf{w}\|_q \le \|\nabla\mathbf{v}\|_{\frac{3q}{3-q}}\|\nabla\mathbf{w}\|_3 \le \|\nabla\mathbf{v}\|_{\frac{3q}{3-q}}(\|\nabla^2\mathbf{w}\|_p + \|\nabla\mathbf{w}\|_{\frac{3q}{3-q}})$$
$$\|\nabla\mathbf{f}\,\mathbf{v}\|_q \le \|\mathbf{v}\|_{\frac{3q}{3-2q}}\|\nabla\mathbf{f}\|_{\frac{3}{2}} \le \|\mathbf{v}\|_{\frac{3q}{3-2q}}(\|\nabla\mathbf{f}\|_q + \|\nabla\mathbf{f}\|_p)$$
$$\|\nabla\mathbf{f}\,\mathbf{w}\|_{k-1,p} \le \|\nabla\mathbf{f}\|_{k-1,p}\|\mathbf{w}\|_{k-1,\infty}$$
$$|\nabla\mathbf{f}\,\mathbf{w}|_{1,q} \le \|\nabla^2\mathbf{f}\|_q\|\mathbf{w}\|_\infty + \|\nabla\mathbf{f}\|_q\|\nabla\mathbf{w}\|_\infty\,.$$

□

Theorems IV.2.1, IV.2.2 and IV.2.4 imply the following estimates

Corollary 2.1 *Let* $\mathbf{T}$, p *solves (1.10) and (1.11), respectively. Let* $2 < p < \infty$, $1 < q < 2$, $k \ge 1$.

(i) Let $\|\nabla\mathbf{w}\|_{k,p}$ *be sufficiently small. Then*

$$\|\mathbf{T}\|_{k,p} \le C\|\nabla\mathbf{w}\|_{k,p} \tag{2.22}$$

$$\|\nabla p\|_{k-1,p} \le C\|\nabla s\|_{k-1,p}\,. \tag{2.23}$$

(ii) Let $\|\nabla\mathbf{w}\|_{C^1}$ *be sufficiently small. Then*

$$\|\mathbf{T}\|_{\frac{3q}{3-q}} \le C\|\nabla\mathbf{w}\|_{\frac{3q}{3-q}} \tag{2.24}$$

$$\|\nabla\mathbf{T}\|_{1,q} \le C\|\nabla^2\mathbf{w}\|_{1,q} \tag{2.25}$$

$$\|\nabla p\|_{1,q} \le C\|\nabla s\|_{1,q}\,. \tag{2.26}$$

Proof: To show (i), we have to verify that $\|\nabla\mathbf{w}\|_{C^{k-1}} + \|\nabla\mathbf{w}\|_{k,p}$ are sufficiently small. But $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and so $\|\nabla\mathbf{w}\|_{C^{k-1}} \le C\|\nabla\mathbf{w}\|_{k,p}$. The estimates (2.22)–(2.23) follow directly from the above mentioned Theorems. Let us only emphasize that for $q < 2$ we can apply Theorem IV.2.4 and get (2.25) while for $p > 2$ we only have (2.22).

□

Lemma 2.3 *We have for* $\mathbf{w}$*,* s *satisfying the assumptions of Corollary 2.1*

$$\|\nabla p \nabla \mathbf{w}\|_q \leq C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},s)|]_0 |||(\mathbf{w},s)|||_0 \tag{2.27}$$

$$\|\nabla p \nabla \mathbf{w}\|_q \leq C\beta^{-\frac{1}{3}-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle_{\beta,q} |||(\mathbf{w},s)|||_0 \tag{2.28}$$

$$\|\nabla p \nabla \mathbf{w}\|_{k-1,p} \leq C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},s)|]_k^2 \tag{2.29}$$

$$|\nabla p \nabla \mathbf{w}|_{1,q} \leq C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},s)|]_k |||(\mathbf{w},s)|||_0 \tag{2.30}$$

$$\|\nabla \mathbf{T} \nabla \mathbf{w}\|_q \leq C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_0 |||(\mathbf{w},\cdot)|||_0 \tag{2.31}$$

$$\|\nabla \mathbf{T} \nabla \mathbf{w}\|_q \leq C\beta^{-\frac{1}{3}-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle_{\beta,q} |||(\mathbf{w},\cdot)|||_0 \tag{2.32}$$

$$\|\nabla \mathbf{T} \nabla \mathbf{w}\|_{k,p} \leq C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_k^2 \tag{2.33}$$

$$|\nabla \mathbf{T} \nabla \mathbf{w}|_{1,q} \leq C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},\cdot)|]_1^2 \tag{2.34}$$

$$\|\mathbf{T} \nabla \mathbf{w}\|_q \leq C\beta^{-\frac{1}{3}-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle_{\beta,q}[|(\mathbf{w},\cdot)|]_0 \tag{2.35}$$

$$|\mathbf{T} \nabla \mathbf{w}|_{1,q} \leq C\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},s)|]_0^2\,. \tag{2.36}$$

Proof: The inequalities (2.27)–(2.36) are easy consequences of Corollary 2.1 and the estimates

$$\|\nabla p \nabla \mathbf{w}\|_q \leq \begin{cases} C\|\nabla p\|_q \|\nabla \mathbf{w}\|_\infty \\ C\|\nabla p\|_3 \|\nabla \mathbf{w}\|_{\frac{3q}{3-q}} \end{cases}$$
$$\|\nabla p \nabla \mathbf{w}\|_{k,p} \leq C\|\nabla p\|_{k,p}\|\nabla \mathbf{w}\|_{k,\infty}$$
$$|\nabla p \nabla \mathbf{w}|_{1,q} \leq C(\|\nabla^2 p\|_q \|\nabla \mathbf{w}\|_\infty + \|\nabla p\|_q \|\nabla^2 \mathbf{w}\|_\infty)\,,$$

analogously for $\mathbf{T}$ and $\nabla \mathbf{T}$.

□

We can now start to prove Theorem 2.1. Unlike the threedimensional case, the balls in $S^k_{p,q}$ must be chosen very carefully.

Proof of Theorem 2.1: As for the threedimensional flow we define operator $\mathcal{M} : (\mathbf{w}, s) \to (\mathbf{u}, \pi)$, where $(\mathbf{u}, \pi)$ solves (1.9)–(1.12). We show that $\mathcal{M}$ maps for ε, β sufficiently small and $\alpha \in [\frac{2}{3}; 1)$ the "ball"

$$B^{\varepsilon,\beta} = \Big\{(\mathbf{w}, s) \in S^k_{p,q};\, \langle \mathbf{w}\rangle_{\beta,q} \leq \varepsilon\beta^{2(1-\frac{1}{q})+1}\,, [|\mathbf{w}, s|]_k \leq \beta^{2(1-\frac{1}{q})+\alpha}\Big\}$$

into itself end that $\mathcal{M}$ is a contraction in the norm $|||(\cdot,\cdot)|||_0$. Applying Lemmas 2.2 and 2.3 together with Theorem III.5.1 we have

$$\begin{aligned}\langle \mathbf{u}\rangle_{\beta,q} \leq C\Big[&\beta^{1+2(1-\frac{1}{q})}|\ln\beta|^{-1} + \|\mathbf{f}\|_{1,q}(1+\beta) + \beta^{-1-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle^2_{\beta,q} + \\ &+ \beta^{-\frac{4}{3}-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle^2_{\beta,q}[|(\mathbf{w},s)|]_1 + \beta^{-\frac{2}{3}-4(1-\frac{1}{q})}\langle \mathbf{w}\rangle_{\beta,q}[|(\mathbf{w},s)|]_1^2 + \\ &+ \beta^{\frac{1}{3}-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle_{\beta,q}[|(\mathbf{w},s)|]_1 + \beta^{-\frac{2}{3}}\langle \mathbf{w}\rangle_{\beta,q}(\|\nabla \mathbf{f}\|_p + \|\nabla \mathbf{f}\|_q) + \\ &+ \beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},s)|]_1^2\Big] \leq \varepsilon\beta^{2(1-\frac{1}{q})+1}\end{aligned} \tag{2.37}$$

if ε, β are sufficiently small and $(\mathbf{w}, s) \in B^{\varepsilon,\beta}$.

Next, Theorem III.5.2 reads

$$[|\mathbf{u}, \pi|]_k \leq C\Big[\|\mathbf{f}\|_{1,q} + \|\nabla \cdot \mathbf{g}\|_{1,q} + \beta^{2(1-\frac{1}{q})}(\|\mathbf{f}\|_{k,p} + \|\nabla \cdot \mathbf{g}\|_{k,p} + \beta)\Big]. \quad (2.38)$$

Again, Lemmas 2.2 and 2.3 employed in (2.38) yield

$$[|\mathbf{u}, \pi|]_k \leq C\Big[c(\mathbf{f}) + \beta^{1+2(1-\frac{1}{q})} + \beta^{-1-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle^2_{\beta,q} + \\ +\beta^{-\frac{2}{3}-2(1-\frac{1}{q})}\langle \mathbf{w}\rangle^2_{\beta,q}[|(\mathbf{w}, s)|]_k + \beta^{-4(1-\frac{1}{q})}[|(\mathbf{w}, s)|]^2_k\Big] \leq \varepsilon\beta^{2(1-\frac{1}{q})+\alpha}$$

and $\mathcal{M}$ maps $B^{\varepsilon,\beta}$ into itself.

Next we show that $\mathcal{M}$ is a contraction in $B^{\varepsilon,\beta}$ in the norm $|||(\cdot,\cdot)|||_0$. Let $\mathbf{w}^i$, s^i, $i = 1, 2$, belong to $B^{\varepsilon,\beta}$ and $\mathbf{u}^i, \pi^i$ be the corresponding solutions to (1.9)–(1.12). Denoting $\mathbf{U} = \mathbf{u}^1 - \mathbf{u}^2$, $\Pi = \pi^1 - \pi^2$, $\mathbf{W} = \mathbf{w}^1 - \mathbf{w}^2$ and $S = s^1 - s^2$ we have that $(\mathbf{U}, \Pi)$ solves (1.20)–(1.22) and Theorems III.5.1, III.5.2 imply

$$|||(\mathbf{U}, \Pi)|||_0 \leq C(1+\beta^{2(1-\frac{1}{q})})\|\nabla\cdot\mathbf{g}(\mathbf{f}, \mathbf{T}^1, p^1, \mathbf{w}^1) - \nabla\cdot\mathbf{g}(\mathbf{f}, \mathbf{T}^2, p^2, \mathbf{w}^2)\|_q. \quad (2.39)$$

We first estimate $\mathbf{T}^1 - \mathbf{T}^2$ and $p^1 - p^2$ solving (1.22) and (1.21), respectively.

$$\|\nabla(p^1 - p^2)\|_q \leq C\Big[\|\nabla S\|_q + \|\nabla\mathbf{W}\nabla p^2\|_q + \|\mathbf{W}\nabla^2 p^2\|_q\Big] \leq \\ \leq C\Big[\|\nabla S\|_q + \|\nabla\mathbf{W}\|_{\frac{3q}{3-q}}\|\nabla p^2\|_3 + \|\mathbf{W}\|_{\frac{3q}{3-2q}}\|\nabla^2 p^2\|_{\frac{3}{2}}\Big] \leq \\ \leq C|||(\mathbf{W}, S)|||_0\Big[\beta^{-2(1-\frac{1}{q})} + (\beta^{-\frac{1}{3}-2(1-\frac{1}{q})} + \beta^{-\frac{2}{3}-2(1-\frac{1}{q})})[|\mathbf{w}^2, s^2|]_k\Big]. \quad (2.40)$$

Completely analogously

$$\|\nabla(\mathbf{T}^1 - \mathbf{T}^2)\|_q \leq \\ \leq C|||(\mathbf{W}, S)|||_0\Big[\beta^{-2(1-\frac{1}{q})} + (\beta^{-\frac{1}{3}-2(1-\frac{1}{q})} + \beta^{-\frac{2}{3}-2(1-\frac{1}{q})})[|\mathbf{w}^2, s^2|]_k\Big]. \quad (2.41)$$

Moreover, we have also

$$\|\mathbf{T}^1 - \mathbf{T}^2\|_{\frac{3q}{3-q}} \leq C\Big[\|\nabla\mathbf{W}\|_{\frac{3q}{3-q}} + \|\mathbf{W}\nabla\mathbf{T}^2\|_{\frac{3q}{3-q}}\Big] \leq \\ \leq C\Big[\|\nabla\mathbf{W}\|_{\frac{3q}{3-q}} + \|\mathbf{W}\|_{\frac{3q}{3-2q}}\|\nabla\mathbf{T}^2\|_{3-q}\Big] \leq \\ \leq C|||(\mathbf{W}, s)|||_0\Big[\beta^{-\frac{1}{3}} + \beta^{-\frac{2}{3}-2(1-\frac{1}{q})}[|\mathbf{w}^2, s^2|]_0\Big]. \quad (2.42)$$

Now, from Lemmas 2.2 and 2.3 we can estimate (2.39) as follows

$$|||(\mathbf{U}, \Pi)|||_0 \leq C\Big[\beta^{-2(1-\frac{1}{q})-1}|||(\mathbf{W}, S)|||_0(\langle\mathbf{w}^1\rangle_{\beta,q} + \langle\mathbf{w}^2\rangle_{\beta,q}) + \\ +\beta^{-2(1-\frac{1}{q})-\frac{4}{3}}|||(\mathbf{W}, S)|||_0(\langle\mathbf{w}^1\rangle_{\beta,q} + \langle\mathbf{w}^2\rangle_{\beta,q})^2 + \\ +\beta^{-2(1-\frac{1}{q})-\frac{2}{3}}|||(\mathbf{W}, S)|||_0(\langle\mathbf{w}^1\rangle_{\beta,q} + \langle\mathbf{w}^2\rangle_{\beta,q}) + \\ +\beta^{-\frac{2}{3}}|||(\mathbf{W}, S)|||_0(\|\nabla\mathbf{f}\|_q + \|\nabla\mathbf{f}\|_p) + \|\nabla(p^1 - p^2)\|_q\|\nabla\mathbf{w}^1\|_\infty + \\ +\|\nabla\mathbf{W}\|_{\frac{3q}{3-2q}}\|\nabla p^2\|_{\frac{3}{2}} + \|\nabla(\mathbf{T}^1 - \mathbf{T}^2)\|_q\|\nabla\mathbf{w}^1\|_\infty + \\ +\|\nabla\mathbf{W}\|_{\frac{3q}{3-2q}}\|\nabla\mathbf{T}^2\|_{\frac{3}{2}} + \|\mathbf{T}^1 - \mathbf{T}^2\|_{\frac{3q}{3-q}}\|\nabla^2\mathbf{w}^1\|_3 + \|\nabla^2\mathbf{W}\|_q\|\mathbf{T}^2\|_\infty\Big].$$

The most restrictive term,

$$\beta^{-2(1-\frac{1}{q})-1}|||(\mathbf{W},S)|||_0(\langle \mathbf{w}^1\rangle_{\beta,q}+\langle \mathbf{w}^2\rangle_{\beta,q}),$$

comes from the convective term. Using (2.40)–(2.42) we finally have ε, β sufficiently small

$$|||(\mathbf{U},\Pi)|||_0 \le C\Big(\varepsilon + o(\beta)\Big)|||(\mathbf{W},S)|||_0 \le \kappa |||(\mathbf{W},S)|||_0$$

with $\kappa \in (0;1)$. The operator $\mathcal{M}$ is a contraction in $S_k^{p,q}$ in the norm $|||(\cdot,\cdot)|||_0$ and Theorem 0.1 finishes the proof.

□

Remark 2.1 Using the same procedure as above in two and three dimensions, we could establish the existence of solutions to the problem (I.4.20) under the assumption that η^P, β and certain norms of $\mathbf{f}$ are sufficiently small. But due to the presence of linear terms on the right hand side we are not able to control the asymptotic behaviour of such solutions and therefore we shall not study this model.

V.3 Plane flow of second grade fluid

As already mentioned in Chapter I, our technique does not allow to show the asymptotic structure of velocity and pressure field of second grade fluid flowing past an obstacle. Nevertheless, we show at least existence of solution to the problem (I.4.27) in Sobolev spaces in two dimensions. The proof is taken from [Po]. See also [Vi], where more restrictive assumptions on $\mathbf{f}$ are used.

We start form the reformulation (I.4.28)–(I.4.29) and prove

Theorem 3.1 *Let $k \ge 1$, $q \in (1;\frac{6}{5})$ and $\|\mathbf{f}\|_{k,q}$ be sufficiently small. Then there exists β_0 such that for all $\beta \in (0;\beta_0]$ there exists at least one strong solution to (I.4.28)–(I.4.29). Moreover, $\nabla^2\mathbf{v} \in W^{k,q}(\Omega)$ $\nabla\mathbf{v} \in L^{\frac{3q}{3-q}}(\Omega)$ and $\mathbf{v}-\mathbf{v}_\infty \in L^{\frac{3q}{3-2q}}(\Omega)$, $\nabla p \in W^{k,q}(\Omega)$.*

We denote

$$[(\mathbf{u},s)]_k = \beta^{2(1-\frac{1}{q})}(\|\nabla^2\mathbf{u}\|_{k,q}+\|\nabla s\|_{k,q}) \tag{3.1}$$

and $\langle \mathbf{u}\rangle_{\beta,q}$ will be as in (2.3). As in the preceding sections we first need some auxiliary lemmas.

Lemma 3.1 *Let $\mathbf{u}$ has finite norms $\langle\cdot\rangle_{\beta,q}$ and $[(\cdot,\cdot)]_0$. Let $\mathbf{u}=-(\beta,0)$ on $\partial\Omega$. Then*[5]

$$\|\mathbf{u}\|_\infty \le C([(\mathbf{u},\cdot)]_0\beta^{-2(1-\frac{1}{q})})^{\frac{3-2q}{q}}\Big[(\langle\mathbf{u}\rangle_{\beta,q}\beta^{-\frac{2}{3}})^{\frac{3(q-1)}{q}}+\beta^{\frac{3(q-1)}{q}}\Big]. \tag{3.2}$$

[5] we could also use Lemma 2.1

Proof: We denote by $\mathbf{w}$ the function which is equal to $\mathbf{u}$ inside of Ω and $-(\beta,0)$ outside of Ω. The function $\mathbf{w}$ belongs to $W^{1,q}(\mathbb{R}^2)$ and the interpolation inequality from Theorem VIII.1.12 gives us

$$\|\mathbf{w}\|_\infty \le C\|D\mathbf{w}\|^a_{s,\mathbb{R}^2}\|\mathbf{w}\|^{1-a}_{r,\mathbb{R}^2}\,,$$

where $0 = a(\frac{1}{s}-\frac{1}{2}) + (1-a)\frac{1}{r}$. We put $r = \frac{3q}{3-2q}$ and $s = \frac{2q}{2-q}$; so $a = \frac{3-2q}{q}$. As $\mathbf{w} = \mathbf{u}$ on Ω and $\nabla\mathbf{w} = \mathbf{0}$ outside of Ω, we have

$$\|\mathbf{u}\|_\infty \le C\Big[\|\mathbf{u}\|^{\frac{3(q-1)}{q}}_{\frac{3q}{3-2q},\Omega} + \beta^{\frac{3(q-1)}{q}}\Big]\|\nabla\mathbf{u}\|^{\frac{3-2q}{q}}_{\frac{2q}{2-q},\Omega}\,.$$

The inequality (3.2) follows from the definitions of the norms.

□

We next estimate the quadratic terms on the right hand side of (I.4.29).

Lemma 3.2 *Let* $\mathbf{u}$ *be sufficiently smooth. Then we have the following estimates*[6] *with* C *independent of* $\mathbf{u}$ *and* β

$$\begin{aligned}
\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_q &\le \langle\mathbf{u}\rangle^2_{\beta,q}\beta^{-1-2(1-\frac{1}{q})}\,,\\
|(\mathbf{u}\cdot\nabla)\mathbf{u}|_{k,q} &\le C[(\mathbf{u},\cdot)]_0^{\frac{3-2q}{q}}[(\mathbf{u},\cdot)]_{k-1}\beta^{-2(1-\frac{1}{q})(\frac{3-q}{q})}\cdot\\
&\quad\cdot\;[\langle\mathbf{u}\rangle^{3(1-\frac{1}{q})}_{\beta,q}\beta^{-2(1-\frac{1}{q})} + \beta^{3(1-\frac{1}{q})}]\,,\quad k\ge 1\\
\|\mathbf{u}\nabla^k\mathbf{u}\|_q &\le C[(\mathbf{u},\cdot)]_0^{\frac{3-2q}{q}}[(\mathbf{u},\cdot)]_{k-2}\beta^{-2(1-\frac{1}{q})\frac{3-q}{q}}\cdot\\
&\quad\cdot\;[\langle\mathbf{u}\rangle^{3(1-\frac{1}{q})}_{\beta,q}\beta^{-2(1-\frac{1}{q})} + \beta^{3(1-\frac{1}{q})}]\,,\quad k\ge 2\\
|\mathbf{u}\nabla^2\mathbf{u}|_{k,q} &\le C[(\mathbf{u},\cdot)]_0^{\frac{3-2q}{q}}[(\mathbf{u},\cdot)]_{k}\beta^{-2(1-\frac{1}{q})\frac{3-q}{q}}\cdot\\
\cdot[\langle\mathbf{u}\rangle^{3(1-\frac{1}{q})}_{\beta,q}&\beta^{-2(1-\frac{1}{q})} + \beta^{3(1-\frac{1}{q})}] + C[(\mathbf{u},\cdot)]^2_{k-1}\beta^{-4(1-\frac{1}{q})}\,,\quad k\ge 1\\
\|\nabla\mathbf{u}\|^2_{2q} &\le C\langle\mathbf{u}\rangle^{6(1-\frac{1}{q})}_{\beta,q}[(\mathbf{u},\cdot)]_0^{\frac{6-4q}{q}}\beta^{-6(1-\frac{1}{q})\frac{2-q}{q}}\\
\|\nabla\mathbf{u}\nabla^k\mathbf{u}\|_q &\le C[(\mathbf{u},\cdot)]_1[(\mathbf{u},\cdot)]_{k-2}\beta^{-4(1-\frac{1}{q})}\,,\quad k\ge 2\\
|\nabla\mathbf{u}\nabla^2\mathbf{u}|_{k,q} &\le C[(\mathbf{u},\cdot)]^2_k\beta^{-4(1-\frac{1}{q})}\,,\quad k\ge 1\\
\|\nabla^2\mathbf{u}\|^2_{2q} &\le C[(\mathbf{u},\cdot)]^2_1\beta^{-4(1-\frac{1}{q})}\\
||\nabla\mathbf{u}|^2|_{k,q} &\le C[(\mathbf{u},\cdot)]^2_{k-1}\beta^{-4(1-\frac{1}{q})}\\
\|\nabla^k s\nabla\mathbf{u}\|_q &\le C[(\mathbf{u},s)]_1[(\mathbf{u},s)]_{k-1}\beta^{-4(1-\frac{1}{q})}\,,\quad k\ge 1\\
\|\nabla s\nabla^k\mathbf{u}\|_q &\le C[(\mathbf{u},s)]_1[(\mathbf{u},s)]_{k-1}\beta^{-4(1-\frac{1}{q})}\,,\quad k\ge 2\\
|\nabla s\nabla\mathbf{u}|_{k,q} &\le C[(\mathbf{u},s)]^2_k\beta^{-4(1-\frac{1}{q})}\,,\quad k\ge 1\,.
\end{aligned}\tag{3.3}$$

[6] Some of the inequalities were already shown in Lemma 2.2. Nevertheless, we shall repeat them as some of them differ due to the use of Lemma 3.1.

Proof: The first inequality has already been shown in Lemma 2.2; see also [Ga2]. The other ones are easy consequences of Lemma 3.1, standard imbedding and interpolation inequalities and definitions of the norms.

□

We are in a position to show that the operator $\mathcal{M}$ maps sufficiently small balls in $W^{1,q}(\Omega)$ into themselves.

Lemma 3.3 *Let $\|\mathbf{f}\|_{k,q}$ and β be sufficiently small. Then there exists $\delta(\beta) > 0$ such that the operator $\mathcal{M}$ maps $B_\delta = \{\mathbf{g} \in W^{1,q}(\Omega); \|\mathbf{g}\|_{1,q} \le \delta\}$ into itself.*

Proof: Let us take $\mathbf{g} \in W^{1,q}(\Omega)$, $1 < q < \frac{6}{5}$, $\|\mathbf{g}\|_{1,q} \le \delta$ small enough (will be precised later). We solve (I.4.28) and use the estimates from Theorems III.5.1 and III.5.2. Now, let us assume (will be demonstrated below) that $\|\nabla \mathbf{u}\|_{C^0}$ is small enough. Let $\mathbf{z}$ be solution of (I.4.29) with the right hand side depending on $(\mathbf{u}, s)$. Then

$$\|\mathbf{z}\|_{1,q} \le C\|\mathbf{F}(\mathbf{u}, s)\|_{1,q} \,.$$

We need therefore to assure the smallness of $\|\nabla \mathbf{u}\|_{C^0}$ and to estimate $\mathbf{F}(\mathbf{u}, s)$ by means of the norms (2.3) and (3.1). Easily

$$\|\nabla \mathbf{u}\|_{C^0} \le C(\|\nabla \mathbf{u}\|_{\frac{3q}{3-q}} + \|\nabla^2 \mathbf{u}\|_{1,q}) \le C(\beta)\|\mathbf{g}\|_{1,q} \tag{3.4}$$

and for δ sufficiently small, $\|\nabla \mathbf{u}\|_{C^0}$ is small. Now from (I.4.29) we see that

$$\begin{aligned}\|\mathbf{F}(\mathbf{u}, s)\|_{1,q} \le\, & C(\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{1,q} + \|D\mathbf{u}D^2\mathbf{u}\|_{1,q} + \beta\|\mathbf{u}D^2\mathbf{u}\|_{1,q} + \\ & + \|DsD\mathbf{u}\|_{1,q} + \|\mathbf{f}\|_{1,q} + \beta^2\|D^2\mathbf{u}\|_{1,q})\,.\end{aligned}$$

Lemma 3.2 reads

$$\begin{aligned}\|\mathbf{F}\|_{1,q} \le\, & C\Big\{\langle \mathbf{u}\rangle^2_{\beta,q}\beta^{-1-2(1-\frac{1}{q})} + \langle \mathbf{u}\rangle^{6(1-\frac{1}{q})}_{\beta,q}[(\mathbf{u}, s)]_0^{\frac{6-4q}{q}}\beta^{-6(1-\frac{1}{q})\frac{2-q}{q}} + \\ & + \langle \mathbf{u}\rangle^{3(1-\frac{1}{q})}_{\beta,q}([(\mathbf{u}, s)]_0^{\frac{3-q}{q}} + [(\mathbf{u}, s)]_1^{\frac{3-q}{q}})\beta^{-2(1-\frac{1}{q})\frac{3}{q}}(1+\beta) + \\ & + ([(\mathbf{u}, s)]_0^{\frac{3-q}{q}} + [(\mathbf{u}, s)]_1^{\frac{3-q}{q}})\beta^{-2(1-\frac{1}{q})\frac{5q-6}{q}}(1+\beta) + \\ & + ([(\mathbf{u}, s)]_1^2 + [(\mathbf{u}, s)]_1[(\mathbf{u}, s)]_0(1+\beta))\beta^{-4(1-\frac{1}{q})} + \\ & + \|\mathbf{f}\|_{1,q} + [(\mathbf{u}, s)]_1\beta^{2-2(1-\frac{1}{q})}\Big\}\,.\end{aligned}$$

Employing Theorems III.5.1, III.5.2 and IV.2.1 we get finally (we assume $|\ln\beta| > 1$)

$$\begin{aligned}\|\mathbf{z}\|_{1,q} \le\, & C\|\mathbf{F}\|_{1,q} \le C\Big\{\|\mathbf{g}\|^2_{1,q}[\beta^{-1-2(1-\frac{1}{q})} + \beta^{-6(1-\frac{1}{q})\frac{2-q}{q}} + \\ & + \beta^{-2(1-\frac{1}{q})\frac{3}{q}}(1+\beta) + \beta^{-4(1-\frac{1}{q})}(1+\beta)] + \\ & + \|\mathbf{g}\|_{1,q}^{\frac{(3-q)}{q}}\beta^{-2(1-\frac{1}{q})\frac{6-5q}{q}}(1+\beta) + \|\mathbf{g}\|_{1,q}\beta^{\frac{2}{q}} + \\ & + \beta^{1+2(1-\frac{1}{q})}|\ln\beta|^{-2} + \beta^{2-2(1-\frac{1}{q})\frac{6-5q}{q}} + \\ & + \beta^{2-2(1-\frac{1}{q})\frac{3-2q}{q}}(1+\beta) + \beta^2(1+\beta) + \|\mathbf{f}\|_{1,q}\Big\}\,.\end{aligned}$$

Therefore assuming $\delta = \varepsilon\beta^{1+2(1-\frac{1}{q})}$,

$$\|\mathbf{z}\|_{1,q} \le \varepsilon\beta^{1+2(1-\frac{1}{q})} = \delta\,.$$

Let us emphasize that

$$\|\mathbf{g}\|_{1,q}^2\beta^{-1-2(1-\frac{1}{q})} \le C\varepsilon^2\beta^{1+2(1-\frac{1}{q})} \le \frac{1}{10}\varepsilon\beta^{1+2(1-\frac{1}{q})}$$

for ε small enough and

$$\beta^{1+2(1-\frac{1}{q})}|\ln\beta|^{-2} \le \frac{1}{10}\varepsilon\beta^{1+2(1-\frac{1}{q})}$$

for β small enough. Lemma 3.3 is proved.

□

Now it remains to show that the operator $\mathcal{M}$ is a contraction in the space $L^q(\Omega)$. It means we are about to show that there exists δ small enough such that for all g_1, $g_2 \in B_\delta$ there exists $\kappa \in (0,1)$ such that

$$\|\mathcal{M}\mathbf{g}_1 - \mathcal{M}\mathbf{g}_2\|_q \le \kappa\|\mathbf{g}_1 - \mathbf{g}_2\|_q\,.$$

Let us first reformulate the problems (I.4.28) and (I.4.29). We have easily

$$\begin{gathered}
-\Delta(\mathbf{u}_1-\mathbf{u}_2) + \varrho\frac{\beta}{\mu}\frac{\partial\mathbf{u}_1-\mathbf{u}_2}{\partial x_1} + \nabla(s_1-s_2) = g_1-g_2\\
\nabla\cdot(\mathbf{u}_1-\mathbf{u}_2) = 0\\
\mathbf{u}_1-\mathbf{u}_2 = \mathbf{0} \qquad \text{at } \partial\Omega\\
\mathbf{u}_1-\mathbf{u}_2 \to \mathbf{0} \qquad \text{as } |\mathbf{x}|\to\infty
\end{gathered} \tag{3.5}$$

$$\begin{gathered}
\mu(\mathbf{z}_1-\mathbf{z}_2) + \alpha_1((\mathbf{u}_1+\mathbf{v}_\infty)\cdot\nabla)(\mathbf{z}_1-\mathbf{z}_2) =\\
= \mathbf{F}(\mathbf{u}_1,s_1) - \mathbf{F}(\mathbf{u}_2,s_2) - \alpha_1(\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla\mathbf{z}_2 \equiv \mathbf{G}\,,
\end{gathered} \tag{3.6}$$

where

$$\begin{aligned}
\mathbf{F}(\mathbf{u}_1,s_1) - \mathbf{F}(\mathbf{u}_2,s_2) = {} & -\varrho((\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla)\mathbf{u}_1 - \varrho(\mathbf{u}_2\cdot\nabla)(\mathbf{u}_1-\mathbf{u}_2) +\\
& + \alpha_1\nabla\cdot\{(\nabla(\mathbf{u}_1-\mathbf{u}_2))^T[\nabla\mathbf{u}_1 + (\nabla\mathbf{u}_1)^T] +\\
& + (\nabla\mathbf{u}_2)^T[\nabla(\mathbf{u}_1-\mathbf{u}_2) + (\nabla(\mathbf{u}_1-\mathbf{u}_2))^T] +\\
& + \varrho\frac{\beta}{\mu}\frac{\partial\mathbf{u}_1}{\partial x_1}\otimes(\mathbf{u}_1-\mathbf{u}_2) + \varrho\frac{\beta}{\mu}\frac{\partial(\mathbf{u}_1-\mathbf{u}_2)}{\partial x_1}\otimes\mathbf{u}_2 -\\
& - (s_1-s_2)(\nabla\mathbf{u}_1)^T - s_2(\nabla(\mathbf{u}_1-\mathbf{u}_2))^T\} +\\
& + \alpha_1\frac{\varrho\beta^2}{\mu}\frac{\partial^2(\mathbf{u}_1-\mathbf{u}_2)}{\partial x_1^2}\,.
\end{aligned} \tag{3.7}$$

Our aim is to show that $\|\mathbf{z}_1-\mathbf{z}_2\|_q \le \kappa\|\mathbf{g}_1-\mathbf{g}_2\|_q$ with $\kappa < 1$. For (3.5) we have

$$\begin{aligned} \langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\beta,q} &\le C\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ [(\mathbf{u}_1 - \mathbf{u}_2, s_1 - s_2)]_0 &\le C\|\mathbf{g}_1 - \mathbf{g}_2\|_q \,, \end{aligned} \tag{3.8}$$

while for (3.6)

$$\|\mathbf{z}_1 - \mathbf{z}_2\|_q \le \frac{1}{\mu - \alpha\vartheta_1}\|\mathbf{G}\|_q \,. \tag{3.9}$$

Similarly as in Lemma 3.3 we can show that ϑ_1 is small if δ is small enough.

We start to estimate $\mathbf{G}$ in $L^q(\Omega)$ by means of $\langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\beta,q}$ and $[\mathbf{u}_1 - \mathbf{u}_2]_0$. The constants in the estimates will depend on $\langle \mathbf{u}_i \rangle_{\beta,q}$ and $[\mathbf{u}_i]_1$ and will be small for δ small. We shall give the estimates of the terms on the right hand side of (3.7).

$$\begin{aligned} &\|((\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla)\mathbf{z}_2\|_q \le \|\mathbf{u}_1 - \mathbf{u}_2\|_\infty \|\nabla \mathbf{z}_2\|_q \le \\ &\le \delta\beta^{-2(1-\frac{1}{q})\frac{3-q}{3}} [\mathbf{u}_1 - \mathbf{u}_2]_0^{\frac{3-2q}{q}} \langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_0^{3(1-\frac{1}{q})} \le \\ &\le \varepsilon\beta^{1-2(1-\frac{1}{q})\frac{3-2q}{q}} \|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned}$$

Let us note that for $\beta \in (1, \frac{6}{5})$ the exponent by β is strictly positive.

$$\begin{aligned} \|((\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla)\mathbf{u}_1\|_q &\le \beta^{-1-2(1-\frac{1}{q})} \langle \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\beta,q} \langle \mathbf{u}_1 \rangle_{\beta,q} \le \\ &\le C(|\ln\beta|^{-1} + \varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned}$$

The same result holds also for the term $\mathbf{u}_2 \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2)$.

$$\begin{aligned} &\beta\|\mathbf{u}_2\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q \le \beta\|\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q \|\mathbf{u}_2\|_\infty \le \\ &\le C\beta^{2-2(1-\frac{1}{q})\frac{3-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q + \beta^2(1+\varepsilon^{\frac{3-2q}{q}})\|\mathbf{g}_1 - \mathbf{g}_2\|_q \end{aligned}$$

Completely analogously we can estimate

$$\beta\|(\mathbf{u}_1 - \mathbf{u}_2)D^2\mathbf{u}_1\|_q \le \beta^{2-2(1-\frac{1}{q})\frac{3-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \,.$$

Moreover

$$\beta^2\|\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q \le C\beta^{\frac{2}{q}}\|\mathbf{g}_1 - \mathbf{g}_2\|_q \,.$$

All the other terms can be estimated by the same term

$$\begin{aligned} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\nabla^2\mathbf{u}_i\|_q &\le \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{\frac{2q}{2-q}} \|\nabla^2\mathbf{u}_i\|_2 \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ \|\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\nabla\mathbf{u}_i\|_q &\le \|\nabla^2(\mathbf{u}_1 - \mathbf{u}_2)\|_q \|\nabla\mathbf{u}_i\|_\infty \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ \|\nabla(s_1 - s_2)\nabla\mathbf{u}_1\|_q &\le \|\nabla(s_1 - s_2)\|_q \|\nabla\mathbf{u}_1\|_\infty \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \\ \|\nabla s_2\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_q &\le \|\nabla s_2\|_2 \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{\frac{2q}{2-q}} \le C\beta^{\frac{2-q}{q}}(1+\varepsilon)\|\mathbf{g}_1 - \mathbf{g}_2\|_q \,. \end{aligned}$$

From the calculations above we conclude

Lemma 3.4 *Let β, ε be small enough, $\delta = \varepsilon\beta^{1+2(1-\frac{1}{q})}$. Then there exists $\kappa \in (0,1)$ such that*

$$\|\mathcal{M}\mathbf{g}_1 - \mathcal{M}\mathbf{g}_2\|_q \le \kappa\|\mathbf{g}_1 - \mathbf{g}_2\|_q$$

for all $\mathbf{g}_1$, $\mathbf{g}_2 \in B_\delta$.

Analogously we can proceed for $k \ge 2$. Combining Lemmas 3.3 and 3.4 with Theorem 0.1 we finish the proof of Theorem 3.1.

VI

Weighted estimates

Using a version of the Banach fixed point theorem (see Theorem V.0.1) we proved the existence of solutions to the system (V.1.6)–(V.1.8) in Sobolev spaces. This chapter will be devoted to the study of weighted estimates of this solutions in order to show that the solutions obey certain asymptotic structure. Due to the construction of solutions it is enough to verify that the operator $\mathcal{M}$ defined in Theorems V.1.1–V.1.3 and V.2.1 maps balls in certain weighted spaces into themselves. Namely, it will be an easy matter to see that such balls have non–empty intersection with balls used in the above mentioned theorems. Then, taking $(\mathbf{w}^0, s^0)$ in this intersection, we have that any $(\mathbf{w}^i, s^i)$, $i \in \mathbb{N}$, defined

$$(\mathbf{w}^i, s^i) = \mathcal{M}(\mathbf{w}^{i-1}, s^{i-1})\,, \quad i = 1, 2, \dots$$

remains in this intersection and due to the weak compactness of such sets we have the same result for the solution; due to the uniqueness[1] of the fixed point constructed in Theorem V.0.1 we therefore get that solution constructed in the last chapter have the asymptotic structure implied by the weighted spaces.

A fundamental role in this weighted estimates will be played by the integral representation of solutions to the modified Oseen problem (see Section III.4) and by the weighted estimates obtained in Section II.3 for the Oseen kernels. Let us recall that due to the similar asymptotic properties of the fundamental solutions to the modified Oseen problem and to the (classical) Oseen problem the estimates from Section II.3 are applicable.

We shall combine these estimates with the weighted estimates to the transport equation (see Theorem IV.2.6) and also with some results from Chapter V. As usually we shall study separately the threedimensional and plane flows.

VI.1 Threedimensional flow

Let us recall that the weight

$$\mu_b^{a,\omega}(\mathbf{x};\beta) = |\mathbf{x}|^{\omega}(1 + |\beta\mathbf{x}|)^{a-\omega}(1 + s(\beta\mathbf{x}))^b \tag{1.1}$$

behaves outside the unit ball as the weight $\nu_b^a(\mathbf{x};\beta)$, see Section II.3. We define

$$\begin{gathered} V_\beta = \Big\{\mathbf{u} \in L^\infty(\Omega; \mu_1^{1,\omega}(\cdot\,;\beta)); \nabla\mathbf{u}, \nabla^2\mathbf{u} \in L^r(\Omega; \mu_{\frac{3}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{3}{r},\omega}(\cdot\,;\beta)), \\ \pi, \nabla\pi \in L^r(\Omega; \mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta))\Big\} \end{gathered} \tag{1.2}$$

[1]in sufficiently small balls

with the norm

$$\|(\mathbf{u},\pi)\|_{V_\beta} = \|\mathbf{u}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega} + \\ +\|\nabla\mathbf{u};\nabla^2\mathbf{u}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega} + \|\pi;\nabla\pi\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega}\,, \tag{1.3}$$

where $r \in (1;\infty)$ is a sufficiently large power and $\omega > 0$ will be precised later. Our aim is to show that the operator $\mathcal{M}$ (defined below) maps sufficiently small balls in V_β into itself for β sufficiently small. We define $\mathcal{M} : V_\beta \mapsto V_\beta$

$$\left.\begin{aligned} A(\mathbf{u}) + \beta\frac{\partial\mathbf{u}}{\partial x_1} + \nabla\pi = \nabla\cdot\boldsymbol{\mathcal{G}}(\mathbf{f},\mathbf{w},p,\mathbf{T}) \\ \nabla\cdot\mathbf{u} = 0 \end{aligned}\right\}\ \text{in } \Omega \\ \mathbf{u} = -\beta\mathbf{e}_1 \quad \text{at } \partial\Omega \\ \mathbf{u}\to\mathbf{0} \quad \text{as } |\mathbf{x}|\to\infty\,, \tag{1.4}$$

$$p + ((\mathbf{w}+\mathbf{v}_\infty)\cdot\nabla)p = s\,, \tag{1.5}$$

$$\mathbf{T} + ((\mathbf{w}+\mathbf{v}_\infty)\cdot\nabla)\mathbf{T} + \mathbf{G}(\nabla\mathbf{w},\mathbf{T}) = 2\mathbf{D}(\mathbf{w})\,, \tag{1.6}$$

where

$$\boldsymbol{\mathcal{G}}(\mathbf{f},\mathbf{w},p,\mathbf{T}) = \mathbf{h} + \mathbf{F}(\nabla\mathbf{w},\mathbf{T}) + p(\nabla\mathbf{w})^T - ((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{w} - \\ -\mathbf{w}\otimes\mathbf{w} - \beta\Big(\frac{\partial\mathbf{w}}{\partial x_1}\otimes\mathbf{w} + ((\mathbf{w}\cdot\nabla)\mathbf{w})\otimes\mathbf{e}_1\Big) + \mathbf{f}\otimes(\mathbf{w}+\beta\mathbf{e}_1)\,, \tag{1.7}$$

$$\nabla\cdot\mathbf{h} = \mathbf{f}\,. \tag{1.8}$$

From Theorem IV.2.6 we easily get

Lemma 1.1 *Let $\|\mathbf{w}\|_{C^2}$ and β be sufficiently small.*[2] *Then for any $1 < r < \infty$ and any $0 \le \omega \le a$, $0 \le b$ and p, $\mathbf{T}$ solution to (1.5) and (1.6), respectively, we have*

$$\|p\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \le C\|s\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \tag{1.9}$$

$$\|p\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \le C\|s\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \tag{1.10}$$

$$\|\mathbf{T}\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \le C\|\nabla\mathbf{w}\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \tag{1.11}$$

$$\|\mathbf{T}\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \le C\|\nabla\mathbf{w}\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))}\,. \tag{1.12}$$

Proof: From Theorem IV.2.6 it follows that we have only to verify that $\|\nabla\ln\mu_b^{a,\omega}(\cdot\,;\beta)\|_{C^1}$ is independent of β for β sufficiently small. But ($\mathbf{y} = \beta\mathbf{x}$)

$$|\nabla\ln\mu_b^{a,\omega}(\mathbf{x};\beta)| \le \\ \le \frac{\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{a-1-\omega}(1+s(\beta\mathbf{x}))^b}{(1+\beta|\mathbf{x}|)^{a-\omega}(1+s(\beta|\mathbf{x}|))^b|\mathbf{x}|^\omega} + \\ + \frac{\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{a-\omega}(1+s(\beta\mathbf{x}))^{b-1}\nabla_{\mathbf{y}}s(\mathbf{y})}{(1+\beta|\mathbf{x}|)^{a-\omega}(1+s(\beta|\mathbf{x}|))^b|\mathbf{x}|^\omega} + \\ + \frac{\omega|\mathbf{x}|^{\omega-1}(1+|\beta\mathbf{x}|)^{a-\omega}(1+s(\beta\mathbf{x}))^b}{(1+\beta|\mathbf{x}|)^{a-\omega}(1+s(\beta|\mathbf{x}|))^b|\mathbf{x}|^\omega} \le C\Big(\frac{1}{|\mathbf{x}|} + \beta\Big)$$

[2] We can replace this condition by $\|\mathbf{w}\|_{C^1} + \|\nabla^2\mathbf{w}\|_p + \beta$ small, $p > 3$.

with C independent of β (recall that $B_{\frac{1}{2}}(\mathbf{0}) \subset \Omega^c$). Analogously for higher derivatives. The proof is complete.

□

Remark 1.1

(i) There exist $C_i = C_i(\Omega, a, b, r)$, $i = 1, 2$, independent of β such that for any a, $b \geq 0$, $\beta \leq 1$ and any $g \in W^{1,r}(\Omega; \eta_b^a)$

$$C_1 \|g\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \leq \Big[\|g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} + \\ + \|\nabla g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))}\Big] \leq C_2 \|g\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \,. \tag{1.13}$$

In order to verify (1.13) it is sufficient to show that

$$\|g\|_{r,(\nabla\mu_b^{a,\omega}(\cdot\,;\beta))} \leq C \|g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \,. \tag{1.14}$$

Then namely

$$\|g\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \leq \|g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} + \|\nabla g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} + \\ + \|g\|_{r,(\nabla\mu_b^{a,\omega}(\cdot\,;\beta))} \leq c_1 (\|g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} + \|\nabla g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))})$$

and

$$\|\nabla g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \leq \|\nabla g \mu_b^{a,\omega}(\cdot\,;\beta)) + g \nabla \mu_b^{a,\omega}(\cdot\,;\beta))\|_r + \\ + \|g\|_{r,(\nabla\mu_b^{a,\omega}(\cdot\,;\beta))} \leq c_2 \|g\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \,.$$

To show (1.14) we proceed as in the proof of Lemma 1.1. Indeed,

$$\int_\Omega |g(\mathbf{x})|^r \nabla(\mu_{br}^{ar,\omega r}(\mathbf{x};\beta))\mathrm{d}\mathbf{x} \leq C \int_\Omega |g(\mathbf{x})|^r \beta \Big[ar\mu_{br}^{ar-1,\omega r}(\mathbf{x};\beta) + \\ + br \nabla_{\mathbf{y}} s(\mathbf{y}) \mu_{br-1}^{ar,\omega r}(\mathbf{x};\beta) + \frac{\omega r}{\beta |\mathbf{x}|} (\mu_{br}^{ar,\omega r}(\mathbf{x};\beta))\Big]\mathrm{d}\mathbf{x} \leq \\ \leq C \int_\Omega \Big[\frac{|g(\mathbf{x})|^r}{|\mathbf{x}|} \mu_{br}^{ar,\omega r}(\mathbf{x};\beta) + |g(\mathbf{x})|^r \mu_{br}^{ar,\omega r}(\mathbf{x};\beta)\beta\Big]\mathrm{d}\mathbf{x} \leq \\ \leq C \|g\|^r_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \,,$$

where we used the fact that $\mathbf{0} \in \Omega^c$ and $|\nabla_{\mathbf{y}} s| = \sqrt{\frac{2s(\mathbf{y})}{|\mathbf{y}|}}$.

(ii) We have for $r > N$

$$\|g \mu_b^{a,\omega}(\mathbf{x};\beta)\|_\infty \leq C \|g \mu_b^{a,\omega}(\mathbf{x};\beta)\|_{1,r} \tag{1.15}$$

and therefore, by (1.13) also

$$\|g \mu_b^{a,\omega}(\mathbf{x};\beta)\|_\infty \leq C(\|g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} + \|\nabla g\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))}) \,. \tag{1.16}$$

From Theorem III.4.1 and Corollary III.4.1 we have

$$\begin{aligned} u_j(\mathbf{x}) = \int_\Omega \frac{\partial \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k}\Big[h_{ik} + F_{ik}(\nabla \mathbf{w}, \mathbf{T}) + p\frac{\partial w_k}{\partial y_i} - w_l w_k \frac{\partial w_i}{\partial y_l} - \\ - w_i w_k - \beta\Big(w_k \frac{\partial w_i}{\partial y_1} + w_l \frac{\partial w_i}{\partial y_l}\delta_{1k}\Big) + f_i(w_k + \beta\delta_{ik})\Big](\mathbf{y})\mathrm{d}\mathbf{y} + \\ + \int_{\partial\Omega}\Big[-\beta\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{ik} + u_i(\mathbf{y})T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) + \\ + \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},\pi)(\mathbf{y}) + \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}_{ik}(\mathbf{y})\Big]n_k(\mathbf{y})\mathrm{d}_{\mathbf{y}}S \end{aligned} \tag{1.17}$$

for $i, j = 1, 2, 3$.

Let us suppose that

$$\|(\mathbf{w}, s)\|_{V_\beta} \le \delta = \varepsilon\beta^a\,, \tag{1.18}$$

where a will be precised later. Moreover, let also

$$\|(\mathbf{w}, s)\|_{V_{k_i}} \le \varepsilon\beta^\alpha\,, \quad i = 1, 2, 3\,, \tag{1.19}$$

where α can be taken in $[\frac{3}{4}; 1)$ (see Theorems V.1.i), $k \ge 2$, $i = 1, 2, 3$. Let us recall that

$$\begin{aligned} \|(\mathbf{w}, s)\|_{V_{k_1}} &= \beta^{\frac{1}{4}}\|\mathbf{w}\|_4 + \|\nabla\mathbf{w}\|_{k,2} + \|s\|_{k,2} \\ \|(\mathbf{w}, s)\|_{V_{k_2}} &= \beta^{\frac{1}{4}}\|\mathbf{w}\|_4 + \|\nabla\mathbf{w}\|_2 + \|s\|_2 + \|\nabla^2\mathbf{w}\|_{k-1,p} + \|\nabla s\|_{k-1,p} \\ \|(\mathbf{w}, s)\|_{V_{k_3}} &= \beta^{\frac{1}{2}}\|\mathbf{w}\|_{\frac{2q}{2-q}} + \beta^{\frac{1}{4}}\|\nabla\mathbf{w}\|_{\frac{4q}{4-q}} + \|\nabla s\|_q + \|\nabla^2\mathbf{w}\|_{k-1,p} + \|\nabla s\|_{k-1,p}\,. \end{aligned}$$

We start to estimate the L^∞–norm of $\mathbf{u}\mu_1^{1,\omega}(\cdot\,;\beta)$. Let us denote by u_j^V the part of u_j which corresponds to the volume integrals, u_j^S the part corresponding to the surface integrals. As $\nabla\boldsymbol{\mathcal{O}}^\mu \sim \eta^{-\frac{3}{2}}_{-\frac{3}{2}}(\cdot\,;\beta)$, we apply Theorem II.3.10 in the estimate of the volume terms.

We have

$$\|\mathbf{u}^V\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta))} = \|\mathbf{u}^V\mu_1^{1,\omega}(\cdot\,;\beta)\|_\infty \le C\beta^{-1+(k-1)\omega}\|\boldsymbol{\mathcal{G}}\|_{\infty,(\mu_1^{\frac{3}{2},k\omega}(\cdot\,;\beta))}\,,$$

$\omega \ge 0$, $(k-1)\omega \le 1$, $k \ge 0$, where we extended $\boldsymbol{\mathcal{G}}$ by zero outside of Ω. We estimate each term in (1.17) separately. We assume for a moment that $\mathbf{f}$ and $\mathbf{h}$ are sufficiently smooth and decay sufficiently fast at infinity; we collect the precise assumptions in the main theorem.

$$\|\mathbf{h}\|_{\infty,(\eta_1^{\frac{3}{2}}(\cdot\,;\beta))} \le \frac{\varepsilon}{20}\beta^{a+1+\omega} \tag{1.20}$$

due to the above mentioned assumptions. Next

$$\begin{aligned} \|\mathbf{F}(\nabla\mathbf{w}, \mathbf{T})\|_{\infty,(\mu_1^{\frac{3}{2},2\omega}(\cdot\,;\beta))} &\le \|\mathbf{T}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\|\nabla\mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))} \le \\ &\le C\|\mathbf{T}\|_{1,r,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\|\nabla\mathbf{w}\|_{1,r,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\,. \end{aligned}$$

Applying Lemma 1.1 together with Remark 1.1 we get for $r \geq 4$ $(\frac{3}{2} - \frac{3}{r} \geq \frac{3}{4})$

$$\|\mathbf{F}(\nabla\mathbf{w}, \mathbf{T})\|_{\infty,(\mu_1^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2a}\,. \tag{1.21}$$

Completely analogously

$$\begin{aligned} &\|p\nabla\mathbf{w}\|_{\infty,(\mu_1^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \leq \\ &\leq \|p\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\|\nabla\mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2a}\,. \end{aligned} \tag{1.22}$$

Further

$$\||\mathbf{w}|^2\nabla\mathbf{w}\|_{\infty,(\mu_1^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \leq \|\mathbf{w}\|^2_{\infty,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\|\nabla\mathbf{w}\|_\infty\,.$$

But (see (1.19))

$$\|\nabla\mathbf{w}\|_\infty \leq C\begin{Bmatrix} \|\nabla\mathbf{w}\|_{2,2} & i=1 \\ \|\nabla\mathbf{w}\|_2 + \|\nabla^2\mathbf{w}\|_p & i=2 \\ \|\nabla\mathbf{w}\|_{\frac{4q}{4-q}} + \|\nabla^2\mathbf{w}\|_p & i=3 \end{Bmatrix} \leq \varepsilon\beta^\alpha$$

and therefore

$$\||\mathbf{w}|^2\nabla\mathbf{w}\|_{\infty,(\mu_1^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \leq C\varepsilon^3\beta^{2a+\alpha}\,. \tag{1.23}$$

Next we have

$$\||\mathbf{w}|^2\|_{\infty,(\mu_1^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \leq \|\mathbf{w}\|^2_{\infty,(\mu_{\frac{1}{2}}^{\frac{3}{4},\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2a}\,. \tag{1.24}$$

Easily also for $r \geq 4$

$$\begin{aligned} &\|\beta\mathbf{w}\nabla\mathbf{w}\|_{\infty,(\mu_1^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \leq \\ &\leq \beta\|\mathbf{w}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta))}\|\nabla\mathbf{w}\|_{\infty,(\mu_0^{\frac{1}{2},\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2a+1} \end{aligned} \tag{1.25}$$

and finally

$$\begin{aligned} \|\mathbf{f}\,\mathbf{w}\|_{\infty,(\mu_1^{\frac{3}{2},\omega}(\cdot\,;\beta))} &+ \|\beta\mathbf{f}\|_{\infty,(\eta_1^{\frac{3}{2}}(\cdot\,;\beta))} \leq \|\mathbf{w}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta))}\|\mathbf{f}\|_{\infty,(\eta_0^{\frac{1}{2}}(\cdot\,;\beta))} + \\ &+\beta\|\mathbf{f}\|_{\infty(\eta_1^{\frac{3}{2}}(\cdot\,;\beta))} \leq C(\varepsilon\beta^a + \beta)\beta \end{aligned} \tag{1.26}$$

due to the assumptions on $\mathbf{f}$. We can summarize now

$$\|\mathbf{u}^V\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2a-1+\omega} + o(\beta^{2a-1+\omega})\,. \tag{1.27}$$

Therefore we see that we need $a \geq 1 - \omega$.

Next we continue with the surface integrals. We denote the surface terms in (1.17) by (1)–(4) and the corresponding parts of $\mathbf{u}^S$ by $\mathbf{u}^{S,1}$–$\mathbf{u}^{S,4}$. We distinguish three situations

a) $|\mathbf{x}| \leq 1$

b) $1 \le |\mathbf{x}| \le \frac{1}{\beta}$, $(\beta < 1)$

c) $\beta|\mathbf{x}| > 1$.

In the case a) we shall not use the integral representation; we rather apply the results from the previous chapter and get for $i = 1$

$$\|\mathbf{u}\|_{\infty,\Omega_1} \le C\|\mathbf{u}\|_{2,2,\Omega_1} \le C(\|\mathbf{u}\|_{2,\Omega_1} + \|\nabla\mathbf{u}\|_{1,2,\Omega_1})$$

and for $i = 2$

$$\|\mathbf{u}\|_{\infty,\Omega_1} \le C(\|\mathbf{u}\|_{2,\Omega_1} + \|\nabla^2\mathbf{u}\|_{p,\Omega_1}) .$$

Applying the Friedrichs inequality (see Theorem VIII.1.10) on Ω_1 we have

$$\|\mathbf{u}\|_{\infty,\Omega_1} \le \left\{ \begin{array}{l} C(\|\beta\mathbf{e}_1\|_{1,(\partial\Omega} + \|\nabla\mathbf{u}\|_{1,2,\Omega_1}) \\ C(\|\beta\mathbf{e}_1\|_{1,(\partial\Omega} + \|\nabla\mathbf{u}\|_{2,\Omega_1} + \|\nabla^2\mathbf{u}\|_{p,\Omega_1}) \end{array} \right\} \le C(\beta + \varepsilon\beta^\alpha)$$

and therefore, as $\mu_1^{1,\omega}(\mathbf{x};\beta) \sim 1$ on Ω_1

$$\|\mathbf{u}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega_1} \le C\varepsilon\beta^\alpha + o(\beta^\alpha) .$$

Combining this with the volume integrals we have

$$\|\mathbf{u}^S\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega_1} \le C\varepsilon\beta^\alpha + o(\beta^\alpha) . \tag{1.28}$$

Next we continue with the case b); we shall use from now the integral representation. We have

$$\begin{gathered} |\mathbf{u}^{S,1}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| \le \beta\mu_1^{1,\omega}(\mathbf{x};\beta) \int_{\partial\Omega} |\boldsymbol{\mathcal{O}}^\mu(\mathbf{x}-\mathbf{y};\beta)||\mathbf{u}(\mathbf{y})|\mathrm{d}_{\mathbf{y}}S \le \\ \le \beta\mu_1^{1,\omega}(\mathbf{x};\beta)\Big[\beta|\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)| + \beta\int_{\partial\Omega} |\boldsymbol{\mathcal{O}}^\mu(\mathbf{x}-\mathbf{y};\beta) - \boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)|\mathrm{d}_{\mathbf{y}}S\Big] \le \\ \le C\beta^3|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))\cdot \\ \cdot\Big[|\boldsymbol{\mathcal{O}}^\mu(\beta\mathbf{x};1)| + \beta\int_{\partial\Omega} |\nabla\boldsymbol{\mathcal{O}}^\mu((\mathbf{x}-t\mathbf{y})\beta;1)|\mathrm{d}_{\mathbf{y}}S\Big] . \end{gathered}$$

Using $|\mathbf{x} - t\mathbf{y}| \le |\mathbf{x}| - |\mathbf{y}| \le \frac{|\mathbf{x}|}{2}$ and the fact that $|\beta\mathbf{x}| \le 1$ we have

$$|\mathbf{u}^{S,1}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| \le C(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))(\beta^2|\mathbf{x}|^{\omega-1} + C\beta^2|\mathbf{x}|^{\omega-2})$$

and therefore $(\omega < 1)$

$$\|\mathbf{u}^{S,1}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le C\beta^2 . \tag{1.29}$$

Next

$$\begin{gathered} |\mathbf{u}^{S,2}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| \le C\beta\mu_1^{1,\omega}(\mathbf{x};\beta)(|\nabla\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)| + |\mathbf{e}(\mathbf{x})| + \\ + |\nabla^2\boldsymbol{\mathcal{O}}^\mu(\mathbf{x}-t\mathbf{y};\beta)| + |\nabla\mathbf{e}(\mathbf{x}-t\mathbf{y})|) \le C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))\cdot \\ \cdot\Big[\frac{\beta^2}{|\beta\mathbf{x}|^2} + \frac{1}{|\mathbf{x}|^2} + \frac{C\beta^3}{|\beta\mathbf{x}|^3} + \frac{C}{|\mathbf{x}|^3}\Big] \le \\ \le C\beta(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))(|\mathbf{x}|^{-2+\omega} + |\mathbf{x}|^{-3+\omega}) . \end{gathered}$$

As $\omega < 1$, we get
$$\|\mathbf{u}^{S,2}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}_1} \le C\beta + o(\beta)\,. \tag{1.30}$$

The third term can be estimated as follows[3]
$$\begin{aligned}|\mathbf{u}^{S,3}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| &\le C\mu_1^{1,\omega}(\mathbf{x};\beta)\Big(|\mathcal{O}^{\mu}(\mathbf{x};\beta)|\int_{\partial\Omega}(|\nabla\mathbf{u}|+|\pi|)\mathrm{d}S+\\ &+\int_{\partial\Omega}(|\nabla\mathbf{u}|+|\pi|)(|\mathcal{O}^{\mu}(\mathbf{x}-\mathbf{y};\beta)-\mathcal{O}^{\mu}(\mathbf{x};\beta)\Big)\mathrm{d}S\le\\ &\le C|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))\Big[\frac{1}{|\mathbf{x}|}(\|\nabla\mathbf{u}\|_{1,2,\Omega_1}+\|\pi\|_{1,2,\Omega_1})+\\ &+\frac{1}{|\mathbf{x}|^2}(\|\nabla\mathbf{u}\|_{1,2,\Omega_1}+\|\pi\|_{1,2,\Omega_1})\Big]\,.\end{aligned}$$

Recalling that $\|\nabla\mathbf{u}\|_{1,2,\Omega_1}+\|\pi\|_{1,2,\Omega_1}\le\varepsilon\beta^{\alpha}$ (see Theorem V.1.1) we get
$$\|\mathbf{u}^{S,3}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}_1} \le C\varepsilon\beta^{\alpha}\,. \tag{1.31}$$

Finally
$$\begin{aligned}|\mathbf{u}^{S,4}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| &\le C|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))\cdot\\ &\cdot(|\mathcal{O}^{\mu}(\mathbf{x};\beta)|+|\nabla\mathcal{O}^{\mu}\Big(\frac{\mathbf{x}}{2};\beta\Big)|)\int_{\partial\Omega}|\mathcal{G}\cdot\mathbf{n}|\mathrm{d}S\,.\end{aligned}$$

We easily get that (see Theorems V.1.1 and Remark VIII.3.6)[4]
$$\begin{aligned}\int_{\partial\Omega}|\mathcal{G}\cdot\mathbf{n}|\mathrm{d}S &\le C\Big[\|\mathbf{h}\|_{2,\Omega_1}+\|\nabla\cdot\mathbf{h}\|_{2,\Omega_1}+\|\mathbf{T}\nabla\mathbf{w}\|_{1,2,\Omega_1}+\|p\nabla\mathbf{w}\|_{2,\Omega_1}+\\ &+\|\nabla p\nabla\mathbf{w}\|_{2,\Omega_1}+\beta^2\|\nabla\mathbf{w}\|_{1,2}+\\ &+\beta^2+\beta\|\mathbf{f}\|_{1,2,\Omega_1}\Big]\le C(\mathbf{f})+C\varepsilon^2\beta^{2\alpha}+o(\beta^{2\alpha})\end{aligned} \tag{1.32}$$

and therefore
$$\|\mathbf{u}^{S,4}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}_1} \le C\varepsilon^2\beta^{2\alpha}+o(\beta^{2\alpha})\,. \tag{1.33}$$

Collecting (1.28)–(1.33) we get
$$\|\mathbf{u}^{S}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}_1} \le C\varepsilon\beta^{\alpha}+o(\beta^{\alpha})\,. \tag{1.34}$$

Next we continue with the case $\mathbf{x}\in\Omega^{\frac{1}{\beta}}$. We study again the four terms separately.
$$\begin{aligned}|\mathbf{u}^{S,1}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| &\le C\beta^2\mu_1^{1,\omega}(\mathbf{x};\beta)(|\mathcal{O}^{\mu}(\mathbf{x};\beta)|+|\nabla\mathcal{O}^{\mu}\Big(\frac{\mathbf{x}}{2};\beta\Big)|)\le\\ &\le C\beta^2|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))\cdot\\ &\cdot\Big[\frac{\beta}{(1+|\beta\mathbf{x}|)(1+s(\beta\mathbf{x}))}+\frac{\beta^2}{(1+|\beta\mathbf{x}|)^{\frac{3}{2}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}}}\Big]\,.\end{aligned}$$

---

[3]If $i=2$, then the norms of $\mathbf{u}$ and π are replaced by $\|\nabla\mathbf{u}\|_{2,\Omega_1}+\|\nabla^2\mathbf{u}\|_{p,\Omega_1}+\|\pi\|_{2,\Omega_1}+\|\nabla\pi\|_{p,\Omega_1}$ and instead of Theorem V.1.1 we apply Theorem V.1.2, similarly for $i=3$.

[4]similarly also in the case of Theorem V.1.2 and Theorem V.1.3.

Therefore
$$\|\mathbf{u}^{S,1}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le C\beta^{3-\omega} + o(\beta^{3-\omega})\,. \tag{1.35}$$

The second term we estimate as follows

$$|\mathbf{u}^{S,2}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| \le C\beta\mu_1^{1,\omega}(\mathbf{x};\beta)\Big[|\nabla\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)| + |\mathbf{e}(\mathbf{x})| +$$
$$+|\nabla^2\boldsymbol{\mathcal{O}}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)| + |\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)|\Big] \le C\beta^3|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))\cdot$$
$$\cdot\Big[\frac{1}{(1+|\beta\mathbf{x}|)^{\frac{3}{2}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}}} + \frac{1}{|\beta\mathbf{x}|^2} + \frac{\beta}{(1+|\beta\mathbf{x}|)^2(1+s(\beta\mathbf{x}))^2} + \frac{\beta}{|\beta\mathbf{x}|^3}\Big]$$

and so
$$\|\mathbf{u}^{S,2}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le C\beta^{3-\omega} + o(\beta^{3-\omega})\,. \tag{1.36}$$

The third term gives us

$$|\mathbf{u}^{S,3}(\mathbf{x})\mu_1^{1,\omega}(\mathbf{x};\beta)| \le C|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega}(1+s(\beta\mathbf{x}))\cdot$$
$$\cdot\Big(|\nabla\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)| + |\mathbf{e}(\mathbf{x})| + |\nabla^2\boldsymbol{\mathcal{O}}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)| + |\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)|\Big)\int_{\partial\Omega}(|\nabla\mathbf{u}| + |\pi|)\mathrm{d}S\,.$$

As $\int_{\partial\Omega}(|\nabla\mathbf{u}|+|\pi|)\mathrm{d}S \le \|\nabla\mathbf{u}\|_{1,2,\Omega_1} + \|\pi\|_{1,2,\Omega_1} \le \varepsilon\beta^\alpha$, we have[5]
$$\|\mathbf{u}^{S,3}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le C\varepsilon\beta^{1-\omega+\alpha} + o(\beta^{1-\omega+\alpha})\,. \tag{1.37}$$

Finally, as in (1.32) we estimate

$$\int_{\partial\Omega}|\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\mathrm{d}S \le C\Big[\|\mathbf{h}\|_{2,\Omega_1} + \|\nabla\cdot\mathbf{h}\|_{2,\Omega_1} + \|\mathbf{T}\nabla\mathbf{w}\|_{1,2,\Omega_1} + \|p\nabla\mathbf{w}\|_{1,2,\Omega_1} +$$
$$\beta^2\|\nabla\mathbf{w}\|_{1,2,\Omega_1} + \beta^2 + \beta\|\mathbf{f}\|_{1,2,\Omega_1}\Big] \le C(\mathbf{f}) + C\varepsilon^2\beta^{2\alpha} + o(\beta^{2\alpha})$$

and therefore

$$\|\mathbf{u}^{S,4}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le C\varepsilon^2\beta^{1-\omega+2\alpha} + o(\beta^{1-\omega+2\alpha})\,. \tag{1.38}$$

Collecting (1.35)–(1.38) we have

$$\|\mathbf{u}^{S}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le C\varepsilon\beta^{1-\omega+\alpha} + o(\beta^{1-\omega+\alpha})\,. \tag{1.39}$$

So, (1.27), (1.28), (1.34) and (1.39) imply for ε, β and the right hand side sufficiently small (we choose $a = 1-\omega$)

$$\|\mathbf{u}\|_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta)),\Omega} \le \frac{1}{5}\varepsilon\beta^{1-\omega}\,.$$

Let us recall that $\alpha \in [\frac{3}{4};1)$ i.e. for any $0<\omega<1$ we may take α in such a way that $1-\omega<\alpha<1$.

[5] see footnote above

Next we study weighted estimates for first and second gradient of velocity. We shall study together also the estimates for pressure and its gradient. We have from Theorems III.4.1 and III.4.2

$$
\begin{aligned}
D^\alpha u_j(\mathbf{x}) &= \mathcal{A}_j^{(1,\alpha)}(\boldsymbol{\mathcal{G}}) + \int_\Omega D^\alpha_{\mathbf{x}} \frac{\partial \mathcal{N}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k} \mathcal{G}_{ik}(\mathbf{y}) \mathrm{d}\mathbf{y} + \\
&+ \int_{\partial\Omega} \Big[-\beta D^\alpha \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) u_i(\mathbf{y}) \delta_{1k} + u_i(\mathbf{y}) D^\alpha T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) + \\
&+ D^\alpha \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) T_{ik}(\mathbf{u},p)(\mathbf{y}) + D^\alpha \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) \mathcal{G}_{ik} \Big] n_k(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S
\end{aligned} \tag{1.40}
$$

for $|\alpha| = 1$ and

$$
\begin{aligned}
D^\alpha u_j(\mathbf{x}) &= \mathcal{A}_j^{(2,\alpha)}(\nabla \cdot \boldsymbol{\mathcal{G}}) + \int_\Omega D^\alpha_{\mathbf{x}} \mathcal{N}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) \frac{\partial \mathcal{G}_{ik}}{\partial y_k}(\mathbf{y}) \mathrm{d}\mathbf{y} + \\
&+ \int_{\partial\Omega} \Big[-\beta D^\alpha \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) u_i(\mathbf{y}) \delta_{1k} + u_i(\mathbf{y}) D^\alpha T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}, e_j)(\mathbf{x}-\mathbf{y};\beta) + \\
&+ D^\alpha \mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta) T_{ik}(\mathbf{u},p)(\mathbf{y}) \Big] n_k(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S
\end{aligned} \tag{1.41}
$$

for $|\alpha| = 2$. For the pressure

$$
\begin{aligned}
\pi(\mathbf{x}) &= \text{v.p.} \int_\Omega \frac{\partial e_i}{\partial x_k}(\mathbf{x}-\mathbf{y}) \mathcal{G}_{ik}(\mathbf{y}) \mathrm{d}\mathbf{y} + c_{ik} \mathcal{G}_{ik}(\mathbf{x}) + \\
&+ \int_{\partial\Omega} \Big[-\beta e_i(\mathbf{x}-\mathbf{y}) u_i(\mathbf{y}) \delta_{1l} + u_i(\mathbf{y}) \mathcal{T}_{il}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + \\
&+ e_i(\mathbf{x}-\mathbf{y}) T_{il}(\mathbf{u},\pi)(\mathbf{y}) + e_i(\mathbf{x}-\mathbf{y}) \mathcal{G}_{il}(\mathbf{y}) \Big] n_l(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S
\end{aligned} \tag{1.42}
$$

$$
\begin{aligned}
\frac{\partial \pi(\mathbf{x})}{\partial x_j} &= \text{v.p.} \int_\Omega \frac{\partial e_i}{\partial x_j}(\mathbf{x}-\mathbf{y}) \frac{\partial \mathcal{G}_{ik}}{\partial y_k}(\mathbf{y}) \mathrm{d}\mathbf{y} + c_{ij} \frac{\partial \mathcal{G}_{ik}}{\partial x_k}(\mathbf{x}) + \\
&+ \int_{\partial\Omega} \Big[-\beta \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} u_i(\mathbf{y}) \delta_{1l} + u_i(\mathbf{y}) \frac{\partial \mathcal{T}_{il}}{\partial x_j}(\mathbf{e})(\mathbf{x}-\mathbf{y}) + \\
&+ \frac{\partial e_i(\mathbf{x}-\mathbf{y})}{\partial x_j} T_{il}(\mathbf{u},\pi)(\mathbf{y}) \Big] n_l(\mathbf{y}) \mathrm{d}_{\mathbf{y}} S + c_{ij} \frac{\partial \mathcal{G}_{ik}}{\partial x_k}(\mathbf{x}) \,,
\end{aligned} \tag{1.43}
$$

where $\boldsymbol{\mathcal{T}}(\mathbf{e})$ is defined in (III.4.16), $\boldsymbol{\mathcal{G}}$ in (1.7), $\mathcal{A}_j^{1,\alpha}$ and $\mathcal{A}_j^{2,\alpha}$ are singular integral operators satisfying the assumptions of Theorem II.3.5 and $D^2\mathcal{N}^\mu_{ij} = D^2\mathcal{O}^\mu_{ij} - D^2\mathcal{S}^\mu_{ij}$ are weakly singular operators such that $D^2\mathcal{N}^\mu_{ij}(\mathbf{x}) \sim \eta_{-1}^{-2}(\mathbf{x})$ for $|\mathbf{x}|$ large.

We start with the estimates of the volume terms. We have from Corollary II.3.3 and Theorem II.3.8 that for $k \geq 0$

$$
\begin{aligned}
\|D^2 \boldsymbol{\mathcal{O}}^\mu * f\|_{r,(\mu_{\frac{3}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{3}{r},\omega}(\cdot;\beta))} &\leq C\beta^{(k-1)\omega} \|f\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,k\omega}(\cdot;\beta))} \\
\|D\mathbf{e} * f\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot;\beta))} &\leq C\beta^{(k-1)\omega} \|f\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot;\beta))}
\end{aligned} \tag{1.44}
$$

for any f such that the norms on the right hand side are finite and f has support outside of the origin[6], $\delta > 0$ sufficiently small. The convolutions on the left hand side are to be understood in the following sense

$$
\begin{aligned}
(D^2 \boldsymbol{\mathcal{O}}^\mu * f)(\mathbf{x}) &= \mathcal{A}(f) + cf(\mathbf{x}) + (\mathcal{N}^\mu * f)(\mathbf{x}) \\
(D\mathbf{e} * f)(\mathbf{x}) &= \text{v.p.} \int_{\mathbb{R}^N} \nabla \mathbf{e}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \mathrm{d}\mathbf{y} + cf(\mathbf{x}) \,,
\end{aligned} \tag{1.45}
$$

[6]see Chapter II, Remark II.3.4

the convolution on the right hand side of (1.45) is now defined in the classical sense, for a.a. $\mathbf{x} \in \mathbb{R}^N$. We need therefore estimates of all terms in contained in $\boldsymbol{\mathcal{G}}$ and $\nabla \cdot \boldsymbol{\mathcal{G}}$ in $L^r(\Omega; \mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,k\omega}(\cdot\,;\beta))$ and $L^r(\Omega; \mu_0^{2-\frac{4}{r},k\omega}(\cdot\,;\beta))$. We estimate each term separately. We have

$$\left.\begin{array}{c} \|\mathbf{h}\|_{r,(\eta_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta}(\cdot\,;\beta))} \\ \|\mathbf{f}\|_{r,(\eta_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta}(\cdot\,;\beta))} \\ \|\mathbf{h}\|_{r,(\eta_0^{2-\frac{4}{r}}(\cdot\,;\beta))} \\ \|\mathbf{f}\|_{r,(\eta_0^{2-\frac{4}{r}}(\cdot\,;\beta))} \end{array}\right\} \leq \frac{\varepsilon}{10}\beta\,. \tag{1.46}$$

Next

$$\|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} \leq \|\mathbf{T}\|_{\infty,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))} \|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))}\,.$$

Now using Remark 1.1 (ii) we have for $r \geq 4$

$$\|\mathbf{T}\|_{\infty,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))} \leq C\big(\|\mathbf{T}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))} + \|\nabla\mathbf{T}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))}\big)$$

i.e., employing Lemma 1.1,

$$\begin{aligned} \|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} &\leq C\|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))}\cdot \\ \cdot\Big[\|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))} &+ \|\nabla^2\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))}\Big] \leq C\varepsilon^2\beta^{2-2\omega}\,. \end{aligned} \tag{1.47}$$

Completely analogously we have

$$\begin{aligned} \|\nabla\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}r}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} &+ \|\mathbf{T}\nabla^2\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} \leq \\ \leq C\Big[\|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))} &+ \|\nabla^2\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{3}{4r}}^{1-\frac{1}{r},\omega}(\cdot\,;\beta))}\Big]^2 \leq C\varepsilon^2\beta^{2-2\omega}\,. \end{aligned} \tag{1.48}$$

The same kind of estimates gives us

$$\begin{aligned} \|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} &+ \\ +\|\nabla(\mathbf{T}\nabla\mathbf{w})\|_{r,(\mu_{1-\frac{1}{r}}^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} &\leq C\varepsilon^2\beta^{2-2\omega} \end{aligned} \tag{1.49}$$

$$\begin{aligned} \|p(\nabla\mathbf{w})^T\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} &+ \|p(\nabla\mathbf{w})^T\|_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} + \\ +\|\nabla p(\nabla\mathbf{w})^T\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} &+ \|\nabla p(\nabla\mathbf{w})^T\|_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} \leq \\ &\leq C\varepsilon^2\beta^{2-2\omega}\,. \end{aligned} \tag{1.50}$$

The trilinear term can be estimated very easily; namely using

$$\||\mathbf{w}|^2\nabla^k\mathbf{w}\|_r \leq \|\nabla^k\mathbf{w}\|_r\|\mathbf{w}\|_\infty^2 \quad (k=1,2)$$
$$\||\nabla\mathbf{w}|^2\mathbf{w}\|_r \leq \|\nabla\mathbf{w}\|_r\|\nabla\mathbf{w}\|_{1,r}\|\mathbf{w}\|_\infty$$

we get as above

$$\begin{aligned}
&\| |\mathbf{w}|^2 \nabla \mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,3\omega}(\cdot\,;\beta))} + \| |\mathbf{w}|^2 \nabla \mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},3\omega}(\cdot\,;\beta))} + \\
&+\|\nabla(|\mathbf{w}|^2 \nabla \mathbf{w})\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,3\omega}(\cdot\,;\beta))} + \|\nabla(|\mathbf{w}|^2 \nabla \mathbf{w})\|_{r,(\mu_0^{2-\frac{4}{r},3\omega}(\cdot\,;\beta))} \leq \\
&\leq C\varepsilon^3 \beta^{3-3\omega}\,.
\end{aligned} \tag{1.51}$$

A little bit another technique must be used in order to estimate the convective term. We have for $r \geq 4$ and δ sufficiently small

$$\begin{aligned}
\| |\mathbf{w}|^2\|^r_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} &= \int_\Omega |\mathbf{w}|^{2r} |\mathbf{x}|^{2\omega r} (1+|\beta\mathbf{x}|)^{2r(1-\omega)-\frac{5}{2}+\delta r}(1+s(\beta\mathbf{x})^{r-\frac{3}{2}} \mathrm{d}\mathbf{x} \\
&\leq \|\mathbf{w}\|^{2r}_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta))} \int_\Omega (1+|\beta\mathbf{x}|)^{-\frac{5}{2}+\delta r}(1+s(\beta\mathbf{x}))^{-r-\frac{3}{2}} \mathrm{d}\mathbf{x} \leq \\
&\leq C\beta^{-3} \|\mathbf{w}\|^{2r}_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta))} \\
\| |\mathbf{w}|^2\|^r_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} &= \\
= \int_\Omega |\mathbf{w}|^{2r} |\mathbf{x}|^{2\omega r} (1+|\beta\mathbf{x}|)^{2r-4-2\omega r} \mathrm{d}\mathbf{x} &\leq C\beta^{-3} \|\mathbf{w}\|^{2r}_{\infty,(\mu_1^{1,\omega}(\cdot\,;\beta))}
\end{aligned}$$

and using the obvious estimate

$$\|\mathbf{w}\cdot\nabla\mathbf{w}\|_r \leq \|\mathbf{w}\|_\infty \|\nabla\mathbf{w}\|_r$$

we end up with

$$\begin{aligned}
\|\mathbf{w}\otimes\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} &\leq C\varepsilon^2\beta^{2-2\omega-\frac{3}{r}} \\
\|(\mathbf{w}\cdot\nabla)\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} &\leq C\varepsilon^2\beta^{2-2\omega} \\
\|\mathbf{w}\otimes\mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} &\leq C\varepsilon^2\beta^{2-2\omega-\frac{3}{r}} \\
\|(\mathbf{w}\cdot\nabla)\mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} &\leq C\varepsilon^2\beta^{2-2\omega}\,.
\end{aligned} \tag{1.52}$$

The next term can be estimated very easily

$$\begin{aligned}
&\|\beta\mathbf{w}\cdot\nabla\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} + \|\beta\nabla(\mathbf{w}\cdot\nabla)\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,2\omega}(\cdot\,;\beta))} + \\
&+\|\beta\mathbf{w}\cdot\nabla\mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} + \|\beta\nabla(\mathbf{w}\cdot\nabla)\mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},2\omega}(\cdot\,;\beta))} \leq \\
&\leq C\varepsilon^2\beta^{3-2\omega}\,.
\end{aligned} \tag{1.53}$$

Finally, for the last term we have

$$\begin{aligned}
\|\mathbf{f}\,\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,\omega}(\cdot\,;\beta))} + \beta^{-\omega}\|\beta\mathbf{f}\|_{r,(\eta_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta}(\cdot\,;\beta))} &\leq C\varepsilon\beta \\
\|\nabla\mathbf{f}\,\mathbf{w}\|_{r,(\mu_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta,\omega}(\cdot\,;\beta))} + \beta^{-\omega}\|\beta\nabla\mathbf{f}\|_{r,(\eta_{1-\frac{3}{2r}}^{2-\frac{5}{2r}+\delta}(\cdot\,;\beta))} &\leq C\varepsilon\beta \\
\|\mathbf{f}\,\mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta))} + \beta^{-\omega}\|\beta\mathbf{f}\|_{r,(\eta_0^{2-\frac{4}{r}}(\cdot\,;\beta))} &\leq C\varepsilon\beta \\
\|\nabla\mathbf{f}\,\mathbf{w}\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta))} + \beta^{-\omega}\|\beta\nabla\mathbf{f}\|_{r,(\eta_0^{2-\frac{4}{r}}(\cdot\,;\beta))} &\leq C\varepsilon\beta\,.
\end{aligned} \tag{1.54}$$

Therefore, combining (1.44) with (1.46)–(1.54) we have for ε and β sufficiently small

$$\begin{aligned} &\|\nabla \mathbf{u}^V\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta))} + \|\nabla^2 \mathbf{u}^V\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta))} + \\ &+\|\pi^V\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta))} + \|\nabla \pi^V\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta))} \le \\ &\le \frac{\varepsilon}{20}\beta^{1-\omega}\,. \end{aligned} \tag{1.55}$$

Next we have to estimate the boundary terms. As above, we distinguish three cases, i.e. $|\mathbf{x}| \le 1$, $1 < |\mathbf{x}| \le \frac{1}{\beta}$ and $|\mathbf{x}| > \frac{1}{\beta}$. In order to estimate the second gradient of $\mathbf{u}$ (first gradient of pressure) we have to require some more regularity on $\mathbf{f}$ — either $\mathbf{f} \in W^{k,2}(\Omega) \cap D^{-1,2}(\Omega)$, $k \ge 3$ (and use Theorem V.1.1) or $\mathbf{f} \in W^{k,p}(\Omega) \cap D^{-1,2}(\Omega) \cap L^2(\Omega)$, $k \ge 2$, $p \in (3;4]$ (and use Theorem V.1.2) or finally $\mathbf{f} \in W^{1,q}(\Omega) \cap W^{k,r}(\Omega)$, $k \ge 1$, $q \in (1;\frac{6}{5}]$ (and use Theorem V.1.3). In the first case we have

$$\begin{aligned} &\|\nabla \mathbf{u}, \nabla^2 \mathbf{u}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega_1} \le C\|\nabla \mathbf{u}, \nabla^2\mathbf{u}\|_{\infty,\Omega_1} \le \\ &\le C\|\nabla \mathbf{u}, \nabla^2 \mathbf{u}\|_{2,2,\Omega_1} \le C\|\nabla \mathbf{u}\|_{3,2} \le C\varepsilon\beta^\alpha\,, \quad \alpha \in \Big[\frac{3}{4};1\Big)\,. \end{aligned} \tag{1.56}$$

Analogously for the second case

$$\begin{aligned} &\|\nabla \mathbf{u}, \nabla^2 \mathbf{u}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega_1} \le C\|\nabla \mathbf{u}, \nabla^2\mathbf{u}\|_{\infty,\Omega_1} \le \\ &\le C\|\nabla \mathbf{u}, \nabla^2 \mathbf{u}\|_{1,p,\Omega_1} \le C\|\nabla \mathbf{u}\|_{2,p} \le C\varepsilon\beta^\alpha\,, \quad \alpha \in \Big[\frac{3}{4};1\Big)\,. \end{aligned} \tag{1.57}$$

In the last case we have directly ($r > 3 > \frac{3q}{3-q}$, $1 < q < \frac{3}{2}$)

$$\begin{aligned} &\|\nabla \mathbf{u}, \nabla^2 \mathbf{u}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega_1} \le C\|\nabla \mathbf{u}, \nabla^2\mathbf{u}\|_{r,\Omega_1} \le \\ &\le C\|\nabla^2 \mathbf{u}\|_{r,\Omega_1} + \|\nabla \mathbf{u}\|_{\frac{3q}{3-q}} \le C(\|\nabla^2\mathbf{u}\|_{r,\Omega_1} + \|\nabla^2 \mathbf{u}\|_{q,\Omega}) \end{aligned} \tag{1.58}$$

as $\nabla \mathbf{u} \in L^{\frac{4q}{4-q}}(\Omega)$. Therefore we get in all three cases for r sufficiently large ($r > 3$)

$$\|\nabla \mathbf{u}, \nabla^2 \mathbf{u}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega_1} \le C\varepsilon\beta^\alpha\,, \quad \alpha \in \Big[\frac{3}{4};1\Big)\,. \tag{1.59}$$

Exactly in the same way we can show[7]

$$\|\pi, \nabla\pi\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega_1} \le C\varepsilon\beta^\alpha\,, \quad \alpha \in \Big[\frac{3}{4};1\Big)\,. \tag{1.60}$$

As analogous estimates are valid for $\nabla \mathbf{u}^V$, $\nabla^2\mathbf{u}^V$, π^V and $\nabla\pi^V$, we get inequalities (1.59) and (1.60) for the surface parts $\nabla \mathbf{u}^S$, $\nabla^2 \mathbf{u}^S$, π^S and $\nabla \pi^S$.

Let us now consider the case $1 < |\mathbf{x}| \le \frac{1}{\beta}$ for $\beta < 1$. As above, we denote the corresponding surface integrals by $\nabla \mathbf{u}^{S,i}$ and $\pi^{S,i}$, $i = 1,2,3,4$ and

[7] We use the fact that the pressure tend to zero for $|\mathbf{x}| \to \infty$ and therefore the Sobolev–Poincaré inequality holds

$\nabla^2 \mathbf{u}^{S,i}$, $\nabla \pi^{S,i}$, $i = 1,2,3$, respectively. We shall estimate each term separately, analogously as in the case of L^∞–estimates. Moreover, it is enough to estimate only $\nabla \mathbf{u}$ and π in the corresponding weighted spaces. Then $\nabla^2 \mathbf{u}$ and $\nabla \pi$ can be estimated easily by the same terms — the asymptotic structure of higher gradients of $\boldsymbol{\mathcal{O}}^\mu$ and $\mathbf{e}$ is better than those of the lower gradients.

$$|\nabla \mathbf{u}^{S,1}(\mathbf{x}) \mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\mathbf{x};\beta)| \le$$
$$\le C\beta^4 |\mathbf{x}|^\omega (1+|\beta \mathbf{x}|)^{\frac{3}{2}-\omega-\frac{3}{r}} (1+s(\beta \mathbf{x}))^{\frac{3}{2}-\frac{1}{r}} \Big(\frac{1}{|\beta \mathbf{x}|^2} + \frac{\beta}{|\beta \mathbf{x}|^3} \Big)$$

i.e.

$$\begin{aligned} \|\nabla \mathbf{u}^{S,1}\|^r_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot;\beta)),\Omega^1_{\frac{1}{\beta}}} &\le C\beta^{(4-\omega)r} \int_{\Omega^1_{\frac{1}{\beta}}} (|\beta \mathbf{x}|^{(\omega-2)r} + \beta^r |\beta \mathbf{x}|^{(\omega-3)r}) \cdot \\ &\cdot (1+|\beta \mathbf{x}|)^{(\frac{3}{2}-\omega-\frac{3}{r})r} (1+s(\beta \mathbf{x}))^{\frac{3}{2}r-1} \mathrm{d}\mathbf{x} \le \\ &\le C\beta^{(4-\omega)r-3} \int_\beta^1 |\mathbf{y}|^{\omega r-2r+2} \mathrm{d}|\mathbf{y}| \le C\beta^{2r} . \end{aligned} \tag{1.61}$$

Analogously

$$\begin{aligned} |\nabla \mathbf{u}^{S,2}(\mathbf{x}) \mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\mathbf{x};\beta)| &\le C\beta |\mathbf{x}|^\omega (1+|\beta \mathbf{x}|)^{\frac{3}{2}-\omega-\frac{3}{r}} (1+s(\beta \mathbf{x}))^{\frac{3}{2}-\frac{1}{r}} \cdot \\ &\cdot \Big(\frac{\beta^3}{|\beta \mathbf{x}|^3} + \frac{1}{|\mathbf{x}|^3} + \frac{\beta^4}{|\beta \mathbf{x}|^4} + \frac{1}{|\mathbf{x}|^4} \Big) , \end{aligned}$$

i.e.

$$\begin{aligned} \|\nabla \mathbf{u}^{S,2}\|^r_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot;\beta)),\Omega^1_{\frac{1}{\beta}}} &\le C\beta^{(4-\omega)r} \int_{\Omega^1_{\frac{1}{\beta}}} (|\beta \mathbf{x}|^{(\omega-3)r} + \beta^r |\beta \mathbf{x}|^{(\omega-4)r}) \cdot \\ &\cdot (1+|\beta \mathbf{x}|)^{(\frac{3}{2}-\omega-\frac{4}{r})r} (1+s(\beta \mathbf{x}))^{r-1} \mathrm{d}\mathbf{x} \le \\ &\le C\beta^{(4-\omega)r-3} \int_\beta^1 |\mathbf{y}|^{\omega r-3r+2} \mathrm{d}|\mathbf{y}| \le C\beta^{r} . \end{aligned} \tag{1.62}$$

$$\begin{aligned} |\nabla \mathbf{u}^{S,3}(\mathbf{x}) \mu^{\frac{3}{2}-\frac{3}{r},\omega}_{1-\frac{1}{r}}(\mathbf{x};\beta)| &\le C\beta |\mathbf{x}|^\omega (1+|\beta \mathbf{x}|)^{\frac{3}{2}-\omega-\frac{3}{r}} (1+s(\beta \mathbf{x}))^{\frac{3}{2}-\frac{1}{r}} \cdot \\ &\cdot \Big(|\nabla \boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)| + |\nabla^2 \boldsymbol{\mathcal{O}}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)| \Big) \int_{\partial\Omega} (|\nabla \mathbf{u}| + |\pi|) \mathrm{d}S . \end{aligned}$$

As in the part $|\mathbf{x}| < 1$ we get either

$$\int_{\partial\Omega} (|\nabla \mathbf{u}| + |\pi|) \mathrm{d}S \le C(\|\nabla \mathbf{u}\|_{1,2,\Omega_1} + \|\pi\|_{1,2,\Omega_1}) \quad (\mathbf{f} \in D_0^{-1,2}(\Omega) \cap W^{1,2}(\Omega))$$

or

$$\int_{\partial\Omega} (|\nabla \mathbf{u}| + |\pi|) \mathrm{d}S \le C(\|\nabla^2 \mathbf{u}\|_{q,\Omega} + \|\nabla \pi\|_{q,\Omega}) \quad (\mathbf{f} \in W^{1,q}(\Omega)\,, 1 < q \le \frac{6}{5}) .$$

Then (see (1.18))

$$\|\nabla \mathbf{u}^{S,3}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot;\beta)),\Omega^1_{\frac{1}{\beta}}} \le C\varepsilon \beta^\alpha . \tag{1.63}$$

Finally, the last term can be estimated

$$|\nabla \mathbf{u}^{S,4}(\mathbf{x})\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\mathbf{x};\beta)| \leq C|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{\frac{3}{2}-\omega-\frac{3}{r}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}-\frac{1}{r}} \cdot \\ \cdot(|\nabla \boldsymbol{\mathcal{O}}^{\mu}(\mathbf{x};\beta)| + |\nabla^2 \boldsymbol{\mathcal{O}}^{\mu}\Big(\frac{\mathbf{x}}{2};\beta\Big)|)\int_{\partial\Omega} |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\mathrm{d}S\,.$$

The surface integral can be estimated (see (1.32))

$$\int_{\partial\Omega} |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\mathrm{d}S \leq C(\mathbf{f}) + C\varepsilon^2\beta^{2\alpha} + o(\beta^{2\alpha})$$

and therefore

$$\|\nabla \mathbf{u}^{S,4}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \leq C\varepsilon^2\beta^{2\alpha}\,. \tag{1.64}$$

Next we estimate the pressure. We have

$$|\pi^{S,1}(\mathbf{x})\mu_0^{2-\frac{4}{r},\omega}(\mathbf{x};\beta)| \leq C\beta^2|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{2-\omega-\frac{4}{r}}\Big(\frac{1}{|\mathbf{x}|^2}+\frac{1}{|\mathbf{x}|^3}\Big)$$

i.e.

$$\|\pi^{S,1}\|^r_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \leq \\ \leq C\beta^{2r}\int_1^{\frac{1}{\beta}}(|\mathbf{x}|^{\omega r-2r+2} + |\mathbf{x}|^{\omega r-3r+2})\mathrm{d}|\mathbf{x}| \leq C\beta^{2r}\,. \tag{1.65}$$

Next

$$|\pi^{S,2}(\mathbf{x})\mu_0^{2-\frac{4}{r},\omega}(\mathbf{x};\beta)| \leq C\beta|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{2-\omega-\frac{4}{r}}(|\mathbf{x}|^{-2}+|\mathbf{x}|^{-3}+|\mathbf{x}|^{-4})$$

and

$$\|\pi^{S,2}\|^r_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \leq \\ \leq C\beta^{r}\int_1^{\frac{1}{\beta}}(|\mathbf{x}|^{(\omega-2)r} + |\mathbf{x}|^{(\omega-3)r} + |\mathbf{x}|^{(\omega-4)r})|\mathbf{x}|^2\mathrm{d}|\mathbf{x}| \leq C\beta^{r}\,. \tag{1.66}$$

The last two terms can be estimated as above

$$\|\pi^{S,3}\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|\pi^{S,4}\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \leq \\ \leq C\int_{\partial\Omega}(|\nabla\mathbf{u}| + |\pi| + |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|)\mathrm{d}S \leq C(\varepsilon\beta^{\alpha} + \varepsilon^2\beta^{2\alpha})\,. \tag{1.67}$$

We can overcome to the last part where $|\mathbf{x}| > \frac{1}{\beta}$. We have

$$|\nabla \mathbf{u}^{S,1}(\mathbf{x})\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\mathbf{x};\beta)| \leq C\beta^2|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{\frac{3}{2}-\omega-\frac{3}{r}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}-\frac{1}{r}} \cdot \\ \cdot\Big[\frac{\beta^2}{(1+|\beta\mathbf{x}|)^{\frac{3}{2}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}}} + \frac{\beta^3}{(1+|\beta\mathbf{x}|)^2(1+s(\beta\mathbf{x}))^2}\Big]$$

and so

$$\|\nabla \mathbf{u}^{S,1}\|^r_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq$$

$$\leq C\beta^{(4-\omega)r} \int_{\frac{1}{\beta}}^{\infty} (1+|\beta\mathbf{x}|)^{-3-\omega r} |\beta\mathbf{x}|^{\omega r} (1+s(\beta\mathbf{x}))^{-1} |\mathbf{x}|^2 \mathrm{d}|\mathbf{x}| \leq \tag{1.68}$$

$$\leq C\beta^{(4-\omega)r-3} \int_1^{\infty} (1+|\mathbf{y}|)^{-3}(1+s(\mathbf{y}))^{-1} \mathrm{d}|\mathbf{y}| \leq C\beta^{(4-\omega)r-3}\,.$$

$$|\nabla \mathbf{u}^{S,2}(\mathbf{x})\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\mathbf{x};\beta)| \leq C\beta^4 |\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{\frac{3}{2}-\omega-\frac{3}{r}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}-\frac{1}{r}}\cdot$$

$$\cdot\Big[\frac{1}{(1+|\beta\mathbf{x}|)^2(1+s(\beta\mathbf{x}))^2} + \frac{1}{|\beta\mathbf{x}|^3} + \frac{\beta}{(1+|\beta\mathbf{x}|)^{\frac{5}{2}}(1+s(\beta\mathbf{x}))^{\frac{5}{2}}} + \frac{\beta}{|\beta\mathbf{x}|^4}\Big]$$

and therefore easily

$$\|\nabla \mathbf{u}^{S,2}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq C\beta^{(4-\omega)-\frac{3}{r}}\,. \tag{1.69}$$

Again the most restrictive is the third term. But

$$(|\nabla \mathbf{u}^{S,3}(\mathbf{x})| + |\nabla \mathbf{u}^{S,4}(\mathbf{x})|)\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\mathbf{x};\beta) \leq$$

$$\leq C\beta^2|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{\frac{3}{2}-\omega-\frac{3}{r}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}-\frac{1}{r}} \int_{\partial\Omega} (|\nabla\mathbf{u}| + |\pi| + |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|)\mathrm{d}S\cdot$$

$$\cdot\Big[\frac{1}{(1+|\beta\mathbf{x}|)^{\frac{3}{2}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}}} + \frac{\beta}{(1+|\beta\mathbf{x}|)^2(1+s(\beta\mathbf{x}))^2}\Big]$$

and therefore easily

$$\|\nabla \mathbf{u}^{S,3}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|\nabla \mathbf{u}^{S,4}\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq C\beta^{2+\alpha-\omega-\frac{3}{r}}\,. \tag{1.70}$$

Let us finally estimate the boundary terms for the pressure. Easily

$$|\pi^{S,1}(\mathbf{x})\mu_0^{2-\frac{4}{r},\omega}(\mathbf{x};\beta)| \leq C\beta^2|\mathbf{x}|^{\omega}(1+|\beta\mathbf{x}|)^{2-\omega-\frac{4}{r}}\Big[\frac{1}{|\mathbf{x}|^2} + \frac{1}{|\mathbf{x}|^3}\Big]$$

and therefore

$$\|\pi^{S,1}\|^r_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq C\beta^{(4-\omega)r-3}\int_1^{\infty} |\mathbf{y}|^{-2}\mathrm{d}|\mathbf{y}| \leq C\beta^{(4-\omega)r-3}\,. \tag{1.71}$$

The other terms can be estimated in the same way

$$\|\pi^{S,2}\|^r_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq C\beta^{(4-\omega)r-3}\int_1^{\infty} |\mathbf{y}|^{-2}\mathrm{d}|\mathbf{y}| \leq C\beta^{(4-\omega)r-3} \tag{1.72}$$

(here we use that the lower order term in $\mathcal{T}_{il}$ is $\beta\mathbf{e}$, see (III.4.16))

$$\|\pi^{S,3}\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|\pi^{S,4}\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq$$

$$\leq C\beta^{(2-\omega)}\Big(\int_{B^{\frac{1}{\beta}}(\mathbf{0})} |\beta\mathbf{x}|^{-4}\mathrm{d}\mathbf{x}\Big)^{\frac{1}{r}} \int_{\partial\Omega}(|\nabla\mathbf{u}| + |\pi| + |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|)\mathrm{d}S \tag{1.73}$$

$$\leq C\varepsilon^2\beta^{2-\omega-\frac{3}{r}}\,.$$

Summarizing the estimates (1.56)–(1.73) we get for ε, β sufficiently small

$$\|\nabla \mathbf{u}^S\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega} + \|\nabla^2 \mathbf{u}^S\|_{r,(\mu^{\frac{3}{2}-\frac{3}{r},\omega}_{\frac{3}{2}-\frac{1}{r}}(\cdot\,;\beta)),\Omega} \leq \frac{1}{10}\varepsilon\beta^{1-\omega}$$
$$\|\pi^S\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega} + \|\nabla \pi^S\|_{r,(\mu_0^{2-\frac{4}{r},\omega}(\cdot\,;\beta)),\Omega} \leq \frac{1}{10}\varepsilon\beta^{1-\omega}$$

and therefore

Theorem 1.1 *Let* $\mathbf{f} = \nabla\cdot\mathbf{h}$ *and one of the following conditions be satisfied*

(i) $\mathbf{h} \in L^2(\Omega)$, $\mathbf{f} \in W^{k,2}(\Omega)$, $k \geq 3$

(ii) $\mathbf{h} \in L^2(\Omega)$, $\mathbf{f} \in L^2(\Omega)\cap W^{k,p}(\Omega)$, $k \geq 2$, $p \in (3;4]$

(iii) $\mathbf{h} \in L^1_{loc}(\overline{\Omega})$, $\mathbf{f} \in W^{1,q}(\Omega)\cap W^{k,r}(\Omega)$, $q = \frac{6}{5}$ *if* $k = 1$, $q = \frac{4}{3}$ *if* $k \geq 2$

with the corresponding norms sufficiently small. Let $\Omega \in C^{k+1}$ *be an exterior domain in* $\mathbb{R}^3$*. Moreover let*

$$\mathbf{h}, \mathbf{f}, \nabla\mathbf{f} \in L^\infty(\Omega; \eta_1^2(\cdot)) \tag{1.74}$$

and let $\beta = |\mathbf{v}_\infty|$ *and* $\|\mathbf{h}, \mathbf{f}, \nabla\mathbf{f}\|_{\infty,(\eta_1^2(\cdot))}$ *be sufficiently small.*

Then $(\mathbf{v}, p)$*, solution to the problem (I.4.14)–(I.4.15) constructed in Theorems V.1.1–V.1.3 has the following asymptotic properties*

$$\begin{aligned} \mathbf{u} = \mathbf{v} - \mathbf{v}_\infty &\in L^\infty(\Omega; \eta_1^1(\cdot)) \\ \nabla\mathbf{v}, \nabla^2\mathbf{v} &\in L^r(\Omega; \eta^{\frac{3}{2}-\frac{3}{r}}_{\frac{3}{2}-\frac{1}{r}}(\cdot)) \\ p, \nabla p &\in L^r(\Omega; \eta_0^{2-\frac{4}{r}}(\cdot))\,, \end{aligned} \tag{1.75}$$

where $r \in [4;\infty)$ *is in the cases (i) and (ii) arbitrary while in the case (iii) corresponds to the integrability of the right hand side.*

Proof: It follows from the calculations done above. Let us only note that whatever regularity we get for π, the same has also p. Finally the condition $\mathbf{h}, \mathbf{f} \in L^r(\Omega; \eta^{2-\frac{5}{2r}+\delta}_{1-\frac{3}{2r}}(\cdot)) \cap L^r(\Omega; \eta_0^{2-\frac{4}{r}}(\cdot))$ follows easily as e.g.

$$\begin{aligned} \|\mathbf{f}\|^r_{r,(\eta^{2-\frac{5}{2r}+\delta}_{1-\frac{3}{2r}}(\cdot))} &\leq \int_\Omega |\mathbf{f}|^r \eta^{2r-\frac{5}{2}+\delta r}_{r-\frac{3}{2}}(\mathbf{x})\mathrm{d}\mathbf{x} \leq \\ \leq \|\mathbf{f}\|^r_{\infty(\eta_1^2(\cdot))} &\int_\Omega |\mathbf{x}|^{-\frac{5}{2}+\delta r}(1+s(\mathbf{x}))^{-\frac{3}{2}}\mathrm{d}\mathbf{x} \leq C\|\mathbf{f}\|^r_{\infty(\eta_1^2(\cdot))}\,. \end{aligned}$$

□

Remark 1.2 Going through the calculation before Theorem 1.1 it is an easy matter to show that if $\mathbf{h}, \mathbf{f}, \nabla\mathbf{f} \in L^\infty(\Omega; \eta^2_{\frac{1}{2}}(\cdot))$, then $\nabla\mathbf{v}$, $\nabla^2\mathbf{v} \in L^r(\Omega; \eta^{\frac{3}{2}-\frac{3}{r}}_{1-\frac{1}{r}}(\cdot))$, while for $\mathbf{h}, \mathbf{f}, \nabla\mathbf{f} \in L^\infty(\Omega; \eta^{\frac{3}{2}}_{\frac{1}{2}}(\cdot))$we have $\nabla\mathbf{v}$, $\nabla^2\mathbf{v} \in L^r(\Omega; \eta^{\frac{3}{2}-\frac{3}{r}}_{1-\frac{1}{r}}(\cdot))$ and p, $\nabla p \in L^r(\Omega; \eta^{\frac{3}{2}-\frac{4}{r}}_{\frac{1}{2}}(\cdot))$. In both cases, $\mathbf{v} - \mathbf{v}_\infty \in L^\infty(\Omega; \eta_1^1(\cdot))$.

Moreover, (1.74) implies evidently $\mathbf{h} \in L^r(\Omega)$, $\mathbf{f} \in W^{1,r}(\Omega)$ for all $r > 1$ and in particular, we can apply Theorem V.1.3 in order to construct solution with the required asymptotic properties without any other assumption on higher gradients of $\mathbf{f}$. Nevertheless, we prefer to keep the formulation of Theorem 1.1 as done above; theorems from Section V.1 then give eventually some additional information about the regularity of the constructed solution.

VI.2 Plane flow

This section is devoted to the study of plane flow of the viscoelastic fluid. As in the preceding section, we study the asymptotic properties of solution to I.4.14–I.4.15 constructed in Theorem V.2.1.

Unlike the threedimensional case, the fundamental Oseen tensor (and fundamental solution to the modified Oseen problem) has more complicated structure — $\mathcal{O}^\mu_{11}$ differs from $\mathcal{O}^\mu_{ij}$, $i+j \geq 3$. As a consequence we expect different asymptotic behaviour in u_1 and u_2, similarly as in Chapter II for the flow in the whole $\mathbb{R}^2$. This must be taken into consideration in the choice of spaces. Moreover, unlike the threedimensional case (cf. [Sm]) we shall not get exactly the same structure for the solution as the fundamental solution has; we loose one logarithm term. We denote for the sake of notational convenience

$$\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta) = \mu_0^{1,\omega}(\mathbf{x};\beta)|\ln(2+|\beta\mathbf{x}|)|^{-1}\,.$$

Let us consider

$$\begin{aligned} V_\beta = \Big\{(\mathbf{u},\pi);\, & u_1 \in L^\infty(\Omega;\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta))\,,\, u_2 \in L^\infty(\Omega;\overline{\mu}_0^{1,\omega}(\cdot\,;\beta))\,, \\ & \nabla\mathbf{u}, \nabla^2\mathbf{u} \in L^r(\Omega;\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta))\,, \\ & \pi, \nabla\pi \in L^r(\Omega;\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta))\Big\} \end{aligned} \tag{2.1}$$

together with the norm

$$\begin{aligned} \|(\mathbf{u},\pi)\|_{V_\beta} = {} & \|u_1\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} + \|u_2\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta))} + \\ & + \|\nabla\mathbf{u},\nabla^2\mathbf{u}\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta))} + \|\pi,\nabla\pi\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta))}\,. \end{aligned} \tag{2.2}$$

As in the threedimensional case, our aim is to show that for $|\mathbf{v}_\infty| = \beta$ and for the right hand side sufficiently small, the operator $\mathcal{M} : V_\beta \mapsto V_\beta$ defined by (1.4)–(1.8) maps sufficiently small balls into itself. The power $r > 2$ will be specified later on. Nevertheless, the idea is to have r as large as possible.

Lemma 2.1 *Let $\|\mathbf{w}\|_{C^2} + \beta$ be sufficiently small.*[8] *Then for $1 < r < \infty$, $0 \leq \omega \leq a$, $0 \leq b$ we have*

$$\begin{aligned} \|p\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} &\leq C\|s\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \\ \|p\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} &\leq C\|s\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \\ \|\mathbf{T}\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} &\leq C\|\nabla\mathbf{w}\|_{r,(\mu_b^{a,\omega}(\cdot\,;\beta))} \\ \|\mathbf{T}\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))} &\leq C\|\nabla\mathbf{w}\|_{1,r,(\mu_b^{a,\omega}(\cdot\,;\beta))}\,. \end{aligned} \tag{2.3}$$

[8] As in the threedimensional case, we can weaken the conditions; see Lemma 1.1.

Proof: As Lemma 1.1, it is an easy consequence of Theorem IV.2.6.

□

Remark 1.1 holds also in this case, so the weighted $W^{1,r}(\Omega)$ norm is equivalent to the weighted $L^r(\Omega)$ norm of the function and its gradient (see (1.13)) and the Sobolev imbedding theorem can be used (see (1.15) and (1.16)). We shall again suppose that

$$\|(\mathbf{w}, s)\|_{V_\beta} \le \delta = \varepsilon\beta^{1-\omega}$$

and show that

$$\|(\mathbf{u}, \pi)\|_{V_\beta} \le \delta$$

for ε, β sufficiently small. Moreover, from Theorem V.2.1 we have also

$$\langle \mathbf{w} \rangle_{\beta,q} \le \varepsilon\beta^{2(1-\frac{1}{q})+1} \tag{2.4}$$

(see (V.2.3))

$$[|(\mathbf{w}, s)|]_k \le \varepsilon\beta^{2(1-\frac{1}{q})+\alpha}, \alpha \in \Big[\frac{2}{3}; 1\Big) \tag{2.5}$$

(see (V.2.4)).

The main tool will be again L^q–weighted estimates ($q \in (1; \infty]$) of the Oseen potentials. We therefore use again the integral representation. Unlike the threedimensional case we must distinguish very carefully the components of $\mathcal{O}^\mu_{ij}$. We start from (1.17), now for $j = 1, 2$. We first consider $j = 1$ and study the L^∞–weighted estimates of u_1. Let us recall that we have to study separately $\frac{\partial \mathcal{O}^\mu_{11}}{\partial y_2}$, then $\frac{\partial \mathcal{O}^\mu_{11}}{\partial y_1}$, $\frac{\partial \mathcal{O}^\mu_{12}}{\partial y_2}$, $\frac{\partial \mathcal{O}^\mu_{21}}{\partial y_2}$ and finally the third group is formed by all the other first derivative of $\boldsymbol{\mathcal{O}}^\mu$ (see Theorems II.3.17–II.3.19).

Therefore we have[9] due to Theorem II.3.17 and Theorem II.3.18

$$\begin{aligned} &\|u_1^V\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \le \\ &\le C\beta^{(k-1)\omega-1}\Big[\|\mathcal{G}_{12}\|_{\infty,(\mu_{\frac{1}{2}}^{1,k\omega}(\cdot\,;\beta))} + \|\mathcal{G}_{11}; \mathcal{G}_{21}; \mathcal{G}_{22}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2}+\delta,k\omega}(\cdot\,;\beta))}\Big], \end{aligned} \tag{2.6}$$

where $\delta > 0$, can be taken arbitrarily small, $0 \le \omega < \frac{1}{6}$, $k = 0, 1, 2, 3$. The presence of δ is caused by the logarithmic factor in Theorem II.3.18, $A = B = \frac{1}{2}$. In fact, we can here estimate all terms in $L^\infty(\Omega; \mu_{\frac{1}{2}}^{1,k\omega})$. We have namely

$$\|\mathbf{h}\|_{\infty,(\eta^1_{\frac{1}{2}}(\cdot\,;\beta))} \le \frac{\varepsilon}{30}\beta \tag{2.7}$$

due to the assumptions. Next, for $r \ge 4$ $(1 - \frac{2}{r} \ge \frac{1}{2})$

$$\begin{aligned} \|\mathbf{F}(\nabla\mathbf{w}, \mathbf{T})\|_{\infty,(\mu_{\frac{1}{2}}^{1,2\omega}(\cdot\,;\beta))} &\le \|\mathbf{T}\|_{\infty,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \|\nabla\mathbf{w}\|_{\infty,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \le \\ &\le C\|\mathbf{T}\|_{1,r,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \|\nabla\mathbf{w}\|_{1,r,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \le \\ &\le C\Big[\|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} + \|\nabla^2\mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))}\Big]^2 \le C\varepsilon^2\beta^{2-2\omega}. \end{aligned} \tag{2.8}$$

[9]We again use the notation $\mathbf{u}^V$, $\mathbf{u}^{S,i}$, see Section VI.1.

Analogously, for $r \geq 5$

$$\begin{aligned} &\|p\nabla \mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{1,2\omega}(\cdot\,;\beta))} \leq \\ &\leq \|p\|_{\infty,(\mu_0^{\frac{2}{5},\omega}(\cdot\,;\beta))} \|\nabla \mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{3}{5},\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2-2\omega}\,. \end{aligned} \tag{2.9}$$

As usually, the trilinear term causes no problems

$$\begin{aligned} \||\mathbf{w}|^2\nabla \mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{1,2\omega}(\cdot\,;\beta))} &\leq \|\mathbf{w}\|^2_{\infty,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \|\nabla \mathbf{w}\|_{1,p} \leq \\ &\leq C\varepsilon^3\beta^{2-2\omega+\alpha}\,, \end{aligned} \tag{2.10}$$

where $p > 2$ is taken from the definition of $[|(\cdot,\cdot)|]_k$ (see (V.2.4)). We can also easily estimate the convective term

$$\|w_i w_k\|_{\infty,(\mu_{\frac{1}{2}}^{1,2\omega}(\cdot\,;\beta))} \leq \|\mathbf{w}\|^2_{\infty,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2-2\omega}\,. \tag{2.11}$$

Further

$$\begin{aligned} &\|\beta \mathbf{w}\nabla \mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{1,2\omega}(\cdot\,;\beta))} \leq \beta\|\mathbf{w}\|_{\infty,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \cdot \\ &\cdot\Big(\|\nabla \mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} + \|\nabla^2 \mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{1}{2},\omega}(\cdot\,;\beta))}\Big) \leq C\varepsilon^2\beta^{3-2\omega} \end{aligned} \tag{2.12}$$

and finally

$$\begin{aligned} \|\mathbf{f}\,\mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{1,\omega}(\cdot\,;\beta))} + \|\beta \mathbf{f}\|_{\infty,(\eta_{\frac{1}{2}}^1(\cdot))} &\leq \|\mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \|\mathbf{f}\|_{\infty(\eta_0^{\frac{1}{2}}(\cdot))} + \\ &+\beta\|\mathbf{f}\|_{\infty,(\eta_{\frac{1}{2}}^1(\cdot))} \leq \frac{\varepsilon}{30}\beta^{2-\omega} \end{aligned} \tag{2.13}$$

due to the assumptions on the right hand side. Collecting (2.6)–(2.13) yields

$$\|u_1^V\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta))} \leq \frac{1}{10}\varepsilon\beta^{1-\omega} \tag{2.14}$$

for ε, β sufficiently small.

Next we estimate the L^∞–weighted norm of u_2^V. As in the integral representation the term $\frac{\partial \mathcal{O}_{11}^\mu}{\partial y_2}$ does not appear, we have

$$\|u_2^V\|_{\infty,(\mu_0^{1,\omega}(\cdot\,;\beta)P|\ln(2+2|\beta\mathbf{x}|)|^{-1})} \leq C\beta^{(k-1)\omega-1}\|\boldsymbol{\mathcal{G}}\|_{\infty,(\mu_0^{1,k\omega}(\cdot\,;\beta))}\,, \tag{2.15}$$

where P is a polynomial of first or second order (see Remark II.3.9). The most delicate term in $\boldsymbol{\mathcal{G}}$ will be w_1w_1. We therefore write (2.15) in a bit different way, namely

$$\begin{aligned} \|u_2^V\|_{\infty,(\mu_0^{1,\omega}(\cdot\,;\beta)P|\ln(2+2|\beta\mathbf{x}|)|^{-1})} &\leq C\beta^{\omega-1}\|w_1w_1\|_{\infty,(\mu_0^{1,2\omega}(\cdot\,;\beta))} + \\ &+C\beta^{(k-1)\omega-1}\|\boldsymbol{\mathcal{G}}'\|_{\infty,(\mu_0^{1+\delta,k\omega}(\cdot\,;\beta))}\,, \end{aligned} \tag{2.16}$$

where $\delta > 0$, $\mathcal{G}'_{ij} = \mathcal{G}_{ij} + w_1w_1\delta_{1i}\delta_{1j}$. Therefore the power of the polynomial P is determined by the term $\frac{\partial \mathcal{O}_{12}^\mu}{\partial y_1} * (w_1w_1)$. From Tab.3 and Tab.4 Chapter II we see ($c = 2$, $d = 0$, $a = 1$, $b = 1$) that the logarithmic term comes from the domain Ω_1 — i.e. the power is 1.

Moreover

$$\|w_1 w_1\|_{\infty,(\mu_0^{1,2\omega}(\cdot\,;\beta))} \le \|w_1\|^2_{\infty,(\mu_0^{\frac{1}{2},\omega}(\cdot\,;\beta))} \le C\varepsilon^2\beta^{2-2\omega}\,. \tag{2.17}$$

So we have

$$\|u_2^V\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta))} \le C\varepsilon^2\beta^{1-\omega} + C\beta^{(k-1)\omega-1}\|\boldsymbol{\mathcal{G}}'\|_{\infty,(\mu_0^{1+\delta,k\omega}(\cdot\,;\beta))}\,. \tag{2.18}$$

We can now estimate all the other terms in the weighted L^∞–spaces, analogously as in (2.7)–(2.13). It can be easily checked that the estimates (2.7)–(2.13) were not "optimal". We only have to restrict a little bit more the values of r, namely $r > 5$. We get

$$\|u_2^V\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta))} \le \frac{1}{10}\varepsilon\beta^{1-\omega}\,. \tag{2.19}$$

We can continue with the estimates of the boundary terms. Similarly as in the threedimensional case we distinguish three cases ($\beta > 1$)

(i) $|\mathbf{x}| \le 1$

(ii) $1 < |\mathbf{x}| \le \frac{1}{\beta}$

(iii) $\frac{1}{\beta} < |\mathbf{x}|$

and denote $u_j^{S,i}$, $i = 1,2,3,4$, the corresponding surface integrals in (1.17).

First, let $|\mathbf{x}| \le 1$. Then

$$\|\mathbf{u}\|_{\infty,\Omega_1} \le C\|\mathbf{u}\|_{2,q,\Omega_1} \le C(\|\mathbf{u}\|_{q,\Omega_1} + \|\nabla^2\mathbf{u}\|_{q,\Omega_1}) \le \\ \le C(\|\mathbf{u}\|_{\frac{2q}{2-q},\Omega_1} + \|\nabla^2\mathbf{u}\|_{q,\Omega_1})\,.$$

Now, employing the Friedrichs inequality (see Theorem VIII.1.10) and Lemma VIII.1.12

$$\|\mathbf{u}\|_{\infty,\Omega_1} \le C\Big(\|\mathbf{u}\|_{1,(\partial\Omega)} + \|\nabla\mathbf{u}\|_{\frac{2q}{2-q},\Omega} + \|\nabla^2\mathbf{u}\|_{q,\Omega_1}\Big) \le C(\beta + \|\nabla^2\mathbf{u}\|_{q,\Omega})$$

and therefore

$$\|\mathbf{u}\|_{\infty,\Omega_1} \le C(\beta + \varepsilon\beta^\alpha)\,.$$

Taking $\alpha > 1-\omega$, using (2.14) and (2.19) we get for β sufficiently small ($\mu_b^{a,\omega}(\mathbf{x};\beta) \sim 1$ in Ω_1)

$$\|u_1^S\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega_1} + \|u_2^S\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega_1} \le \frac{1}{10}\varepsilon\beta^{1-\omega}\,. \tag{2.20}$$

Next, let $1 < |\mathbf{x}| \le \frac{1}{\beta}$. We have[10]

$$
\begin{gathered}
|u_1^{S,1}(\mathbf{x})\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\mathbf{x};\beta)| + |u_2^{S,1}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)| \le \\
\le \beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{\frac{1}{2}-\omega}(1+s(\beta\mathbf{x}))^{\frac{1}{2}}\Big[|\mathcal{O}_{11}^\mu(\mathbf{x};\beta)| + C|\nabla\mathcal{O}_{11}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)|\Big] + \\
+\beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega}|\ln(2+|\beta\mathbf{x}|)|^{-1}\Big[|\mathcal{O}_{12}^\mu(\mathbf{x};\beta)| + C|\nabla\mathcal{O}_{12}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)|\Big] \le \\
\le C\beta^2|\mathbf{x}|^\omega\Big(|\ln|\beta\mathbf{x}|| + \frac{C}{|\mathbf{x}|}\Big) \le C\beta^{2-\omega},
\end{gathered}
$$

where we used the fact that $|\beta\mathbf{x}|^\omega|\ln|\beta\mathbf{x}||$ is bounded for $|\beta\mathbf{x}| \le 1$. Therefore

$$
\|u_1^{S,1}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot;\beta)),\Omega_{\frac{1}{\beta}}} + \|u_2^{S,1}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega_{\frac{1}{\beta}}} \le C\beta^{2-\omega}. \tag{2.21}
$$

The second term can be again estimated very easily

$$
\begin{gathered}
|u_1^{S,2}(\mathbf{x})\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\mathbf{x};\beta)| + |u_2^{S,2}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)| \le \\
\le C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{\frac{1}{2}-\omega}(1+s(\beta\mathbf{x}))^{\frac{1}{2}}\Big[|\nabla\mathcal{O}_{11}^\mu(\mathbf{x};\beta)| + \\
+|\nabla^2\mathcal{O}_{11}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)| + |\mathbf{e}(\mathbf{x})| + |\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)|\Big] + \\
+C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega}|\ln(2+|\beta\mathbf{x}|)|^{-1}\Big[|\nabla\mathcal{O}_{12}^\mu(\mathbf{x};\beta)| + |\nabla^2\mathcal{O}_{12}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)| + \\
+|\mathbf{e}(\mathbf{x})| + |\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)|\Big] \le C\beta\Big(\frac{1}{|\mathbf{x}|^{1-\omega}} + \frac{1}{|\mathbf{x}|^{2-\omega}}\Big) \le C\beta
\end{gathered}
$$

and therefore

$$
\|u_1^{S,2}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot;\beta)),\Omega_{\frac{1}{\beta}}} + \|u_2^{S,2}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta)),\Omega_{\frac{1}{\beta}}} \le C\beta. \tag{2.22}
$$

Unlike the threedimensional case, we must proceed very carefully in the estimate of the third term. We assume $\mathbf{u} = {}^I\mathbf{u} + {}^{II}\mathbf{u}$ where ${}^I\mathbf{u}$ solves the Oseen problem with zero right hand side and non–zero boundary condition while ${}^{II}\mathbf{u}$ solves the Oseen problem with zero boundary conditions and non–zero right hand side. Then we have

$$
|u_j^{S,3}(\mathbf{x})| \le \Big|\int_{\partial\Omega}\mathcal{O}_{ij}^\mu(\mathbf{x}-\mathbf{y};\beta)\Big[T_{ik}({}^I\mathbf{u},{}^I\pi) + T_{ik}({}^{II}\mathbf{u},{}^{II}\pi)\Big](\mathbf{y})n_k(\mathbf{y})\mathrm{d}_{\mathbf{y}}S\Big|.
$$

From Lemma III.5.1 we have

$$
\Big|\int_{\partial\Omega}T_{ik}({}^I\mathbf{u},{}^I\pi)(\mathbf{y})n_k(\mathbf{y})\mathrm{d}_{\mathbf{y}}S\Big| \le C|\ln\beta|^{-1}\|\mathbf{u}_*\|_{2-\frac{1}{q},q,(\partial\Omega)} \le C\beta|\ln\beta|^{-1} \tag{2.23}
$$

[10]We write only the most restrictive terms; for u_1 it is $\mathcal{O}_{11}^\mu$, for u_2 then $\mathcal{O}_{12}^\mu$.

and so

$$\begin{aligned}|{}^I u_1^{S,3}(\mathbf{x})\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\mathbf{x};\beta)| + |{}^I u_2^{S,3}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)| \le \\ C|\mathcal{O}^\mu(\mathbf{x};\beta)|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{\frac{1}{2}-\omega}\Big[(1+|\beta\mathbf{x}|)^{\frac{1}{2}}|\ln(2+|\beta\mathbf{x}|)|^{-1}+ \\ +(1+s(\beta\mathbf{x}))^{\frac{1}{2}}\Big]\Big|\int_{\partial\Omega} T_{ik}({}^I\mathbf{u},{}^I\pi)(\mathbf{y})n_k(\mathbf{y})\mathrm{d}_{\mathbf{y}}S\Big|+ \\ +C|\nabla\mathcal{O}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)||\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{\frac{1}{2}-\omega}\Big[(1+|\beta\mathbf{x}|)^{\frac{1}{2}}|\ln(2+|\beta\mathbf{x}|)|^{-1}+ \\ +(1+s(\beta\mathbf{x}))^{\frac{1}{2}}\Big]\int_{\partial\Omega}(|\nabla\mathbf{u}|+|p|)\mathrm{d}S \le \\ \le C\Big[\ln|\beta\mathbf{x}||\mathbf{x}|^\omega\beta|\ln\beta|^{-1} + \frac{1}{|\mathbf{x}|^{1-\omega}}\beta^{-2(1-\frac{1}{q})}[|(\mathbf{u},\pi)|]_0\Big]\,.\end{aligned}$$

Therefore

$$\begin{aligned}\|{}^I u_1^{S,3}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|{}^I u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le \\ \le C|\ln\beta|^{-1}(\beta^{1-\omega}+\beta)\,,\end{aligned} \tag{2.24}$$

where we used the fact that

$$\int_{\partial\Omega}(|\nabla\mathbf{u}|+|\pi|)\mathrm{d}S \le C(\|\nabla\mathbf{u}\|_{\frac{2q}{2-q},\Omega_1} + \|\nabla^2\mathbf{u}\|_{q,\Omega_1} + \|\pi\|_{\frac{2q}{2-q},\Omega_1} + \|\nabla\pi\|_{q,\Omega_1})\,.$$

Moreover, due to the asymptotic properties of π

$$\|\pi\|_{\frac{2q}{2-q},\Omega_1} \le C\|\pi\|_{\frac{2q}{2-q},\Omega} \le C\|\nabla\pi\|_{q,\Omega} \tag{2.25}$$

and

$$\int_{\partial\Omega}(|\nabla\mathbf{u}|+|\pi|)\mathrm{d}S \le C\beta^{-2(1-\frac{1}{q})}[|({}^I\mathbf{u},{}^I\pi)|]_0 \le C\beta|\ln\beta|^{-1} \tag{2.26}$$

(see Theorem III.5.1, recall that ${}^I\mathbf{f}=\mathbf{0}$).

Analogously we have

$$\begin{aligned}\|{}^{II} u_1^{S,3}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|{}^{II} u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le \\ \le C\beta^{-\omega}\beta^{-2(1-\frac{1}{q})}[|({}^{II}\mathbf{u},{}^{II}\pi)|]_0 \le C\beta^{-\omega-2(1-\frac{1}{q})}\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_q\,.\end{aligned}$$

We estimate $\nabla\cdot\boldsymbol{\mathcal{G}}$ in $L^q(\Omega)$, using (2.4) and (2.5).

$$\begin{aligned}\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_q \le C\Big[\|\mathbf{f}\|_q + \|\nabla\mathbf{T}\nabla\mathbf{w}\|_q + \|\mathbf{T}\nabla^2\mathbf{w}\|_q + \|\nabla p\nabla\mathbf{w}\|_q + \\ +\||\mathbf{w}|^2\nabla^2\mathbf{w}\|_q + \|\mathbf{w}|\nabla\mathbf{w}|^2\|_q + \||\nabla\mathbf{w}|^2\|_q + \|\mathbf{w}\nabla^2\mathbf{w}\|_q + \\ +\beta\|\nabla\mathbf{f}\|_q + \|\nabla\mathbf{f}\,\mathbf{w}\|_q + \|\mathbf{f}\nabla\mathbf{w}\|_q + \|(\mathbf{w}\cdot\nabla)\mathbf{w}\|_q\Big]\,.\end{aligned}$$

The most restrictive term is the last one. But (see Lemma V.2.2, inequality (V.2.6))

$$\|(\mathbf{w}\cdot\nabla)\mathbf{w}\|_q \le \beta^{-1-2(1-\frac{1}{q})}\langle\mathbf{w}\rangle^2_{\beta,q} \le \varepsilon^2\beta^{1+2(1-\frac{1}{q})}\,.$$

Applying Lemma V.2.2 also on other terms we get

$$\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_q \le C\Big[\varepsilon^2\beta^{1+2(1-\frac{1}{q})} + \varepsilon^2\beta^{2\alpha} + \varepsilon^3\beta^{\frac{4}{3}+2(1-\frac{1}{q})} + c(\mathbf{f})\Big] \tag{2.27}$$

i.e.

$$\|{}^{II}u_1^{S,3}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|{}^{II}u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le C\varepsilon^2\beta^{1-\omega}\,. \tag{2.28}$$

The last term can be estimated as follows

$$\|u_1^{S,4}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|u_2^{S,4}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le C\beta^{-\omega}\int_{\partial\Omega}|\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\mathrm{d}S\,.$$

Using the fact that

$$\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}\mathrm{d}S \le C(\|\mathbf{g}\|_q + \|\nabla\cdot\mathbf{g}\|_q)$$

for any $q>1$ and $\mathbf{g}\in\tilde{H}_q$ (see Remark VIII.3.6) we have

$$\begin{aligned}&\int_{\partial\Omega}|\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\mathrm{d}S \le\\ &\le C\Big[\|\mathbf{h}\|_{q,\Omega_1} + \|\mathbf{f}\|_q + (1+\beta)\beta^{-4(1-\frac{1}{q})}[|(\mathbf{w},s)|]_0^2 + \beta^2 + \beta\|\mathbf{f}\|_{1,q}\Big]\end{aligned} \tag{2.29}$$

and therefore

$$\|u_1^{S,4}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|u_2^{S,4}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le C\beta^{2\alpha-\omega}\,. \tag{2.30}$$

Combining (2.21), (2.22), (2.24), (2.28) and (2.30) yield

$$\|u_1^{S}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|u_2^{S}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le \frac{\varepsilon}{10}\beta^{1-\omega}\,. \tag{2.31}$$

Now, let $|\mathbf{x}|>\frac{1}{\beta}$. We study again the four terms separately.

$$\begin{aligned}|u_1^{S,1}(\mathbf{x})\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\mathbf{x};\beta)| &\le C\beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{\frac{1}{2}-\omega}(1+s(\beta\mathbf{x}))^{\frac{1}{2}}\cdot\\ &\cdot(|\mathcal{O}_{11}^\mu(\mathbf{x};\beta)| + |\nabla\mathcal{O}_{11}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)|) \le C(\beta^{2-\omega}+\beta^{3-\omega})\,.\end{aligned}$$

Analogously for $|u_2^{S,1}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)|$ the significant term is $\mathcal{O}_{12}^\mu$ and therefore for $\beta<1$

$$\|u_1^{S,1}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|u_2^{S,1}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le C\beta^{2-\omega}\,. \tag{2.32}$$

The second term is estimated similarly

$$\begin{aligned}&|u_1^{S,2}(\mathbf{x})\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\mathbf{x};\beta)| + |u_2^{S,2}(\mathbf{x})\overline{\mu}_0^{1,\omega}(\mathbf{x};\beta)| \le\\ &\le C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{\frac{1}{2}-\omega}\Big[(1+s(\beta\mathbf{x}))^{\frac{1}{2}} + (1+|\beta\mathbf{x}|)^{\frac{1}{2}-\omega}|\ln(2+|\beta\mathbf{x}|)|^{-1}\Big]\cdot\\ &\cdot\Big[|\nabla\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)| + |\mathbf{e}(\mathbf{x})| + |\nabla^2\boldsymbol{\mathcal{O}}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)| + |\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)|\Big] \le\\ &\le C\beta^{2-\omega}\Big(\frac{1}{(1+|\beta\mathbf{x}|)^{\frac{1}{2}}(1+s(\beta\mathbf{x}))^{\frac{1}{2}}} + \frac{(1+s(\beta\mathbf{x}))^{\frac{1}{2}}}{(1+|\beta\mathbf{x}|)^{\frac{1}{2}}} + |\ln(2+|\beta\mathbf{x}|)|^{-1}\Big)\end{aligned}$$

and therefore

$$\|u_1^{S,2}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|u_2^{S,2}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq C\beta^{2-\omega}\,. \tag{2.33}$$

In the estimate of the third term we must proceed as above — divide the surface integral into two parts where the first corresponds to the part with zero right-hand side, the other one to zero boundary condition. We have

$$\|u_1^{S,3}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq$$
$$\leq C\beta^{-\omega}\Big[\Big|\int_{\partial\Omega}\mathbf{T}({}^I\mathbf{u},{}^I\pi)\cdot\mathbf{n}\mathrm{d}S\Big| + \beta^{-2(1-\frac{1}{q})}\|\nabla\cdot\boldsymbol{\mathcal{G}}\|_q\Big]$$

and (2.23) together with (2.27) yield

$$\begin{aligned}\|u_1^{S,3}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|u_2^{S,3}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} &\leq \\ \leq C\beta^{1-\omega}|\ln\beta|^{-1}(\varepsilon^2+1)\,.\end{aligned} \tag{2.34}$$

The fourth term is then estimated by

$$\|u_1^{S,4}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|u_2^{S,4}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \leq \beta^{-\omega}\int_{\partial\Omega}|\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\mathrm{d}S\,.$$

But $\int_{\partial\Omega}|\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|\mathrm{d}S$ is estimated by (2.29); combining this with (2.20), (2.31)–(2.34) we have

$$\|u_1^{S}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega} + \|u_2^{S}\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega} \leq \frac{1}{10}\varepsilon\beta^{1-\omega}\,. \tag{2.35}$$

Recalling the estimates of the volume parts we finally have

$$\|u_1\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot\,;\beta)),\Omega} + \|u_2\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot\,;\beta)),\Omega} \leq \frac{1}{5}\varepsilon\beta^{1-\omega}\,. \tag{2.36}$$

We continue with the higher gradients of the velocity and the pressure, the L^r–weighted estimates. As in three dimensions we get the integral representation for higher gradients with the right hand side defined in (1.7). Nevertheless, for the first gradient we must be more careful.

$$\begin{aligned} D^\alpha u_j(\mathbf{x}) = \mathcal{A}_j^{(1,\alpha)}(\boldsymbol{\mathcal{G}}') + \int_\Omega D^\alpha_{\mathbf{x}}\frac{\partial\mathcal{N}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)}{\partial x_k}\mathcal{G}'_{ik}\mathrm{d}\mathbf{y} + \\ + \int_\Omega\frac{\partial\mathcal{O}^\mu_{1j}}{\partial x_1}(\mathbf{x}-\mathbf{y};\beta)(w_1D^\alpha_{\mathbf{y}}w_1)(\mathbf{y})\mathrm{d}\mathbf{y} + \int_{\partial\Omega}\Big[-\beta D^\alpha_{\mathbf{x}}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)u_i(\mathbf{y})\delta_{1k} + \\ + u_i(\mathbf{y})D^\alpha_{\mathbf{x}}T_{ik}(\boldsymbol{\mathcal{O}}^\mu_{\cdot j}e_j)(\mathbf{x}-\mathbf{y};\beta) + D^\alpha_{\mathbf{x}}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)T_{ik}(\mathbf{u},\pi)(\mathbf{y}) + \\ + D^\alpha_{\mathbf{x}}\mathcal{O}^\mu_{ij}(\mathbf{x}-\mathbf{y};\beta)\mathcal{G}'_{ik}\Big]n_k(\mathbf{y})\mathrm{d}_{\mathbf{y}}S\,,\end{aligned} \tag{2.37}$$

where $\mathcal{G}'_{ik} = \mathcal{G}_{ik} + w_iw_k\delta_{1i}\delta_{1k}$. (This follows from the fact that $\eta^\alpha_\beta \in A_r$, the Muckenhoupt class, for $\beta < \frac{r-1}{2}$ and we could not get the estimate with the weight $\mu_1^{1,\omega}(\cdot\,;\beta)$).

We start with the volume integrals.

Applying Theorem II.3.8 and Corollary II.3.6 we get

$$\begin{aligned} \|D^2\mathcal{O}^\mu * f\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta))} &\le C\|D^2\mathcal{O}^\mu * f\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r},\omega}(\cdot\,;\beta))} \le \\ &\le C\beta^{(k-1)\omega}\|f\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta,k\omega}(\cdot\,;\beta))}, \quad 0<\delta<\frac{1}{2r} \\ \|D\mathbf{e} * f\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta))} &\le C\beta^{(k-1)\omega}\|f\|_{r,(\mu_0^{1-\frac{3}{r},k\omega}(\cdot\,;\beta))}\,. \end{aligned} \tag{2.38}$$

(The convolutions are to be understood in the sense as shown in (1.45)). As in the threedimensional case, we estimate all terms in $\mathcal{G}'$ and $\nabla\cdot\mathcal{G}'$ in the corresponding L^r–weighted spaces.

$$\|\mathbf{h}\|_{r,(\eta_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta}(\cdot\,;\beta))} \le \|\mathbf{h}\|_{\infty,(\eta_{\frac{1}{2}}^{\frac{3}{2}}(\cdot\,;\beta))} \le \frac{\varepsilon}{20}\beta \tag{2.39}$$

$$\|\mathbf{f}\|_{r,(\eta_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta}(\cdot\,;\beta))} \le \|\mathbf{f}\|_{\infty,(\eta_{\frac{1}{2}}^{\frac{3}{2}}(\cdot\,;\beta))} \le \frac{\varepsilon}{20}\beta \tag{2.40}$$

$$\|\mathbf{h}\|_{r,(\eta_0^{1-\frac{3}{r}}(\cdot\,;\beta))} \le \|\mathbf{h}\|_{\infty,(\eta_0^{1}(\cdot\,;\beta))} \le \frac{\varepsilon}{20}\beta \tag{2.41}$$

$$\|\mathbf{f}\|_{r,(\eta_0^{1-\frac{3}{r}}(\cdot\,;\beta))} \le \|\mathbf{f}\|_{\infty,(\eta_0^{1}(\cdot\,;\beta))} \le \frac{\varepsilon}{20}\beta \tag{2.42}$$

and all the terms are sufficiently small due to the assumptions on the right hand side.

Next, using Remark 1.1 (ii) we easily estimate for $r > 2$

$$\begin{gathered} \|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \le \\ \le C\Big[\|\mathbf{T}\|_{r,(\mu_{\frac{1}{4}}^{\frac{3}{4},\omega}(\cdot\,;\beta))} + \|\nabla\mathbf{T}\|_{r,(\mu_{\frac{1}{4}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\Big]\|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{3}{4},\omega}(\cdot\,;\beta))} \le \\ \le C\Big[\|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{3}{4},\omega}(\cdot\,;\beta))} + \|\nabla^2\mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\Big]^2 \end{gathered}$$

and therefore

$$\|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \le C\varepsilon^2\beta^{2(1-\omega)}\,. \tag{2.43}$$

Analogously

$$\begin{gathered} \|\nabla\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot\,;\beta))} + \|\mathbf{T}\nabla^2\mathbf{w}\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot\,;\beta))} \le \\ \le C\Big[\|\nabla^2\mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{3}{4},\omega}(\cdot\,;\beta))} + \|\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{4}}^{\frac{3}{4},\omega}(\cdot\,;\beta))}\Big]^2 \le C\varepsilon^2\beta^{2(1-\omega)} \end{gathered} \tag{2.44}$$

$$\|\mathbf{T}\nabla\mathbf{w}\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot\,;\beta))} + \|\nabla(\mathbf{T}\nabla\mathbf{w})\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot\,;\beta))} \le C\varepsilon^2\beta^{2(1-\omega)} \tag{2.45}$$

$$\begin{aligned}\|p(\nabla\mathbf{w})^T\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot;\beta))} + \|\nabla p(\nabla\mathbf{w})^T\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot;\beta))} \le \\ +\|p(\nabla\mathbf{w})^T\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot;\beta))} + \|\nabla p(\nabla\mathbf{w})^T\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot;\beta))} \le \\ \le C\varepsilon^2\beta^{2(1-\omega)}\,.\end{aligned} \tag{2.46}$$

Next, using

$$\begin{aligned}\||\mathbf{w}|^2\nabla^k\mathbf{w}\|_r \le \|\nabla^k\mathbf{w}\|_r\|\mathbf{w}\|_\infty^2 \\ \||\nabla\mathbf{w}|^2\mathbf{w}\|_r \le \|\nabla\mathbf{w}\|_r\|\nabla\mathbf{w}\|_{1,r}\|\mathbf{w}\|_\infty\,, \quad r>2\end{aligned}$$

we get as above

$$\begin{aligned}\||\mathbf{w}|^2\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot;\beta))} + \|\nabla(|\mathbf{w}|^2\nabla\mathbf{w})\|_{r,(\mu_{\frac{1}{2}}^{\frac{3}{2},2\omega}(\cdot;\beta))} + \\ +\||\mathbf{w}|^2\nabla\mathbf{w}\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot;\beta))} + \|\nabla(|\mathbf{w}|^2\nabla\mathbf{w})\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot;\beta))} \le \\ \le C\varepsilon^3\beta^{3(1-\omega)}\,.\end{aligned} \tag{2.47}$$

Concerning the convective term we have now to distinguish two cases. If $i\cdot j \neq 1$ we have easily (see Lemma II.3.2)

$$\begin{aligned}\|w_iw_j\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta,2\omega}(\cdot;\beta))} \le C\|w_2\|_{\infty,(\overline{\mu}_0^{1,\omega}(\cdot;\beta))}\|w_1\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot;\beta))}\cdot \\ \cdot\Big(\int_\Omega(1+|\beta\mathbf{x}|)^{-2+\delta}(1+s(\beta\mathbf{x}))^{-1}|\ln((2+|\beta\mathbf{x}|)|^r\mathrm{d}\mathbf{x}\Big)^{\frac{1}{r}} \le \\ \le C\varepsilon^2\beta^{2-2\omega}\beta^{-\frac{2}{r}}\,.\end{aligned} \tag{2.48}$$

But for $i=j=1$ we do not apply the Green theorem (see Theorem VIII.1.15 and so we estimate by means of Theorem II.3.29

$$\begin{aligned}\Big\|\frac{\partial\mathcal{O}_{1j}^\mu}{\partial y_1}*(w_1\nabla w_1)\Big\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot;\beta))} \le \\ \le C\Big\|\frac{\partial\mathcal{O}_{1j}^\mu}{\partial y_1}*(w_1\nabla w_1)\Big\|_{r,(\mu_{\frac{1}{2}-\frac{1}{2r}}^{\frac{3}{2}-\frac{5}{2r},\omega}(\cdot;\beta))} \le \\ \le C\beta^{-1+\omega}\|w_1\nabla w_1\|_{r,(\mu_{\frac{1}{2}-\frac{1}{2r}}^{\frac{3}{2}-\frac{2}{r},2\omega}(\cdot;\beta))} \le \\ \le C\beta^{-1+\omega}\|w_1\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot;\beta))}\|\nabla w_1\|_{r,(\mu_0^{1-\frac{2}{r},\omega}(\cdot;\beta))} \le \\ \le C\varepsilon^2\beta^{1-\omega}\,.\end{aligned} \tag{2.49}$$

The other estimates are easier

$$\begin{aligned}\|\mathbf{w}\cdot\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r},2\omega}(\cdot;\beta))} \le \\ \le \|\mathbf{w}\|_{\infty,(\mu_{\frac{1}{2}}^{\frac{1}{2},\omega}(\cdot;\beta))}\|\nabla\mathbf{w}\|_{r,(\mu_0^{1-\frac{2}{r},\omega}(\cdot;\beta))} \le C\varepsilon^2\beta^{2(1-\omega)}\end{aligned} \tag{2.50}$$

$$\begin{gathered} \|w_i w_j\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot\,;\beta))} \leq \\ \leq \|\mathbf{w}\|^2_{\infty,(\mu_0^{\frac{1}{2},\omega}(\cdot\,;\beta))} \Big(\int_\Omega (1+|\beta\mathbf{x}|)^{-3}\mathrm{d}\mathbf{x}\Big)^{\frac{1}{r}} \leq C\varepsilon^2\beta^{2(1-\omega)-\frac{2}{r}} \end{gathered} \tag{2.51}$$

$$\begin{gathered} \|\mathbf{w}\cdot\nabla\mathbf{w}\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot\,;\beta))} \leq \\ \leq \|\mathbf{w}\|_{\infty,(\mu_0^{\frac{1}{2},\omega}(\cdot\,;\beta))} \|\nabla\mathbf{w}\|_{r,(\mu_0^{1-\frac{2}{r},\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta^{2(1-\omega)}\,. \end{gathered} \tag{2.52}$$

Finally, the last terms can be estimated easily combining the estimates above

$$\begin{gathered} \|\beta\mathbf{w}\cdot\nabla\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta,2\omega}(\cdot\,;\beta))} + \|\beta\nabla(\mathbf{w}\cdot\nabla\mathbf{w})\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta,2\omega}(\cdot\,;\beta))} + \\ + \|\beta\mathbf{w}\cdot\nabla\mathbf{w}\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot\,;\beta))} + \|\beta\nabla(\mathbf{w}\cdot\nabla\mathbf{w})\|_{r,(\mu_0^{1-\frac{3}{r},2\omega}(\cdot\,;\beta))} \leq \\ \leq C\varepsilon^2\beta^{3-2\omega} \end{gathered} \tag{2.53}$$

$$\begin{gathered} \|\mathbf{f}\,\mathbf{w}\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta,2\omega}(\cdot\,;\beta))} + \|\nabla\mathbf{f}\,\mathbf{w})\|_{r,(\mu_{\frac{1}{2}-\frac{1}{r}}^{\frac{3}{2}-\frac{2}{r}+\delta,2\omega}(\cdot\,;\beta))} + \\ + \|\mathbf{f}\,\mathbf{w}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta))} + \|\nabla\mathbf{f}\,\mathbf{w}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta))} \leq C\varepsilon^2\beta\,. \end{gathered} \tag{2.54}$$

Therefore, combining (2.39)–(2.54) we have for ε, β sufficiently small

$$\begin{gathered} \|\nabla\mathbf{u}^V\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta))} + \|\nabla^2\mathbf{u}^V\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta))} + \\ \|\pi^V\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta))} + \|\nabla\pi^V\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta))} \leq \frac{\varepsilon}{10}\beta^{1-\omega}\,. \end{gathered} \tag{2.55}$$

Next we continue with the boundary terms. We denote as usually the corresponding surface integral by $\nabla\mathbf{u}^{S,i}$, $i=1,2,3,4$ and distinguish three cases

(i) $|\mathbf{x}| \leq 1$

(ii) $1 < |\mathbf{x}| \leq \frac{1}{\beta}$

(iii) $\frac{1}{\beta} < |\mathbf{x}|$.

As the weighted estimates are equivalent to the standard L^q–norms for $|\mathbf{x}| \leq 1$ and

$$\begin{gathered} \|\nabla\mathbf{u},\nabla^2\mathbf{u}\|_{r,\Omega_1} \leq C(\|\nabla^2\mathbf{u}\|_{1,p} + \|\nabla\mathbf{u}\|_{\frac{2q}{2-q},\Omega_1}) \leq \\ \leq C(\|\nabla^2\mathbf{u}\|_{1,p} + \|\nabla^2\mathbf{u}\|_q) \leq C\varepsilon\beta^\alpha \\ \|\pi,\nabla\pi\|_{r,\Omega_1} \leq C(\|\nabla\pi\|_{1,p} + \|\pi\|_{\frac{2q}{2-q},\Omega_1}) \leq \\ \leq C(\|\nabla\pi\|_{1,p} + \|\nabla\pi\|_q) \leq C\varepsilon\beta^\alpha\,, \end{gathered}$$

we have

$$\begin{gathered} \|\nabla\mathbf{u}^S,\nabla^2\mathbf{u}^S\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega_1} + \|\pi^S,\nabla\pi^S\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta)),\Omega_1} \leq \\ \leq C\varepsilon\beta^\alpha\,, \quad \alpha\in\Big[\frac{2}{3};1\Big)\,. \end{gathered} \tag{2.56}$$

Now let $1 < |\mathbf{x}| \leq \frac{1}{\beta}$, $\beta < 1$. Clearly as in the threedimensional case it is enough to get estimates for $\nabla\mathbf{u}^S$, π^S; those for $\nabla^2\mathbf{u}^S$, $\nabla\pi^S$ are much easier.

$$|\nabla \mathbf{u}^{S,1}(\mathbf{x})\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\mathbf{x};\beta)| \leq$$
$$\leq C\beta^3|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{2}{r}}(1+s(\beta\mathbf{x}))^{1-\frac{1}{r}}\Big(\frac{1}{|\beta\mathbf{x}|}+\frac{\beta}{|\beta\mathbf{x}|^2}\Big)$$

and so

$$\begin{aligned}\|\nabla \mathbf{u}^{S,1}\|^r_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} &\leq C\beta^{(3-\omega)r}\int_{\Omega^1_{\frac{1}{\beta}}}(|\beta\mathbf{x}|^{(\omega-1)r}+\beta^r|\beta\mathbf{x}|^{(\omega-2)r})\cdot\\ &\cdot(1+|\beta\mathbf{x}|)^{(1-\omega-\frac{2}{r})r}(1+s(\beta\mathbf{x}))^{r-1}\mathrm{d}\mathbf{x}\leq \\ &\leq C\beta^{3r-\omega r-2}\int_\beta^1|\mathbf{y}|^{\omega r-r+2}\mathrm{d}|\mathbf{y}|\leq C\beta^{2r}\,.\end{aligned} \tag{2.57}$$

Analogously

$$\begin{aligned}\|\nabla \mathbf{u}^{S,2}\|^r_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} &\leq \beta^{(3-\omega)r}\int_{\Omega^1_{\frac{1}{\beta}}}(|\beta\mathbf{x}|^{(\omega-2)r}+\beta^r|\beta\mathbf{x}|^{(\omega-3)r})\cdot\\ &\cdot(1+|\beta\mathbf{x}|)^{(1-\omega-\frac{2}{r})r}(1+s(\beta\mathbf{x}))^{r-1}\mathrm{d}\mathbf{x}\leq C\beta^{2r}\,.\end{aligned} \tag{2.58}$$

Combining the estimates in the threedimensional case with (2.26) and (2.29) we get

$$\begin{aligned}&\|\nabla \mathbf{u}^{S,3}\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}}+\|\nabla \mathbf{u}^{S,4}\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}}\leq\\ &\leq C|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{2}{r}}(1+s(\beta\mathbf{x}))^{1-\frac{1}{r}}\Big[|\nabla\boldsymbol{\mathcal{O}}^\mu(\mathbf{x};\beta)|+\\ &+|\nabla^2\boldsymbol{\mathcal{O}}^\mu\Big(\frac{\mathbf{x}}{2};\beta\Big)|\Big]\int_{\partial\Omega}(|\nabla\mathbf{u}|+|\pi|+|\boldsymbol{\mathcal{G}}|)\mathrm{d}S\leq C\Big[\|\nabla^2\mathbf{u}\|_q+\\ &+\|\nabla\pi\|_q+c(\mathbf{f})+\beta^2+[|(\mathbf{w},s)|]_0^2\beta^{-4(1-\frac{1}{q})}\Big]\leq C\beta^\alpha\,,\end{aligned} \tag{2.59}$$

i.e.

$$\|\nabla\mathbf{u}^S,\nabla^2\mathbf{u}^S\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}}\leq C\beta^\alpha\,. \tag{2.60}$$

Next we estimate the pressure terms in $\Omega^1_{\frac{1}{\beta}}$

$$|\pi^{S,1}(\mathbf{x})\mu_0^{1-\frac{3}{r},\omega}(\mathbf{x};\beta)|\leq C\beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{3}{r}}\Big(|\mathbf{e}(\mathbf{x})|+|\nabla\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)|\Big)$$

and so

$$\|\pi^{S,1}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}}\leq C\beta^2\,. \tag{2.61}$$

Analogously

$$|\pi^{S,2}(\mathbf{x})\mu_0^{1-\frac{3}{r},\omega}(\mathbf{x};\beta)|\leq$$
$$\leq C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{3}{r}}\Big(\beta|\mathbf{e}(\mathbf{x})|+|\nabla\mathbf{e}(\mathbf{x})|+|\nabla^2\mathbf{e}\Big(\frac{\mathbf{x}}{2}\Big)|\Big)$$

and

$$\|\pi^{S,2}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}}\leq C\beta\,. \tag{2.62}$$

The terms $\pi^{S,3}$ and $\pi^{S,4}$ can be estimated as $\nabla\mathbf{u}^{S,3}$ and $\nabla\mathbf{u}^{S,4}$; therefore

$$\begin{aligned}&\|\pi^{S,3}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} + \|\pi^{S,4}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le \\ &\le C\int_{\partial\Omega}(|\nabla\mathbf{u}| + |\pi| + |\boldsymbol{\mathcal{G}}\cdot\mathbf{n}|)\mathrm{d}S \le C\beta(\varepsilon\beta^\alpha + \varepsilon^2\beta^{2\alpha})\,.\end{aligned} \tag{2.63}$$

Summarizing (2.61)–(2.63) yields

$$\|\pi^S\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot\,;\beta)),\Omega^1_{\frac{1}{\beta}}} \le \frac{\varepsilon}{20}\beta^{1-\omega}\,. \tag{2.64}$$

Finally let $|\mathbf{x}| > \frac{1}{\beta}$. Now

$$\begin{aligned}&|\nabla\mathbf{u}^{S,1}(\mathbf{x})\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\mathbf{x};\beta)| \le C\beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{2}{r}}(1+s(\beta\mathbf{x}))^{1-\frac{1}{r}}\cdot \\ &\cdot\Big[\frac{\beta}{(1+|\beta\mathbf{x}|)(1+s(\beta\mathbf{x}))} + \frac{\beta^2}{(1+|\beta\mathbf{x}|)^{\frac{3}{2}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}}}\Big]\end{aligned}$$

and

$$\begin{aligned}&\|\nabla\mathbf{u}^{S,1}\|^r_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le \\ &\le C\beta^{(3-\omega)r}\int_{\frac{1}{\beta}}^\infty(1+|\beta\mathbf{x}|)^{-2}(1+s(\beta\mathbf{x}))^{-1}|\mathbf{x}|\mathrm{d}|\mathbf{x}| \le C\beta^{(3-\omega)r-2}\,.\end{aligned} \tag{2.65}$$

$$\begin{aligned}&|\nabla\mathbf{u}^{S,2}(\mathbf{x})\mu_{1-\frac{1}{r}}^{1-\frac{3}{r},\omega}(\mathbf{x};\beta)| \le \beta^3|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{2}{r}}(1+s(\beta\mathbf{x}))^{1-\frac{1}{r}}\cdot \\ &\cdot\Big[\frac{1}{(1+|\beta\mathbf{x}|)^{\frac{3}{2}}(1+s(\beta\mathbf{x}))^{\frac{3}{2}}} + \frac{1}{|\beta\mathbf{x}|^2} + \\ &+\frac{\beta}{(1+|\beta\mathbf{x}|)^2(1+s(\beta\mathbf{x}))^2} + \frac{\beta}{|\beta\mathbf{x}|^3}\Big]\end{aligned}$$

and

$$\begin{aligned}&\|\nabla\mathbf{u}^{S,2}\|^r_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le \\ &\le C\beta^{(3-\omega)r-2}\int_{B^1(\mathbf{0})}|\mathbf{y}|^{-2}(1+s(\mathbf{y}))^{-1}\mathrm{d}\mathbf{y} \le C\beta^{(3-\omega)r-2}\,.\end{aligned} \tag{2.66}$$

Analogously

$$\begin{aligned}&\|\nabla\mathbf{u}^{S,3}\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} + \|\nabla\mathbf{u}^{S,4}\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega^{\frac{1}{\beta}}} \le \\ &\le C\beta^{1-\omega-\frac{2}{r}}\Big(\int_{B^1(\mathbf{0})}|\mathbf{y}|^{-2}(1+s(\mathbf{y}))^{-1}\mathrm{d}\mathbf{y}\Big)^{\frac{1}{r}}\int_{\partial\Omega}(|\nabla\mathbf{u}| + |\pi| + |\boldsymbol{\mathcal{G}}'|)\mathrm{d}S \le \\ &\le C\beta^{1+\alpha-\omega-\frac{2}{r}}\end{aligned} \tag{2.67}$$

and collecting (2.56), (2.60), (2.65)–(2.67)

$$\|\nabla\mathbf{u}^S\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega} + \|\nabla^2\mathbf{u}^S\|_{r,(\mu_{1-\frac{1}{r}}^{1-\frac{2}{r},\omega}(\cdot\,;\beta)),\Omega} \le \frac{\varepsilon}{10}\beta^{1-\omega} \tag{2.68}$$

for β, ε sufficiently small, α sufficiently close to 1. The pressure terms are treated in the same way

$$|\pi^{S,1}(\mathbf{x})\mu_0^{1-\frac{3}{r},\omega}(\mathbf{x};\beta)| \le \beta^2|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{3}{r}}\Big(\frac{1}{|\mathbf{x}|}+\frac{1}{|\mathbf{x}|^2}\Big)$$

i.e.

$$\begin{gathered}\|\pi^{S,1}\|^r_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot;\beta)),\Omega^{\frac{1}{\beta}}} \le \\ \le C\beta^{(3-\omega)r}\int_{|\mathbf{x}|>\frac{1}{\beta}}(|\beta\mathbf{x}|^{-3}+\beta|\beta\mathbf{x}|^{-r-3})\mathrm{d}\mathbf{x} \le C\beta^{(3-\omega)r-2}\,.\end{gathered} \tag{2.69}$$

Next

$$\begin{gathered}|\pi^{S,2}(\mathbf{x})\mu_0^{1-\frac{3}{r},\omega}(\mathbf{x};\beta)| \le \\ \le C\beta|\mathbf{x}|^\omega(1+|\beta\mathbf{x}|)^{1-\omega-\frac{3}{r}}\Big[\frac{\beta}{|\mathbf{x}|}+\frac{1}{|\mathbf{x}|^2}+\frac{1}{|\mathbf{x}|^3}\Big]\end{gathered}$$

and

$$\|\pi^{S,2}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot;\beta)),\Omega^{\frac{1}{\beta}}} \le C\beta^{(3-\omega)-\frac{2}{r}}\,. \tag{2.70}$$

Finally

$$\begin{gathered}\|\pi^{S,3}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot;\beta)),\Omega^{\frac{1}{\beta}}} + \|\pi^{S,4}\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot;\beta)),\Omega^{\frac{1}{\beta}}} \le \\ \le C\beta^{(1-\omega-\frac{2}{r})}\Big(\int_{B^1(\mathbf{0})}|\mathbf{y}|^{-3}\mathrm{d}\mathbf{y}\Big)^{\frac{1}{r}}\int_{\partial\Omega}(|\nabla\mathbf{u}|+|\pi|+|\boldsymbol{\mathcal{G}}'|)\mathrm{d}S \le \\ \le C\beta^{1+\alpha-\omega-\frac{2}{r}}\,.\end{gathered} \tag{2.71}$$

Collecting (2.56), (2.64), (2.69)–(2.71)

$$\|\pi^S\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot;\beta)),\Omega} + \|\nabla\pi^S\|_{r,(\mu_0^{1-\frac{3}{r},\omega}(\cdot;\beta)),\Omega} \le \frac{1}{10}\varepsilon\beta^{1-\omega}\,, \tag{2.72}$$

where ε, β must be assumed sufficiently small. We have proved

Theorem 2.1 *Let* $\mathbf{f} = \nabla\cdot\mathbf{h}$ *and let* $\mathbf{h} \in L^1_{loc}(\overline{\Omega})$, $\mathbf{f} \in W^{2,q}(\Omega)\cap W^{k,p}(\Omega)$, $q \in (1;\frac{6}{5})$, $k \ge 2$ *with the norms sufficiently small. Let* $\Omega \in C^{k+1}$ *be an exterior domain in* $\mathbb{R}^2$*. Moreover, let*

$$\mathbf{h},\mathbf{f},\nabla\mathbf{f} \in L^\infty(\Omega;\eta_{\frac{1}{2}}^{\frac{3}{2}}(\cdot)) \tag{2.73}$$

and let $\beta = |\mathbf{v}_\infty|$ *and* $\|\mathbf{h},\mathbf{f},\nabla\mathbf{f}\|_{\infty,(\eta_{\frac{1}{2}}^{\frac{3}{2}}(\cdot))}$ *be sufficiently small.*

Then $(\mathbf{v},p)$*, solution to the problem (I.4.14)–(I.4.15) constructed in Theorem V.2.1 has the following asymptotic properties*

$$\begin{gathered}u_1 = v_1-\beta\mathbf{e}_1 \in L^\infty(\Omega;\eta_{\frac{1}{2}}^{\frac{1}{2}}(\cdot)) \\ v_2 \in L^\infty(\Omega;\eta_0^1(\cdot)|\ln(2+\cdot)|^{-1}) \\ \nabla\mathbf{v},\nabla^2\mathbf{v} \in L^r(\Omega;\eta_{1-\frac{1}{r}}^{1-\frac{2}{r}}(\cdot)) \\ p,\nabla p \in L^r(\Omega;\eta_0^{1-\frac{3}{r}}(\cdot))\,,\end{gathered} \tag{2.74}$$

where $r \in (5;\infty)$.

Remark 2.1 From (2.74) it follows that our solution has almost the same asymptotic behaviour ar the fundamental solution to the Oseen problem (for u_2 up to a logarithmic term, for $\nabla\mathbf{u}$, p up to a very small power for r sufficiently large). If we are not interested in the precise asymptotic structure of $\nabla\mathbf{v}$, we can weaken the assumptions on $\mathbf{f}$. Namely for $\mathbf{h}, \mathbf{f}, \nabla\mathbf{f} \in L^\infty(\Omega; \eta^1_{\frac{1}{2}}(\cdot))$ (indeed, with a sufficiently small norm) we would get $(2.74)_{1,2,4}$ and instead of $(2.74)_3$ only $\nabla\mathbf{v}, \nabla^2\mathbf{v} \in L^r(\Omega; \eta^{1-\frac{2}{r}}_{\frac{1}{2}-\frac{1}{r}}(\cdot))$.

Using Remark 1.1 (i.e. the imbedding theorem) we easily see that we have $\nabla\mathbf{v} \in L^\infty(\Omega; \eta^{1-\frac{2}{r}}_{1-\frac{1}{r}}(\cdot))$, i.e. in other words, $\nabla\mathbf{v} \in L^\infty(\Omega; \eta^{1-2\varepsilon}_{1-\varepsilon}(\cdot))$ for any $\varepsilon > 0$. Analogously $p \in L^\infty(\Omega; \eta^{1-\varepsilon}_0(\cdot))$ for all $\varepsilon > 0$. Unlike the threedimensional case, the assumptions (2.73) are not sufficient in order to assure the existence of solution; we must add an assumption on the second gradient of $\mathbf{f}$.

VII

Axially symmetric flow of the ideal and viscous fluid in $\mathbb{R}^3$

This chapter is devoted to the study of non–steady axially symmetric flow in the whole $\mathbb{R}^3$. We especially deal with the ideal incompressible fluid (i.e. the Euler equations, see (I.2.5)) and the viscous incompressible Newtonian fluid (i.e. the Navier–Stokes equations, see (I.3.6)). In Section VII.1 we study the viscous fluid and obtain for axially symmetric data (the right hand side and the initial condition) that the corresponding solution exists on any compact subinterval of $[0;\infty)$ and is regular as the right hand side and initial condition allow. Next, using this result we get some estimates independent of viscosity, pass with it to zero and obtain analogous result for the Euler equations. Let us mention that the part dealing with the viscous fluid is taken from [LeMaNePo] while the other part has not been published yet.

Definition 0.1 *A scalar function φ written in cylindrical coordinates is called axially symmetric if it is independent of θ, i.e. $\varphi = \varphi(r, z)$.*

A vector function $\xi = (\xi_r, \xi_\theta, \xi_z)$ is called axially symmetric if $\xi_\theta \equiv 0$ and ξ_r and ξ_z are axially symmetric.

VII.1 Viscous fluid

In the early thirties J. Leray studies the Cauchy problem for the Navier–Stokes equations (see [Ler]) and shows that in two spatial dimensions there exists uniquely determined solution while in three dimensions he only shows the existence of a "turbulent" solution (in fact, weak) and the question of its uniqueness (in the class of weak solutions) as well as its regularity remains open. In fifties and sixties, Hopf and Ladyzhenskaya (see [Ho] and [Lad1]) extend his results to the boundary value problems. However, the situation remains the same; in two spatial dimensions the solution is unique and regular as the data of the problem allow, in three dimensions only the existence of weak solutions was established.

J. Leray in [Ler] even proposed a possible construction of a singular solution for regular data. This construction was recently excluded in [NeRuSv]; see also [MaNePoSc] or even for a larger class of solutions [Ts]. The question of regularity[1] and uniqueness of weak solutions in three space dimensions remains still one of the most fundamental problems in mathematical theory of fluid dynamics.

[1]Evidently, here we speak about global-in-time regularity. The local-in-time regularity for sufficiently smooth data can be shown easily, see also Theorem 1.1.

However, if only axially symmetric flows are permitted then it is possible to show global-in-time existence of regular solution (see Theorem 1.2 below). As well-known, this solution is unique even in the class of all weak solutions considered for axially symmetric data only (see Theorem 1.3 below).

At the world congress of mathematics in Moscow (1966) O.A. Ladyzhenskaya presented new apriori estimates concerning the axially symmetric flow of viscous fluid in the whole $\mathbb{R}^3$. The proof of existence of global-in-time regular solution to the Cauchy problem for the Navier–Stokes equations for axially symmetric data was then presented in [Lad2]; the same idea was used by Uchovskii and Yudovich in [UcYu], the latter was even published some months sooner than [Lad2] and contained also the study of axially symmetric flow of the incompressible ideal fluid. Surprisingly, their results do not seem to be known in the wide Navier-Stokes community, situation which might be caused by the technicalities occurring within the proof (for example a special basis in cylindrical coordinates on bounded balls with increasing radius is constructed in order to define convenient approximations).

The aim of this paper is to give another proof, following an elementary and clear method. We first consider a viscous fluid; this enables us to build our proof on known (and nowadays standard) results on existence, uniqueness and regularity of a weak solution to the evolutionary Stokes system. Starting from this, we present (in Subsection VII.1.1) local-in-time existence and uniqueness of smooth axially symmetric solution to the Navier-Stokes system.

Subsection VII.1.2 is then devoted to the derivation of some “global” estimates, which allow us to extend the smooth solution to arbitrary time interval. We wish to emphasize that the crucial trick in this procedure is due to Ladyzhenskaya.

VII.1.1 Axially symmetric solution on a short time interval

The Navier-Stokes equations in $\mathbb{R}^3$, written in cartesian coordinates x_1, x_2, x_3, have a non-dimensional form

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0 \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f}\,, \end{aligned} \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2, v_3) : (0,\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$ and $p : (0,\infty)\times\mathbb{R}^3 \to \mathbb{R}$ are unknowns and $\mathbf{f} = (f_1, f_2, f_3) : (0,\infty) \to \mathbb{R}^3$ is prescribed. System (1.1) is completed by an initial condition

$$\mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x})\,,\ \mathbf{x}\in\mathbb{R}^3\,, \qquad \text{where } \nabla\cdot\mathbf{v}_0 = 0\,.$$

In cylindrical coordinates given by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$, equations (1.1) are transformed into the system

$$
\begin{aligned}
&\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{r} v_\theta \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{1}{r} v_\theta^2 + \frac{\partial p}{\partial r} - \\
&\qquad -\nu \Big[\frac{1}{r}\frac{\partial}{\partial r}\big(r \frac{\partial v_r}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \Big] = f_r \\
&\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{1}{r} v_\theta \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{1}{r} v_\theta v_r + \frac{1}{r}\frac{\partial p}{\partial \theta} - \\
&\qquad -\nu \Big[\frac{1}{r}\frac{\partial}{\partial r}\big(r \frac{\partial v_\theta}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \Big] = f_\theta \qquad (1.2) \\
&\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{1}{r} v_\theta \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} + \frac{\partial p}{\partial z} - \\
&\qquad -\nu \Big[\frac{1}{r}\frac{\partial}{\partial r}\big(r \frac{\partial v_z}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \Big] = f_z \\
&\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0\,.
\end{aligned}
$$

If ξ stands instead of $\mathbf{v}$, $\mathbf{v}_0$ or $\mathbf{f}$ above, then by $(\xi_r, \xi_\theta, \xi_z)$ we mean the vector $(\xi_1 \cos\theta + \xi_2 \sin\theta, -\xi_1 \sin\theta + \xi_2 \cos\theta, \xi_3)$.

The objective of this subsection is to show that if $\mathbf{v}_0$ and $\mathbf{f}$ are axially symmetric, then there exist a $t > 0$ and the axially symmetric solution $(\mathbf{v}, p)$ of (1.1) defined on $(0, t)$ satisfying the initial condition. First, we deal with the evolutionary Stokes system.

Lemma 1.1 *Let $T \in (0; \infty)$, $I = (0; T)$ and let $k \geq 1$, $k \in \mathbb{N}$. Let us assume that $\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^3)$ and $\mathbf{F} \in L^2(I; W^{k-1,2}(\mathbb{R}^3))$ are axially symmetric. Then there exists exactly one weak (and also strong) solution to the Stokes problem (in $I \times \mathbb{R}^3$)*

$$
\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{F}\,, \qquad \nabla \cdot \mathbf{v} = 0\,,
$$
$$
\mathbf{v}(0, \mathbf{x}) = \mathbf{v}_0(\mathbf{x})\,, \qquad \mathbf{x} \in \mathbb{R}^3\,,
$$

such that $\mathbf{v} \in L^\infty(I; W^{k,2}(\mathbb{R}^3)) \cap L^2(I; W^{k+1}(\mathbb{R}^3))$, $\frac{\partial \mathbf{v}}{\partial t} \in L^2(I; W^{k-1}(\mathbb{R}^3))$ and $\nabla p \in L^2(I; W^{k-1,2}(\mathbb{R}^3))$ $(k \geq 2)$.

Moreover, $\mathbf{v}$ and p are axially symmetric.

Proof: The existence and uniqueness as well as the energy estimates are classical, in the whole space we can directly apply the difference-quotient method to obtain the following energy inequalities ($J = (0; t)$, $t \in I$)

$$
\begin{aligned}
\|\mathbf{v}\|^2_{L^\infty(J;L^2(\mathbb{R}))} &\leq \|\mathbf{v}(0)\|_2^2 + \frac{c}{\nu}\int_0^t \|\mathbf{F}(\tau)\|_2^2 d\tau \\
\|D^k\mathbf{v}\|^2_{L^\infty(J;L^2(\mathbb{R}))} &\leq \|D^k\mathbf{v}(0)\|_2^2 + \frac{c}{\nu}\int_0^t \|D^{k-1}\mathbf{F}(\tau)\|_2^2 d\tau \\
\nu \int_0^t \|D^3\mathbf{v}(\tau)\|_2^2 d\tau &\leq \frac{c}{\nu}\int_0^t \|\nabla\mathbf{F}(\tau)\|_2^2 d\tau + \|D^2\mathbf{v}(0)\|_2^2 \\
\int_0^t \|D^k \frac{\partial \mathbf{v}}{\partial t}(\tau)\|_2^2 d\tau &\leq \frac{c}{\nu}\int_0^t \|D^k\mathbf{F}(\tau)\|_2^2 d\tau + c\|D^{k+1}\mathbf{v}(0)\|_2^2\,,\ k \geq 0\,.
\end{aligned}
\qquad (1.3)
$$

Moreover, the pressure p solves

$$
\Delta p = \nabla \cdot \mathbf{F}\,,
$$

which leads to the regularity for p.

It remains to show that the solution is axially symmetric. Transforming the Stokes system into the cylindrical coordinates we obtain

$$\begin{aligned}\frac{\partial v_r}{\partial t} - \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_r}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\Big] + \frac{\partial p}{\partial r} = F_r \\ \frac{\partial v_\theta}{\partial t} - \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_\theta}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\Big] + \frac{1}{r}\frac{\partial p}{\partial \theta} = F_\theta \\ \frac{\partial v_z}{\partial t} - \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\Big] + \frac{\partial p}{\partial z} = F_z \\ \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0\,.\end{aligned}$$

We see that all the differential operators on the left-hand side of this system commute with the operator $\frac{\partial}{\partial\theta}$. Denoting

$$\mathbf{u} \equiv \Big(\frac{\partial v_r}{\partial \theta}, \frac{\partial v_\theta}{\partial \theta}, \frac{\partial v_z}{\partial \theta}\Big)\,, \qquad q \equiv \frac{\partial p}{\partial \theta}$$

and using the assumptions on axial symmetry of $\mathbf{F}$ and $\mathbf{v}_0$, we obtain (after returning to the cartesian coordinates)

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla q = \mathbf{0}\,, &\qquad \nabla\cdot\mathbf{u} = 0\,, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{0}\,, &\qquad \mathbf{x}\in\mathbb{R}^3\,.\end{aligned}$$

Therefore, thanks to the uniqueness of square integrable solutions to the Stokes system, $\mathbf{u} \equiv \mathbf{0}$ and $q \equiv 0$. Substituting back this fact into the equation for $\mathbf{v}$ and p in the cylindrical coordinates we get

$$\begin{aligned}\frac{\partial v_r}{\partial t} - \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_r}{\partial r}) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\Big] + \frac{\partial p}{\partial r} = F_r \\ \frac{\partial v_\theta}{\partial t} - \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_\theta}{\partial r}) + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2}\Big] = 0 \\ \frac{\partial v_z}{\partial t} - \nu\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r}) + \frac{\partial^2 v_z}{\partial z^2}\Big] + \frac{\partial p}{\partial z} = F_z \\ \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0\end{aligned}$$

and $\mathbf{v}_0 = ((v_0)_r, 0, (v_0)_z)$. Since v_θ occurs only in the second equation above, again the uniqueness argument implies that $v_\theta \equiv 0$. Thus the solution is necessarily axially symmetric.

□

We now construct axially symmetric solution to the full Navier-Stokes system. Let $t > 0$ and $J = (0;t)$. We set

$$X = X(t) \equiv \Big\{\mathbf{u} \in L^\infty(J; W^{2,2}(\mathbb{R}^3)); \mathbf{u} \text{ axially symmetric}\Big\}\,.$$

Further, let $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$ and $\mathbf{f} \in L^2(0,\infty; W^{1,2}(\mathbb{R}^3))$ be axially symmetric, $\nabla\cdot\mathbf{v}_0 = 0$. Take $\mathbf{v} \in X$ and define an operator $S : X \mapsto X$ in such a way that $\mathbf{u} \equiv S(\mathbf{v})$ solves the evolutionary Stokes system with the initial value $\mathbf{v}_0$ and the right-hand side $\mathbf{f} - v_k \frac{\partial \mathbf{v}}{\partial x_k}$. Notice that $\mathbf{f} - v_k \frac{\partial \mathbf{v}}{\partial x_k}$ is axially symmetric.[2] Consequently, by Lemma 1.1, we observe that $\mathbf{u} \in X(t)$ for all $t \in (0;\infty)$ and p is axially symmetric.

We will show that for t sufficiently small S is a contraction in X. The Banach fixed point theorem gives then the existence of unique solution to the Navier–Stokes equations on $(0;t)$; the solution is moreover strong as belongs to X.

In the sequel we will frequently use the classical interpolation inequality

$$\|\mathbf{z}\|_4 \le \|\mathbf{z}\|_2^{1/4}\|\mathbf{z}\|_6^{3/4} \le c\|\mathbf{z}\|_2^{1/4}\|\nabla \mathbf{z}\|_2^{3/4},$$

and also two inequalities of Agmon's type (see Theorem VIII.1.12)

$$\begin{aligned} \|\mathbf{z}\|_\infty &\le c\,\|\mathbf{z}\|_2^{1/4}\|\nabla^2\mathbf{z}\|_2^{3/4} \\ \|\mathbf{z}\|_\infty &\le c\,\|\nabla\mathbf{z}\|_2^{1/2}\|\nabla^2\mathbf{z}\|_2^{1/2}. \end{aligned} \tag{1.4}$$

Theorem 1.1 *Let $\mathbf{f} \in L^2_{loc}(0,\infty;W^{1,2}(\mathbb{R}^3))$ and $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$ (divergence free) be axially symmetric. Then there exists exactly one solution $(\mathbf{v},p)$ such that $\mathbf{v} \in L^\infty(J;W^{2,2}(\mathbb{R}^3)) \cap L^2(J;W^{3,2}(\mathbb{R}^3))$, $\frac{\partial \mathbf{v}}{\partial t} \in L^2(J;W^{1,2}(\mathbb{R}^3))$, $\nabla p \in L^\infty(J;L^2(\mathbb{R}^3))$ solving the Navier–Stokes equations on (possibly short) time interval $J=(0;t)$. Moreover, $\mathbf{v}$ and p are axially symmetric.*

Proof: It is easy to see, with help of (1.3), that for $C \equiv \|\mathbf{v}_0\|_{2,2}^2 + 1$ there is a $t_0 > 0$ such that $\|S(\mathbf{v})\|_X \le C$ whenever $\|\mathbf{v}\|_X \le C$. Then it remains to verify that $S : X \mapsto X$ is a contraction. For this purpose fix $\mathbf{v}^i \in X$, $\|\mathbf{v}^i\|_X \le C$, $i = 1,2$. As $S(\mathbf{v}^2) - S(\mathbf{v}^1)$ satisfies the Stokes system with zero initial condition and the right-hand side $\mathbf{g} \equiv -\mathbf{v}_k^2 \frac{\partial \mathbf{v}^2}{\partial x_k} + \mathbf{v}_k^1 \frac{\partial \mathbf{v}^1}{\partial x_k}$, it is sufficient to estimate $\mathbf{g}$ and $\nabla\mathbf{g}$ in $L^2(J;L^2(\mathbb{R}^3))$ (cf. (1.3)). We have (with help of (1.4))

$$\begin{aligned} \int_0^t \|\nabla\mathbf{g}(\tau)\|_2^2 d\tau \le & \int_0^t \|\,|\nabla(\mathbf{v}^2-\mathbf{v}^1)|(|\nabla\mathbf{v}^2| + |\nabla\mathbf{v}^1|)(\tau)\|_2^2\, \mathrm{d}\tau + \\ & + \int_0^t \Big(\|(\mathbf{v}^2-\mathbf{v}^1)(D^2\mathbf{v}^2)(\tau)\|_2^2 + \|\mathbf{v}^1 D^2(\mathbf{v}^2-\mathbf{v}^1)(\tau)\|_2^2\Big)\mathrm{d}\tau \le \\ & \le \int_0^t \|\nabla(\mathbf{v}^2-\mathbf{v}^1)(\tau)\|_{1,2}^2(\|\nabla\mathbf{v}^2(\tau)\|_{1,2}^2 + \|\nabla\mathbf{v}^1(\tau)\|_{1,2}^2)\mathrm{d}\tau + \\ & + \int_0^t \|(\mathbf{v}^2-\mathbf{v}^1)(\tau)\|_{2,2}^2\|D^2\mathbf{v}^2(\tau)\|_2^2 d\tau + \\ & + \int_0^t \|D^2(\mathbf{v}^2-\mathbf{v}^1)(\tau)\|_2^2\|\mathbf{v}^1(\tau)\|_{2,2}^2\mathrm{d}\tau \le \\ & \le K(C)\,t\|\mathbf{v}^2-\mathbf{v}^1\|_X^2 . \end{aligned}$$

[2]Indeed, transforming the convective term into the cylindrical coordinates (cf.(1.2)) we have

$$\begin{aligned} K_r &= v_r\frac{\partial v_r}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \\ K_\theta &= v_r\frac{\partial v_\theta}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \\ K_z &= v_r\frac{\partial v_z}{\partial r} + \frac{1}{r}v_\theta\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}. \end{aligned}$$

It is an easy matter to see that, for $\mathbf{v}$ being axially symmetric, the convective term is again axially symmetric.

Analogously we get the L^2–estimates of $\mathbf{g}$ and again it is an easy matter to see that for $t > 0$ sufficiently small ($t \le t_0$) we obtain an estimate of the type

$$\|S(\mathbf{v}^2) - S(\mathbf{v}^1)\|_X \le \kappa \|\mathbf{v}^2 - \mathbf{v}^1\|_X$$

with $\kappa < 1$.

Banach fixed point theorem then gives the existence and uniqueness of $\mathbf{v} \in X$ solving the Navier-Stokes system. From (1.3) we obtain the additional regularity for $\mathbf{v}$ stated in Theorem 1.1. Similarly, there exists p such that $\nabla p \in L^\infty(J; L^2(\mathbb{R}^3)) \cap L^2(J; W^{1,2}(\mathbb{R}^3))$. The proof of theorem is complete.

□

VII.1.2 Global axially symmetric solution

Provided that $\mathbf{v}_0$ and $\mathbf{f}$ are axially symmetric we know, by Theorem 1.1, that there is a $t > 0$ and an axially symmetric solution $(\mathbf{v}, p)$ defined on $(0; t)$ and solving

$$\begin{aligned} &\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} + \frac{\partial p}{\partial r} - \nu \Big[\frac{1}{r}\frac{\partial}{\partial r}\big(r \frac{\partial v_r}{\partial r}\big) + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\Big] = f_r \\ &\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} + \frac{\partial p}{\partial z} - \nu \Big[\frac{1}{r}\frac{\partial}{\partial r}\big(r \frac{\partial v_z}{\partial r}\big) + \frac{\partial^2 v_z}{\partial z^2}\Big] = f_z \\ &\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0\,. \end{aligned} \tag{1.5}$$

We denote by t^* supremum of all $t > 0$ for which Theorem 1.1 holds, i.e.

$$t^* = \sup \Big\{ t;\ \text{there is an axially symmetric solution to (1.1) on } (0;t) \text{ belonging to } X \Big\}\,.$$

Then either $t^* = \infty$ or $t^* < \infty$. The aim of this section is to exclude the latter case. Let us assume that $t^* < \infty$. Then necessarily[3]

$$\limsup_{t \to t^*-} \|\mathbf{v}(t)\|_{W^{2,2}(\mathbb{R}^3)} = \infty\,. \tag{1.6}$$

(Otherwise we could define $\mathbf{v}$ at t^* by the limit and take it as a new initial value. As $\mathbf{v}(t^*) \in W^{2,2}(\mathbb{R}^3)$, we could extend, by Theorem 1.1, $(\mathbf{v}, p)$ behind t^*, which would contradict to the definition of t^*.)

Let $t < t^*$ be arbitrary, $I \equiv (0; t)$ and $(\mathbf{v}, p)$ be a solution on I given by Theorem 1.1. Because of regularity we can take curl of (1.5). Thanks to the axial symmetry the vector $\mathbf{w} \equiv \nabla \times \mathbf{v}$ has the only nonzero component w_θ given by $w_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$. For lucidity, we denote w_θ by ω. We see that ω solves

$$\frac{\partial \omega}{\partial t} + v_r \frac{\partial \omega}{\partial r} + v_z \frac{\partial \omega}{\partial z} - \frac{v_r}{r}\omega - \nu \Big[\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{1}{r}\frac{\partial \omega}{\partial r} - \frac{\omega}{r^2}\Big] = g\,, \tag{1.7}$$

where $g \equiv (\nabla \times \mathbf{f})_\theta$.

We will need the following lemma on equivalence of norms for ω and $\mathbf{v}$.

[3]Note that $\mathbf{v} \in C(0, t; W^{2,2}(\mathbb{R}^3))$ for all $t \in (0; t^*)$. This is a direct consequence of the facts that $\mathbf{v} \in L^2(0, t; W^{3,2}(\mathbb{R}^3))$ and $\frac{\partial \mathbf{v}}{\partial t} \in L^2(0, t; W^{1,2}(\mathbb{R}^3))$, see Theorem VIII.1.21.

Lemma 1.2 *Let* $\mathbf{v}$ *be a smooth, divergence free, axially symmetric vector field and* $\omega \equiv (\nabla \times \mathbf{v})_\theta$. *Then*

(i) $\|\omega\|_2$ *is equivalent to* $\|\nabla\mathbf{v}\|_2$,

(ii) $\|\nabla\omega\|_2 + \|\frac{\omega}{r}\|_2$ *is equivalent to* $\|D^2\mathbf{v}\|_2$.

(iii) $\|D^2\omega\|_2 + \|\frac{\partial}{\partial r}(\frac{\omega}{r})\|_2 + \|\frac{\partial}{\partial z}(\frac{\omega}{r})\|_2 \leq C\,\|D^3\mathbf{v}\|_2$.

Proof: Lemma is proved in Appendix, even for a more general settings; see Theorem VIII.4.1.

□

Now we would like to multiply (1.7) successively by $\frac{\omega}{r^2}$, ω and $\frac{\partial\omega}{\partial t}$ and integrate over $\mathbb{R}^3$ with the aim to derive apriori estimates for ω which, combined with Lemma 1.2 (ii), would yield the contradiction to (1.6). Although the multiplication by $\frac{\omega}{r^2}$ is the key step in the proof of Ladyzhenskaya and also Uchovskii and Yudovich, we do not know if $\frac{\omega}{r^2} \in L^2(I;L^2(\mathbb{R}^3)$ here. However, we can multiply (1.7) by $\frac{\omega}{r^{2-\varepsilon}}$ with $\varepsilon > 0$ arbitrarily small, as follows from the next lemma.

Lemma 1.3 *Let* $\omega = (\nabla \times \mathbf{u})_\theta$, $\mathbf{u} \in X(t)$. *Then*

(i) $\frac{\omega}{r^{2-\varepsilon}}$ *and* $\frac{1}{r^{1-\varepsilon}}\frac{\partial\omega}{\partial r}$ *belong to* $L^2(I;L^2(\mathbb{R}^3))$ *for all* $\varepsilon > 0$;

(ii) let $g_1(\eta) \equiv \int_{-\infty}^{\infty}(r^\delta|\frac{\omega}{r}|^2)(\eta,z)\,dz$ *and* $g_2(\eta) \equiv \int_{-\infty}^{\infty}(r^\delta|\frac{\partial\omega}{\partial r}|^2)(\eta,z)\,dz$, *then* g_1 *and* g_2 *are bounded for any* $\delta \in (0,2)$.

Proof: To prove (i) we first observe that, by Lemma 1.2, $\frac{\omega}{r}$ and $\frac{\partial}{\partial r}(\frac{\omega}{r})$ belong to $L^2(I;L^2(\mathbb{R}^3))$. We then define $g \in W^{1,2}(\mathbb{R}^3)$ in such a way that $g = \frac{\omega}{r}$ for $r < 1$, $g = 0$ for $r > 2$ and $\|g\|_{1,2} \leq C\|\frac{\omega}{r}\|_{1,2}$. For $\varepsilon > 0$ fixed, the Hardy inequality (see Theorem VIII.1.16) yields

$$\int_{-\infty}^{\infty}\int_0^{\infty} r^{-1+2\varepsilon}|g|^2 \mathrm{d}r\,\mathrm{d}z \leq$$
$$\leq \Big(\frac{2}{\varepsilon}\Big)^2 \int_{-\infty}^{\infty}\int_0^{\infty} r^{1+2\varepsilon}\Big|\frac{\partial g}{\partial r}\Big|^2 \mathrm{d}r\,\mathrm{d}z \leq C(\varepsilon)\Big\|\nabla\frac{\omega}{r}\Big\|_2^2 .$$

Moreover, for $r > 1$ we have

$$\int_{-\infty}^{\infty}\int_1^{\infty}\Big|\frac{\omega}{r^{2-\varepsilon}}\Big|^2 r\mathrm{d}r\,\mathrm{d}z \leq \int_{-\infty}^{\infty}\int_1^{\infty}\Big|\frac{\omega}{r}\Big|^2 r\mathrm{d}r\,\mathrm{d}z\,.$$

In a very similar way we can show that

$$\Big\|\frac{1}{r^{1-\varepsilon}}\frac{\partial\omega}{\partial r}\Big\|_2 \leq C(\varepsilon)\Big\|\frac{\partial\omega}{\partial r}\Big\|_{1,2} .$$

Thus (i) is proved. To verify (ii), let $\eta > 0$ and $\delta \in (0;2)$. Then

$$g_1(\eta) = -\int_{-\infty}^{\infty}\int_\eta^{\infty}\frac{\partial}{\partial r}\Big(r^\delta\Big|\frac{\omega}{r}\Big|^2\Big)\mathrm{d}r\,\mathrm{d}z \leq$$
$$\leq \int_{-\infty}^{\infty}\int_\eta^{\infty}\delta r^{-1+\delta}\Big|\frac{\omega}{r}\Big|^2 + 2r^\delta\Big|\frac{\omega}{r}\Big|\Big|\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\Big|\mathrm{d}r\mathrm{d}z \leq$$
$$\leq \Big\|\frac{\omega}{r^{2-\delta/2}}\Big\|_2^2 + \Big\|\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\Big\|_2\Big\|\frac{\omega}{r^{2-\delta}}\Big\|_2 .$$

Thus $g_1(\eta)$ is bounded for all η due to (i) and Lemma 1.2 (iii). The boundedness of g_2 is proved analogously starting from $-\int_{-\infty}^{\infty}\int_{\eta}^{\infty}\frac{\partial}{\partial r}(r^{\delta}|\frac{\partial\omega}{\partial r}|^2)\mathrm{d}r\mathrm{d}z$ and using Lemma 1.2 (iii).

□

Corollary 1.1 *For every $\varepsilon > 0$*

$$\lim_{\eta\to 0+}\int_{-\infty}^{\infty}\Big(\frac{\partial\omega}{\partial r}\frac{\omega}{r^{1-\varepsilon}}\Big)(\eta,z)\,dz = 0\,.$$

Proof: For fixed $\varepsilon > 0$, we have

$$\int_{-\infty}^{\infty}\Big(\frac{\partial\omega}{\partial r}\frac{\omega}{r^{1-\varepsilon}}\Big)(\eta,z)\,\mathrm{d}z \le$$
$$\le \Big(\int_{-\infty}^{\infty}\Big(\Big|\frac{\partial\omega}{\partial r}\Big|^2 r^{\varepsilon/2}\Big)(\eta,z)\,dz\Big)^{1/2}\Big(\int_{-\infty}^{\infty}\Big(\Big|\frac{\omega}{r}\Big|^2 r^{\varepsilon/2}\Big)(\eta,z)\,dz\Big)^{1/2}\eta^{\varepsilon/2}\,,$$

which gives the assertion thanks to Lemma 1.3 (ii).

□

Now, we are going to multiply (1.7) by $\frac{\omega}{r^{2-\varepsilon}}$ and integrate over $\mathbb{R}^3$ with the aim to let finally $\varepsilon\to 0^+$. The integration over $\mathbb{R}^3$ is clearly allowed as all integrals are finite; for example (by $\int$ we mean $\int_{-\infty}^{\infty}\int_0^{\infty}$ in what follows) it holds

$$\Big|\int\frac{1}{r}\frac{\partial\omega}{\partial r}\frac{\omega}{r^{2-\varepsilon}}r\mathrm{d}r\mathrm{d}z\Big| \le \Big\|\frac{1}{r^{1-\varepsilon/2}}\frac{\partial\omega}{\partial r}\Big\|_2\Big\|\frac{\omega}{r^{2-\varepsilon/2}}\Big\|_2\,,$$

and the right-hand side is finite due to Lemma 1.3 (i).

Lemma 1.4 *Let $\mathbf{v}_0\in W^{2,2}(\mathbb{R}^3)$ and $\mathbf{f}\in L^2_{loc}(0,\infty;W^{1,2}(\mathbb{R}^3))$ be axially symmetric and let $t<t^*$. Then it holds*

$$\Big\|\frac{\omega(t)}{r}\Big\|_2^2 \le C(\mathbf{v}_0,\mathbf{f}) \tag{1.8}$$

$$\|\omega(t)\|_2^2 + \nu\int_0^t\Big(\|\nabla\omega(\tau)\|_2^2 + \Big\|\frac{\omega(\tau)}{r}\Big\|_2^2\Big)\,d\tau \le C(\mathbf{v}_0,\mathbf{f}) \tag{1.9}$$

$$\int_0^t\Big\|\frac{\partial\omega(\tau)}{\partial t}\Big\|_2^2\,d\tau + \nu\Big(\|\nabla\omega(t)\|_2^2 + \Big\|\frac{\omega(t)}{r}\Big\|_2^2\Big) \le C(\mathbf{v}_0,\mathbf{f})\,, \tag{1.10}$$

where $C(\mathbf{v}_0,\mathbf{f})$ denotes a quantity depending on $\|\mathbf{v}_0\|_{2,2}$ and $\int_0^T\|\mathbf{f}(t)\|_{1,2}^2dt$, $t^*<T<\infty$, arbitrary.

Proof: We will split the proof into three steps.

Step 1. In order to prove (1.8) we multiply (1.7) by $\frac{\omega}{r^{2-\varepsilon}}$ with $\varepsilon>0$ small and integrate over $\mathbb{R}^3$ with respect to the measure $r\mathrm{d}r\mathrm{d}z$, which is allowed due to Lemma 1.3 (i). We will obtain

$$\frac{1}{2}\frac{d}{dt}\int\Big|\frac{\omega}{r^{1-\varepsilon/2}}\Big|^2 r\mathrm{d}r\mathrm{d}z + \nu\int\Big(\Big|\nabla\Big(\frac{\omega}{r^{1-\varepsilon/2}}\Big)\Big|^2 + (\varepsilon-\frac{\varepsilon^2}{4})\Big|\frac{\omega}{r^{2-\varepsilon/2}}\Big|^2\Big)r\mathrm{d}r\mathrm{d}z =$$
$$= \int g\frac{\omega}{r^{2-\varepsilon}}r\mathrm{d}r\mathrm{d}z + \frac{\varepsilon}{2}\int\frac{v_r}{r}\frac{\omega^2}{r^{2-\varepsilon}}r\mathrm{d}r\mathrm{d}z\,. \tag{1.11}$$

Indeed, the term including $\frac{\partial \omega}{\partial t}$ is elementary. (We can use e.g. the theorem on derivative of integral depending on a parameter.) The convective term gives

$$\begin{aligned}
\int \Big(v_r \frac{\partial \omega}{\partial r} + v_z \frac{\partial \omega}{\partial z} - \frac{v_r \omega}{r}\Big) \frac{\omega}{r^{2-\varepsilon}}\, r \mathrm{d}r\mathrm{d}z = \\
= \int \Big(\frac{v_r}{2} \frac{\partial(\omega^2)}{\partial r} \Big(\frac{1}{r^{1-\varepsilon}}\Big) + \frac{v_z}{2} \frac{\partial(\omega^2)}{\partial z} \Big(\frac{1}{r^{1-\varepsilon}}\Big) - \frac{v_r}{r} \frac{\omega^2}{r^{1-\varepsilon}}\Big) \mathrm{d}r\mathrm{d}z = \\
= -\frac{1}{2} \int \Big(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\Big) \frac{\omega^2}{r^{1-\varepsilon}} \mathrm{d}r\mathrm{d}z - \frac{\varepsilon}{2} \int \frac{v_r}{r} \frac{\omega^2}{r^{1-\varepsilon}} \mathrm{d}r\mathrm{d}z = \\
= -\frac{\varepsilon}{2} \int \frac{v_r}{r} \frac{\omega^2}{r^{1-\varepsilon}} \mathrm{d}r\mathrm{d}z\,. \qquad \text{(due to } (1.5)_3)
\end{aligned}$$

Let us note that the boundary terms disappear due to the integrability (at infinity) and due to an analogue of Corollary 1.1 (at $r = 0$). The elliptic term requires precise investigations

$$\begin{aligned}
-\nu \int \Big(\frac{\partial^2 \omega}{\partial r^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{\omega}{r^2}\Big) \frac{\omega}{r^{2-\varepsilon}} r \mathrm{d}r\mathrm{d}z = -\nu \Big[\int_{-\infty}^{\infty} \frac{\partial \omega}{\partial r} \frac{\omega}{r^{1-\varepsilon}} \, \mathrm{d}z\Big]_{r=0}^{r=\infty} + \\
+\nu \int \Big(\Big|\frac{\partial}{\partial r}\Big(\frac{\omega}{r^{1-\varepsilon/2}}\Big)\Big|^2 + \Big|\frac{\partial}{\partial z}\Big(\frac{\omega}{r^{1-\varepsilon/2}}\Big)\Big|^2 + \varepsilon(1 - \frac{\varepsilon}{4}) \Big|\frac{\omega}{r^{2-\varepsilon/2}}\Big|^2\Big) \, r \mathrm{d}r\mathrm{d}z\,,
\end{aligned}$$

and the boundary term vanishes due to Corollary 1.1 and due to the integrability of ω.

Now we can estimate the right-hand side of (1.11). Since

$$\begin{aligned}
\int \Big(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\Big) \frac{\omega}{r^{2-\varepsilon}} r \, \mathrm{d}r\mathrm{d}z = -\int \Big(f_r \frac{\partial}{\partial z}\Big(\frac{\omega}{r^{1-\varepsilon}}\Big) - f_z \frac{\partial}{\partial r}\Big(\frac{\omega}{r^{1-\varepsilon}}\Big)\Big) \mathrm{d}r\mathrm{d}z = \\
= -\int \Big(f_r r^{\varepsilon/2} \frac{\partial}{\partial z}\Big(\frac{\omega}{r^{1-\varepsilon/2}}\Big) - f_z r^{\varepsilon/2} \frac{\partial}{\partial r}\Big(\frac{\omega}{r^{1-\varepsilon/2}}\Big) - \frac{\varepsilon}{2} f_z r^{\varepsilon/2} \frac{\omega}{r^{2-\varepsilon/2}}\Big) \mathrm{d}r\mathrm{d}z\,,
\end{aligned}$$

we have by the Hardy and Young inequalities (see Theorem VIII.1.16 and Lemma VIII.1.1)

$$\begin{aligned}
\Big|\int g \frac{\omega}{r^{2-\varepsilon}} \, r \mathrm{d}r\mathrm{d}z\Big| \le \Big\|\frac{\mathbf{f}}{r^{1-\varepsilon/2}}\Big\|_2 \Big(\Big\|\nabla \frac{\omega}{r^{1-\varepsilon/2}}\Big\|_2 + \frac{\varepsilon}{2} \Big\|\frac{\omega}{r^{2-\varepsilon/2}}\Big\|_2\Big) \le \\
\le \frac{\nu}{2} \Big\|\nabla \frac{\omega}{r^{1-\varepsilon/2}}\Big\|_2^2 + \frac{\varepsilon}{4} \Big\|\frac{\omega}{r^{2-\varepsilon/2}}\Big\|_2^2 + c \, \|\mathbf{f}\|_{1,2}^2\,.
\end{aligned}$$

Further, by means of (1.4), we have

$$\begin{aligned}
\varepsilon \Big|\int \frac{v_r}{r} \frac{\omega^2}{r^{2-\varepsilon}} r \mathrm{d}r\mathrm{d}z\Big| \le \varepsilon \|v_r\|_\infty \Big\|\frac{\omega}{r^{2-\varepsilon/2}}\Big\|_2 \Big\|\frac{\omega}{r^{1-\varepsilon/2}}\Big\|_2 \le \\
\le \frac{\varepsilon}{4} \Big\|\frac{\omega}{r^{2-\varepsilon/2}}\Big\|_2^2 + \varepsilon \, c \, \|\nabla \mathbf{v}\|_2 \|D^2 \mathbf{v}\|_2 \Big\|\frac{\omega}{r^{1-\varepsilon/2}}\Big\|_2^2\,.
\end{aligned}$$

Putting all calculations together and integrating the result with respect to time we obtain for all $\tau \in (0; t)$

$$\Big\|\frac{\omega(\tau)}{r^{1-\varepsilon/2}}\Big\|_2^2 \le c(\mathbf{f}, \mathbf{v}_0) + \varepsilon \int_0^\tau \|\nabla \mathbf{v}(s)\|_{1,2}^2 \Big\|\frac{\omega(s)}{r^{1-\varepsilon/2}}\Big\|_2^2 \mathrm{d}s\,.$$

The Gronwall inequality (see Theorem VIII.1.20) then implies

$$\Big\|\frac{\omega(\tau)}{r^{1-\varepsilon}}\Big\|_2^2 \le c(\mathbf{f}, \mathbf{v}_0) \exp\Big(\varepsilon \int_0^t \|D^2 \mathbf{v}(\tau)\|_2^2 \, \mathrm{d}\tau\Big)\,.$$

The right-hand side is finite by the assumption on $\mathbf{v}$, which allows to pass to the limit as $\varepsilon \to 0$ at the right-hand side. As $|\frac{\omega}{r^{1-\varepsilon/2}}|$ is bounded by $|\frac{\omega}{r}|$ for $r \in (0;1)$, and by $|\omega|$ for $r \geq 1$, we can let ε tend to 0 at the left-hand side by the Lebesgue dominated theorem, and we obtain (1.8).

Step 2. The next estimate (1.9) is obtained by multiplying (1.7) by ω (and integrating over $\mathbb{R}^3$). The elliptic term gives

$$-\nu \int \Big(\frac{\partial^2\omega}{\partial r^2} + \frac{\partial^2\omega}{\partial z^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{\omega}{r^2}\Big)\omega r \mathrm{d}r\mathrm{d}z = \\ = \nu \int \Big[\Big(\frac{\partial\omega}{\partial r}\Big)^2 + \frac{\omega^2}{r^2} + \Big(\frac{\partial\omega}{\partial z}\Big)^2\Big] r\mathrm{d}r\mathrm{d}z = \nu\Big(\|\nabla\omega\|_2^2 + \Big\|\frac{\omega}{r}\Big\|_2^2\Big).$$

Since

$$\int \frac{\partial\omega}{\partial t}\,\omega\, r\mathrm{d}r\mathrm{d}z = \frac{1}{2}\frac{d}{dt}\|\omega\|_2^2$$

and

$$\int g\omega r\mathrm{d}r\mathrm{d}z \leq 2\int |\mathbf{f}|\Big(|\nabla\omega| + \Big|\frac{\omega}{r}\Big|\Big) r\mathrm{d}r\mathrm{d}z \leq 2\|\mathbf{f}\|_2\Big(\|\nabla\omega\|_2 + \Big\|\frac{\omega}{r}\Big\|_2\Big),$$

we can concentrate ourselves on the estimate of the convective term. We have

$$\int \Big(v_r\frac{\partial\omega}{\partial r}\omega + +v_z\frac{\partial\omega}{\partial z}\omega - \frac{v_r\omega^2}{r}\Big) r\mathrm{d}r\mathrm{d}z = \\ = \int \Big[v_r\frac{\partial}{\partial r}\Big(\frac{\omega^2}{2}\Big) + v_z\frac{\partial}{\partial z}\Big(\frac{\omega^2}{2}\Big) - \frac{v_r\omega^2}{r}\Big] r\mathrm{d}r\mathrm{d}z = \\ = \int \frac{\omega^2}{2}\Big(-\frac{\partial v_r}{\partial r} - \frac{\partial v_z}{\partial z} - 2\frac{v_r}{r} - \frac{v_r}{r}\Big) r\mathrm{d}r\mathrm{d}z = -\int \frac{v_r\omega^2}{r} r\mathrm{d}r\mathrm{d}z\,.$$

Adding all computations, integrating over $(0;t)$, using the Agmon inequality (1.4), the above shown estimate (1.8) and Lemma 1.2 (i), (ii) we obtain

$$\begin{aligned} \|\omega(t)\|_2^2 + C\int_0^t \|D^2\mathbf{v}(\tau)\|_2^2\mathrm{d}\tau + \frac{\nu}{4}\int_0^t \Big(\|\nabla\omega(\tau)\|_2^2 + \Big\|\frac{\omega}{r}\Big\|_2^2\mathrm{d}\tau\Big) \leq \\ \leq C\int_0^t \int |\omega(\tau)|^2|v_r|\mathrm{d}r\mathrm{d}z + C(\mathbf{v}_0,\mathbf{f}) \leq \\ \leq \int_0^t \|\mathbf{v}(\tau)\|_\infty\Big\|\frac{\omega(\tau)}{r}\Big\|_2\|\omega(\tau)\|_2\mathrm{d}\tau + C(\mathbf{v}_0,\mathbf{f}) \leq \\ \leq C(\mathbf{v}_0,\mathbf{f})\int_0^t \|\nabla\mathbf{v}\|_2^{1/2}\|D^2\mathbf{v}\|_2^{1/2}\|\omega\|_2\mathrm{d}\tau + C(\mathbf{v}_0,\mathbf{f}) \leq \\ \leq \frac{C}{8}\int_0^t \|D^2\mathbf{v}\|_2^2\mathrm{d}\tau + C(\mathbf{v}_0,\mathbf{f})\int_0^t \|\nabla\mathbf{v}\|_2^2\mathrm{d}\tau + C(\mathbf{v}_0,\mathbf{f}) \leq \\ \leq \frac{C}{8}\int_0^t \|D^2\mathbf{v}\|_2^2\mathrm{d}\tau + C(\mathbf{v}_0,\mathbf{f}), \end{aligned} \tag{1.12}$$

where we use (at the last step) the classical first energy estimates

$$\|\mathbf{v}(t)\|_2^2 + \int_0^t \|\nabla\mathbf{v}(\tau)\|_2^2\mathrm{d}\tau \leq \|\mathbf{v}_0\|_2^2 + c\int_0^t \|\mathbf{f}(\tau)\|_2^2\mathrm{d}\tau\,. \tag{1.13}$$

Step 3. The last estimate (1.10) is obtained similarly. Now we multiply (1.7) by $\frac{\partial\omega}{\partial t}$. It yields

$$\begin{aligned}
\Big\|\frac{\partial\omega}{\partial t}\Big\|_2^2 &+ \frac{1}{2}\frac{d}{dt}\nu\Big(\|\nabla\omega\|_2^2 + \Big\|\frac{\omega}{r}\Big\|_2^2\Big) \le \\
&\le \Big|\int\Big(v_r\frac{\partial\omega}{\partial r} + v_z\frac{\partial\omega}{\partial z} - \frac{v_r\omega}{r} + g\Big)\frac{\partial\omega}{\partial t}\, r\mathrm{d}r\mathrm{d}z\Big| \le \\
&\le \frac{1}{2}\Big\|\frac{\partial\omega}{\partial t}\Big\|_2^2 + \int|\mathbf{v}|^2\Big(|\nabla\omega|^2 + \Big|\frac{\omega}{r}\Big|^2\Big)\, r\mathrm{d}r\mathrm{d}z + \|\mathbf{f}\|_{1,2}^2 \le \\
&\le \frac{1}{2}\Big\|\frac{\partial\omega}{\partial t}\Big\|_2^2 + \|\mathbf{v}\|_\infty^2\Big(\|\nabla\omega\|_2^2 + \Big\|\frac{\omega}{r}\Big\|_2^2\Big) + \|\mathbf{f}\|_{1,2}^2\,.
\end{aligned}$$

We must only pay attention to the elliptic term, as we do not have enough regularity ($\frac{\partial}{\partial t}\nabla\omega \notin L^2(I;L^2(\mathbb{R}^3)$ generally). Nevertheless, using Theorem VIII.1.19 we can prove that

$$\begin{aligned}
&\int_0^t\int\Big(\frac{\partial^2\omega}{\partial r^2} + \frac{\partial^2\omega}{\partial z^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{\omega}{r^2}r\Big)\mathrm{d}r\mathrm{d}z\mathrm{d}\tau = \\
&= \frac{1}{2}\Big(\|\nabla\omega(t)\|_2^2 + \Big\|\frac{\omega}{r}(t)\Big\|_2^2 - \|\nabla\omega(0)\|_2^2 - \Big\|\frac{\omega}{r}(0)\Big\|_2^2\Big)\,.
\end{aligned}$$

Again, by (1.4), (1.12) and (1.13) we see that

$$\int_0^t \|\mathbf{v}(\tau)\|_\infty^2 \le C\Big(\|\mathbf{v}_0\|_{2,2}^2, \int_0^t \|\mathbf{f}\|_{1,2}^2\mathrm{d}\tau\Big)\,.$$

The Gronwall inequality finishes the proof of (1.10). Lemma 1.4 is proved.

□

The task to exclude (1.6) is now very easy. By the equivalence of the norm (cf. Lemma 1.2 (i), (ii)) we see that (1.8)–(1.10) can be rewritten as

$$\|\mathbf{v}(t)\|_{2,2}^2 \le C\Big(\|\mathbf{v}_0\|_2^2, \int_0^t \|\mathbf{f}\|_{1,2}^2\mathrm{d}\tau\Big)$$

valid for all $t < t^*$. Passing to the lim sup at the left-hand side we obtain

$$\limsup_{t\to t^*-}\|\mathbf{v}(t)\|_{2,2}^2 < \infty.$$

Thus (1.6) does not hold and consequently $t^* = \infty$. We have proved

Theorem 1.2 *Let $T \in (0;\infty)$ be arbitrary, and let $\mathbf{v}_0 \in W^{2,2}(\mathbb{R}^3)$, $\nabla\cdot\mathbf{v}_0 = 0$, and $\mathbf{f} \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$ be axially symmetric. Then there exists (global) axially symmetric solution to the Navier-Stokes equations (1.1) satisfying*

$$\begin{aligned}
&\mathbf{v} \in L^\infty(0,T;W^{2,2}(\mathbb{R}^3)) \cap L^2(0,T;W^{3,2}(\mathbb{R}^3))\,, \\
&\frac{\partial\mathbf{v}}{\partial t} \in L^2(0,T;W^{1,2}(\mathbb{R}^3))\,.
\end{aligned}$$

An easy consequence of Theorem 1.2 is the following statement.

Theorem 1.3 *Let $\mathbf{v}_0$ and $\mathbf{f}$ be as in Theorem 1.2. Then global axially symmetric solution to (1.1) given by Theorem 1.1 is unique in the class of weak solutions to (1.1).*

Proof: Compare with [CoFo], Chapt. 10 or [Se].

□

VII.2 Ideal fluid

Next we study the incompressible Euler equations, i.e. the system (cf. (I.2.5))

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{v} = 0 \end{aligned} \tag{2.1}$$

completed by the initial condition

$$\mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x})\,, \quad \mathbf{x}\in\mathbb{R}^3 \tag{2.2}$$

with $\nabla\cdot\mathbf{v}_0 = 0$. For the incompressible Euler equations similar problems as for the Navier–Stokes equations appear. The local-in-time existence is known for quite a long time (see [Lic]), in two dimensions we have that the solution with finite initial energy exists for all time (see e.g. [BeKaMa] or [Maj]). But in three dimensions it is still not clear whether solutions with finite initial energy may blow up or not (see e.g. [Maj] for a discussion of the numerical experiments in this context). Nevertheless, there exists a precise characterisation of the time instant, when the the energy blows up. Namely, denoting $\mathbf{w} = \nabla\times\mathbf{v}$, we have (see [BeKaMa])

Lemma 2.1 *The interval $[0,T^*)$ with $T^*<\infty$ is a maximal interval of smooth existence if and only if the energy accumulates so rapidly that*

$$\int_0^t \|\mathbf{w}(\tau)\|_\infty d\tau \to \infty \ \text{as } t\to T^*\,. \tag{2.3}$$

Here $[0,T)$ is a maximal interval of smooth existence provided the function $\mathbf{v}\in C([0;T);W^{s,2}(\mathbb{R}^N))$ and $\|\mathbf{v}(t)\|_{s,2}\to\infty$ as $t\to T$, $s>\frac{N}{2}+1$.

We shall use this lemma in order to show that for axially symmetric data (2.3) cannot happen and therefore the solution (axially symmetric) exists globally in time, is smooth and therefore unique.

Let us also recall that similar result have been proved by Uchovskii and Yudovich in [UcYu] by means of a quite different technique. Further, in the paper of Beale, Kato and Majda [BeKaMa] another approach can be found. They prove the global existence of smooth solution under different assumptions (the non–negativity of the initial condition).

Let us start to study (2.1) under the assumption of the axial symmetry of the data. First, let us present some auxiliary lemmas.

Lemma 2.2 *Let $\mathbf{v}\in W^{4,2}(\mathbb{R}^3)$ be axially symmetric. Then $\frac{\omega}{r}\in L^\infty(\mathbb{R}^3)$, where $\omega = (\nabla\times\mathbf{v})_\theta$.*

Proof: From Lemma VIII.4.17 in Appendix we have

$$\left\|\frac{\omega}{r}\right\|_\infty \le C\|D^2\mathbf{v}\|_\infty$$

The rest follows from the imbedding theorem,

$$\|D^2\mathbf{v}\|_\infty \le C\|D^2\mathbf{v}\|_{2,2}\,.$$

□

Definition 2.1 *We put*

$$W_\infty^{3,2,ax} = \overline{W^{4,2,ax}(\mathbb{R}^3)}^{\|\cdot\|_{W_\infty^{3,2}}},$$

where, for axially symmetric function $\mathbf{v}$*,*

$$\|\mathbf{v}\|_{W_\infty^{3,2}} = \|\mathbf{v}\|_{3,2} + \left\|\frac{\omega}{r}\right\|_\infty$$

and $W^{4,2,ax}(\mathbb{R}^3)$ *denotes the set of all axially symmetric functions belonging to* $W^{4,2}(\mathbb{R}^3)$*.*

Remark 2.1

$$W_\infty^{3,2,ax} \subset \left\{\mathbf{u} \in W^{3,2}(\mathbb{R}^3); \mathbf{u} \text{ axially symmetric}, \left\|\frac{\omega}{r}\right\|_\infty < +\infty\right\},$$

the opposite inclusion being not clear.

We shall assume from now that[4]

$$\mathbf{v}_0 \in W_\infty^{3,2,ax}, \quad \mathbf{f} \in L^2_{loc}(0,\infty; W_\infty^{3,2,ax}) \tag{2.4}$$

and denote by v_0^δ, $\mathbf{f}^\delta$ their approximations (in the space variables) in the sense of Definition 2.1. We have

Lemma 2.3 *Let* $\mathbf{v}_0^\delta \in W^{4,2}(\mathbb{R}^3)$*,* $\mathbf{f}^\delta \in L^2_{loc}(0,\infty; W^{4,2}(\mathbb{R}^3))$ *be the approximations of the data. Then there exists unique*

$$\begin{aligned} \mathbf{v}^{\delta,\nu} &\in L^2_{loc}(0,\infty; W^{5,2}(\mathbb{R}^3)) \cap L^\infty_{loc}(0,\infty; W^{4,2}(\mathbb{R}^3)) \\ \frac{\partial \mathbf{v}^{\delta,\nu}}{\partial t} &\in L^2_{loc}(0,\infty; W^{3,2}(\mathbb{R}^3)) \\ \nabla p^{\delta,\nu} &\in L^2_{loc}(0,\infty; W^{3,2}(\mathbb{R}^3)), \end{aligned} \tag{2.5}$$

solution (axially symmetric) to the Navier–Stokes equations (1.1).

Proof: The proof is completely analogous to the proof of Theorems 1.1 and 1.2. Using this method we get solution which belongs to $L^2_{loc}(0,\infty; W^{3,2}(\mathbb{R}^3)) \cap L^\infty_{loc}(0,\infty; W^{2,2}(\mathbb{R}^3))$ and which is axially symmetric. The higher regularity follows from the fact that the above proved regularity implies the full regularity of the solution to the Navier–Stokes equations, see e.g. [Te] or [He].

□

The next aim is to obtain some estimates which are independent of the viscosity.

[4]The assumptions on the time integrability of $\mathbf{f}$ can be further weaken, nevertheless, we shall not do it.

Lemma 2.4 *Under the assumptions of Lemma 2.3 we have on a sufficiently short time interval* $J = (0, \bar{t})$, $\bar{t} = \bar{t}(\mathbf{v}_0, \mathbf{f})$,

$$\begin{aligned} \|\mathbf{v}\|_{L^\infty(0,\bar{t};W^{4,2}(\mathbb{R}^3))} &\leq C(\mathbf{v}_0^\delta, \mathbf{f}^\delta, \bar{t}) \\ \Big\|\frac{\partial \mathbf{v}}{\partial t}\Big\|_{L^2(0,\bar{t};W^{3,2}(\mathbb{R}^3))} &\leq C(\mathbf{v}_0^\delta, \mathbf{f}^\delta, \bar{t}) \\ \|\nabla p\|_{L^2(0,\bar{t};W^{3,2}(\mathbb{R}^3))} &\leq C(\mathbf{v}_0^\delta, \mathbf{f}^\delta, \bar{t}) \,, \end{aligned} \tag{2.6}$$

where the constant C *does not depend on the viscosity* ν. *Moreover,*

$$\begin{aligned} \|\mathbf{v}\|_{L^\infty(0,\bar{t};W^{3,2}(\mathbb{R}^3))} &\leq C(\mathbf{v}_0, \mathbf{f}, \bar{t}) \\ \Big\|\frac{\partial \mathbf{v}}{\partial t}\Big\|_{L^2(0,\bar{t};W^{2,2}(\mathbb{R}^3))} &\leq C(\mathbf{v}_0, \mathbf{f}, \bar{t}) \\ \|\nabla p\|_{L^2(0,\bar{t};W^{2,2}(\mathbb{R}^3))} &\leq C(\mathbf{v}_0, \mathbf{f}, \bar{t}) \,, \end{aligned} \tag{2.7}$$

where the constant C *can be taken independent of* ν *and* δ.

Proof: We multiply first the equation $(1.1)_2$ by $\mathbf{v}$ and integrate over $\mathbb{R}^3$:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_2 + \nu\|\nabla \mathbf{v}\|_2^2 \leq \|\mathbf{f}\|_2\|\mathbf{v}\|_2$$

i.e.

$$\|\mathbf{v}\|_{L^\infty(0,\bar{t};L^2(\mathbb{R}^3))} \leq C\|\mathbf{f}\|_{L^1(0,\bar{t};L^2(\mathbb{R}^3))} + \|\mathbf{v}_0\|_2 \,.$$

Analogously we proceed for higher derivatives. Let us mention only the highest derivatives. We use the fact that $(\nabla \cdot \mathbf{v} = 0)$

$$\begin{aligned} \int_{\mathbb{R}^3} \nabla^k(\mathbf{v}\cdot\nabla\mathbf{v})\nabla^k\mathbf{v}\mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^3} |\nabla^k\mathbf{v}|^2|\nabla\mathbf{v}|\mathrm{d}\mathbf{x} + \\ + \int_{\mathbb{R}^3} |\nabla^k\mathbf{v}||\nabla^{k-1}\mathbf{v}||\nabla^2\mathbf{v}|\mathrm{d}\mathbf{x} + \ldots + \int_{\mathbb{R}^3} |\nabla^k\mathbf{v}|^2|\nabla\mathbf{v}|\mathrm{d}\mathbf{x} \,, \end{aligned}$$

for $k = 3$

$$\int_{\mathbb{R}^3} |\nabla^3\mathbf{v}|^2|\nabla\mathbf{v}|\mathrm{d}\mathbf{x} \leq \|\nabla^3\mathbf{v}\|_2^2\|\nabla\mathbf{v}\|_\infty \leq C\|\nabla\mathbf{v}\|_{2,2}^3$$

and for $k = 4$

$$\int_{\mathbb{R}^3} |\nabla^4\mathbf{v}|^2|\nabla\mathbf{v}|\mathrm{d}\mathbf{x} \leq \|\nabla^4\mathbf{v}\|_2^2\|\nabla\mathbf{v}\|_\infty \leq C\|\nabla\mathbf{v}\|_{3,2}^3 \,.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_{4,2}^2 \leq C\|\mathbf{v}\|_{4,2}^3 + \|\mathbf{f}^\delta\|_{4,2}\|\mathbf{v}\|_{4,2} \tag{2.8}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|_{3,2}^2 \leq C\|\mathbf{v}\|_{3,2}^3 + \|\mathbf{f}^\delta\|_{3,2}\|\mathbf{v}\|_{3,2} \,. \tag{2.9}$$

On the interval where $\|\mathbf{v}(t)\|_{4,2} \geq 1$ we have

$$\frac{d}{dt}\frac{1}{\|\mathbf{v}(t)\|_{4,2}} \geq -C - C_1\|\mathbf{f}^\delta\|_{4,2}$$

and the first inequality follows by integration over sufficiently short time interval. Next, taking the divergence of $(2.1)_1$

$$\Delta p = \nabla \cdot \mathbf{f} + \nabla \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v})$$

and

$$\begin{aligned}\|\nabla p\|_{L^2(0,\bar{t};W^{3,2}(\mathbb{R}^3))} &\le C\|\mathbf{f}\|_{L^2(0,\bar{t};W^{3,2}(\mathbb{R}^3))}+\\ +\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{L^2;0,\bar{t};W^{3,2}(\mathbb{R}^3))} &\le C(\mathbf{f}^\delta, \mathbf{v}_0^\delta, \bar{t})\,.\end{aligned}$$

Analogously, directly from the equation, we have the estimate for the time derivative. Now, using (2.9) we can easily obtain (2.7); namely $\|\mathbf{v}_0^\delta\|_{3,2} \le C\|\mathbf{v}_0\|_{3,2}$ and $\|\mathbf{f}^\delta\|_{L^2(0,\bar{t};W^{3,2}(\mathbb{R}^3))} \le C\|\mathbf{f}\|_{L^2(0,\bar{t};W^{3,2}(\mathbb{R}^3))}$ with the constant independent of δ. The proof is finished.

□

We may now pass with with $\nu \to 0^+$. So we get solution to the Euler equations with the above mentioned regularity.[5]

Lemma 2.5 *Let $\mathbf{v}_0^\delta \in W^{4,2}(\mathbb{R}^3)$, $\mathbf{f}^\delta \in L^2_{loc}(0,\infty;W^{4,2}(\mathbb{R}^3))$, be axially symmetric. Then there exists $\bar{t} > 0$ such that on $(0,\bar{t})$ there exists solution axially symmetric to the Euler equations and the estimates (2.6) and (2.7) hold.*

Up to now we did not use the crucial fact that the data and the solution are axially symmetric. Let us again denote by ω the only nonzero (the θ) component of the $\nabla \times \mathbf{v}$, cf. Section VII.1. We have

$$\frac{\partial \omega}{\partial t} + v_r\frac{\partial \omega}{\partial r} + v_z\frac{\partial \omega}{\partial z} - \frac{v_r}{r}\omega = g\,, \quad g = (\nabla \times \mathbf{f})_\theta\,, \tag{2.10}$$

together with the continuity equation

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0\,.$$

Let us first recall the regularity of ω on $I = (0;\bar{t})$. We apply Theorem VIII.4.1 together with Lemma 2.4 and standard imbedding theorems. First we start with estimates which are δ–independent.

$$\begin{aligned}
&\frac{\partial \omega}{\partial t} \in L^2(I;L^2(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)) && \frac{v_r}{r} \in L^\infty(I;L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3))\\
&\frac{\partial \omega}{\partial r} \in L^\infty(I;L^2(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)) && \frac{\partial}{\partial r}\frac{\omega}{r} \in L^\infty(I;L^2(\mathbb{R}^3))\\
&\frac{\partial}{\partial t}\frac{\partial \omega}{\partial r} \in L^2(I;L^2(\mathbb{R}^3)) && \frac{\omega}{r} \in L^\infty(I;L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3))\\
&\mathbf{v}, \nabla\mathbf{v} \in L^\infty(I;L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)) && \frac{g}{r} \in L^2(I;L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3))\,.\\
&g \in L^2(I;L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3))
\end{aligned} \tag{2.11}$$

[5] As the equation contains nonlinear term and we study the equation on a non–compact domain, we must proceed a bit more carefully. We multiply the equation by a smooth function with compact support and pass to the limit in this equality. Applying the Cantor diagonal argument we get a subsequence which converges weakly in $L^2(0,\bar{t};W^{4,2}(\mathbb{R}^3)$ and strongly in $L^2(0,\bar{t};W^{3,2}(B_R)$ for all $R > 0$ (due to the Lions–Aubin lemma, see e.g. [Lio]) to a function, satisfying the Euler equations a.e. in $(0;\bar{t}) \times \mathbb{R}^3$ together with the above mentioned regularity.

The following estimates blow up if $\delta \to 0^+$

$$\begin{aligned} &\frac{\partial \omega}{\partial t} \in L^2(I; L^\infty(\mathbb{R}^3)) \qquad \frac{\partial}{\partial t}\frac{\partial \omega}{\partial r} \in L^2(I; L^6(\mathbb{R}^3)) \\ &\frac{\partial \omega}{\partial r} \in L^\infty(I; L^\infty(\mathbb{R}^3)) \qquad \frac{\partial}{\partial r}\frac{\omega}{r} \in L^\infty(I; L^6(\mathbb{R}^3))\,. \end{aligned} \tag{2.12}$$

Now let $I = (0;t)$ with $t < t^*$,

$$t^* = \sup\Big\{t > 0; \exists\, \mathbf{v} \text{ axially symmetric },\mathbf{v} \in L^\infty(I; W^{4,2}(\mathbb{R}^3)),$$
$$\frac{\partial \mathbf{v}}{\partial t} \in L^\infty(I; W^{3,2}(\mathbb{R}^3)) \text{ solving (2.1) in } I \times \mathbb{R}^3\Big\}\,.$$

By a contradiction argument we shall show that necessarily $t^* = \infty$. Let $t^* < \infty$. Then Lemma 2.1 implies

$$\int_0^{t^*} \|\omega(\tau)\|_\infty \mathrm{d}\tau = \infty\,.$$

In what follows, we shall exclude this possibility by showing that

$$\|\omega\|_{L^\infty(0,t;L^\infty(\mathbb{R}^3))} \le C\,,$$

where the constant C remains bounded for $t \to t^*$. We multiply the equation (2.10) by $\frac{1}{r}|\frac{\omega}{r}|^{p-1}\mathrm{sign}\,\frac{\omega}{r}$ and integrate $r\mathrm{d}r\mathrm{d}z$. We have[6] for $p \ge 2$

$$\int \frac{\partial \omega}{\partial t}\Big|\frac{\omega}{r}\Big|^{p-1}\mathrm{sign}\,\frac{\omega}{r}\mathrm{d}r\mathrm{d}z + \int \Big(v_r\frac{\partial \omega}{\partial r} + v_z\frac{\partial \omega}{\partial z} - \frac{v_r}{r}\omega\Big)\Big|\frac{\omega}{r}\Big|^{p-1}\mathrm{sign}\,\frac{\omega}{r}\mathrm{d}r\mathrm{d}z =$$
$$= \int g\Big|\frac{\omega}{r}\Big|^{p-1}\mathrm{sign}\,\frac{\omega}{r}\mathrm{d}r\mathrm{d}z\,. \tag{2.13}$$

Due to the regularity of $\mathbf{v}$ on I we easily verify that all integrals in (2.13) are finite. Moreover, let us show that the convective term is equal to zero. We have namely

$$\begin{aligned} &\int \Big(v_r\frac{\partial \omega}{\partial r} + v_z\frac{\partial \omega}{\partial z} - \frac{v_r}{r}\omega\Big)\Big|\frac{\omega}{r}\Big|^{p-1}\mathrm{sign}\,\frac{\omega}{r}\mathrm{d}r\mathrm{d}z = \\ &= \frac{1}{p}\int \Big[v_r\frac{\partial}{\partial r}\Big|\frac{\omega}{r}\Big|^p + v_z\frac{\partial}{\partial z}\Big|\frac{\omega}{r}\Big|^p\Big] r\mathrm{d}r\mathrm{d}z = \\ &= \int_{-\infty}^{\infty}\Big[v_r\Big|\frac{\omega}{r}\Big|^p\Big]_0^\infty \mathrm{d}z - \frac{1}{p}\int \Big(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\Big)\Big|\frac{\omega}{r}\Big|^p r\mathrm{d}r\mathrm{d}z = 0\,. \end{aligned}$$

We used the fact that $\nabla \cdot \mathbf{v} = 0$ (see $(2.1)_2$) and similarly as in Section VII.1 we may show that the boundary terms disappear. We have

Lemma 2.6 *Let $\mathbf{v}_0^\delta$, $\mathbf{f}^\delta$ be as above. Then*

$$\frac{1}{p}\frac{d}{dt}\int \Big|\frac{\omega}{r}\Big|^p r\,dr\,dz \le \int g\Big|\frac{\omega}{r}\Big|^{p-1} r\,dr\,dz \tag{2.14}$$

and

$$\Big\|\frac{\omega}{r}\Big\|_{L^\infty(I;L^p(\mathbb{R}^3))} \le C(\mathbf{v}_0) + \Big\|\frac{g}{r}\Big\|_{L^1(I;L^p(\mathbb{R}^3))} \qquad \forall p \in [2;\infty]\,. \tag{2.15}$$

[6] As in Section VII.1, $\int f\mathrm{d}r\mathrm{d}z$ denotes $\int_{-\infty}^{\infty}\int_0^\infty f(r,z)\mathrm{d}r\mathrm{d}z$.

Proof: The inequality (2.14) was shown above. Now

$$\frac{1}{p}\frac{d}{dt}\Big\|\frac{\omega}{r}\Big\|_p^p \le \Big\|\frac{g}{r}\Big\|_p \Big\|\frac{\omega}{r}\Big\|_p^{p-1},$$

i.e.

$$\frac{d}{dt}\Big\|\frac{\omega}{r}\Big\|_p \le \Big\|\frac{g}{r}\Big\|_p,$$

integrating over the time interval I and passing with p to infinity we get the result.

□

Remark 2.2

(i) Analogously to the case when Ω is bounded we have that if $\|f\|_p \le C$ with C independent of p, then $f \in L^\infty(\mathbb{R}^N)$ and $\|f\|_\infty \le C$. Moreover, if $f \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $\limsup_{q\to\infty} \|f\|_q \le \|f\|_\infty$. The proofs are similar to those in bounded domains (cf. e.g. [KuFuJo]).

(ii) The above shown inequality enables us to pass with $\delta \to 0^+$ and use only the information $\mathbf{v}_0 \in W^{3,2,ax}_\infty$, $\mathbf{f} \in L^2(I; W^{3,2,ax}_\infty)$. We get again the existence of strong solutions to the Euler equations and the inequality from Lemma 2.6 holds true.

Now we multiply (2.10) by $|\omega|^{p-1}\operatorname{sign}\omega$ and integrate $r\mathrm{d}r\mathrm{d}z$. We have

$$\int \frac{\partial\omega}{\partial t}|\omega|^{p-1}\operatorname{sign}\omega r\mathrm{d}r\mathrm{d}z + \int \Big(v_r\frac{\partial\omega}{\partial r} + v_z\frac{\partial\omega}{\partial z} - \frac{v_r}{r}\omega\Big)|\omega|^{p-1}\operatorname{sign}\omega r\mathrm{d}r\mathrm{d}z =$$
$$= \int g|\omega|^{p-1}\operatorname{sign}\omega r\mathrm{d}r\mathrm{d}z\,. \tag{2.16}$$

We can again easily verify that all the integrals are finite and that we may apply the Green theorem on the convective term. So we get

$$\int \Big(v_r\frac{\partial\omega}{\partial r} + v_z\frac{\partial\omega}{\partial z} - \frac{v_r}{r}\omega\Big)|\omega|^{p-1}\operatorname{sign}\omega r\mathrm{d}r\mathrm{d}z =$$
$$= \frac{1}{p}\int \Big(v_r\frac{\partial}{\partial r}|\omega|^p + v_z\frac{\partial}{\partial z}|\omega|^p\Big) r\mathrm{d}r\mathrm{d}z -$$
$$- \int v_r|\omega|^p r\mathrm{d}r\mathrm{d}z = -\int v_r|\omega|^p \mathrm{d}r\mathrm{d}z\,.$$

Now

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p \le \int |v_r||\omega|^{p-1}\Big|\frac{\omega}{r}\Big| r\mathrm{d}r\mathrm{d}z + \int |g||\omega|^{p-1} r\mathrm{d}r\mathrm{d}z \le$$
$$\le \|\mathbf{v}\|_p\|\omega\|_p^{p-1}\Big\|\frac{\omega}{r}\Big\|_{L^\infty(I;L^\infty(\mathbb{R}^3))} + \|g\|_p\|\omega\|_p^{p-1}$$

i.e.

$$\frac{d}{dt}\|\omega\|_p \le C\|\mathbf{v}\|_p + \|g\|_p\,. \tag{2.17}$$

We first use (2.17) for $p = 4$. We have (see Theorem VIII.1.12)

$$\|\mathbf{v}\|_4 \le C\|\mathbf{v}\|_2^a\|\nabla\mathbf{v}\|_4^{1-a} \le C_1\|\mathbf{v}\|_2^a\|\omega\|_4^{1-a}$$

with $a = \frac{3}{7}$ and therefore, using the first energy inequality $\|\mathbf{v}\|_{L^\infty(I;L^2(\mathbb{R}^3))} \le C$,

$$\frac{d}{dt}\|\omega\|_4 \le C_2\|\omega\|_4^{1-a} + \|g\|_4 \,.$$

So we have[7]

$$\|\omega\|_{L^\infty(I;L^4(\mathbb{R}^3))} \le C(\mathbf{f}, \mathbf{v}_0) \,. \tag{2.18}$$

Now

$$\|\mathbf{v}\|_\infty \le C\|\mathbf{v}\|_2^{\frac{1}{7}}\|\nabla\mathbf{v}\|_4^{\frac{6}{7}} \le C_1\|\mathbf{v}\|_2^{\frac{1}{7}}\|\omega\|_4^{\frac{6}{7}} \,,$$

(see Theorems VIII.1.12 and VIII.4.1) and therefore

$$\|\mathbf{v}\|_{L^\infty(I;L^\infty(\mathbb{R}^3))} \le C \,.$$

We return to the inequality (2.17)

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p \le \int |v_r||\omega|^{p-1}\Big|\frac{\omega}{r}\Big| r\mathrm{d}r\mathrm{d}z + \int |g||\omega|^{p-1} r\mathrm{d}r\mathrm{d}z \le$$
$$\le \|\mathbf{v}\|_{L^\infty(I;L^\infty(\mathbb{R}^3))}\|\omega\|_p^{p-1}\Big\|\frac{\omega}{r}\Big\|_p + \|g\|_p\|\omega\|_p^{p-1} \,,$$

i.e.

$$\frac{d}{dt}\|\omega\|_p \le C\Big\|\frac{\omega}{r}\Big\|_p + \|g\|_p \,.$$

Integrating the inequality over the time interval I we get

$$\|\omega\|_{L^\infty(I;L^p(\mathbb{R}^3))} \le C(\mathbf{f}, \mathbf{v}_0)$$

and in particular the constant does not depend on p. We may pass with p to ∞ to get

$$\|\omega\|_{L^\infty(I;L^\infty(\mathbb{R}^3))} \le C(\mathbf{f}, \mathbf{v}_0) \,,$$

which excludes the possibility of an blow up. We have therefore

Theorem 2.1 *Let* $\mathbf{f} \in L^2_{loc}(0,\infty; W^{3,2,ax}_\infty)$, $\mathbf{v}_0 \in W^{3,2,ax}_\infty$ *with* $\nabla\cdot\mathbf{v}_0 = 0$. *Then there exists solution to the incompressible Euler equations (2.1)–(2.2) on any compact subinterval of* $(0;\infty)$. *This solution is regular and unique in the class of all weak solutions to the Euler equations. More precisely, we have*

$$\mathbf{v} \in L^\infty_{loc}(0,\infty; W^{3,2}(\mathbb{R}^3))$$
$$\frac{\partial\mathbf{v}}{\partial t} \in L^2_{loc}(0,\infty; W^{2,2}(\mathbb{R}^3))$$
$$\nabla p \in L^2_{loc}(0,\infty; W^{2,2}(\mathbb{R}^3)) \,.$$

Proof: We have only to show the uniqueness, the rest being proved above. Let us assume that $\mathbf{u}$ is another weak solution to the same data; using the mollification of the difference $\mathbf{w}_\varepsilon = (\mathbf{u}-\mathbf{v})_\varepsilon$ as a test function, we easily get after passing with $\varepsilon \to 0^+$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|\mathbf{w}|^2\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^3}|\mathbf{w}|^2|\nabla\mathbf{v}|\mathrm{d}\mathbf{x} \le \|\nabla\mathbf{v}\|_{L^\infty(I;L^\infty(\mathbb{R}^3))}\|\mathbf{w}\|_2^2 \,.$$

As $\mathbf{w}(0) = \mathbf{0}$, the Gronwall inequality finishes the proof.

□

[7] If $\|\omega(\tau)\|_4 \le 1$, the inequality is trivial; if $\|\omega(\tau)\|_4 > 1$, we have $\|\omega(\tau)\|_4^{1-a} \le \|\omega(\tau)\|_4$ and we can apply the Gronwall inequality.

VIII

Appendix

VIII.1 Function spaces, basic inequalities

Let $B_R(\mathbf{x})$ denote an open ball with diameter R centered at $\mathbf{x}$,

$$B_R(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^N; |\mathbf{y}-\mathbf{x}| < R\} \tag{1.1}$$

and $B^R(\mathbf{x})$ the exterior part to a closed ball,

$$B^R(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^N; |\mathbf{y}-\mathbf{x}| > R\}\,. \tag{1.2}$$

If $\mathbf{x} = \mathbf{0}$, we shall write usually only B_R and B^R, respectively.

Let $\Omega \subset \mathbb{R}^N$ be a bounded or an unbounded domain. In the latter $\Omega = \mathbb{R}^N$ or Ω is an exterior domain. It means that there exists $\mathcal{O} = \Omega^c$, compact, simply connected set, such that $\Omega = \mathbb{R}^N \backslash \mathcal{O}$. Without loss of generality we shall suppose that $B_{\frac{1}{2}} \subset \mathcal{O} \subset B_1$. For Ω unbounded we denote

$$\begin{aligned} \Omega_R &= \Omega \cap B_R \\ \Omega^R &= \Omega \cap B^R \\ \Omega_{R_2}^{R_1} &= \Omega_{R_2} \setminus \Omega^{R_1}\,. \end{aligned} \tag{1.3}$$

The bounded domain Ω is called a domain of class C^0 (a domain with continuous boundary) if there exist $\alpha > 0$ and M cartesian systems of coordinates $(x_{r_1}, \dots, x_{r_{N-1}}, x_{r_N}) = (\mathbf{x}'_r, x_{r_N})$, $r = 1, \dots, M$ and M functions $a_r(\mathbf{x}'_r)$, continuous on

$$\Delta_r = \{\mathbf{x}'_r; |x_{r_i}| < \alpha, i = 1, 2, \dots, N-1\}\,, r = 1, 2, \dots, M$$

such that for all $\mathbf{x} \in \partial\Omega$ there exists $r \in \{1, 2, \dots, M\}$ and $\mathbf{x}'_r \in \Delta_r$, $\mathbf{x} = T_r(\mathbf{x}'_r, a_r(\mathbf{x}'_r))$, $T_r : X_r \mapsto X$. Moreover, we suppose that there exists $\beta > 0$ such that if

$$\begin{aligned} V_r^+ &= \{(\mathbf{x}'_r, x_{r_N}); \mathbf{x}'_r \in \Delta_r, a_r(\mathbf{x}'_r) < x_{r_N} < a_r(\mathbf{x}'_r) + \beta\} \\ V_r^- &= \{(\mathbf{x}'_r, x_{r_N}); \mathbf{x}'_r \in \Delta_r, a_r(\mathbf{x}'_r) - \beta < x_{r_N} < a_r(\mathbf{x}'_r)\}\,, \end{aligned}$$

then $T_r(V_r^+) \subset \Omega$ and $T_r(V_r^-) \subset \mathbb{R}^N \setminus \overline{\Omega}$.

If in addition $a_r \in C^{k,\mu}(\Delta_r)$[1], $r = 1, 2, \dots M$, then we say that $\Omega \in C^{k,\mu}$, $k \geq 0$, $\mu \in (0; 1]$.

For Ω an exterior domain we say that $\Omega \in C^{k,\mu}$ if the domain $\operatorname{int}\mathcal{O} \in C^{k,\mu}$.

[1] see below for the definition of Hölder–continuous functions

By $u(\mathbf{x})$ we denote a scalar–valued function from Ω to $\mathbb{R}$. Vector– and tensor–valued functions are printed boldfaced, i.e.

$$\mathbf{v}(\mathbf{x}) = (v_1(\mathbf{x}), \dots, v_k(\mathbf{x}))\,.$$

We also use the summation convention i.e. we sum up over twice repeated indices, from 1 to N. For example, the divergence of a vector field $\mathbf{v}$ will be written as

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i}\,,$$

while for the tensor field

$$\nabla \cdot \mathbf{T} = \Big\{ \frac{\partial T_{ij}}{\partial x_j} \Big\}_{i=1}^{N}\,.$$

Next, for φ a scalar field

$$\nabla \varphi = \Big(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_N} \Big)\,,$$

while for a vector field

$$\nabla \mathbf{v} = \Big\{ \frac{\partial v_i}{\partial x_j} \Big\} \quad i = 1, \dots, k,\ j = 1, \dots, N\,.$$

The curl of $\mathbf{v}$ will be denoted by

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v}\,.$$

By $\mathbf{u} \cdot \mathbf{v}$ we understand the usual scalar product of two vector fields, while

$$\nabla \mathbf{u} : \nabla \mathbf{v} = \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}$$

denotes the scalar product of two tensor valued functions. Moreover,

$$\frac{\partial}{\partial y_i} f(\mathbf{x} - \mathbf{y}) = \frac{\partial}{\partial z_j} f(\mathbf{z}) \Big|_{\mathbf{z} = \mathbf{x} - \mathbf{y}} \frac{\partial z_j}{\partial y_i} = -\frac{\partial}{\partial z_i} f(\mathbf{z}) \Big|_{\mathbf{z} = \mathbf{x} - \mathbf{y}}\,.$$

We recall one useful inequality

Lemma 1.1 (Young)
There exists $C = C(\varepsilon, p)$ such that for any $p \in (1; \infty)$, any a, $b \in \mathbb{R}^+$ and any $\varepsilon > 0$

$$ab \le \varepsilon a^p + C b^{p'}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$.

and the classical Lax–Milgram theorem on the existence of solutions to an abstract problem

Theorem 1.1 *Let $a(u, v)$ be a bilinear, continuous and V–elliptic form on a Hilbert space V. Let $f \in V^*$, the dual space to V. Then there exists exactly one solution $u \in V$ to the problem*

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V\,.$$

Proof: See e.g. [Ne].

□

VIII.1.1 Continuous and continuously differentiable functions. Hölder–continuous functions

Let Ω be as above. Then we denote

$$\begin{aligned} C^0(\Omega) &= C(\Omega) = \{u : \Omega \mapsto \mathbb{R}; u \text{ is continuous on } \Omega\} \\ C^0(\overline{\Omega}) &= C(\overline{\Omega}) = \{u \in C(\Omega); u \text{ is bounded and uniformly} \\ &\qquad \text{continuous on } \Omega\} \\ C^k(\Omega) &= \{u \in C(\Omega); D^\alpha u \in C(\Omega)\, \forall |\alpha| \le k\}\,, k \in \mathbb{N} \\ C^k(\overline{\Omega}) &= \{u \in C^k(\Omega); D^\alpha u \in C(\overline{\Omega})\, \forall |\alpha| \le k\}\,, k \in \mathbb{N} \\ C^\infty(\Omega) &= \bigcap_{k \in \mathbb{N}_0} C^k(\Omega) \\ C^\infty(\overline{\Omega}) &= \bigcap_{k \in \mathbb{N}_0} C^k(\overline{\Omega})\,. \end{aligned} \tag{1.4}$$

All derivatives are understood in the classical sense. Let us recall that $u \in C(\overline{\Omega})$ if and only if there exists a uniquely determined continuous extension of u up to the boundary.

We denote for $k \in \mathbb{N}_0$ and $u \in C^k(\overline{\Omega})$

$$\|u\|_{C^k(\overline{\Omega})} \equiv \sum_{|\alpha| \le k} \|D^\alpha u\|_{C^0(\overline{\Omega})}\,, \tag{1.5}$$

where $\|u\|_{C^0(\overline{\Omega})} \equiv \sup_{\mathbf{x} \in \Omega} |u(\mathbf{x})|$. Then $C^k(\overline{\Omega})$ becomes a Banach space with the norm $\|\cdot\|_{C^k(\overline{\Omega})}$.

Let

$$\operatorname{supp} u = \overline{\{\mathbf{x} \in \Omega; u(\mathbf{x}) \neq 0\}}\,. \tag{1.6}$$

Then

$$C_0^k(\Omega) = \{u \in C^k(\Omega); \operatorname{supp} u \subset \Omega_R\} \tag{1.7}$$

for some R sufficiently large. For Ω bounded (1.7) means that $\operatorname{supp} u \subset \Omega$. Moreover, if Ω is an exterior domain, we denote

$$C_0^k(\overline{\Omega}) = \{u \in C^k(\overline{\Omega}); \operatorname{supp} u \subset \overline{\Omega}_R\} \tag{1.8}$$

for some $R > \operatorname{diam} \Omega^c$ and some $k \in \mathbb{N}_0$ or $k = \infty$.

For $u \in C^k(\overline{\Omega})$ we take

$$H_{\alpha,\mu}(u) = \sup_{\substack{\mathbf{x} \neq \mathbf{y} \\ \mathbf{x},\mathbf{y} \in \Omega}} \frac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\mu}\,, \tag{1.9}$$

$|\alpha| \le k$, $\mu \in (0;1]$. Then $C^{k,\mu}(\overline{\Omega})$ denotes the set of all functions from $C^k(\overline{\Omega})$ such that

$$\sum_{|\alpha|=k} H_{\alpha,\mu}(u) < \infty\,.$$

The space $C^{k,\mu}(\overline{\Omega})$ is a Banach space equipped with the norm

$$\|u\|_{C^{k,\mu}(\overline{\Omega})} = \sum_{|\alpha|=k} H_{\alpha,\mu}(u) + \|u\|_{C^k(\overline{\Omega})}\,. \tag{1.10}$$

Remark 1.1 If $k = 0$ and $\mu \in (0;1)$, we usually call the functions from $C^{0,\mu}(\overline{\Omega})$ Hölder–continuous, while for $\mu = 1$ Lipschitz–continuous. See e.g. [KuFuJo] for more details.

Let us furthermore note that we shall not distinguish between $C^k(\Omega)$ and $C^k(\Omega)^N$. It means that we shall write

$$\mathbf{u} \in C^k(\Omega)\,,$$

which means that $u_i \in C^k(\Omega)$ for $i = 1, 2, \dots, N$. The same holds also for other spaces defined above and below.

VIII.1.2 Lebesgue spaces

Throughout the whole thesis, all integrals are understood in the Lebesgue sense.
Let $1 \leq q < \infty$. Then

$$L^q(\Omega) = \{u \text{ measurable}; \int_\Omega |u(\mathbf{x})|^q \mathrm{d}\mathbf{x} < \infty\}\,. \tag{1.11}$$

The standard assumption $u = v \iff u(\mathbf{x}) = v(\mathbf{x})$ a.e. in Ω yields us a Banach space (if $q = 2$ a Hilbert space) equipped with the norm

$$\|u\|_{q,\Omega} = \Big(\int_\Omega |u(\mathbf{x})|^q \mathrm{d}\mathbf{x}\Big)^{\frac{1}{q}}\,. \tag{1.12}$$

If no misunderstanding can appear, we skip Ω in the norm. For $q = \infty$ we denote

$$L^\infty(\Omega) = \{u \text{ measurable }; |u(\mathbf{x})| \leq K \text{ a.e. in } \Omega\} \tag{1.13}$$

and

$$\begin{aligned}\|u\|_{\infty,\Omega} &= \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |u(\mathbf{x})| = \inf_{\substack{E\subset\Omega\\|E|=0}} \sup_{\mathbf{x}\in\Omega\setminus E} |u(\mathbf{x})| = \\ &= \inf_{\alpha\in\mathbb{R}} \{|u(\mathbf{x})| \leq \alpha \text{ a.e. in } \Omega\}\,.\end{aligned} \tag{1.14}$$

We have the following classical result

Lemma 1.2 (Hölder's inequality)
Let $u \in L^q(\Omega)$, $v \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$ ($q' = 1$ if $q = \infty$). Then $uv \in L^1(\Omega)$ and

$$\|uv\|_1 \leq \|u\|_q \|v\|_{q'}\,. \tag{1.15}$$

Using the preceding lemma we can easily demonstrate

Lemma 1.3 (Interpolation in q)
Let $u \in L^p(\Omega) \cap L^q(\Omega)$, $1 \leq p < q \leq \infty$. Then $u \in L^r(\Omega)$ for all $r \in [p;q]$ and

$$\|u\|_r \leq \|u\|_p^\alpha \|u\|_q^{1-\alpha} \tag{1.16}$$

with $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

Let us moreover recall that for $1 \le q < \infty$

$$L^q(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{q,\Omega}}$$

and therefore $L^q(\Omega)$ is separable for $1 \le q < \infty$. Moreover, using the following lemma which characterizes linear continuous functionals on the Lebesgue spaces we may easily show that $L^q(\Omega)$ is reflexive for $1 < q < \infty$.

Lemma 1.4 (Riesz)
Let $F \in (L^q(\Omega))^*$, $1 \le q < \infty$. Then there exists exactly one $f \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$ such that for all $g \in L^q(\Omega)$

$$\langle F, g\rangle = \int_\Omega f g d\mathbf{x}\,.$$

Moreover

$$\|F\|_{(L^q(\Omega))^*} = \|f\|_{q'}\,.$$

Proof: See e.g. [KuFuJo].

□

We denote

$$L^q_{loc}(\Omega) = \{u;\, u \in L^q(K)\ \forall K \subset \Omega, K \text{ compact}\}\,. \tag{1.17}$$

Especially for Ω exterior domain

$$L^q_{loc}(\overline{\Omega}) = \{u;\, u \in L^q(\Omega_R)\ \forall R > \operatorname{diam}\Omega^c\}\,. \tag{1.18}$$

Finally, let g be a measurable non–negative function on Ω. We say that u belongs to the weighted L^q–space ($u \in L^q_{(g)}(\Omega)$) if

$$\int_\Omega |u(\mathbf{x})|^q g(\mathbf{x})\mathrm{d}\mathbf{x} < \infty\,.$$

We denote by

$$\|u\|_{q,(g),\Omega} = \Big(\int_\Omega |u(\mathbf{x})|^q g(\mathbf{x})\mathrm{d}\mathbf{x}\Big)^{\frac{1}{q}} \tag{1.19}$$

and by

$$\mathcal{L}^q_{(g)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{q,(g),\Omega}}\,. \tag{1.20}$$

VIII.1.3 Sobolev spaces

By $W^{k,p}(\Omega)$ we understand the set of all functions from $L^p(\Omega)$ such that all distributional derivatives[2] up to the order k belong to $L^p(\Omega)$, $1 \le p \le \infty$. As usually, putting

$$u = v \Longleftrightarrow u(\mathbf{x}) = v(\mathbf{x}) \text{ a.e. in } \Omega$$

[2]see Section VIII.4

we get a Banach space (for $p = 2$ a Hilbert space) equipped with the norm

$$\|u\|_{k,q,\Omega} = \Big(\sum_{|\alpha| \leq k} \|D^\alpha u\|_q^q \Big)^{\frac{1}{q}} . \tag{1.21}$$

The spaces $W^{k,p}(\Omega)$ are separable for $k \geq 0$, $p \in [1;\infty)$ and reflexive for $k \geq 0$, $p \in (1;\infty)$. We have

Lemma 1.5 *Let Ω be a bounded domain of class C^0, $1 \leq p < \infty$, $k \in \mathbb{N}$. Then*

$$W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p,\Omega}} .$$

Proof: See e.g. [Ne].

□

For Ω unbounded we have

Lemma 1.6 *Let Ω be an exterior domain of class C^0, $1 \leq p < \infty$, $k \in \mathbb{N}$. Then*

$$W^{k,p}(\Omega) = \overline{C_0^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p,\Omega}} .$$

Proof: It is an easy consequence of Lemma 1.5 and properties of the Lebesgue integral.

□

Remark 1.2 The space $W^{k,\infty}(\Omega)$ is isometrically isomorphic with $C^{k-1,1}(\overline{\Omega})$.

For Ω sufficiently regular we can always extend a function from $W^{k,p}(\Omega)$ onto the whole $\mathbb{R}^N$ in such a way that it remains in the same regularity class in $\mathbb{R}^N$.

Lemma 1.7 *Let Ω be bounded or exterior domain of class $C^{0,1}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$. Then there exists operator E from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^N)$ such that*

(i) $(Eu)(\mathbf{x}) = u(\mathbf{x})$, $\mathbf{x} \in \Omega$

(ii) $\|Eu\|_{k,p,\mathbb{R}^N} \leq C(k,p,\Omega)\|u\|_{k,p,\Omega}$

(iii) Eu has compact support in $\mathbb{R}^N$ if Ω is bounded

Proof: See [St].

□

Remark 1.3 The assertion of Lemma 1.7 holds true also for the spaces $C^k(\overline{\Omega})$ or

$$X^p_{k,s}(\Omega) = \{u \in C^s(\overline{\Omega}); D^\alpha u \in L^p(\Omega),\ s < |\alpha| \leq k\} ,$$
$$\|u\|_{X^p_{k,s}} = \|u\|_{C^s(\overline{\Omega})} + \Big(\sum_{s<|\alpha|\leq k} \|D^\alpha u\|_p^p \Big)^{\frac{1}{p}} .$$

For $\Omega \in C^{0,1}$ there exists again operator $E : C^k(\overline{\Omega}) \mapsto C^k(\mathbb{R}^N)$ ($X^p_{k,s}(\Omega) \mapsto X^p_{k,s}(\mathbb{R}^N)$) such that the properties (i) and (iii) remains true and we replace the norms in $W^{k,p}$ by the norms in C^k ($X^p_{k,s}$). The proof follows easily from the proof of Lemma 1.7 presented in [St].

The following two lemmas are fundamental in the study of imbedding from $W^{k,p}(\Omega)$ into $L^q(\Omega)$ and $C^{0,\mu}(\overline{\Omega})$. For the proofs see e.g. [Ev] or [KuFuJo].

Lemma 1.8 *Let $1 \le p < N$. Then there exists $C = C(p,N)$ such that*

$$\|u\|_{\frac{Np}{N-p},\mathbb{R}^N} \le C\|\nabla u\|_{p,\mathbb{R}^N}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Lemma 1.9 *Let $p > N$. Then there exists $C = C(p,N)$ such that*

$$\|u\|_{C^{0,1-\frac{N}{p}}(\mathbb{R}^N)} \le C\|u\|_{1,p,\mathbb{R}^N}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Combining Lemmas 1.8, 1.9 together with Lemmas 1.7, 1.6, 1.5 and 1.3 we easily obtain

Theorem 1.2 (Imbedding I)
Let $\Omega \in C^{0,1}$ be a bounded or an exterior domain. Let $kp < N$. Then there exists a constant $C = C(\Omega, N, k, p, q)$ such that

$$\|u\|_{q,\Omega} \le C\|u\|_{k,p,\Omega}$$

for all $u \in W^{k,p}(\Omega)$, $q \in [p; p^*]$, where $p^* = \frac{Np}{N-kp}$; it means that

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)\,.$$

Remark 1.4 If Ω is bounded, then $q \in [1; p^*]$.

Theorem 1.3 (Imbedding II)
Let $\Omega \in C^{0,1}$, $p \ge 1$, $kp > N$. Set

$$\mu \begin{cases} = k - \dfrac{N}{p} & \qquad k - \dfrac{N}{p} < 1 \\ < 1 & \text{if} \quad k - \dfrac{N}{p} = 1 \\ = 1 & \qquad k - \dfrac{N}{p} > 1\,. \end{cases}$$

Then $W^{k,p}(\Omega) \hookrightarrow C^{0,\mu}(\overline{\Omega})$, i.e. there exists $C = C(N,k,p,\Omega)$ and a representative $\overline{u} = u$ a.e. in Ω such that

$$\|\overline{u}\|_{C^{0,\mu}(\overline{\Omega})} \le C\|u\|_{k,p,\Omega}\,.$$

Remark 1.5 If $kp = N$, then it can be shown that $W^{k,p}(\Omega)$ is not imbedded into $L^\infty(\Omega)$, see e.g. [KuFuJo]. We have therefore only $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1; \infty)$ for Ω bounded, $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$, $q \in [p; \infty)$ for Ω exterior. See e.g. [KuFuJo] for further details.

For Ω bounded we can even show that in certain situations the imbedding is compact.

Theorem 1.4 (Imbedding III)
Let $\Omega \in C^{0,1}$ be a bounded domain in $\mathbb{R}^N$. Then we have

(i) for $kp < N$, $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1; p^*)$

(ii) for $kp = N$, $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q \in [1; \infty)$

(iii) for $kp > N$, $W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$.

Proof: See e.g. [Ne].

□

Let Ω be a bounded or an exterior domain of class $C^{0,1}$. We define[3]

$$L^q(\partial\Omega) = \Big\{ u \text{ measurable on } \partial\Omega; \sum_{r=1}^{M} \|{}_r u\|^q_{q,(\Delta_r)} = \sum_{r=1}^{M} \int_{\Delta_r} |{}_r u(\mathbf{x}'_r)|^q \mathrm{d}\mathbf{x}'_r < \infty \Big\} .$$

Then $L^q(\partial\Omega)$ is a Banach space equipped with the norm[4]

$$\|u\|_{q,(\partial\Omega)} = \Big(\sum_{r=1}^{M} \int_{\Delta_r} |{}_r u(\mathbf{x}'_r)|^q \mathrm{d}\mathbf{x}'_r \Big)^{\frac{1}{q}} .$$

We have

Theorem 1.5 (Traces I)
Let $\Omega \in C^{0,1}$ be a bounded or an exterior domain, $kp < N$, $p \in [1; \infty)$. Then there exists operator $T : W^{k,p}(\Omega) \mapsto L^q(\partial\Omega)$, $q \in [1; p^{\#}]$, $p^{\#} = \frac{p(N-1)}{N-kp}$ such that

(i) $\|Tu\|_{q,(\partial\Omega)} \le C(q, N, \Omega, k, p) \|u\|_{k,p,\Omega}$

(ii) $Tu = u/_{\partial\Omega}$ for $u \in C^\infty(\overline{\Omega})$.

---

[3]see the definition of a domain with smooth boundary;

$${}_r u(\mathbf{x}'_r) := u(T_r(\mathbf{x}'_r, a_r(\mathbf{x}'_r)))$$

[4]Let $\overline{\Omega} \subset \cup_{r=1}^{M} V_r^+ \cup V_{M+1}$, see the definition of a domain with smooth boundary. Then we define

$$I_1(u) = \sum_{r=1}^{M} \int_{\Lambda_r} |{}_r u|^q \mathrm{d}S = \sum_{r=1}^{M} \int_{\Delta_r} |{}_r u|^q \Big(1 + \sum_{i-1}^{N-1} \Big(\frac{\partial a_r}{\partial x_{r_i}} \Big)^2 \Big)^{\frac{1}{2}} \mathrm{d}\mathbf{x}'_r .$$

It is easily seen that thanks to the fact that $|\nabla a_r| \le C$ a.e. on Δ_r

$$I_1(u) \sim I_2(u) = \sum_{r=1}^{M} \int_{\Delta_r} |{}_r u|^q \mathrm{d}\mathbf{x}'_r .$$

Proof: See e.g. [Ne].

□

Remark 1.6 If $kp = N$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ for all $q \in [1;\infty)$; if $kp > N$, then from Theorem 1.3 $W^{k,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ for all $q \in [1;\infty]$.

In general, it is not true that the range of T coincides with $L^{p^\#}(\partial\Omega)$ for $kp < N$. It is possible to show that the range of T is closed subset of $L^{p^\#}(\partial\Omega)$ and can be characterized using the spaces with non–integer derivatives.

Let $s \in (0;1)$ and $q \in [1;\infty)$. We put

$$\langle\langle u \rangle\rangle_{s,q,(\partial\Omega)} = \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^q}{|\mathbf{y}-\mathbf{x}|^{N+qs-1}} \mathrm{d}S_{\mathbf{y}}\mathrm{d}S_{\mathbf{x}} \tag{1.22}$$

and denote by $W^{s,q}(\partial\Omega)$ the set of all functions from the space $L^q(\partial\Omega)$ such that $\langle\langle u \rangle\rangle_{s,q,(\partial\Omega)} < \infty$. We get a Banach space with the norm

$$\|u\|_{s,q,(\partial\Omega)} = \|u\|_{q,(\partial\Omega)} + \langle\langle u \rangle\rangle_{s,q,(\partial\Omega)}\,. \tag{1.23}$$

Similarly for $s > 1$ we define $W^{s,q}(\partial\Omega)$. Let $\Omega \in C^{[s],1}$, $[s]$ being the integer part of s. Let ${}_ru(\mathbf{x}'_r)$ be as above. We denote by $W^{s,q}(\partial\Omega)$ the set of all measurable functions on $\partial\Omega$ such that

$$\|u\|_{s,q,(\partial\Omega)} = \Big(\sum_{r=1}^{M} \|{}_ru\|^q_{s,q,(\Delta_r)}\Big)^{\frac{1}{q}} < \infty\,,$$

where

$$\|{}_ru\|_{s,q,(\Delta_r)} = \Big(\sum_{0\le|\alpha|\le[s]} \|D^\alpha{}_ru\|^q_{q,(\Delta_r)}\Big)^{\frac{1}{q}} + \langle\langle {}_ru \rangle\rangle_{s,q,(\Delta_r)}$$
$$\langle\langle {}_ru \rangle\rangle_{s,q,(\Delta_r)} = \Big(\sum_{|\alpha|=[s]} \int_{\Delta_r}\int_{\Delta_r} \frac{|D^\alpha{}_ru(\mathbf{y}') - D^\alpha{}_ru(\mathbf{x}')|^q}{|\mathbf{y}'-\mathbf{x}'|^{N+q(s-[s])-1}} \mathrm{d}\mathbf{y}'\mathrm{d}\mathbf{x}'\Big)^{\frac{1}{q}}\,.$$

The proof of the following theorems can be found e.g. in [Ne].

Theorem 1.6 (Traces II)
Let $p > 1$, $k \in \mathbb{N}$, $\Omega \in C^{k-1,1}$ be a bounded or an exterior domain. Then there exists a unique continuous linear mapping $\mathcal{T}_k : W^{k,p}(\Omega) \mapsto \prod_{l=0}^{k-1} W^{k-l-\frac{1}{p},p}(\partial\Omega)$ such that

$$\mathcal{T}_k u = \Big(u, \frac{\partial u}{\partial n}, \dots, \frac{\partial^{k-1}u}{\partial n^{k-1}}\Big)$$

for all $u \in C^\infty(\overline{\Omega})$; there exists $C = C(k,p,\Omega,N)$ such that

$$\sum_{l=0}^{k-1} \|(\mathcal{T}_k u)_l\|_{k-l-\frac{1}{p},p,(\partial\Omega)} \le C\|u\|_{k,p,\Omega}\,. \tag{1.24}$$

Remark 1.7 Evidently, (1.24) can be replaced by

$$\sum_{l=0}^{k-1} \|(\mathcal{T}_k u)_l\|_{k-l-\frac{1}{p},p,(\partial\Omega)} \leq C(k,p,\partial\Omega,V,N)\|u\|_{k,p,V}\,, \tag{1.25}$$

where $V = \Omega_R$ with $R > \operatorname{diam}\Omega^c$ for Ω exterior and $V = \cup_{r=1}^{M} V_r^+$ with V_r^+ from the definition of a domain with smooth boundary.

Theorem 1.7 (Inverse theorem on traces)
Let $p > 1$, $k \in \mathbb{N}$, $\Omega \in C^{k,1}$ be a bounded or an exterior domain. Then there exists a continuous linear mapping $T_k : \prod_{l=0}^{k-1} W^{k-l-\frac{1}{p},p}(\partial\Omega) \mapsto W^{k,p}(\Omega)$ such that for each $(u_0, u_1, \ldots, u_{k-1}) \in \prod_{l=0}^{k-1} W^{k-l-\frac{1}{p},p}(\partial\Omega)$; $T_k(u_0, u_1, \ldots, u_{k-1}) = v$ implies $T\Big(\frac{\partial^l v}{\partial n^l}\Big) = u_l$ on $\partial\Omega$, $l = 0, 1, \ldots, k-1$, T defined in Theorem 1.5. It means that there exists a constant $C = C(k,p,\Omega,N)$ such that

$$\|v\|_{k,p,\Omega} \leq C \sum_{l=0}^{k-1} \|u_l\|_{k-l-\frac{1}{p},p,(\partial\Omega)}\,. \tag{1.26}$$

If Ω is an exterior domain, then v can be chosen with bounded support.

Remark 1.8 If $k = 1$, we can take $\Omega \in C^{0,1}$. See e.g. [Ne] or [KuFuJo].

As an easy consequence of Theorems 1.6 and 1.7 we have

Corollary 1.1 *Let $\Omega \in C^{k,1}$ be a domain in $\mathbb{R}^N$. If $k \geq l$ and $\frac{1}{p} \geq \frac{1}{q} - \frac{k-l}{N}$, then*

$$W^{k-\frac{1}{q},q}(\partial\Omega) \hookrightarrow W^{l-\frac{1}{p},p}(\partial\Omega)\,.$$

Proof: Let $u \in W^{k-\frac{1}{q},q}(\partial\Omega)$. Then there exists $v \in W^{k,q}(\Omega)$ such that $\mathcal{T}_k v = (u, 0, \ldots, 0)$. But $W^{k,q}(\Omega) \hookrightarrow W^{l,p}(\Omega)$ and therefore $(\mathcal{T}_k v)_0 = u \in W^{l-\frac{1}{p},p}(\partial\Omega)$. Inequalities (1.24) and (1.26) finishes the proof.

□

We denote by

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,p,\Omega}}\,.$$

Let us note that for $\Omega = \mathbb{R}^N$, $W_0^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$. Otherwise we have

Theorem 1.8 *Let $\Omega \in C^{k,1}$. Then*

$$W_0^{k,p}(\Omega) = \{u \in W^{k,p}(\Omega)\,;\, Tu = T\frac{\partial u}{\partial n} = \ldots = T\frac{\partial^{k-1} u}{\partial n^{k-1}} = 0 \text{ in the sense of traces}\}\,.$$

Proof: See e.g. [Ne].

□

Remark 1.9 If $k = 1, 2$, then we can take $\Omega \in C^{0,1}$, for $k = 3$ it is enough to have $\Omega \in C^{1,1}$; see [Ne].

Lemma 1.8 implies

Lemma 1.10 *Let Ω be a domain. Then there exists $C = C(N, q)$ such that for all $u \in W_0^{1,q}(\Omega)$*

$$\|u\|_{\frac{Nq}{N-q}} \le C\|\nabla u\|_q ,$$

$1 \le q < N$. *If moreover Ω is bounded, then*

$$\|u\|_q \le C|\Omega|^{\frac{1}{N}} \|\nabla u\|_q$$

for all $1 \le q < \infty$.

Applying Theorem 1.4 (i) we can show

Theorem 1.9 (Poincaré)
Let $\Omega \in C^{0,1}$ be a bounded domain. Then there exists $C = C(\Omega, N, p, q)$ such that

$$\Big(\int_\Omega |u - \int_\Omega u d\mathbf{x}|^q d\mathbf{x}\Big)^{\frac{1}{q}} \le C\|\nabla u\|_p$$

for all $u \in W^{1,p}(\Omega)$ and $q \in [1; \frac{Np}{N-p}]$ if $p < N$, $q \in [1; \infty)$ if $p \ge N$.

Theorem 1.10 (Friedrichs)
Let $\Omega \in C^{0,1}$ be a bounded domain, $\Gamma \subset \partial\Omega$ be a part of boundary with positive Lebesgue $N-1$–dimensional measure. Then there exists $C = C(\Omega, p, N, q)$ such that

$$\|u\|_{q,\Omega} \le C(\|\nabla u\|_p + \int_\Gamma |u| dS) ,$$

for all $u \in W^{1,p}(\Omega)$ and $q \in [1; \frac{Np}{N-p}]$ if $p < N$, $q \in [1; \infty)$ if $p \ge N$.

Theorem 1.11 (Interpolation in s)
Let $\Omega \in C^{0,1}$ be a bounded or an exterior domain, $1 \le r < \infty$. Then

$$\|\nabla w\|_r \le C\|w\|_r^{\frac{1}{2}} \|w\|_{2,r}^{\frac{1}{2}} \tag{1.27}$$

for all $w \in W^{2,r}(\Omega)$. Especially, if $\Omega = \mathbb{R}^N$, then

$$\|\nabla w\|_r \le C\|w\|_r^{\frac{1}{2}} \|\nabla^2 w\|_r^{\frac{1}{2}} . \tag{1.28}$$

Proof: The inequality (1.28) in case $\Omega = \mathbb{R}^N$ is proved in [Mar]. If $\Omega \in C^{0,1}$, we can extend the function from $W^{2,r}(\Omega)$ onto $\mathbb{R}^N$ due to Lemma 1.7. Let us recall that

$$\begin{aligned} \|w\|_{r,\mathbb{R}^N} &\le C\|w\|_{r,\Omega} \\ \|\nabla^2 w\|_{r,\mathbb{R}^N} &\le C\|w\|_{2,r,\Omega} . \end{aligned} \tag{1.29}$$

Then (1.27) follows easily from (1.28) and (1.29).

□

From Lemma 1.1 and Theorem 1.11 we get

Corollary 1.2 *Let $w \in W^{2,r}(\Omega)$. Then for all $\varepsilon > 0$ there exists $C(\varepsilon)$ such that*

$$\|\nabla w\|_r \le \varepsilon \|w\|_{2,r} + C(\varepsilon)\|w\|_r \,.$$

The following interpolation inequalities are proved in [Mar].

Theorem 1.12 *Let $\nabla w \in L^s(\mathbb{R}^N)$, $w \in L^q(\mathbb{R}^N)$, $N \ge 2$, $s \in [1;\infty]$, $q \ge 1$. Then there exists $C > 0$ such that for $a \in [0;1]$ and $s \in [1;N)$*

$$\|w\|_r \le \begin{cases} C\|\nabla w\|_s^a \|w\|_q^{1-a} & q \le \frac{Ns}{N-s} \\ C\|\nabla w\|_s^{1-a}\|w\|_q^a & q \ge \frac{Ns}{N-s}\,, \end{cases}$$

$$r \in \begin{cases} [q; \frac{Ns}{N-s}) & \frac{1}{r} = a(\frac{1}{s} - \frac{1}{N}) + (1-a)\frac{1}{q} & q \le \frac{Ns}{N-s} \\ [\frac{Ns}{N-s}; q] & \frac{1}{r} = (1-a)(\frac{1}{s} - \frac{1}{N}) + \frac{a}{q} & q \ge \frac{Ns}{N-s}\,. \end{cases}$$

Moreover, for $s \in [N;\infty]$, $r \ge q$ $(N \ge 1)$

$$\|w\|_r \le C\|\nabla w\|_s^a \|w\|_q^{1-a}\,, \quad a \in [0;1)$$

and $\frac{1}{r} = a(\frac{1}{s} - \frac{1}{N}) + (1-a)\frac{1}{q}$.

Theorem 1.13 *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, $\Omega \in C^{0,1}$ be an exterior domain. Let $\nabla w \in L^s(\Omega)$, $w \in L^q(\Omega)$. Then there exists $C = C(r,s,q,a) > 0$ such that*

$$\|w\|_r \le C\|\nabla w\|_s^a \|w\|_q^{1-a}\,,$$

where if $s \in [1;N)$, then

$$r \in \begin{cases} [q; \frac{Ns}{N-s}) & \frac{1}{r} = a(\frac{1}{s} - \frac{1}{N}) + (1-a)\frac{1}{q} & q \le \frac{Ns}{N-s} \\ [\frac{Ns}{N-s}; q] & \frac{1}{r} = (1-a)(\frac{1}{s} - \frac{1}{N}) + \frac{a}{q} & q \ge \frac{Ns}{N-s}\,, \end{cases}$$

$a \in [0;1]$ and if $s \in [N;\infty)$, then $r \in [q;\infty)$ and $\frac{1}{r} = a(\frac{1}{s} - \frac{1}{N}) + (1-a)\frac{1}{q}$, $a \in [0;1)$.

Remark 1.10 Theorem 1.13 does not hold for $r = \infty$. Nevertheless (see [Mar] Remark 2.3)

$$\|w\|_\infty \le C\|\nabla w\|_s^a \|w\|_q^{1-a} + C(\varepsilon)\|\nabla w\|_s^{a-\varepsilon}\|w\|_q^{1-a+\varepsilon}$$

for all $\varepsilon \in (0;a]$.

Similarly to the Lebesgue spaces we denote

$$W^{k,p}_{loc} = \{u;\, u \in W^{k,p}(K);\, \forall K \subset \Omega\,,\, K \text{ compact}\}\,.$$

Next we shall characterize the dual spaces to $W_0^{k,p}(\Omega)$; We shall denote them by $(W_0^{k,p}(\Omega))^*$. Let us consider in $(W_0^{k,p}(\Omega))^*$ the linear subspace constituted by functionals of the form $\langle \mathcal{G}, u\rangle = (f,u) = \int_\Omega f u \mathrm{d}\mathbf{x}$, $f \in L^{q'}(\Omega)$. We set

$$\|f\|_{-k,p'} = \sup_{\|u\|_{k,p} \le 1} |\langle \mathcal{G}, u\rangle| \tag{1.30}$$

and denote by $W_0^{-k,p'}(\Omega)$ the space obtained by completing $L^{p'}(\Omega)$ in the norm (1.30). Then

Theorem 1.14 *The spaces* $W_0^{-k,p'}(\Omega)$ *and* $(W_0^{k,p}(\Omega))^*$, $1 < q < \infty$, *are algebraically and isometrically isomorphic.*

Proof: See [Lax] or [Mir].

□

Let us finish this subsection by recalling two useful results.

Combining the imbedding and trace theorem we can prove the following generalization of the classical Green formula (see also e.g. [Ne])

Theorem 1.15 *Let* $\Omega \in C^{0,1}$ *be a bounded domain,* $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$ *with* $\frac{1}{p} + \frac{1}{q} \leq \frac{N+1}{N}$, *for* $N > p \geq 1$, $N > q \geq 1$, *with* $q > 1$ *for* $p \geq N$ *and with* $p > 1$ *for* $q \geq N$. *Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} v \, d\mathbf{x} = \int_{\partial\Omega} u v n_i \, dS - \int_\Omega u \frac{\partial v}{\partial x_i} \, d\mathbf{x} \,,$$

where $\mathbf{n} = (n_1, \ldots, n_N)$ *is the outer normal to* Ω.

Finally, the Hardy inequality is an important tool in the weighted estimates (see [HaLiPo])

Theorem 1.16 (Hardy's inequality)
Let $f \in C^1([0;\infty))$. *Then*

$$\int_0^\infty |f(t)|^p t^{\varepsilon - p} \, dt \leq \Big(\frac{p}{|\varepsilon - p + 1|} \Big)^p \int_0^\infty |f'(t)|^p t^{\varepsilon} \, dt \,,$$

which holds for $\varepsilon > p - 1$ *if* $f(\infty) = 0$, *for* $\varepsilon < p - 1$ *if* $f(0) = 0$.

VIII.1.4 Homogeneous Sobolev spaces

In exterior domains we often meet situations when the classical Sobolev spaces are not applicable. We therefore introduce the homogeneous Sobolev spaces

$$\begin{aligned} D^{m,q}(\Omega) &= \{u \in L^1_{loc}(\Omega); D^\alpha u \in L^q(\Omega), \ \forall |\alpha| = m\} \\ D_0^{m,q}(\Omega) &= \overline{C_0^\infty(\Omega)}^{|\cdot|_{m,q,\Omega}} \,, \end{aligned} \tag{1.31}$$

where

$$|u|_{m,q} = \Big(\sum_{|\alpha| = m} \|D^\alpha u\|_q^q \Big)^{\frac{1}{q}} \,. \tag{1.32}$$

We easily have that if $u \in D^{m,q}(\Omega)$, then $u \in W^{m,q}_{loc}(\Omega)$ and, for $\Omega \in C^{0,1}$, also $u \in W^{m,q}_{loc}(\overline{\Omega})$. Especially for $\Omega \in C^{0,1}$, bounded, the spaces $W^{m,q}(\Omega)$ and $D^{m,q}(\Omega)$ coincides.

Assuming $u_1 = u_2$ whenever $|u_1 - u_2|_{m,q} = 0$ we get[5]

[5] it means that $u_1 = u_2$ whenever they differ by a polynomial of degree $m - 1$ a.e. in Ω

Lemma 1.11 *$\{D^{m,q}(\Omega); |\cdot|_{m,q}\}$ and $\{D_0^{m,q}(\Omega); |\cdot|_{m,q}\}$ are Banach spaces which are separable for $1 \le q < \infty$ and reflexive for $1 < q < \infty$.*

Proof: See e.g. [Ga1].

□

Next we shall study the asymptotic structure of functions from $D^{1,q}(\Omega)$ with Ω an exterior domain. By $\int_{S_N} f(R,\omega)\mathrm{d}\omega$ we understand surface integral over the unit sphere S_N.

Lemma 1.12 *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be an exterior locally lipschitzian domain and let $u \in D^{1,q}(\Omega)$. Let $1 \le q < N$. Then there exists a unique $u^* \in L^q(S_N)$ such that*

$$\lim_{R\to\infty} \int_{S_N} |u(R,\omega) - u^*(R,\omega)|^q \, d\omega = 0\,.$$

Moreover, for

$$u_0 = (N\omega_N)^{-1} \int_{S_N} u^*(R,\omega)\, d\omega\,, \quad w = u - u_0 \quad \omega_N = |S_N|_{N-1}$$

we have for all $R > diam\, \Omega^c$

$$\int_{S_N} |w(R,\omega)|^q \, d\omega \le C(q,N) R^{q-N} \int_{\Omega^R} |\nabla u|^q \, d\mathbf{x}$$

and $w \in L^s(\Omega)$, $s = \frac{Nq}{N-q}$,

$$\|w\|_s \le C(q,N)|w|_{1,q}\,.$$

If $q \ge N$, then

$$\int_{S_N} |u(R,\omega)|^q \, d\omega = h(R)o(1) \qquad as\ R \to \infty\,,$$

where $h(R) = (\ln R)^{N-1}$ if $q = N$, while $h(R) = R^{q-N}$ if $q > N$.

Proof: See [Ga1].

□

Next we have

Theorem 1.17 *Let $\Omega \subset \mathbb{R}^N$ be an exterior domain and let*

$$u \in D^{1,r}(\Omega) \cap D^{1,q}(\Omega)\,,\ 1 \le r < \infty\,,\ N < q < \infty\,.$$

If $r < N$, then there exists $u_0 \in \mathbb{R}$ such that

$$\lim_{|\mathbf{x}|\to\infty} |u(\mathbf{x}) - u_0| = 0$$

uniformly; if $r = N$

$$\lim_{|\mathbf{x}|\to\infty} \frac{|u(\mathbf{x})|}{(\ln|\mathbf{x}|)^{\frac{N-1}{N}}} = 0$$

uniformly. Finally, if

$$u \in D^{1,q}(\Omega)\,, \quad N < q < \infty\,,$$

it holds that

$$\lim_{|\mathbf{x}|\to\infty} \frac{|u(\mathbf{x})|}{|\mathbf{x}|^{\frac{q-N}{q}}} = 0$$

uniformly.

Proof: See [Ga1].

□

Remark 1.11 The condition $u \in D^{1,r}(\Omega)$, $1 \le r < N$ can be replaced by $u - u_0 \in L^q(\Omega)$ for some $q \in [1;\infty)$.

Now, let us study the question of approximation of functions from $D^{1,q}(\Omega)$ by functions from $C_0^\infty(\overline{\Omega})$. We want to give the conditions, under which the space $C_o^\infty(\overline{\Omega})$ is dense in $D^{1,q}$, i.e.

$$\forall u \in D^{1,q}(\Omega) \exists u_n \in C_0^\infty(\overline{\Omega}) \quad |u_n - u|_{1,q} \to 0 \text{ for } n \to \infty\,. \tag{1.33}$$

Lemma 1.13 *Let* $\Omega \in C^{0,1}$ *be an exterior domain. Then the condition (1.33) holds for* $q \ge N$*. For* $1 \le q < N$ *the condition (1.33) holds if and only if the constant* u_0 *from Lemma 1.12 is zero. Moreover, for* $1 \le q < N$, $u \in D_0^{1,q}(\Omega)$ *if and only if the trace*[6] $Tu = 0$ *and* $u_0 = 0$*, while for* $q \ge N$ $u \in D_0^{1,q}(\Omega)$ *if and only if the trace* $Tu = 0$*.*

Proof: See [Ga1].

□

Let $1 < q < \infty$. By $(D_0^{1,q}(\Omega))^*$ we denote the dual space to $D_0^{1,q}(\Omega)$ ($\Omega \subset \mathbb{R}^N$, either an exterior domain or $\mathbb{R}^N$). We consider the functional

$$\langle \mathcal{G}, u\rangle = (f,u) = \int_\Omega f u \mathrm{d}\mathbf{x}\,, \tag{1.34}$$

$f \in C_0(\Omega)$, Ω exterior domain. We have by Lemma 1.10

$$|\langle \mathcal{G}, u\rangle| \le C\|f\|_{q'}|u|_{1,q}$$

and so we set

$$|\mathcal{G}|_{-1,q'} = \sup_{|u|_{1,q}=1} |\langle \mathcal{G}, u\rangle|$$

[6] see Theorem 1.5

and by $D_0^{-1,q'}(\Omega)$ we denote the completition of $C_0(\Omega)$ in this norm. We can show for $1 < q < \infty$ (see [Lax]) that $D_0^{-1,q'}(\Omega)$ and $(D_0^{1,q}(\Omega))^*$ are isometrically and topologically isomorphic.

In the case of $\Omega = \mathbb{R}^N$ we have for $q < N$

$$|\langle \mathcal{G}, u\rangle| \leq \|f\|_{\frac{Nq'}{N+q'}} |u|_{\frac{Nq}{N-q}} \leq C\|f\|_{\frac{Nq'}{N+q'}} |u|_{1,q}$$

and we proceed as above. If $q \geq N$ the elements of $D_0^{1,q}(\mathbb{R}^N)$ are equivalent classes determined by functions that may differ by constants. Therefore the functions f must satisfy

$$\int_{\mathbb{R}^N} f \mathrm{d}\mathbf{x} = 0\,.$$

Then ($\operatorname{supp} f \subset B_R$ for some $R > 0$)

$$|\langle \mathcal{G}, u\rangle| = \Big|\int_{B_R} fu \mathrm{d}\mathbf{x}\Big| = \Big|\int_{B_R} f(u+C) \mathrm{d}\mathbf{x}\Big| \leq \|f\|_{q',B_R}\|u+C\|_{q,B_R}\,.$$

Choosing C in such a way that

$$\int_{B_R} (u+C)\mathrm{d}\mathbf{x} = 0\,,$$

we can apply the Poincaré inequality (see Theorem 1.9) to obtain

$$|\langle \mathcal{G}, u\rangle| \leq C\|f\|_{q',\mathbb{R}^N}\|u\|_{1,q,B_R} \leq C\|f\|_{q',\mathbb{R}^N}\|u\|_{1,q,\mathbb{R}^N}\,.$$

We have

Theorem 1.18 *Let $\Omega \subseteq \mathbb{R}^N$ be either an exterior, locally lipschitzian domain or the whole $\mathbb{R}^N$. Then, functionals of the form (1.34) are bounded in $(D_0^{1,q}(\Omega))^*$, $1 \leq q < \infty$ with $f \in C_0(\Omega)$ and $\int_\Omega f \mathrm{d}\mathbf{x} = 0$ if $\Omega = \mathbb{R}^N$ and $q \geq N$. Moreover, if $1 < q < \infty$, then their completition $D_0^{-1,q'}(\Omega)$ is isometrically isomorphic to $(D_0^{1,q}(\Omega))^*$.*

VIII.1.5 Bochner spaces. Abstract continuous functions

Let X be a Banach space. We say that $u : I \mapsto X$, strongly measurable function, belongs to $L^p(I;X)$ if

$$\int_I \|u(\tau)\|_X^p \mathrm{d}\tau < \infty\,;$$

here $I \subset \mathbb{R}^m$. We usually assume $I \subset \mathbb{R}$ as such spaces play an important role in the evolutionary equations. Moreover, we say that $u \in C(I;X)$ if for all $t_0 \in I$

$$\lim_{t\to t_0} \|u(t) - u(t_0)\|_X = 0\,.$$

Again, we shall use such spaces for $I \subset \mathbb{R}$. We have in particular

Theorem 1.19 *Let I be a compact interval, $p \in (1;\infty)$, $k \geq 0$. Then*

(i) the space $C_0^\infty(I \times \mathbb{R}^N)$ is dense in $L^p(I; W^{k,p}(\mathbb{R}^N))$

(ii) the space $C^\infty(I; C_0^\infty(\mathbb{R}^N))$ *is dense in* $C(I; W^{k,p}(\mathbb{R}^N))$.

Proof: See [Lio] or [KuFuJo], where also more detailed description of such spaces can be found.

□

In Chapter VII we also need

Theorem 1.20 (Gronwall's inequality)
Let $I \subset \mathbb{R}$ *be an interval,* $\eta \in \mathbb{R}$, $s \in I$, ϱ, $\xi : I \mapsto \mathbb{R}$. *Let* ϱ *be a integrable function,* $\varrho(t) \geq 0$, $\xi(t) \geq 0$ *for* $t \in I$, $\eta \geq 0$. *Let*

$$\xi(t) \leq \eta + \int_s^t \varrho(\sigma)\xi(\sigma)\,d\sigma$$

for all $t \in I$. *Then*

$$\xi(t) \leq \eta \,exp\Big(\int_s^t \varrho(\sigma)\,d\sigma\Big)$$

for $t \in I$.

Proof: See e.g. [Ku].

□

Let $u \in L^p(0,T;X)$. Then $g \in L^p(0,T;X)$ is the weak derivative of u,

$$u' = v\,,$$

provided $\int_0^T \varphi'(t)u(t)\mathrm{d}t = -\int_0^T \varphi(t)u(t)\mathrm{d}t$ for all functions $\varphi \in C_0^\infty(0;T)$. We have

Theorem 1.21 *Suppose* $u \in L^2(0,T;W^{1,2}(\mathbb{R}^N))$, *with the time derivative* $u' \in L^2(0,T;W^{-1,2}(\mathbb{R}^N))$.

(i) Then

$$u \in C([0;T]; L^2(\mathbb{R}^N))$$

(after possible being redefined on a set of measure zero).

(ii) The mapping

$$t \mapsto \|u(t)\|_2^2$$

is absolutely continuous, with

$$\frac{d}{dt}\|u(t)\|_2^2 = 2\langle u'(t), u(t)\rangle$$

for a.e. $t \in (0;T)$.

Proof: See e.g. [Ev]. More general version of such a result can be found e.g. in [GaGrZa].

□

VIII.2 Some remarks on integral operators. Cut–off functions

A general integral operator T with kernel $\tilde{K}$ can be written in the form

$$(Tf)(\mathbf{x}) = \int_\Omega \tilde{K}(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y}\,, \tag{2.1}$$

$\Omega \subseteq \mathbb{R}^N$. We shall study such integral operators only in the case when (2.1) represents convolutions, it means that $\tilde{K}(\mathbf{x},\mathbf{y}) = K(\mathbf{x}-\mathbf{y})$. Two important situations will be considered: either $K(\cdot)$ is locally integrable function, or $K(\cdot)$ has a special form and the convolution (2.1) must be studied in the principal value sense.

Let us start with the former. First, we can without loss of generality take $\Omega = \mathbb{R}^N$.[7] We have

Theorem 2.1 (Young)
Let $K \in L^s(\mathbb{R}^N)$, $1 \le s \le \infty$. If $f \in L^q(\mathbb{R}^N)$, $1 \le q \le \infty$, $\frac{1}{q} \ge 1 - \frac{1}{s}$, then $K * f \in L^r(\mathbb{R}^N)$, $\frac{1}{r} = \frac{1}{s} + \frac{1}{q} - 1$ and

$$\|K * f\|_r \le \|K\|_s \|f\|_q\,. \tag{2.2}$$

The Young inequality (2.2) can be e.g. proved using the Riesz–Thorin interpolation theorem which will be also needed in Section VIII.4. For the proof see e.g. [BeLo] or [StWe].

Theorem 2.2 (Riesz–Thorin)
Let T be an operator such that for some (p_i, q_i), $i = 1,2$, $q_1 \le q_2$,

$$\|Tf\|_{p_i} \le C_i \|f\|_{q_i} \tag{2.3}$$

for all f from some dense subset of $L^{q_i}(\mathbb{R}^N)$[8]. Then T can be continuously extended onto all $L^q(\mathbb{R}^N)$, $q \in [q_1; q_2]$ and

$$\|Tf\|_p \le C_1^t C_2^{1-t} \|f\|_q\,, \tag{2.4}$$

where $\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2}$, $\frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_2}$, $t \in [0;1]$.

Proof of Theorem 2.1: We have evidently

$$\|K * f\|_\infty \le \|K\|_s \|f\|_{s'}$$
$$\|K * f\|_s \le \|K\|_s \|f\|_1\,.$$

Therefore

$$\|K * f\|_r \le \|K\|_s \|f\|_q\,,$$

where

$$\frac{1}{r} = \frac{1-t}{s} \qquad \frac{1}{q} = \frac{t}{s'} + \frac{1-t}{1}\,,$$

[7] Otherwise, we can extend K and f by zero outside of Ω.

[8] e.g. from $C_0^\infty(\mathbb{R}^N)$ if $q_i \in [1;\infty)$, or it contains all characteristic functions of all Lebesgue measurable sets with finite measure if $q_i = \infty$

i.e.

$$\frac{1}{q} = 1 + \frac{1}{r} - \frac{1}{s}\,.$$

The condition $t \in [0;1]$ yields $\frac{1}{q} \geq 1 - \frac{1}{s}$.

□

Next, let

$$K(\mathbf{x}-\mathbf{y}) = \frac{k(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^\lambda}\,, \tag{2.5}$$

where $k(\cdot)$ is a given regular function. We have for $0 < \lambda < N$

Theorem 2.3 *Let $f \in L^q(\mathbb{R}^N)$, $1 < q < \infty$ and K be of the type (2.5) with $\lambda > N(1-\frac{1}{q})$, $0 < \lambda < N$. Then the integral transform defined in (2.1) with $\Omega = \mathbb{R}^N$ belongs to $L^s(\mathbb{R}^N)$, where $\frac{1}{s} = \frac{\lambda}{N} + \frac{1}{q} - 1$. Moreover*

$$\|Tf\|_s \leq C\|f\|_q \tag{2.6}$$

with $C = C(q, N, \lambda)$.

Proof: See e.g. [St].

□

In the case of Ω bounded we get a stronger result

Theorem 2.4 *Let Ω be bounded, K of the type (2.5) and $f \in L^q(\Omega)$, $1 < q < \infty$. Then for $\lambda < N(1-\frac{1}{q})$ the integral transform (2.1) belongs to $C^{0,\mu}(\overline{\Omega})$, where $\mu = \min\{1, N(1-\frac{1}{q}) - \lambda\}$ and*

$$\|Tf\|_{C^{0,\mu}} \leq C_1\|f\|_q$$

with $C_1 = C_1(diam\,\Omega, N, q, \lambda)$. Moreover, if $\lambda = N(1-\frac{1}{q})$, then $Tf \in L^r(\Omega)$ for all $r \in [1;\infty)$ and

$$\|Tf\|_r \leq C_2\|f\|_q$$

with $C_2 = C_2(diam\,\Omega, N, q, \lambda)$.

Proof: See e.g. [Ga1].

□

Another important case, if $\lambda = N$, was discussed in subsection II.3.2, where also another types of singular integral operators are studied (using the Fourier transform). Let us only recall that we must add some assumptions on $k(\cdot)$ and the integral transform (2.1) has to be considered in the principal value sense, i.e.

$$(Tf)(\mathbf{x}) = \text{v.p.}\int_{\mathbb{R}^N} K(\mathbf{x}-\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y} = \lim_{\varepsilon\to 0^+}\int_{B^\varepsilon(\mathbf{x})} K(\mathbf{x}-\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y}\,.$$

Next we study the so–called cut–off functions. Let $\eta(z) \in C_0^\infty(\mathbb{R})$ be such that

$$\operatorname{supp}\eta \subset B_2(0)\,, \eta(z) = 1 \text{ in } B_1(0)\,, \eta(z) \le 1 \text{ in } \mathbb{R}\,.$$

We shall study two kinds of cut–off functions. We put

$$\begin{aligned} \eta_R(\mathbf{x}) &= \eta(R|\mathbf{x}|)\,, \quad R > 0 \\ \zeta_R(\mathbf{x}) &= \eta\Big(\frac{\ln\ln|\mathbf{x}|}{\ln\ln R}\Big)\,, \quad R > \mathrm{e}\,, \end{aligned} \tag{2.7}$$

where we define $\eta(-r) = 1$ for $r \ge 0$. We call the function η_R the usual cut–off function and ζ_R the Sobolev cut–off function.

We shall often use the following property of the cut–off functions

Lemma 2.1 *Let $u \in L^q_{loc}(\Omega)$, $\nabla u \in L^q(\Omega)$ and u_0 be defined in Lemma 1.12. Let Ω be an exterior domain or $\Omega = \mathbb{R}^N$.*

(i) If $1 \le q < N$, then

$$\|(u - u_0)\nabla\eta_R\|_q \to 0 \text{ as } R \to \infty\,. \tag{2.8}$$

(ii) If $q > N$, then

$$\|u\nabla\eta_R\|_q \to 0 \text{ as } R \to \infty\,. \tag{2.9}$$

Proof: Let us start with the case (i). We have for $R > \operatorname{diam}\Omega^c$ due to Lemma 1.12

$$\begin{aligned} \int_{\mathbb{R}^N} |(u-u_0)\nabla\eta_R|^q \mathrm{d}\mathbf{x} &\le \frac{C}{R^q}\int_R^{2R} r^{N-1}\Big(\int_{S_N}|u-u_0|^q \mathrm{d}\omega\Big)\mathrm{d}r \le \\ &\le \frac{C}{R^q}\int_R^{2R} r^{q-1}\Big(\int_{B^r}|\nabla u|^q \mathrm{d}\mathbf{x}\Big)\mathrm{d}r \le \frac{C}{R^q}\int_R^{2R} r^{q-1}\mathrm{d}r\|\nabla u\|^q_{q,B^R} \\ &\le C\|\nabla u\|_{q,B^R} \to 0 \end{aligned}$$

as $R \to \infty$. Next, let $q > N$. Again, applying Lemma 1.12

$$\begin{aligned} \int_{\mathbb{R}^N} |u\nabla\eta_R|^q \mathrm{d}\mathbf{x} &\le \frac{C}{R^q}\int_R^{2R} r^{N-1}\Big(\int_{S_N}|u|^q \mathrm{d}\omega\Big)\mathrm{d}r \le \\ &\le \frac{o(1)}{R^q}\int_R^{2R} r^{q-1}\mathrm{d}r \le o(1) \end{aligned}$$

as $R \to \infty$.

□

Remark 2.1 If $q = N$, then

$$\int_{\mathbb{R}^N} |u\nabla\eta_R|^N \mathrm{d}\mathbf{x} \le o(1)\frac{C}{R^N}\int_R^{2R} r^{N-1}(\ln r)^{N-1}\mathrm{d}r$$

and we cannot control the integral for $R \to \infty$.

Lemma 2.2 *Let $u \in L^q_{loc}(\Omega)$, $\nabla u \in L^q(\Omega)$ and u_0 be defined as in Lemma 1.12. Let Ω be an exterior domain or $\Omega = \mathbb{R}^N$.*

(i) If $1 \le q < N$*, then*

$$\|(u-u_0)\nabla\zeta_R\|_q \to 0 \ \text{as}\ R\to\infty\,. \tag{2.10}$$

(ii) If $q \ge N$*, then*

$$\|u\nabla\zeta_R\|_q \to 0 \ \text{as}\ R\to\infty\,. \tag{2.11}$$

Proof: We have

$$\nabla^k\zeta_R \le \frac{C}{\ln\ln R}\frac{1}{|\mathbf{x}|^k\ln|\mathbf{x}|}$$

for $|\mathbf{x}|$ and R sufficiently large, $k\ge 1$. Let us first consider the case (i). We have for $R > \operatorname{diam}\Omega^c$ due to Lemma 1.12

$$\int_{\mathbb{R}^N} |(u-u_0)\nabla\zeta_R|^q \mathrm{d}\mathbf{x} \le \frac{C}{(\ln\ln R)^q}\int_R^{\mathrm{e}^{\ln^2 R}} r^{N-q-1}(\ln r)^{-q}\cdot$$

$$\cdot\Big(\int_{S_N}|u-u_0|^q\mathrm{d}\omega\Big)\mathrm{d}r \le \frac{C}{(\ln\ln R)^q}\int_R^{\mathrm{e}^{\ln^2 R}} r^{-1}(\ln r)^{-q}\Big(\int_{B^r}|\nabla u|^q\mathrm{d}\mathbf{x}\Big)\mathrm{d}r \le$$

$$\le \frac{C}{(\ln\ln R)^q}\int_{\ln R}^{\ln^2 R} t^{-q}\mathrm{d}t\|\nabla u\|^q_{q,B^R} \to 0$$

as $R\to\infty$. Next, let $q\ge N$. Again, applying Lemma 1.12

$$\int_{\mathbb{R}^N}|u\nabla\zeta_R|^q\mathrm{d}\mathbf{x} \le \frac{C}{(\ln\ln R)^q}\int_R^{\mathrm{e}^{\ln^2 R}} r^{N-1-q}(\ln r)^{-q}\Big(\int_{S_N}|u|^q\mathrm{d}\omega\Big)\mathrm{d}r\,.$$

Now, if $q = N$,

$$\int_{\mathbb{R}^N}|u\nabla\zeta_R|^q\mathrm{d}\mathbf{x} \le \frac{o(1)}{(\ln\ln R)^N}\int_R^{\mathrm{e}^{\ln^2 R}}(\ln r)^{-1}r^{-1}\mathrm{d}r \le$$

$$\le \frac{o(1)}{(\ln\ln R)^{N-1}} \to 0$$

as $R\to\infty$. If $q > N$, then

$$\int_{\mathbb{R}^N}|u\nabla\zeta_R|^q\mathrm{d}\mathbf{x} \le \frac{o(1)}{(\ln\ln R)^q}\int_R^{\mathrm{e}^{\ln^2 R}}(\ln r)^{-q}r^{-1}\mathrm{d}r \le$$

$$\le \frac{o(1)}{(\ln\ln R)^{q-1}}\int_{\ln R}^{\ln^2 R}\frac{1}{t^q} \to 0$$

as $R\to\infty$ ($q > N \ge 2$).

□

VIII.3 The problem $\nabla\cdot\mathbf{v} = f$. Function spaces of hydrodynamics

Before defining the special function spaces with zero divergence, we start with an auxiliary problem. Let $\Omega\subset\mathbb{R}^N$ be a bounded domain, $N\ge 2$. Let $f\in L^q(\Omega)$ be given. We search a vector field $\mathbf{v}:\Omega\mapsto\mathbb{R}^N$ such that

$$\begin{aligned} \nabla\cdot\mathbf{v} &= f \quad \text{in}\ \Omega \\ \mathbf{v} &\in W_0^{1,q}(\Omega) \\ |\mathbf{v}|_{1,q} &\le C\|f\|_q\,, \end{aligned} \tag{3.1}$$

where $C = C(N, q, \Omega)$.

Evidently, the right hand side f must undergo a certain compatibility condition. Namely,

$$\int_\Omega f \mathrm{d}\mathbf{x} = 0\,. \tag{3.2}$$

Since Ω is bounded, we have (see Lemma 1.10) that $(3.1)_3$ is equivalent with

$$\|\mathbf{v}\|_{1,q} \le C\|f\|_q\,. \tag{3.3}$$

Solution to (3.1) is evidently non–unique.

Theorem 3.1 *Let Ω be a bounded domain*[9] *in $\mathbb{R}^N$ of class $C^{0,1}$. Then, given $f \in L^q(\Omega)$, $1 < q < \infty$, there exists solution to the problem (3.1). Furthermore, if $\Omega = B_R(\mathbf{0})$*[10]*, the constant C in $(3.1)_3$ can be taken independently of the size of Ω.*

Proof: See [Bog] or [Ga1].

□

Concerning the regularity of solution we have

Theorem 3.2 *Let $\Omega \in C^{0,1}$ be a bounded domain in $\mathbb{R}^N$, $N \ge 2$. Given*

$$f \in W_0^{m,q}(\Omega)\,, \quad m \ge 0\,, \quad 1 < q < \infty\,,$$

satisfying (3.2), there exists $\mathbf{v} \in W_0^{m+1,q}(\Omega)$ verifying (3.1) and

$$\|\nabla\mathbf{v}\|_{m,q} \le C\|f\|_{m,q}\,, \tag{3.4}$$

where the constant C behaves like the constant in Theorem 3.1. Moreover, if f has compact support in Ω, then $\mathbf{v}$ can be taken also with compact support. Especially, if $f \in C_0^\infty(\Omega)$, then $\mathbf{v} \in C_0^\infty(\Omega)$.

Proof: See [Bog] or [Ga1].

□

Remark 3.1 If $f \in W_0^{m,q}(\Omega) \cap W_0^{m,r}(\Omega)$, $1 < q, r < \infty$, $m \ge 0$, satisfying (3.2), then the solution $\mathbf{v} \in W_0^{m+1,q}(\Omega) \cap W_0^{m+1,r}(\Omega)$ and

$$\begin{aligned} \|\nabla\mathbf{v}\|_{m,q} &\le C\|f\|_{m,q} \\ \|\nabla\mathbf{v}\|_{m,r} &\le C\|f\|_{m,r}\,. \end{aligned} \tag{3.5}$$

[9] The condition can be further weaken; it is enough to take Ω satisfying the cone condition, see e.g. [Bog].

[10] The precise estimate for Ω general is given in [Ga1] or [Bog]; we need such estimate only for Ω a ball.

Lemma 3.1 *Let us consider a generalization to (3.1), namely*

$$\begin{aligned} \nabla \cdot \mathbf{v} &= f \ \textit{in } \Omega \\ \mathbf{v} &\in W^{1,q}(\Omega) \\ \mathbf{v} &= \mathbf{a} \ \textit{at } \partial\Omega\,, \end{aligned} \tag{3.6}$$

$\Omega \in C^{0,1}$, $f \in L^q(\Omega)$, $\mathbf{a} \in W^{1-\frac{1}{q},q}(\partial\Omega)$, $1 < q < \infty$ *satisfying*

$$\int_\Omega f\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} dS\,.$$

Then there exists at least one solution to the problem (3.6) such that

$$\|\mathbf{v}\|_{1,q} \le C(\|f\|_q + \|\mathbf{a}\|_{1-\frac{1}{q},q,(\partial\Omega)})\,. \tag{3.7}$$

Proof: Let us denote by $\mathbf{A}$ an extension of $\mathbf{a}$ onto Ω due to Theorem 1.7. Denote by $\mathbf{u}$ solution to

$$\begin{aligned} \nabla \cdot \mathbf{u} &= f - \nabla \cdot \mathbf{A} \\ \mathbf{u} &\in W_0^{1,q}(\Omega)\,. \end{aligned} \tag{3.8}$$

Due to Theorem 3.1 there exists solution to (3.8) such that

$$\|\mathbf{u}\|_{1,q} \le C(\|f\|_q + \|\nabla \cdot \mathbf{A}\|_q)$$

and as

$$\mathbf{v} = \mathbf{u} + \mathbf{A}$$
$$\|\mathbf{A}\|_{1,q} \le C\|\mathbf{a}\|_{1-\frac{1}{q},q,(\partial\Omega)}\,,$$

the proof is complete.

□

For Ω exterior domain we can skip the condition (3.2). Namely, considering the problem

$$\begin{aligned} \nabla \cdot \mathbf{v} &= f \quad \text{ in } \Omega \\ \mathbf{v} &\in D_0^{1,q}(\Omega) \\ |\mathbf{v}|_{1,q} &\le C\|f\|_q\,, \end{aligned} \tag{3.9}$$

we have

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$ be a $C^{0,1}$–exterior domain. Then for any $f \in L^q(\Omega)$, $1 < q < \infty$, there exists a solution to the problem (3.9).*

Proof: See [Bog] or [Ga1].

□

Remark 3.2 Let us only note that for $f \in C_0^\infty(\Omega)$ we have

$$\mathbf{v} = \nabla\psi + \mathbf{w}\,,$$

where $\Delta\psi = f$ in $\mathbb{R}^N$ and $\mathbf{w}$ has bounded support. Therefore even for such f the solution does not have bounded support and behaves at infinity like $\nabla(\mathcal{E} * f)$, $\mathcal{E}$ being the fundamental solution to the Laplace equation.

Remark 3.3 Analogously to the case Ω bounded we have that for $f \in L^q(\Omega) \cap L^r(\Omega)$ the solution to (3.9) belongs to $D_0^{1,q}(\Omega) \cap D_0^{1,r}(\Omega)$ and

$$|\mathbf{v}|_{1,q} \le C\|f\|_q \quad |\mathbf{v}|_{1,r} \le C\|f\|_r \,. \tag{3.10}$$

Studying the problem

$$\begin{gathered} \nabla \cdot \mathbf{v} = f \text{ in } \Omega \\ \mathbf{v} \in D^{1,q}(\Omega) \\ \mathbf{v} = \mathbf{a} \text{ at } \partial\Omega \\ |\mathbf{v}|_{1,q} \le C(\|f\|_q + \|\mathbf{a}\|_{1-\frac{1}{q},q,(\partial\Omega)}) \end{gathered} \tag{3.11}$$

we can verify that there exists solution to (3.11) such that the constant $C = C(N,q,\Omega)$ in $(3.11)_4$. Let us note that the condition $\int_\Omega f \mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\mathrm{d}S$ is no more required. If $f \equiv 0$ and $\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\mathrm{d}S = 0$, then the solution to (3.11) can be taken with bounded support.

In the theory of incompressible fluids the spaces with zero divergence have a great importance. We denote by

$${}_0\mathcal{D}(\Omega) = \{\mathbf{u} \in C_0^\infty(\Omega); \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\} \tag{3.12}$$

and

$$\widehat{H}_q^1(\Omega) = \{\mathbf{u} \in W_0^{1,q}(\Omega); \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\} \tag{3.13}$$

$$H_q^1(\Omega) = \overline{\{{}_0\mathcal{D}(\Omega)\}}^{\|\cdot\|_{1,q}} \tag{3.14}$$

$$\widehat{\mathcal{D}}_0^{1,q}(\Omega) = \{\mathbf{u} \in D_0^{1,q}(\Omega); \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\} \tag{3.15}$$

$$\mathcal{D}_0^{1,q}(\Omega) = \overline{\{{}_0\mathcal{D}(\Omega)\}}^{|\cdot|_{1,q}} \,. \tag{3.16}$$

We shall show that for Ω bounded or exterior domain with sufficiently smooth boundary, namely $\Omega \in C^{0,1}$, the spaces defined by closure coincides with the spaces $\widehat{H}_q^1(\Omega)$ and $\widehat{\mathcal{D}}_0^{1,q}$, respectively. See also [Ga1] or [LaSo].

Let us start with Ω bounded. We have

Theorem 3.4 *Let $\Omega \in C^{0,1}$ be a bounded domain*[11] *in $\mathbb{R}^N$, $N \ge 2$, $1 \le q < \infty$. Then*

$$\begin{gathered} \widehat{H}_q^1(\Omega) = H_q^1(\Omega) \\ \widehat{\mathcal{D}}_0^{1,q} = \mathcal{D}_0^{1,q}(\Omega)\,. \end{gathered}$$

Proof: As for Ω bounded $\mathcal{D}_0^{1,q}(\Omega) = H_q^1(\Omega)$ and $\widehat{\mathcal{D}}_0^{1,q}(\Omega) = \widehat{H}_q^1(\Omega)$, it is enough to proof the result for $H_q^1(\Omega)$.

Evidently, $H_q^1(\Omega) \subset \widehat{H}_q^1(\Omega)$. Now, let $\mathbf{u} \in \widehat{H}_q^1(\Omega)$ and $\mathbf{u}_k$ be the approximative sequence of $\mathbf{u}$ from $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$, $1 < q < \infty$. Let $f_k = -\nabla\cdot\mathbf{u}_k$. As $\int_\Omega f_k \mathrm{d}\mathbf{x} = 0$ and $f_k \in C_0^\infty(\Omega)$, there exists $\mathbf{v}_k \in C_0^\infty(\Omega)$ such that

$$\begin{gathered} \nabla\cdot\mathbf{v}_k = f_k \\ \|\mathbf{v}_k\|_{1,q} \le C\|f_k\|_q = C\|\nabla\cdot u_k\|_q \end{gathered}$$

[11]The condition can be further weaken, e.g. Ω satisfying the cone condition.

(see Theorem 3.2). Denoting $\mathbf{w}_k = \mathbf{u}_k + \mathbf{v}_k$, then easily $\nabla \cdot \mathbf{w}_k = 0$ in Ω, and

$$\begin{aligned}\|\mathbf{w}_k - \mathbf{u}\|_{1,q} &\le \|\mathbf{u}_k - \mathbf{u}\|_{1,q} + \|\mathbf{v}_k\|_{1,q} \le \\ &\le \|\mathbf{u}_k - \mathbf{u}\|_{1,q} + C\|\nabla \cdot \mathbf{u}_k\|_q \to 0\,.\end{aligned}$$

We have therefore constructed an approximative sequence of $\mathbf{u}$ from ${}_0\mathcal{D}(\Omega)$ and the proof is complete for $1 < q < \infty$.

Now, let $q = 1$. It is sufficient to show that each continuous functional F defined in $\widehat{H}^1_1(\Omega)$, vanishing in $H^1_1(\Omega)$, is identically zero. But since $H^1_q(\Omega) \subset H^1_1(\Omega)$ and $\widehat{H}^1_q(\Omega) = H^1_q(\Omega)$ for $q > 1$, we have that $F \in S$, where

$$S = \{\mathcal{G} \in (\widehat{H}^1_q(\Omega))^*; \langle \mathcal{G}, \mathbf{v}\rangle = 0 \text{ for all } \mathbf{v} \in H^1_q(\Omega)\}$$

and hence $F = 0$.

□

In the case of Ω exterior we must restrict ourselves on $q \in (1; \infty)$. We have again

Theorem 3.5 *Let $\Omega \in C^{0,1}$ be an exterior domain in $\mathbb{R}^N$, $N \ge 2$, $1 < q < \infty$. Then*

$$\widehat{H}^1_q(\Omega) = H^1_q(\Omega)$$
$$\widehat{\mathcal{D}}^{1,q}_0 = \mathcal{D}^{1,q}_0(\Omega)\,.$$

Proof: We again show the assertion only for $H^1_q(\Omega)$; the case $\mathcal{D}^{1,q}_0(\Omega)$ can be treated analogously and is even easier. As above, it is enough to verify that $\widehat{H}^1_q(\Omega) \subset H^1_q(\Omega)$, the opposite inclusion being trivial.

Let $\psi \in C^1(\mathbb{R})$ with $\psi(\xi) = 1$ if $|\xi| \le 1$ and $\psi(\xi) = 0$ if $|\xi| \ge 2$. We set

$$\psi_R(\mathbf{x}) = \psi\Big(\frac{|\mathbf{x}|}{R}\Big)\,, \quad R > \operatorname{diam}\Omega^c\,.$$

Let $\mathbf{v} \in \widehat{H}^1_q(\Omega)$ and $\mathbf{w}^{(R)}$ be solution to

$$\begin{aligned}\nabla \cdot \mathbf{w}^{(R)} &= -\mathbf{v} \cdot \nabla \psi_R \quad \text{in } \Omega^R_{2R} \\ \mathbf{w}^{(R)} &\in W^{1,q}_0(\Omega^R_{2R}) \\ |\mathbf{w}^{(R)}|_{1,q,\Omega^R_{2R}} &\le C_1 \|\mathbf{v} \cdot \nabla \psi_R\|_{q,\Omega^R_{2R}}\,.\end{aligned} \tag{3.17}$$

Evidently,

$$\int_{\Omega^R_{2R}} \mathbf{v} \cdot \nabla \psi_R \mathrm{d}\mathbf{x} = \int_{\partial B^R} \mathbf{v} \cdot \mathbf{n} \mathrm{d}S = 0$$

and thus there exists a solution to (3.17). Moreover, by Lemma 1.10

$$\begin{aligned}\|\mathbf{w}^{(R)}\|_{q,\Omega^R_{2R}} &\le cR|\mathbf{w}^{(R)}|_{1,q,\Omega^R_{2R}} \le \\ &\le c_1 R\|\mathbf{v} \cdot \nabla \psi_R\|_{q,\Omega^R_{2R}} \le C_2\|\mathbf{v}\|_{q,\Omega^R_{2R}}\end{aligned}$$

as $|\nabla\psi_R| \le \frac{C}{R}$. Both C_1 and C_2 are independent of R. We extend the function $\mathbf{w}^{(R)}$ by zero outside of Ω^R_{2R} and denote

$$\mathbf{v}^{(R)} = \psi_R \mathbf{v} + \mathbf{w}^{(R)}\,.$$

As $\mathbf{v}^{(R)} \in \widehat{H}^1_q(\Omega_{2R})$ and $\Omega_{2R} \in C^{0,1}$, for each $\varepsilon > 0$ there exists $\mathbf{v}^{\varepsilon,R} \in {}_0\mathcal{D}(\Omega_{2R})$ such that

$$\|\mathbf{v}^{(R)} - \mathbf{v}^{\varepsilon,R}\|_{1,q,\Omega_{2R}} < \varepsilon\,.$$

Thus

$$\|\mathbf{v} - \mathbf{v}^{\varepsilon,R}\|_{q,\Omega} \le \|\mathbf{v}^{\varepsilon,R} - \mathbf{v}^{(R)}\|_{q,\Omega} + \|\mathbf{v} - \mathbf{v}^{(R)}\|_{q,\Omega} \le$$
$$\le \varepsilon + \|(1-\psi_R)\mathbf{v}\|_{q,\Omega} + \|\mathbf{w}^R\|_{q,\Omega^R_{2R}}\,.$$

Taking R sufficiently large we get for any $\delta > 0$

$$\|\mathbf{v} - \mathbf{v}^{\varepsilon,R}\|_{q,\Omega} < \delta\,,$$

i.e. $\mathbf{v}^{\varepsilon,R} \to \mathbf{v}$ in $L^q(\Omega)$. Analogously we can show

$$|\mathbf{v} - \mathbf{v}^{\varepsilon,R}|_{1,q,\Omega} \to 0$$

as $\varepsilon \to 0^+$, $R \to \infty$, which completes the proof.

□

Remark 3.4 It is an easy matter to see that Theorem 3.5 holds also for $\Omega = \mathbb{R}^N$.

Theorem 3.6 *Let $\Omega \in C^{0,1}$ be a domain in $\mathbb{R}^N$, $N \ge 2$. Assume*

(i) $\mathbf{v} \in H^1_q(\Omega) \cap [\,\cap_{i=1}^k L^{r_i}(\Omega)]$

(ii) $\mathbf{v} \in \mathcal{D}^{1,q}_0(\Omega) \cap [\,\cap_{i=1}^k L^{r_i}(\Omega)]$

for some q, $r_i \in (1;\infty)$. Then there exists a sequence $\mathbf{v}^n \in {}_0\mathcal{D}(\Omega)$ such that

(i) $\lim_{n\to\infty}(\|\mathbf{v}^n - \mathbf{v}\|_{1,q} + \sum_{i=1}^k \|\mathbf{v}^n - \mathbf{v}\|_{r_i}) = 0$

(ii) $\lim_{n\to\infty}(|\mathbf{v}^n - \mathbf{v}|_{1,q} + \sum_{i=1}^k \|\mathbf{v}^n - \mathbf{v}\|_{r_i}) = 0\,.$

Proof: See [Ga1].

□

Next, let us mention the Helmholtz–Weyl decomposition of $L^q(\Omega)$. Namely, we investigate whether

$$L^q(\Omega) = G_q(\Omega) \oplus H_q(\Omega)\,, \tag{3.18}$$

where

$$G_q(\Omega) = \{\mathbf{w} \in L^q(\Omega); \mathbf{w} = \nabla p \text{ for some } p \in W^{1,q}_{loc}(\Omega)\}\,. \tag{3.19}$$

We have

Theorem 3.7 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be either a domain of class C^2 or the whole space. Then the Helmholtz–Weyl decomposition (3.18) holds for any $q \in (1;\infty)$.*

Remark 3.5 If $q = 2$, then Theorem 3.7 remains true for $\Omega \subset \mathbb{R}^N$, any domain.

Proof: See e.g. [Ga1].

□

The space $H_q(\Omega)$ can be further characterized. Let us consider for $\mathbf{u} \in C_0^\infty(\overline{\Omega})$

$$F_{\mathbf{u}}(\omega) = \int_{\partial\Omega} \omega \mathbf{u} \cdot \mathbf{n} \mathrm{d}S\,, \quad \omega \in W^{1-\frac{1}{q'},q'}(\partial\Omega)\,. \tag{3.20}$$

We denote by $\gamma_{\mathbf{u}\cdot\mathbf{n}}$ the linear map that to each $\mathbf{u} \in C_0^\infty(\overline{\Omega})$ prescribes the corresponding form $F_{\mathbf{u}}$, i.e.

$$\gamma_{\mathbf{u}\cdot\mathbf{n}} = F_{\mathbf{u}}\,.$$

We then have

Lemma 3.2 *Let $\Omega \in C^{0,1}$, be a domain in $\mathbb{R}^N$, $N \geq 2$. Then*

$$H_q(\Omega) = \{\mathbf{u} \in L^q(\Omega); \nabla \cdot \mathbf{u} = 0, \gamma_{\mathbf{u}\cdot\mathbf{n}} = 0 \ \textit{at}\ \partial\Omega\}\,.$$

Proof: See [Te].[12]

□

Finally, let

$$\tilde{H}_q(\Omega) = \{\mathbf{u} \in L^1_{loc}(\Omega); \|\mathbf{u}\|_{\tilde{H}_q} < \infty\}\,, \tag{3.21}$$

where

$$\|\mathbf{u}\|_{\tilde{H}_q} = \|\mathbf{u}\|_q + \|\nabla \cdot \mathbf{u}\|_q\,. \tag{3.22}$$

Furthermore, let

$$\tilde{H}_{0,q} = \overline{\{C_0^\infty(\Omega)\}}^{\|\cdot\|_{\tilde{H}_q}}\,. \tag{3.23}$$

Then

Lemma 3.3 *Let $\Omega \in C^{0,1}$ be a domain in $\mathbb{R}^N$, $N \geq 2$. Then*

(i) $\tilde{H}_q = \overline{\{C_0^\infty(\overline{\Omega})\}}^{\|\cdot\|_{\tilde{H}_q}}$

(ii) $\tilde{H}_{0,q} = \{\mathbf{u} \in \tilde{H}_q; \gamma_{\mathbf{u}\cdot\mathbf{n}} = 0 \ \textit{at}\ \partial\Omega\}\,.$

Proof: See [Te].

□

[12] The proof in [Te] is done for $q = 2$. Nevertheless, we may demonstrate the general case $1 < q < \infty$ in the same lines.

Remark 3.6 Let φ be an extension of ω (see (3.20)) in $W^{1,q'}(\Omega)$. Then due to the Green theorem (see Theorem 1.15)

$$|F_{\mathbf{u}}(\omega)| = \Big| \int_\Omega (\mathbf{u} \cdot \nabla\varphi + \varphi\nabla \cdot \mathbf{u})\mathrm{d}\mathbf{x}\Big| \leq \\ \leq \|\mathbf{u}\|_{\tilde{H}_q} \|\varphi\|_{1,q'} \leq C\|\mathbf{u}\|_{\tilde{H}_q} \|\omega\|_{1-\frac{1}{q'},q',(\partial\Omega)}$$

for $u \in C_0^\infty(\overline{\Omega})$. As a consequence of Lemma 3.3 (i) we have that the operator $\gamma_{\mathbf{u}\cdot\mathbf{n}}$ is well defined for $\mathbf{u} \in \tilde{H}_q(\Omega)$ and

$$\|\gamma_{\mathbf{u}\cdot\mathbf{n}}\|_{W^{-\frac{1}{q},q}(\partial\Omega)} \leq C\|\mathbf{u}\|_{\tilde{H}_q} .$$

VIII.4 Distributions. Fourier transform

We denote for Ω a domain

$$\mathcal{D}(\Omega) = C_0^\infty(\Omega), \quad \Omega \subseteq \mathbb{R}^N .$$

We say that

$$\varphi_n \overset{\mathcal{D}}{\to} \varphi$$

if there exists $\Omega' \subset\subset \Omega$ (i.e. $\Omega' \subset \overline{\Omega'} \subset \Omega$) such that $\operatorname{supp}\varphi_n \subset \Omega'$ for all $n \in \mathbb{N}$ and

$$D^\alpha\varphi_n \;\; D^\alpha\varphi , \quad n \to \infty \quad \forall\alpha \in \mathbb{N}^N . \tag{4.1}$$

We say that T is a distribution on Ω ($T \in \mathcal{D}'(\Omega)$) if T is a linear continuous operator on $\mathcal{D}(\Omega)$, i.e. $T : \mathcal{D}(\Omega) \mapsto \mathbb{R}$, T is linear and

$$\langle T, \varphi_k\rangle \to \langle T, \varphi\rangle \quad \text{whenever } \varphi_k \overset{\mathcal{D}}{\to} \varphi .$$

The distribution T is called regular if there exists a function $f \in L^1_{loc}(\Omega)$ such that

$$\langle T, \varphi\rangle = \int_\Omega f\varphi\mathrm{d}\mathbf{x}$$

for all $\varphi \in \mathcal{D}(\Omega)$. We shall denote the regular distribution represented by a function f by T_f.

We say that $T_k \to T$ in $\mathcal{D}'(\Omega)$ if

$$\langle T_k, \varphi\rangle \to \langle T, \varphi\rangle \quad \forall\varphi \in \mathcal{D}(\Omega) .$$

It is possible to show that the space $\mathcal{D}'(\Omega)$ is complete (see e.g. [Vl]).

A very important notion on $\mathcal{D}'(\Omega)$ is the weak derivative. We call a functional $G \in \mathcal{D}'(\Omega)$ the weak derivative of T, $G = D^\alpha T$, if for all $\varphi \in \mathcal{D}(\Omega)$

$$\langle G, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle . \tag{4.2}$$

It is an easy consequence of the definition that any distribution has derivatives of all order; moreover $D^{\alpha+\beta}T = D^{\beta+\alpha}T$ for any α, β multiindices. If the distribution T_f is regular with $f \in C^k(\Omega)$, then

$$D^\alpha T_f = T_{\frac{\partial^{|\alpha|} f}{\partial x^\alpha}}$$

for all $|\alpha| \le k$.

Let $T \in \mathcal{D}'(\Omega)$. We say that T has finite order, if there exist K and m independent of Ω' such that[13]

$$|\langle T, \varphi \rangle| \le K \|\varphi\|_{C^m(\overline{\Omega'})} \quad \forall\, \varphi \in \mathcal{D}(\Omega')\,, \quad \forall\, \Omega' \subset \overline{\Omega'} \subset \Omega\,. \tag{4.3}$$

The smallest m satisfying (4.3) is called the order of T.

We say that T has compact support in Ω (i.e. $T \in \mathcal{E}'(\Omega)$) if there exists a compact set $K \subset \Omega$ such that

$$\langle T, \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{D}(\Omega \setminus K)\,.$$

A special case of such distributions are those with support containing only one point. We easily see that e.g. the Dirac δ–distribution is such an example. On the other side, we have

Lemma 4.1 *Let the support of $T \in \mathcal{D}'(\Omega)$ contain only one point $\mathbf{x} = \mathbf{0}$. Then it can be uniquely written as*

$$T(\mathbf{x}) = \sum_{|\alpha| \le M} C_\alpha D^\alpha \delta(\mathbf{x})\,,$$

where M is the order of T and $C_\alpha \in \mathbb{R}$.

Proof: See e.g. [Vl].

□

For more detailed description of the space $\mathcal{D}'(\Omega)$, see e.g. [Vl]. For our purpose we shall need rather the space of tempered distributions.

The function φ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^N) = \mathcal{S}$, if $\varphi \in C^\infty(\mathbb{R}^N)$ and the function, together with the derivatives of all order, decays at infinity faster than any power of $|\mathbf{x}|$. It means that

$$C_{\alpha,\beta} = \sup_{\mathbf{x} \in \mathbb{R}^N} |\mathbf{x}^\alpha D^\beta \varphi(\mathbf{x})| < \infty \quad \forall \alpha, \beta \in \mathbb{N}^N\,.$$

Evidently, $\mathcal{D}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N)$.

We say that

$$\varphi_k \xrightarrow{\mathcal{S}} 0 \text{ if } \mathbf{x}^\alpha D^\beta \varphi_k(\mathbf{x}) \;\; 0 \text{ as } k \to \infty\,. \tag{4.4}$$

By $\mathcal{S}'$ we denote the space of linear continuous functionals on $\mathcal{S}$; the space $\mathcal{S}'$ is usually called the space of tempered distributions.

Let $p \in \mathbb{N}_0$. We put for $\varphi \in \mathcal{S}$

$$\|\varphi\|_{\mathcal{S}_p} = \sup_{\substack{|\alpha| \le p \\ |\mathbf{x}| \in \mathbb{R}^N}} (1 + |\mathbf{x}|^2)^{\frac{p}{2}} |D^\alpha \varphi(\mathbf{x})|\,. \tag{4.5}$$

[13] In general, not all distributions have finite order. Nevertheless, the relation (4.3) is satisfied for all $T \in \mathcal{D}'(\Omega)$ with K, m depending on Ω'.

We get in such a way a countable number of norms in $\mathcal{S}$. We can close the space $\mathcal{S}$ in the norm $\mathcal{S}_p$; denoting the closure by $\mathcal{S}_p$ we evidently have[14]

$$\mathcal{S}_{p+1} \hookrightarrow \mathcal{S}_p \,, \quad p \in \mathbb{N}_0 \,. \tag{4.6}$$

We say that the tempered distribution T has finite order, if there exists a constant K, independent of φ, such that for all $\varphi \in \mathcal{S}$

$$|\langle T, \varphi \rangle| \le K \|\varphi\|_{\mathcal{S}_p} \tag{4.7}$$

for some $p \in \mathbb{N}_0$. We have

Lemma 4.2 *Every element of $\mathcal{S}'$ has finite order.*

Proof: See [Vl].

□

Due to Lemma 4.2, each tempered distribution can be continuously extended onto some $\mathcal{S}_p$; moreover e.g. a regular distribution[15]

$$\langle T, \varphi \rangle = \int_{\mathbb{R}^N} f \varphi \mathrm{d}\mathbf{x} \tag{4.8}$$

can be often extended onto much larger class of functions in dependence on the integrability properties of f.

By analogy with $\mathcal{D}'$ we can define the weak derivative on $\mathcal{S}'$; we only require that (4.2) holds for any $\varphi \in \mathcal{S}$.

Let $a \in C^\infty(\mathbb{R}^N)$ has at most polynomial behaviour at infinity including all derivatives. We define for $T \in \mathcal{S}'$

$$aT \in \mathcal{S}' : \quad \langle aT, \varphi \rangle = \langle T, a\varphi \rangle \quad \forall \varphi \in \mathcal{S} \,. \tag{4.9}$$

As any tempered distribution has finite order, we can always extend the class of $\mathcal{S}'$–multipliers using the right-hand side of (4.9).

We can further define another type of product. Let $T(\mathbf{x}) \in \mathcal{S}'(\mathbb{R}^n)$, $G(\mathbf{y}) \in \mathcal{S}'(\mathbb{R}^m)$. We define the direct product $T(\mathbf{x}) \times G(\mathbf{y})$ as

$$\langle T(\mathbf{x}) \times G(\mathbf{y}), \varphi \rangle = \langle T(\mathbf{x}), \langle G(\mathbf{y}), \varphi(\mathbf{x}, \mathbf{y}) \rangle \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^{n+m}) \,. \tag{4.10}$$

It is possible to show (see e.g. [Vl]) that $T \times G \in \mathcal{S}'(\mathbb{R}^{n+m})$.

Another very important notion (especially in connection with the Fourier transform, see below) is the convolution. If f, $g \in \mathcal{D}(\mathbb{R}^N)$, then

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^N} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) \mathrm{d}\mathbf{y} = \int_{\mathbb{R}^N} g(\mathbf{x} - \mathbf{z}) f(\mathbf{z}) \mathrm{d}\mathbf{z} \,.$$

We want to extend the convolution to $\mathcal{S}'$. Let us recall that in general it is not possible to define the convolution for any elements from $\mathcal{S}'$.

[14] Moreover, see e.g. [Vl], the imbedding is compact.

[15] Unlike the case $\mathcal{D}'$, we must add some slightly restrictive assumptions on f at infinity; f must behave like some polynomial at infinity i.e. $|f(\mathbf{x})| \le C|\mathbf{x}|^s$ for some $s \in \mathbb{R}$ as $|\mathbf{x}| \to \infty$.

Let us first take T, $G \in \mathcal{D}'(\mathbb{R}^n)$. Let $T(\mathbf{x}) \times G(\mathbf{y})$ admits the extension $\langle T(\mathbf{x}) \times G(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y})\rangle$ on functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$ in the following sense. For any sequences $\eta_k \in \mathcal{D}(\mathbb{R}^{2n})$ tending[16] to 1 there exists

$$\lim_{k\to\infty} \langle T(\mathbf{x}) \times G(\mathbf{y}), \eta_k(\mathbf{x};\mathbf{y})\varphi(\mathbf{x}+\mathbf{y})\rangle$$

and does not depend on the sequence η_k. We call the convolution $T * G$ the functional

$$\begin{aligned}\langle T * G, \varphi\rangle &\equiv \langle T(\mathbf{x}) \times G(\mathbf{y}), \varphi(\mathbf{x}+\mathbf{y})\rangle = \\ &= \lim_{k\to\infty} \langle T(\mathbf{x}) \times G(\mathbf{y}), \eta_k(\mathbf{x};\mathbf{y})\varphi(\mathbf{x}+\mathbf{y})\rangle\,, \varphi \in \mathcal{D}(\mathbb{R}^n)\,.\end{aligned} \tag{4.11}$$

Now, as $\mathcal{S}'(\mathbb{R}^N) \subset \mathcal{D}'(\mathbb{R}^N)$, we can define $T * G$ in $\mathcal{D}'(\mathbb{R}^N)$ in the same way. The question is when $T * G \in \mathcal{S}'(\mathbb{R}^N)$[17].

We have

Lemma 4.3 *Let $T \in \mathcal{S}'(\mathbb{R}^n)$, $G \in \mathcal{E}'(\mathbb{R}^n)$. Then the convolution $T*G \in \mathcal{S}'(\mathbb{R}^n)$ and*

$$\langle T * G, \varphi\rangle = \langle T(\mathbf{x}) \times G(\mathbf{y}), \eta(\mathbf{y})\varphi(\mathbf{x}+\mathbf{y})\rangle\,, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)\,,$$

and $\eta \in \mathcal{D}(\mathbb{R}^n)$, $\eta = 1$ in the neighborhood of $\operatorname{supp} G$.

Lemma 4.4 *Let $T \in \mathcal{S}'(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^n)$. Then the convolution $T*G_g \in \mathcal{S}'(\mathbb{R}^n)$ and*

$$\langle T * G_g, \varphi\rangle = \langle T, g * \varphi(-\mathbf{x})\rangle\,, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$
$$T * G_g = \langle T(\cdot), g(\mathbf{x} - \cdot)\rangle\,.$$

Proof: The proof of both lemmas can be found e.g. in [Vl], where also other cases are studied when the convolution exists; either in $\mathcal{D}'$ or in $\mathcal{S}'$.

□

VIII.4.1 Fourier transform

Let $\varphi \in \mathcal{S}$. Then we define

$$\mathcal{F}(\varphi)(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \varphi(\mathbf{x}) \mathrm{e}^{i(\mathbf{x},\xi)} \mathrm{d}\mathbf{x} \tag{4.12}$$

and

$$\mathcal{F}^{-1}(\varphi)(\mathbf{x}) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \varphi(\xi) \mathrm{e}^{-i(\mathbf{x},\xi)} \mathrm{d}\xi\,. \tag{4.13}$$

We easily check that $\mathcal{F}^{-1}(\varphi)(\mathbf{x}) = \mathcal{F}(\varphi)(-\mathbf{x}) = \mathcal{F}(\psi)(\mathbf{x})$, where $\psi(\xi) = \varphi(-\xi)$ for all $\xi \in \mathbb{R}^N$.

The following properties of the Fourier transform on $\mathcal{S}$ are classical and can be found e.g. in [StWe].[18]

---

[16] i.e. $\forall K \subset \mathbb{R}^{2n}$, compact, there exists $k_0(K) \in \mathbb{N}$ such that $\eta_k(\mathbf{x}) = 1$ for all $k \geq k_0$, $\mathbf{x} \in K$, and η_k, together with all derivatives, is uniformly bounded, $|D^\alpha \eta_k(\mathbf{x})| \leq C_\alpha$, $\mathbf{x} \in \mathbb{R}^{2n}$, $k = 1, 2, \ldots$

[17] generally, of course, for any T, $G \in \mathcal{D}'(\mathbb{R}^N)$, we cannot define $T * G \in \mathcal{D}'(\mathbb{R}^N)$, see e.g. [Vl]

[18] Let us note that the authors use a slightly different definition of the Fourier transform.

Lemma 4.5 *Let f, $g \in \mathcal{S}$. Then we have*

(a) $\mathcal{F}(D^\alpha f)(\xi) = (-i\xi)^\alpha \mathcal{F}(f)(\xi)$

(b) $D^\alpha \mathcal{F}(f)(\xi) = \mathcal{F}((i\mathbf{x})^\alpha f)(\xi)$

(c) $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear isomorphisms of $\mathcal{S}$ onto $\mathcal{S}$ and for all f from $\mathcal{S}$

$$\mathcal{F}^{-1}\mathcal{F}(f) = \mathcal{F}\mathcal{F}^{-1}(f) = f$$

(d) For all f, $g \in \mathcal{S}$

$$\int_{\mathbb{R}^N} \mathcal{F}(f) g \, d\mathbf{x} = \int_{\mathbb{R}^N} f \mathcal{F}(g) \, d\mathbf{x}$$

(e) Parseval equality: for all $f \in \mathcal{S}$

$$\|f\|_2 = \|\mathcal{F}(f)\|_2$$

(f) For all f, $g \in \mathcal{S}$

$$\mathcal{F}(f * g) = (2\pi)^{\frac{N}{2}} \mathcal{F}(f)\mathcal{F}(g)$$

(g) For all $f \in \mathcal{S}$

$$\|\mathcal{F}(f)\|_\infty \le (2\pi)^{-\frac{N}{2}} \|f\|_1 \, .$$

We can now use Lemma 4.5 in order to extend the Fourier transform onto larger spaces. First, due to (g), we easily observe that the Fourier transform as well as its inverse can be defined on $L^1(\mathbb{R}^N)$ by (4.12) and (4.13). We then have (see e.g. [StWe])

Lemma 4.6 *Let $f \in L^1(\mathbb{R}^N)$. Then*

(a) $\mathcal{F}(f)$ is uniformly continuous on $\mathbb{R}^N$

(b) $\mathcal{F}(f)$ 0 as $|\mathbf{x}| \to \infty$

(c) if $\mathcal{F}(f) \in L^1(\mathbb{R}^N)$, then $\mathcal{F}^{-1}\mathcal{F}(f) = \mathcal{F}\mathcal{F}^{-1}(f) = f$ a.e. in $\mathbb{R}^N$.

Remark 4.1 Generally, $f \in L^1(\mathbb{R}^N)$ does not imply $\mathcal{F}(f) \in L^1(\mathbb{R}^N)$ and therefore $\mathcal{F}$ is not a linear isomorphism of $L^1(\mathbb{R}^N)$ onto itself. Nevertheless, some properties of the Fourier transform on $\mathcal{S}$ are kept also for functions from $L^1(\mathbb{R}^N)$; namely (d), (f) and (g). Moreover, if both sides have sense in $L^1(\mathbb{R}^N)$, then also (a) and (b).

We also have on both $\mathcal{S}$ and $L^1(\mathbb{R}^N)$

Lemma 4.7 *The Fourier transform commutes with orthogonal transformations, i.e. for T orthogonal*

$$\mathcal{F}(f)(T\xi) = \mathcal{F}(f(T))(\xi) \quad \forall f \in L^1(\mathbb{R}^N) \, . \tag{4.14}$$

In particular, if f is radially symmetric, then so is $\mathcal{F}(f)$.

Proof: It is an easy consequence of properties of scalar product, see e.g. [StWe].

□

Now, using the Parseval equality (see Lemma 4.5 (e)) we can extend the Fourier transform onto $L^2(\mathbb{R}^N)$. Using the fact that $\mathcal{S}$ is dense in $L^2(\mathbb{R}^N)$, we define for $f \in L^2(\mathbb{R}^N)$

$$\mathcal{F}(f) = \lim_{n\to\infty} \mathcal{F}(f_n)\,, \tag{4.15}$$

where $f_n \to f$ in $L^2(\mathbb{R}^N)$, $f_n \in \mathcal{S}$; analogously also $\mathcal{F}^{-1}(f)$. It is possible to show that the definition does not depend on the approximative sequence and we have (see e.g. [StWe])

Lemma 4.8 *The Fourier transform is an isometric isomorphism of $L^2(\mathbb{R}^N)$ onto itself; we have*

$$\mathcal{F}^{-1}\mathcal{F}(f) = \mathcal{F}\mathcal{F}^{-1}(f) = f$$

a.e. in $\mathbb{R}^N$.

Whenever $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, both definitions coincide.

Remark 4.2 Lemma 4.7 holds also in case $f \in L^2(\mathbb{R}^N)$. Moreover, if $f \in W^{k,2}(\mathbb{R}^N)$, then the assertion (a) from Lemma 4.5 is true; the derivative of f is to be understood in the weak sense. Therefore we have on $W^{k,2}(\mathbb{R}^N)$ the following equivalent norm

$$|||f|||_{k,2} = \|(1+|\xi|^2)^{\frac{k}{2}}\mathcal{F}(f)\|_2\,, \tag{4.16}$$

see e.g. [Ev].

Finally, we use the property (d) and extend the Fourier transform onto $\mathcal{S}'$. Let $T \in \mathcal{S}'$. Then we call the functional $\mathcal{F}(T)$ the Fourier transform of T if

$$\langle \mathcal{F}(T), \varphi\rangle = \langle T, \mathcal{F}(\varphi)\rangle \quad \forall \varphi \in \mathcal{S}\,. \tag{4.17}$$

Evidently, whenever T_f is a regular distribution with $f \in \mathcal{S}$, $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$, then the corresponding definition in $\mathcal{S}$, $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$, respectively, coincides with the definition on $\mathcal{S}'$ in the sense of equality on $\mathcal{S}'$.

We also define on $\mathcal{S}'$ $\mathcal{F}^{-1}(T)$ by

$$\mathcal{F}^{-1}(T) = \mathcal{F}(T(-\mathbf{x}))\,, \quad T \in \mathcal{S}'\,. \tag{4.18}$$

Then we have

Lemma 4.9 *The Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are linear isomorphisms of $\mathcal{S}'$ onto $\mathcal{S}'$. Moreover, for any $T \in \mathcal{S}'$*

$$\mathcal{F}^{-1}(\mathcal{F}(T)) = \mathcal{F}(\mathcal{F}^{-1}(T)) = T\,, \tag{4.19}$$

where the equality (4.19) holds in the sense of $\mathcal{S}'$.

Proof: Since $\mathcal{S}$ is dense in $\mathcal{S}'$ (see e.g. [Vl]), it follows from the assertion (c) of Lemma 4.5.

□

Now, as $L^p(\mathbb{R}^N) \subset \mathcal{S}'$ for all $p \in [1;\infty]$, we have defined the Fourier transform on all Lebesgue spaces; for the values of $p \in [1;2]$ we can show that $\mathcal{F}(T_f)$[19] is a regular function from $L^{p'}(\mathbb{R}^N)$. If $p > 2$, then there exists always a function from $L^p(\mathbb{R}^N)$ such that $\mathcal{F}(T_f)$ is not a regular tempered function (see e.g. [StWe]).

Lemma 4.10 (Hausdorff–Young)
Let $f \in L^p(\mathbb{R}^N)$, $p \in [1;2]$. Then $\mathcal{F}(f) \in L^{p'}(\mathbb{R}^N)$ and

$$\|\mathcal{F}(f)\|_{p',\mathbb{R}^N} \le C(p,N)\|f\|_{p,\mathbb{R}^N} . \tag{4.20}$$

Proof: As $\mathcal{S}$ is dense in $L^p(\mathbb{R}^N)$, $1 \le p \le 2$, it is enough to show (4.20) on $\mathcal{S}$. But due to Lemma 4.5, properties (e) and (g) we have

$$\|\mathcal{F}(f)\|_\infty \le C\|f\|_1$$
$$\|\mathcal{F}(f)\|_2 \le C\|f\|_2$$

and the inequality (4.20) is an easy consequence of the Riesz–Thorin theorem (see Theorem 2.2).

□

Using again the fact that $\mathcal{S}$ is dense in $\mathcal{S}'$, we can easily verify that (the operations like product and derivatives are to be understood in the sense of $\mathcal{S}'$)

Lemma 4.11 *Let $T \in \mathcal{S}'$. Then*

$$D^\alpha \mathcal{F}(T) = \mathcal{F}((i\mathbf{x})^\alpha T)$$
$$\mathcal{F}(D^\alpha T) = (-i\xi)^\alpha \mathcal{F}(T) .$$

Remark 4.3 Using the definition of $\mathcal{F}$ together with Lemma 4.11 we easily verify that (δ is the Dirac distribution)

$$\mathcal{F}(\mathbf{x}^\alpha) = (-i)^{|\alpha|} D^\alpha \mathcal{F}(1) = (-i)^{|\alpha|}(2\pi)^{-\frac{N}{2}} D^\alpha \delta$$
$$\mathcal{F}(D^\alpha \delta) = (-i\xi)^\alpha \mathcal{F}(\delta) = (-i\xi)^\alpha (2\pi)^{-\frac{N}{2}} .$$

Finally, we want to extend the validity of property (f) from Lemma 4.5 to $\mathcal{S}'$. As mentioned above, it is in general not true that for T, $G \in \mathcal{S}'$ the convolution $T * G$ has sense in $\mathcal{S}'$. Nevertheless, we have

[19] for the sake of brevity, we shall write only $\mathcal{F}(f)$ in such a case

Lemma 4.12

(a) Let $T \in \mathcal{S}'$, $G \in \mathcal{E}'$. Then $T * G \in \mathcal{S}'$ and

$$\mathcal{F}(T * G) = (2\pi)^{\frac{N}{2}} \mathcal{F}(T)\mathcal{F}(G)\,, \tag{4.21}$$

where the product is to be understood in the sense of $\mathcal{S}'$.

(b) Let $T \in \mathcal{S}'$, $g \in \mathcal{S}$. Then $T * G_g \in \mathcal{S}'$

$$\mathcal{F}(T * G_g) = (2\pi)^{\frac{N}{2}} \mathcal{F}(g)\mathcal{F}(T)\,, \tag{4.22}$$

where (4.22) holds in the sense of (4.9).

Proof: See e.g. [Vl].

□

In Chapter III we gave a small generalization of (4.21) for T_f, T_g with f, g from some Lebesgue spaces. See also e.g. [StWe].

VIII.4.2 Some applications

This subsection is devoted to some applications of the Fourier transform needed in Chapters II, III and VII. We calculate the Fourier transform of $\mathcal{O}_{ij}$, E^*, $\mathcal{S}_{ij}$ and $\mathcal{E}$, defined in Chapters II and III. We start with the simplest situation.

Lemma 4.13 *Let $\mathcal{E}$ be the fundamental solution to the Laplace equation. Then it holds*[20]

$$\mathcal{F}(D^\alpha \mathcal{E}) = -(2\pi)^{-\frac{N}{2}} \frac{(-i\xi)^\alpha}{|\xi|^2} \tag{4.23}$$

with

(a) $|\alpha| > 0$ for $N = 2$

(b) $|\alpha| \geq 0$ for $N \geq 3$.

Proof: We have that

$$-|\xi|^2 \mathcal{F}(\xi) = (2\pi)^{-\frac{N}{2}}\,. \tag{4.24}$$

(a) Let $N = 2$. Since $\frac{1}{|\xi|^2}$ is not locally integrable function in $\mathbb{R}^2$, we cannot easily divide (4.24) by $|\xi|^2$. We calculate the Fourier transform of the first derivative of $\mathcal{E}$. We have in $\mathcal{S}'$

$$|\xi|^2 \mathcal{F}\Big(\frac{\partial \mathcal{E}}{\partial x_i}\Big) = i\xi_i |\xi|^2 \mathcal{F}(\mathcal{E})\,.$$

Employing (4.24) we have

$$\begin{aligned}\Big\langle -|\xi|^2 \mathcal{F}\Big(\frac{\partial \mathcal{E}}{\partial x_i}\Big), \varphi\Big\rangle &= \langle -|\xi|^2 \mathcal{F}(\mathcal{E}), (i\xi_i)\varphi\rangle = \\ &= \langle (2\pi)^{-1}, i\xi_i\varphi\rangle = \langle (2\pi)^{-1} i\xi_i, \varphi\rangle \quad \forall \varphi \in \mathcal{S}\,.\end{aligned}$$

[20]The derivatives are taken in the weak sense.

Hence

$$|\xi|^2 \mathcal{F}\Big(\frac{\partial \mathcal{E}}{\partial x_i}\Big) = \frac{i\xi_i}{2\pi} \quad \text{in } \mathcal{S}'\,.$$

Now, as $\frac{\xi_i}{|\xi|^2} \in L^1_{loc}(\mathbb{R}^2)$, we have

$$\mathcal{F}\Big(\frac{\partial \mathcal{E}}{\partial x_i}\Big) = \frac{i\xi_i}{|\xi|^2}\frac{1}{2\pi} + H\,,$$

where $H \in \mathcal{S}'$ is supported at $\mathbf{0}$. As any element from $\mathcal{S}'$ has finite order, we get from Lemma 4.4

$$H = \sum_{|\alpha| \leq M} C_\alpha D^\alpha \delta$$

with $M \in \mathbb{N}$. Using the fact that $\frac{\partial \mathcal{E}}{\partial x_i} = \frac{1}{2\pi}\frac{x_i}{|\mathbf{x}|^2}$ in $\mathcal{S}'$ we easily observe that $H = 0$ and

$$\mathcal{F}\Big(\frac{\partial \mathcal{E}}{\partial x_i}\Big) = \frac{i}{2\pi}\frac{\xi_i}{|\xi|^2} \quad \text{in } \mathcal{S}'\,.$$

Now, applying Lemma 4.11 (b) we easily get the result for $N = 2$.

(b) If $N \geq 3$, we can divide directly in (4.24) and get

$$\mathcal{F}(\mathcal{E}) = -(2\pi)^{-\frac{N}{2}}\frac{1}{|\xi|^2} + H\,,$$

where H is again a tempered distribution supported at $\mathbf{0}$; therefore we have $H = \sum_{|\alpha|\leq M} C_\alpha D^\alpha \delta$. We can modify the proof of Lemma III.1.6 to show that

$$\Big|\mathcal{F}^{-1}\Big(\frac{C}{|\xi|^2}\Big)(\mathbf{x})\Big| \leq C|\mathbf{x}|^{2-N}$$

for $\mathbf{x} \neq \mathbf{0}$. Since $\mathcal{E}(\mathbf{x}) \sim |\mathbf{x}|^{2-N}$ as $|\mathbf{x}| \to \infty$ and $|\mathbf{x}| \to 0$, $H = 0$. (See also [Vl] for another argument). The rest is obvious.

□

.

Lemma 4.14 *Let $\boldsymbol{\mathcal{S}}$ be the fundamental Stokes tensor. Then*[21]

$$\mathcal{F}(D^\alpha \mathcal{S}_{ij}) = (2\pi)^{-\frac{N}{2}}(-i\xi)^\alpha\Big[\frac{\delta_{ij}}{|\xi|^2} - \frac{\xi_i\xi_j}{|\xi|^4}\Big] \tag{4.25}$$

with

(a) $|\alpha| > 0$ *for* $N = 2$

(b) $|\alpha| \geq 0$ *for* $N = 3$.

[21] see footnote above

Proof: We proceed as in Lemma 4.13. Recalling that

$$-\Delta \mathcal{S}_{ij} + \frac{\partial e_i}{\partial x_j} = \delta_{ij}\delta$$

and ($r = |\mathbf{x}|$)

$$\begin{aligned} e_i(\mathbf{x}) &= \frac{\partial \mathcal{E}}{\partial x_i} \\ \mathcal{S}_{ij}(\mathbf{x}) &= \begin{cases} \frac{1}{4\pi}\Big[\delta_{ij}\ln\frac{1}{r} + \frac{x_i x_j}{r^2}\Big] & \text{if } N = 2 \\ \frac{1}{8\pi}\Big[\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\Big] & \text{if } N = 3 \end{cases} \end{aligned}$$

we get

$$|\xi|^2 \mathcal{F}(\mathcal{S}_{ij}) = (2\pi)^{-\frac{N}{2}}\Big[\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}\Big].$$

For $N = 2$ we again cannot divide by $|\xi|^2$; nevertheless, we calculate the Fourier transform of the first derivative of $\mathcal{S}_{ij}$ and get (4.25) for $|\alpha| > 0$.

For $N \geq 3$ we can divide by $|\xi|^2$ and argue as in Lemma 4.13.

□

Lemma 4.15 *Let* $\mathcal{O}$ *be the fundamental Oseen tensor. Then*

$$\mathcal{F}(D^\alpha \mathcal{O}_{ij}(\cdot\,;\beta)) = (2\pi)^{-\frac{N}{2}}(-i\xi)^\alpha \frac{\delta_{ij}|\xi|^2 - \xi_i\xi_j}{|\xi|^2(|\xi|^2 - i\beta\xi_1)} \tag{4.26}$$

with $|\alpha| \geq 0$ *and* $N \geq 2$.

Proof: Due to the definition we have

$$-\Big(\Delta - \beta\frac{\partial}{\partial x_1}\Big)\mathcal{O}_{ij} + \frac{\partial e_i}{\partial x_j} = \delta_{ij}\delta\,,$$

where $e_i = \frac{\partial \mathcal{E}}{\partial x_i}$, $\mathcal{O}_{ij}(\cdot\,;\beta)$ see Chapter II. We have

$$(|\xi|^2 - i\beta\xi_1)\mathcal{F}(\mathcal{O}_{ij}) = (2\pi)^{-\frac{N}{2}}\Big(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\Big).$$

Unlike the Stokes problem, we can now divide by $h(\xi) = i\beta\xi_1 - |\xi|^2$ as $\frac{1}{h(\xi)} \in L^1_{loc}(\mathbb{R}^N)$. Let us demonstrate this for $N = 2$; if $N \geq 3$, the proof is much easier. We have

$$\begin{aligned} \int_{B_1(\mathbf{0})} \frac{\mathrm{d}\xi}{|h(\xi)|} &\leq C \int_{B_1(\mathbf{0})} \frac{|\xi|^2 + \beta|\xi_1|}{|\xi|^4 + \beta^2|\xi_1|^2}\mathrm{d}\xi \leq \\ &\leq C_1 \int_0^1 \int_0^{\frac{\pi}{2}} \frac{r^2 + \beta r|\cos\varphi|}{r^4 + r^2\cos^2\varphi} r\mathrm{d}r\mathrm{d}\varphi\,. \end{aligned}$$

Now

$$\begin{aligned} \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\cos\varphi}{r^2 + \cos^2\varphi}\mathrm{d}\varphi\mathrm{d}r &= \int_0^{\frac{\pi}{2}} \operatorname{arctg}\frac{1}{\cos\varphi}\mathrm{d}\varphi \leq C \\ \int_0^1 \int_0^{\frac{\pi}{2}} \frac{r^2}{r^2 + \cos^2\varphi}\mathrm{d}\varphi\mathrm{d}r &= \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \cos\varphi \operatorname{arctg}\frac{1}{\cos\varphi}\mathrm{d}\varphi \leq C\,. \end{aligned}$$

As $\mathcal{O}_{ij}$ 0 for $|\mathbf{x}| \to \infty$, we have (4.26). The proof is complete.

□

Lemma 4.16 *Let E^* be the fundamental solution to the "Oseen problem without pressure". Then*

$$\mathcal{F}(D^\alpha E^*(\cdot\,;\beta)) = (2\pi)^{-\frac{N}{2}} \frac{(-i\xi)^\alpha}{|\xi|^2 - i\beta\xi_1} \tag{4.27}$$

with $|\alpha| \geq 0$ and $N \geq 2$.

Proof: Let us recall that

$$-\Delta E^* + \beta \frac{\partial E^*}{\partial x_1} = \delta$$

i.e.

$$(|\xi|^2 - i\beta\xi_1)\mathcal{F}(E^*) = (2\pi)^{-\frac{N}{2}}\,.$$

Now, arguing as in Lemma 4.15 we get the result.

□

We finish this section by proving a general version of Lemma VII.1.2. For the notion of axially symmetric functions, see Chapter VII.

In what follows $\mathbf{v}$ is a smooth vector field from $\mathbb{R}^3$ into $\mathbb{R}^3$, usually divergence free and axially symmetric. The vector $\mathbf{w} = (w_x, w_y, w_z)$ is $\nabla \times \mathbf{v}$ in cartesian coordinates. The components of its representation in cylindrical coordinates are denoted by w_r, w_θ, w_z. If $\mathbf{v}$ is axially symmetric, then the only non–zero component of $\mathbf{w}$ in cylindrical coordinates (θ–component) is denoted by ω.

Theorem 4.1 *There exist constants C_i, $i = 1, \ldots, 6$, such that for all $\mathbf{v}$ smooth divergence free and axially symmetric vectors and any $p \in (1;\infty)$*[22]

$$C_1(p)\|D\mathbf{v}\|_p \leq \|\omega\|_p \leq C_2\|D\mathbf{v}\|_p \tag{4.28}$$

$$C_3(p)\|D^2\mathbf{v}\|_p \leq \|\nabla\omega\|_p + \Big\|\frac{\omega}{r}\Big\|_p \leq C_4\|D^2\mathbf{v}\|_p \tag{4.29}$$

$$C_5(p)\Big\|\frac{\partial^2\omega}{\partial r^2}\Big\|_p + \Big\|\frac{\partial^2\omega}{\partial r\partial z}\Big\|_p + \Big\|\frac{\partial^2\omega}{\partial z^2}\Big\|_p + \Big\|\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\Big\|_p + \Big\|\frac{1}{r}\frac{\partial\omega}{\partial z}\Big\|_p \leq \|D^3\mathbf{v}\|_p \leq$$

$$\leq C_6\Big\|\frac{\partial^2\omega}{\partial r^2}\Big\|_p + \Big\|\frac{\partial^2\omega}{\partial r\partial z}\Big\|_p + \Big\|\frac{\partial^2\omega}{\partial z^2}\Big\|_p + \Big\|\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\Big\|_p + \Big\|\frac{1}{r}\frac{\partial\omega}{\partial z}\Big\|_p. \tag{4.30}$$

The proof will follow directly from the following two lemmas.

Lemma 4.17 *There exist constants K_i, $i = 1, \ldots, 6$ such that for any smooth axially symmetric vector $\mathbf{v}$ with $\nabla \times \mathbf{v} = \mathbf{w}$ and any $\mathbf{a} \in \mathbb{R}^2$, $a_1^2 + a_2^2 > 0$ we have*

$$K_1|\mathbf{w}(\mathbf{a})| \leq |\omega(\mathbf{a})| \leq K_2|\mathbf{w}(\mathbf{a})| \tag{4.31}$$

$$K_3|D\mathbf{w}(\mathbf{a})| \leq \Big|\frac{\partial\omega(\mathbf{a})}{\partial r}\Big| + \Big|\frac{\partial\omega(\mathbf{a})}{\partial z}\Big| + \Big|\frac{\omega(\mathbf{a})}{r}\Big| \leq K_3|D\mathbf{w}(\mathbf{a})| \tag{4.32}$$

[22]By Du we understand the vector $(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3})$, while $\nabla\omega$ denotes $(\frac{\partial\omega}{\partial r}, \frac{\partial\omega}{\partial z})$.

$$K_5\Big|\frac{\partial^2\omega(\mathbf{a})}{\partial r^2}\Big|+\Big|\frac{\partial^2\omega(\mathbf{a})}{\partial r\partial z}\Big|+\Big|\frac{\partial^2\omega(\mathbf{a})}{\partial z^2}\Big|+\Big|\frac{\partial}{\partial r}\frac{\omega(\mathbf{a})}{r}\Big|+\Big|\frac{\partial}{\partial z}\frac{\omega(\mathbf{a})}{r}\Big|\le|D^2\mathbf{w}(\mathbf{a})|\le$$

$$\le K_6\Big|\frac{\partial^2\omega(\mathbf{a})}{\partial r^2}+\Big|\frac{\partial^2\omega(\mathbf{a})}{\partial r\partial z}\Big|+\Big|\frac{\partial^2\omega(\mathbf{a})}{\partial z^2}\Big|+\Big|\frac{\partial}{\partial r}\frac{\omega(\mathbf{a})}{r}\Big|+\Big|\frac{\partial}{\partial z}\frac{\omega(\mathbf{a})}{r}\Big|, \quad (4.33)$$

where $r=\sqrt{a_1^2+a_2^2}$.

Proof: We have

$$\omega=-w_x\sin\theta+w_y\cos\theta\,,$$
$$0=w_x\cos\theta+w_y\sin\theta\,,$$

i.e. $w_x=-\omega\sin\theta$, $w_y=\omega\cos\theta$, and the inequality (4.31) follows.

Further we have

$$\frac{\partial w_x}{\partial x}=-\frac{\partial\omega}{\partial r}\sin\theta\cos\theta+\frac{\omega}{r}\sin\theta\cos\theta\,,$$
$$\frac{\partial w_x}{\partial y}=-\frac{\partial\omega}{\partial r}\sin^2\theta-\frac{\omega}{r}\cos^2\theta\,,$$
$$\frac{\partial w_y}{\partial x}=\frac{\partial\omega}{\partial r}\cos^2\theta+\frac{\omega}{r}\sin^2\theta\,,$$
$$\frac{\partial w_y}{\partial y}=\frac{\partial\omega}{\partial r}\sin\theta\cos\theta-\frac{\omega}{r}\sin\theta\cos\theta\,,$$
$$\frac{\partial^2 w_x}{\partial x^2}=-\frac{\partial^2\omega}{\partial r^2}\sin\theta\cos^2\theta-\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)(\sin^3\theta-2\sin\theta\cos^2\theta)\,,$$
$$\frac{\partial^2 w_x}{\partial x\partial y}=-\frac{\partial^2\omega}{\partial r^2}\sin^2\theta\cos\theta-\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)(\cos^3\theta-2\sin^2\theta\cos\theta)\,,$$
$$\frac{\partial^2 w_x}{\partial y^2}=-\frac{\partial^2\omega}{\partial r^2}\sin^3\theta-3\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\sin\theta\cos^2\theta\,,$$
$$\frac{\partial^2 w_y}{\partial x^2}=\frac{\partial^2\omega}{\partial r^2}\cos^3\theta+3\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)\sin^2\theta\cos\theta\,,$$
$$\frac{\partial^2 w_y}{\partial x\partial y}=\frac{\partial^2\omega}{\partial r^2}\sin\theta\cos^2\theta+\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)(\sin^3\theta-2\sin\theta\cos^2\theta)\,,$$
$$\frac{\partial^2 w_y}{\partial y^2}=\frac{\partial^2\omega}{\partial r^2}\sin^2\theta\cos\theta+\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)(\cos^3\theta-2\sin^2\theta\cos\theta)\,,$$

i.e.

$$\frac{\partial\omega}{\partial z}=-\frac{\partial w_x}{\partial z}\sin\theta+\frac{\partial w_y}{\partial z}\cos\theta\,,$$
$$\frac{\partial\omega}{\partial r}=\frac{\partial w_y}{\partial x}\cos^2\theta+\frac{\partial w_y}{\partial y}\sin\theta\cos\theta-\frac{\partial w_x}{\partial x}\sin\theta\cos\theta-\frac{\partial w_x}{\partial y}\sin^2\theta\,,$$
$$\frac{\omega}{r}=\frac{\partial w_x}{\partial x}\sin\theta\cos\theta+\frac{\partial w_y}{\partial x}\sin^2\theta-\frac{\partial w_x}{\partial y}\cos^2\theta-\frac{\partial w_y}{\partial y}\cos\theta\sin\theta\,,$$
$$\frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big)=\frac{\partial^2 w_y}{\partial y^2}\cos\theta(\cos^2\theta-\sin^2\theta)+\frac{\partial^2 w_x}{\partial x^2}\sin\theta(3\cos^2\theta-\sin^2\theta)+$$
$$+2\frac{\partial^2 w_y}{\partial x^2}\sin^2\theta\cos\theta\,,$$
$$\frac{\partial^2\omega}{\partial r^2}=\frac{\partial^2 w_y}{\partial y^2}\sin^2\theta\cos\theta-4\frac{\partial^2 w_x}{\partial x^2}\sin\theta\cos^2\theta+$$
$$+\frac{\partial^2 w_y}{\partial x^2}\cos\theta(\cos^2\theta-\sin^2\theta)-\frac{\partial^2 w_x}{\partial y^2}\sin\theta\,,$$

which gives (4.32) and (4.33).

□

The proof of Theorem 4.1 is now split into two parts. First one is simple, we just observe that the second inequalities in (4.31)–(4.33) lead to the second inequalities in (4.28)–(4.30). Second part of the proof of Theorem 4.1 is formulated in the following lemma.

Lemma 4.18 *There exist $C_1(p)$, $C_2(p)$ and $C_3(p)$ such that for any smooth divergence free vectors* $\mathbf{v}$ *and* $p \in (1;\infty)$ *we have*

$$C_1(p)\|D\mathbf{v}\|_p \le \|\mathbf{w}\|_p \tag{4.34}$$

$$C_2(p)\|D^2\mathbf{v}\|_p \le \|D\mathbf{w}\|_p \tag{4.35}$$

$$C_3(p)\|D^3\mathbf{v}\|_p \le \|D^2\mathbf{w}\|_p \tag{4.36}$$

Proof: We use the Marcinkiewicz multiplier theorem (see Theorem II.3.2). As (4.35) is more complicated than (4.34), we just concentrate on (4.35); (4.36) can be proved following lines of the proof below.

We denote by $\mathcal{F}(v_x)$, $\mathcal{F}(v_y)$, $\mathcal{F}(v_z)$ the Fourier transform of the component v_x, v_y, v_z and put

$$\begin{aligned}
A_1 &= \xi_1\xi_2\mathcal{F}(v_x) - \xi_1^2\mathcal{F}(v_y) & A_7 &= \xi_2\xi_3\mathcal{F}(v_x) - \xi_1\xi_3\mathcal{F}(v_y)\\
A_2 &= \xi_1\xi_3\mathcal{F}(v_y) - \xi_1\xi_2\mathcal{F}(v_z) & A_8 &= \xi_3^2\mathcal{F}(v_y) - \xi_2\xi_3\mathcal{F}(v_y)\\
A_3 &= \xi_1\xi_3\mathcal{F}(v_x) - \xi_1^2\mathcal{F}(v_z) & A_9 &= \xi_3^2\mathcal{F}(v_x) - \xi_1\xi_3\mathcal{F}(v_z)\\
A_4 &= \xi_2^2\mathcal{F}(v_x) - \xi_1\xi_2\mathcal{F}(v_y) & A_{10} &= \xi_1^2\mathcal{F}(v_x) + \xi_1\xi_2\mathcal{F}(v_y) + \xi_1\xi_3\mathcal{F}(v_z)\\
A_5 &= \xi_2\xi_3\mathcal{F}(v_y) - \xi_2^2\mathcal{F}(v_z) & A_{11} &= \xi_1\xi_2\mathcal{F}(v_x) + \xi_2^2\mathcal{F}(v_y) + \xi_2\xi_3\mathcal{F}(v_z)\\
A_6 &= \xi_2\xi_3\mathcal{F}(v_x) - \xi_1\xi_2\mathcal{F}(v_z) & A_{12} &= \xi_1\xi_3\mathcal{F}(v_x) + \xi_2\xi_3\mathcal{F}(v_y) + \xi_3^2\mathcal{F}(v_z)
\end{aligned}$$

i.e. $A_1 - A_9$ are (up to the sign) the Fourier transforms of $\nabla\nabla\times\mathbf{v}$, $A_{10} - A_{12}$ are the Fourier transforms of $-\nabla\nabla\cdot\mathbf{v}$. We shall calculate $\mathcal{F}(v_x)$, $\mathcal{F}(v_y)$ and $\mathcal{F}(v_z)$ by means of A_i:

$$\begin{aligned}
\xi_2 A_1 + \xi_3 A_3 + \xi_1 A_{10} &= \xi_1|\xi|^2\mathcal{F}(v_x)\,,\\
\xi_2 A_4 + \xi_1 A_{11} + \xi_3 A_6 &= \xi_2|\xi|^2\mathcal{F}(v_x)\,,\\
\xi_1 A_{12} + \xi_2 A_7 + \xi_3 A_9 &= \xi_3|\xi|^2\mathcal{F}(v_x)\,.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\mathcal{F}(v_x) = \frac{1}{|\xi|^4}(\xi_1\xi_2 A_1 &+ \xi_1\xi_3 A_3 + \xi_2^2 A_4 + \xi_2\xi_3 A_6 + \\
&+ \xi_2\xi_3 A_7 + \xi_3^2 A_9 + \xi_1^2 A_{10} + \xi_1\xi_2 A_{11} + \xi_1\xi_3 A_{12})\,.
\end{aligned}$$

Similar expression we obtain for $\mathcal{F}(v_y)$ and $\mathcal{F}(v_z)$. Denoting by $D^2\mathbf{v}$ the vector of 18 components (all different second derivatives with respect to the spatial variables) and by $\mathbf{A}$ the vector $(A_1 \dots A_{12})$, we have

$$\mathcal{F}(D^2\mathbf{v}) = \mathbf{T}\mathbf{A}\,, \tag{4.37}$$

where

$$\mathbf{T} : R^{12} \mapsto R^{18}$$

has non–zero components of the type $\frac{\xi_1^a \xi_2^b \xi_3^c}{|\xi|^4}$ with $a, b, c \in \mathbb{N}_0$, $a + b + c = 4$, i.e. they are bounded and the l-th order derivatives can be bounded by $C|\xi|^{-l}$. We may apply the Marcinkiewicz multiplier theorem (see Theorem II.3.2) to obtain the desired inequality. The inequality (4.36) follows similarly.

□

Remark 4.4 In Chapter VII, we need the inequality (4.1) only for $p = 2$. This case does not require the use of the multiplier theorem, it follows directly from (4.37) by means of the Parseval equality, see Lemma 4.5 (e).

VIII.5 Modified Stokes problem. Existence of pressure

Finally we shall investigate the modified Stokes problem, i.e.

$$\left.\begin{array}{r} A(\mathbf{u}) + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{array}\right\} \text{ in } \Omega \qquad \mathbf{u} = \mathbf{u}_* \text{ at } \partial\Omega \tag{5.1}$$

and if Ω is an exterior domain,

$$\mathbf{u} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty$$

with $A(\mathbf{u}) = -\Delta\mathbf{u} + \mu\frac{\partial^2 \mathbf{u}}{\partial x_1^2}$, $\mu \in [0; 1)$. We shall not develop a general theory for such a problem; we shall only prove some estimates needed in the theory of the modified Oseen problem which are completely analogous with similar results for the classical Stokes problem.

There are two approaches; we can either use the fact that the system (5.1) is elliptic in the sense of Agmon, Douglis and Nirenberg (see [AgDoNi]) or, under more restrictive assumptions on μ, use the results on the classical Stokes problem (see e.g. [Ga1]). Unfortunately, we were not able to develop the theory for the modified Stokes problem directly, following [Ga1] for the classical Stokes problem.

We denote by

$$a_\mu(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} \mathrm{d}\mathbf{x} - \mu \int_\Omega \frac{\partial \mathbf{u}}{\partial x_1} \cdot \frac{\partial \mathbf{v}}{\partial x_1} \mathrm{d}\mathbf{x} \,,$$

see Chapter III. Then we say that $\mathbf{u}$ is a q–weak solution to (5.1), if

(i) $\mathbf{u} \in D^{1,q}(\Omega)$

(ii) $\mathbf{u}$ is (weakly) divergence free

(iii) $\mathbf{u} = \mathbf{u}_*$ at $\partial\Omega$ in the sense of traces

(iv) for all $\boldsymbol{\varphi} \in {}_0\mathcal{D}(\Omega)$

$$a_\mu(\mathbf{u}, \boldsymbol{\varphi}) = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \,.$$

Moreover, if Ω is an exterior domain, then also

(v)
$$\int_{S_N} |\mathbf{u}(R,\omega)| \mathrm{d}\omega \to 0 \quad \text{as } R \to \infty .$$

As in Chapter II for the Oseen problem, the weak formulation formally excluded the pressure. We shall show now that in fact we can reconstruct the pressure using the weak formulation. We give several results here. Some of them are applicable for the Oseen and modified Oseen problem, another only for the Stokes problem and its modified version. We start with a general lemma from functional analysis.

Lemma 5.1 *Let $A : X \mapsto Y$ is a continuous operator, $D(A) = X$, A^{-1} exists and is continuous. Let X, Y be reflexive Banach spaces. Then*

$$R(A^*) = (\ker A)^\perp ,$$

where $R(A^*)$ denotes the range of the adjoint operator to A and

$$(\ker A)^\perp = \{f \in X^*; \langle f, u\rangle = 0 \ \ \forall u \in \ker A\}$$
$$\ker A = \{u \in X; Au = 0\} .$$

Proof: See e.g. [Tay].

□

Now, using the results from Section VIII.3 we have

Theorem 5.1 *Let Ω be such that the problem*

$$\begin{gathered} \nabla \cdot \mathbf{v} = f \\ \mathbf{v} \in D_0^{1,q}(\Omega) \\ |\mathbf{v}|_{1,q} \leq C\|f\|_q \end{gathered} \tag{5.2}$$

is solvable for all $f \in L^q(\Omega)$ (if Ω bounded, then $\int_\Omega f \mathrm{d}\mathbf{x} = 0$). Let $\mathcal{G}$ be a continuous linear functional on $D_0^{1,q}(\Omega)$, $1 < q < \infty$, such that

$$\langle \mathcal{G}, \mathbf{g}\rangle = 0$$

for all $\mathbf{g} \in \widehat{\mathcal{D}}_0^{1,q}(\Omega)$. Then there exists exactly one $p \in L^{q'}(\Omega)$ (if Ω bounded, then $\int_\Omega p \mathrm{d}\mathbf{x} = 0$) such that

$$\langle \mathcal{G}, \mathbf{g}\rangle = \int_\Omega p \nabla \cdot \mathbf{g} d\mathbf{x} \quad \forall \mathbf{g} \in D_0^{1,q}(\Omega) . \tag{5.3}$$

Proof: Let us consider the linear operator

$$A : \mathbf{v} \in D_0^{1,q}(\Omega) \to \nabla \cdot \mathbf{v} \in L^q(\Omega) .$$

Evidently, A is bounded, linear and $R(A) = L^q(\Omega)$ ($L^q(\Omega)/\mathbb{R}$ if Ω is bounded). We have (see Lemma 5.1)

$$(\ker A)^{\perp} = R(A^*)\,.$$

Since $\ker A = \widehat{\mathcal{D}}_0^{1,q}(\Omega)$, $(\widehat{\mathcal{D}}_0^{1,q}(\Omega))^{\perp} = R(A^*)$ and $\mathcal{G} \in R(A^*)$. As $R(A) = L^q(\Omega)$ ($L^q(\Omega)/\mathbb{R}$ if Ω is bounded), using the Riesz representation theorem (see Lemma 1.4)

$$\langle \mathcal{F}, \mathbf{g}\rangle = \int_\Omega pA\mathbf{g}\mathrm{d}\mathbf{x} = \int_\Omega p\nabla\cdot\mathbf{g}\mathrm{d}\mathbf{x}$$

for all $\mathbf{g} \in D_0^{1,q}(\Omega)$.

□

Corollary 5.1 *Let Ω be a bounded or exterior domain of class $C^{0,1}(\Omega)$ and $N \geq 2$. Then each linear continuous functional on $D_0^{1,q}(\Omega)$, $1 < q < \infty$, which is identically zero on $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$ can be written in the form*

$$\langle \mathcal{F}, \boldsymbol{\psi}\rangle = \int_\Omega p\nabla\cdot\boldsymbol{\psi}d\mathbf{x} \quad \forall\boldsymbol{\psi} \in D_0^{1,q}(\Omega)$$

for $p \in L^{q'}(\Omega)$ ($p \in L^{q'}(\Omega)/\mathbb{R}$ if Ω is bounded).

Corollary 5.2 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a domain and $\mathcal{G}$ be a linear continuous functional on $D_0^{1,q}(\Omega')$, $1 < q < \infty$, which is identically zero on $\widehat{\mathcal{D}}_0^{1,q}(\Omega')$ for all $\Omega' \subset \Omega$ bounded, $\overline{\Omega'} \subset \Omega$. Then there exists $p' \in L^{q'}_{loc}(\Omega)$ such that*

$$\langle \mathcal{F}, \boldsymbol{\psi}\rangle = \int_\Omega p\nabla\cdot\boldsymbol{\psi}d\mathbf{x} \quad \forall\boldsymbol{\psi} \in C_0^\infty(\Omega)\,.$$

We next estimate the pressure for the modified Stokes problem. Let us assume that $\mathbf{f} \in D_0^{-1,q}(\Omega)$, $\Omega \in C^{0,1}$, a bounded or an exterior domain. Let $\mathbf{u}$ be a q–weak solution to (5.1). Then due to the condition (iv) and due to the fact that $\mathbf{f} \in D_0^{-1,q}(\Omega)$ we have that

$$\langle \mathcal{F}, \boldsymbol{\psi}\rangle = a_\mu(\mathbf{u}, \boldsymbol{\psi}) - \langle \mathbf{f}, \boldsymbol{\psi}\rangle$$

is a functional bounded on $D_0^{1,q'}(\Omega)$ which is identically zero on $\widehat{\mathcal{D}}_0^{1,q}(\Omega)$. Applying Corollary 5.1 we get the existence of $p \in L^q(\Omega)$ ($L^q(\Omega)/\mathbb{R}$ if Ω is bounded) such that

$$a_\mu(\mathbf{u}, \boldsymbol{\psi}) - \langle \mathbf{f}, \boldsymbol{\psi}\rangle = \int_\Omega p\nabla\cdot\boldsymbol{\psi}\mathrm{d}\mathbf{x} \quad \forall\boldsymbol{\psi} \in D_0^{1,q'}(\Omega)\,. \tag{5.4}$$

Let now Ω be bounded. We consider the following problem in Ω

$$\begin{gathered}\nabla\cdot\boldsymbol{\psi} = |p|^{q-2}p - \frac{1}{|\Omega|}\int_\Omega |p|^{q-2}p\mathrm{d}\mathbf{x} \\ \boldsymbol{\psi} \in D_0^{1,q'}(\Omega) \\ \|\boldsymbol{\psi}\|_{1,q'} \leq C\|p\|_q^{q-1}\,.\end{gathered} \tag{5.5}$$

Then, using such $\boldsymbol{\psi}$ in (5.4) we get ($\int_\Omega p\mathrm{d}\mathbf{x} = 0$)

$$\|p\|_q^q \leq C(|\mathbf{f}|_{-1,q} + |\mathbf{u}|_{1,q})|\boldsymbol{\psi}|_{1,q'} \leq C\|p\|_q^{q-1}(|\mathbf{f}|_{-1,q} + |\mathbf{u}|_{1,q})$$

and we have

$$\|p\|_q \leq C(|\mathbf{f}|_{-1,q} + |\mathbf{u}|_{1,q})\,. \tag{5.6}$$

If we skip the condition $\int_\Omega p\mathrm{d}\mathbf{x} = 0$, we get instead of (5.6)

$$\inf_{c\in\mathbb{R}} \|p + c\|_q \leq C(|\mathbf{f}|_{-1,q} + |\mathbf{u}|_{1,q})\,. \tag{5.7}$$

If Ω is unbounded, we can instead of (5.5) consider directly

$$\begin{aligned} \nabla\cdot\boldsymbol{\psi} &= |p|^{q-2}p \\ \boldsymbol{\psi} &\in D_0^{1,q'}(\Omega) \\ |\boldsymbol{\psi}|_{1,q'} &\leq C\|p\|_q^{q-1}\,. \end{aligned} \tag{5.8}$$

Proceeding analogously we show again (5.6).

We therefore have

Theorem 5.2 *Let* $\mathbf{u}$ *be a q–weak solution to the modified Stokes problem (5.1) in* $\Omega \in C^{0,1}$ *a bounded or an exterior domain. Let* $\mathbf{f} \in D_0^{-1,q}(\Omega)$*. Then there exists a unique function* $p \in L^q(\Omega)$[23]*, called pressure, such that*

$$a_\mu(\mathbf{u},\boldsymbol{\psi}) - \langle \mathbf{f},\boldsymbol{\psi}\rangle = \int_\Omega p\nabla\cdot\boldsymbol{\psi}d\mathbf{x} \quad \forall\boldsymbol{\psi}\in C_0^\infty(\Omega)\,.$$

Moreover, there exists $C = C(q,N,\Omega)$ *such that* ($\int_\Omega p\mathrm{d}\mathbf{x} = 0$ *if* Ω *is bounded)*

$$\|p\|_q \leq C(|\mathbf{u}|_{1,q} + |\mathbf{f}|_{-1,q})\,.$$

For the Oseen problem and its modified version, Corollary 5.1 is not applicable. We have namely the functional ($\mu = 0$ for the classical Oseen problem)

$$\langle \mathcal{G}_0,\boldsymbol{\psi}\rangle = a_\mu(\mathbf{u},\boldsymbol{\psi}) + \beta(\frac{\partial\mathbf{u}}{\partial x_1}\boldsymbol{\psi}) - \langle\mathbf{f},\boldsymbol{\psi}\rangle$$

and therefore, even for $\mathbf{f}\in D_0^{-1,q}(\Omega)$, we do not have apriori the corresponding pressure $p\in L^q(\Omega)$, only $p \in L^q_{loc}(\Omega)$[24]. Nevertheless, we can get

Theorem 5.3 *Let* $\mathbf{u}$ *be q–weak solution to the modified Oseen problem (III.0.1),* $\Omega \in C^{0,1}$*, a bounded or an exterior domain. Let* $\mathbf{f}\in D_0^{-1,q}(\Omega)$*. Then there exists scalar function* $p \in L^q_{loc}(\Omega)$*, called pressure, such that*

$$a_\mu(\mathbf{u},\boldsymbol{\psi}) + \beta\Big(\frac{\partial\mathbf{u}}{\partial x_1},\boldsymbol{\psi}\Big) - \langle\mathbf{f},\boldsymbol{\psi}\rangle = (p,\nabla\cdot\boldsymbol{\psi})$$

[23] unique up to an additive constant for Ω bounded

[24] see Theorem III.3.7 for the estimates giving the global integrability for the pressure for Ω an exterior domain

for all $\boldsymbol{\psi} \in C_0^\infty(\Omega)$*. Moreover, if* Ω *is bounded, then there exists* $C = C(q, N, \Omega)$ *such that* ($\int_\Omega p d\mathbf{x} = 0$)

$$\|p\|_q \leq C(\|\mathbf{u}\|_{1,q} + |\mathbf{f}|_{-1,q}) \tag{5.9}$$

and if Ω *is an exterior domain, then there exists* $C = C(q, N, \Omega)$ *such that for all* $R > diam\,\Omega^c$

$$\|p\|_{q,\Omega_R/\mathbb{R}} \leq C(\|\mathbf{u}\|_{q,\Omega_R} + |\mathbf{u}|_{1,q} + |\mathbf{f}|_{-1,q})\,. \tag{5.10}$$

Proof: We proceed as for the Stokes problem. The only difference consists in the presence of the term $(\frac{\partial \mathbf{u}}{\partial x_1}, \boldsymbol{\psi})$. If Ω is bounded, we have for $\boldsymbol{\psi}$, solution to the problem (5.5),

$$\Big(\frac{\partial \mathbf{u}}{\partial x_1}, \boldsymbol{\psi}\Big) = \Big(\mathbf{u}, \frac{\partial \boldsymbol{\psi}}{\partial x_1}\Big)$$

and we easily get (5.9). If Ω is unbounded, we can no more control the L^q–norm of $\mathbf{u}$. We therefore consider instead of (5.8) the problem (5.5) with $\Omega := \Omega_R$ and get (5.10).

□

Let us come back to the modified Stokes problem. We shall study existence, uniqueness and regularity of q–weak solutions to the problem (5.1). First we have

Lemma 5.2 *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *of class* $C^{0,1}$*,* $N \geq 2$*,* $\mathbf{f} \in D_0^{-1,2}(\Omega)$*,* $\mathbf{u}_* \in W^{\frac{1}{2},2}(\partial\Omega)$*,* $\int_{\partial\Omega} \mathbf{u}_* \cdot \mathbf{n} dS = 0$*,* $0 \leq \mu < 1$*. Then there exists exactly one* 2*–weak solution to (5.1). Moreover*

$$\|\mathbf{u}\|_{1,2} + \|p\|_{2/\mathbb{R}} \leq C(|\mathbf{f}|_{-1,2} + \|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})\,. \tag{5.11}$$

Proof: Searching the solution in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{w}$$

with $\mathbf{w}$ a divergence free extension of the boundary data we easily get the existence of solution combining results from Section VIII.3 due to the Lax–Milgram theorem (see Theorem 1.1).

□

Lemma 5.3 *Let* Ω *be an exterior domain in* $\mathbb{R}^N$ *of class* $C^{0,1}$*,* $N \geq 3$*,* $\mathbf{f} \in D_0^{-1,2}(\Omega)$*,* $\mathbf{u}_* \in W^{\frac{1}{2},2}(\partial\Omega)$*. Then there exists exactly one* 2*–weak solution to (5.1) such that* $\int_{S_N} |\mathbf{v}| d\omega \to 0$ *as* $R \to \infty$*. Moreover*

$$\|\mathbf{u}\|_{2,\Omega_R} + |\mathbf{u}|_{1,2} + \|p\|_2 \leq C(|\mathbf{f}|_{-1,2} + \|\mathbf{u}_*\|_{\frac{1}{2},2,(\partial\Omega)})\,, \tag{5.12}$$

where $C = C(\Omega, R, N)$*.*

Proof: It is analogous to the proof of Theorem III.3.3. We search the solution in the form

$$\mathbf{u} = \mathbf{v} + \mathbf{w} + \boldsymbol{\sigma},$$

where

$$\boldsymbol{\sigma} = -\nabla \mathcal{E} \int_{\partial\Omega} \mathbf{u}_* \cdot \mathbf{n} \mathrm{d}S,$$

$\mathbf{w}$ is a divergence free extension of $\mathbf{u}_* - \boldsymbol{\sigma}$ with bounded support and $\mathbf{v}$ is a 2–weak solution to

$$\begin{gathered} \left.\begin{aligned} A(\mathbf{v}) + \nabla p = \mathbf{f} - A(\boldsymbol{\sigma} + \mathbf{w}) \\ \nabla \cdot \mathbf{v} = 0 \end{aligned}\right\} \text{ in } \Omega \\ \mathbf{v} = \mathbf{0} \text{ at } \partial\Omega \\ \textstyle\int_{S_N} |\mathbf{v}(R,\omega)| \mathrm{d}\omega \to 0 \text{ as } R \to \infty . \end{gathered} \tag{5.13}$$

Again, the existence of a unique solution to (5.13) can be established using the Lax–Milgram theorem.

□

Next, let us recall that the system (5.1) is elliptic in the sense of Agmon, Douglis and Nirenberg (see [AgDoNi]). We therefore have

Theorem 5.4 *Let $\Omega \in C^{m+2}$, a bounded domain, $\mathbf{f} \in W^{m,q}(\Omega)$, $m \geq 0$, $1 < q < \infty$. Then there exists C, independent of $\mathbf{f}$ and $\mathbf{u}$, such that if $\mathbf{u}$ is the q–weak solution to (5.1) and p the corresponding pressure, then*

$$\|\mathbf{u}\|_{m+2,q} + \|p\|_{m+1,q} \leq C(\|\mathbf{f}\|_{m,q} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)} + \|\mathbf{u}\|_{1,q} + \|p\|_q). \tag{5.14}$$

Theorem 5.5 *Let $\Omega \in C^2$ be a bounded domain, $\mathbf{f} \in W_0^{-1,q}(\Omega)$, $1 < q < \infty$, $\mathbf{u}_* \in W^{1-\frac{1}{q},q}(\partial\Omega)$. Then if $\mathbf{u}$ is a q–weak solution to the modified Stokes problem, then*

$$\|\mathbf{u}\|_{1,q} + \|p\|_{q/\mathbb{R}} \leq C(\|\mathbf{f}\|_{-1,q} + \|\mathbf{u}_*\|_{1-\frac{1}{q},q,(\partial\Omega)} + \|\mathbf{u}\|_q + \|p\|_{-1,q}), \tag{5.15}$$

where p denotes the corresponding pressure from Theorem 5.2.

We get (see e.g. [Ga1] for the classical Stokes problem)

Theorem 5.6 *Let $\Omega \in C^2$ be a bounded domain. Let $\mathbf{u}$ be a q–weak solution to (5.1) with zero data. Then $\mathbf{u} = \mathbf{0}$ and $p = const$ in Ω.*

Proof: If $q = 2$, the result follows from Lemma 5.2. If $q > 2$, then any q–weak solution is also 2–weak solution and $\mathbf{u} = \mathbf{0}$. Finally, if $q < 2$, then due to Theorem 5.4 $\mathbf{u} \in W^{2,q}(\Omega)$, $p \in W^{1,q}(\Omega)$, i.e. $\mathbf{u} \in W^{1,r_1}(\Omega)$, $r_1 = \frac{Nq}{N-q}$. Repeating this argument several times we get after finite number of steps that $\mathbf{u} \in W^{1,2}(\Omega)$ which implies $\mathbf{u} = \mathbf{0}$. Next, easily, $p = $ const.

□

Corollary 5.3 *If* $\mathbf{u}$ *is a q–weak solution to (5.1),* $\Omega \in C^{m+1}$*, bounded domain, then*

$$\|\mathbf{u}\|_{m+2,q} + \|p\|_{m+1,q} \le C(\|\mathbf{f}\|_{m,q} + \|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)})\,. \tag{5.16}$$

Proof: We have to show that $\|\mathbf{u}\|_{1,q}+\|p\|_q \le C(\|\mathbf{f}\|_{m,q}+\|\mathbf{u}_*\|_{m+2-\frac{1}{q},q,(\partial\Omega)})$. Let us assume the contrary, i.e. that there exists $(\mathbf{u}_*)_k$, $\mathbf{f}_k$ and $(\mathbf{u}_k, p_k)$ the corresponding solution to (5.1) such that

$$\|\mathbf{u}_k\|_{1,q} + \|p_k\|_q > k(\|\mathbf{f}_k\|_{m,q} + \|(\mathbf{u}_*)_k\|_{m+2-\frac{1}{q},q,(\partial\Omega)}) \quad \forall k \in \mathbb{N}\,.$$

We can take without loss of generality $\|\mathbf{u}_k\|_{1,q} + \|p_k\|_q = 1$, $\int_\Omega p_k \mathrm{d}\mathbf{x} = 0$. We have due to (5.14), at least for a chosen subsequence,

$$\begin{aligned} \mathbf{u}_k &\rightharpoonup \mathbf{u} \text{ in } W^{m+2,q}(\Omega)\\ p_k &\rightharpoonup p \text{ in } W^{m+1,q}(\Omega)\\ \mathbf{f}_k &\to \mathbf{0} \text{ in } W^{m,q}(\Omega)\\ (\mathbf{u}_*)_k &\to \mathbf{0} \text{ in } W^{m+2-\frac{1}{q},q}(\partial\Omega)\,, \end{aligned}$$

where $(\mathbf{u}, p)$ solves (5.1) with $\mathbf{f} = \mathbf{u}_* = \mathbf{0}$. Due to Theorem 5.6, $\mathbf{u} = 0$, $p =$ const but $\int_\Omega p\mathrm{d}\mathbf{x} = 0$ implies $p = 0$. The compact imbedding $W^{2,q}(\Omega) \hookrightarrow\hookrightarrow W^{1,q}(\Omega)$ ($W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$) yields us $\mathbf{u}_k \to \mathbf{u}$ in $W^{1,q}(\Omega)$, $p_k \to p$ in $L^q(\Omega)$, but $\|\mathbf{u}\|_{1,q} + \|p\|_q = 1$, yielding a contradiction.

□

Next we consider $N \ge 3$ and Ω an exterior domain. We have

Theorem 5.7 *Let* $1 < q < N$*,* $\Omega \in C^2$ *be an exterior domain in* $\mathbb{R}^N$*,* $N \ge 3$*. Let* $\mathbf{f} = \mathbf{u}_* = \mathbf{0}$*. Then the only q–weak solution to (5.1) is such that* $\mathbf{u} = \mathbf{0}$ *and the corresponding pressure* $p = 0$*.*[25]

Sketch of the proof: We shall not give the details of the proof as they are quite technical and long but completely analogous to the classical Stokes problem (see e.g. [No3]). Firstly we construct the fundamental solution to the modified Stokes problem (more precisely, its Fourier transform) and using the Lizorkin multiplier theorem (see Theorem II.3.3) we show L^q–estimates of the solution to the modified Stokes problem in the whole $\mathbb{R}^N$ which are completely analogous with the estimates for the classical Stokes problem.

Next, using the structure of the Fourier transform of the fundamental solution and its derivatives it is an easy matter to verify that the fundamental solution together with all derivatives have the same asymptotic properties as the fundamental solution to the classical Stokes problem.

Then we show that if $\mathbf{u}_1$, $\mathbf{u}_2$ are two (apriori different) solutions to the modified Stokes problem in $\mathbb{R}^N$, $\nabla\mathbf{u}_1 \in L^p(\mathbb{R}^N)$, $\nabla\mathbf{u}_2 \in L^q(\mathbb{R}^N)$, $1 < p, q < N$, then $\mathbf{u}_1 = \mathbf{u}_2 +$ const, $p_1 = p_2 +$ const. The proof is the same as the proof of Lemma III.2.3 for the modified Oseen problem.

[25] We add to p such a constant that $p \in L^q(\Omega)$.

Finally, let $\mathbf{u}$ be a q–weak solution to the modified Stokes problem with the right hand side of bounded support in Ω, $1 < q < N$. Then we easily verify that $\mathbf{u}$ has the same asymptotic properties as the fundamental solution. Now, if $\mathbf{u}$ is a q–weak solution to (5.1) with zero data, we can use as test function $\mathbf{u}\eta_R$ with η_R the usual cut–off function. Due to its asymptotic properties we get $\nabla\mathbf{u} = \mathbf{0}$ a.e. in Ω and since $\mathbf{u} \to \mathbf{0}$ as $|\mathbf{x}| \to \infty$ in some (weak) sense, we have $\mathbf{u} = \mathbf{0}$. Then easily $p = 0$, where we added a constant to p in such a way that $p \in L^q(\Omega)$.

□

As a consequence we have

Corollary 5.4 *Let* $\mathbf{u}$ *be a strong solution to the modified Stokes problem such that* $\nabla^2\mathbf{u} \in L^q(\Omega)$, $\Omega \in C^2$ *exterior domain in* $\mathbb{R}^N$, $1 < q < \frac{N}{2}$ *such that* $\mathbf{u} \to \mathbf{0}$ *as* $|\mathbf{x}| \to \infty$ *in some (eventually weak) sense. Let* $\mathbf{v}$ *be another solution with the same properties. Then* $\mathbf{u} = \mathbf{v}$ *a.e. in* Ω.

Proof: Denote $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Then $\nabla^2\mathbf{w} \in L^q(\Omega)$, solves the modified Stokes problem with zero data. As $\mathbf{w}$ tends to zero as $|\mathbf{x}| \to \infty$ in some sense, we have that $\mathbf{w} \in L^s(\Omega)$, $\nabla\mathbf{u} \in L^r(\Omega)$, $1 < s < \infty$, $1 < r < N$. Applying Theorem 5.7 we see that $\mathbf{w} = \mathbf{0}$. Moreover, if $p_{\mathbf{u}}$, $p_{\mathbf{v}}$ are the corresponding pressures from Theorem 5.2, we have $p_{\mathbf{u}} - p_{\mathbf{v}} = \text{const}$.

□

Remark 5.1 Let us finally note that the estimate (5.14) or, more precisely, (5.16) can be shown without use of the Agmon–Douglis–Nirenberg results. We can namely look at the solution to the modified Stokes problem as a fixed point of the operator $T : \mathbf{w} \mapsto \mathbf{u}$

$$\left.\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} - \mu\frac{\partial^2\mathbf{w}}{\partial x_1^2} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}\right\} \text{ in } \Omega$$
$$\mathbf{u} = \mathbf{u}_* \text{ at } \partial\Omega$$

and for μ sufficiently small we get its existence together with all estimates. Unlike the procedure introduced above, where we assumed only $\mu < 1$, we shall get much stronger restriction on the size of μ.

Conclusions

In the presented work two different problems were solved. In the first part, the questions raised in [Du] and [Vi] were successfully answered. We have shown that, at least for certain classes of non–Newtonian fluids, the velocity and the pressure obey for non–zero velocity prescribed at infinity the same asymptotic properties as the fundamental solution to the Oseen problem; in particular, the velocity shows the existence of a wake region behind the obstacle.

Moreover, we have also verified that under the assumption of sufficiently fast decay of the right hand side, its smallness and the smallness of the velocity prescribed at infinity itself, we can show the precise asymptotic structure not only for the velocity itself, but also for its first gradient.

Such studies made us do precise investigations of weighted estimated of both singular and weakly singular integral operators with Oseen kernels; for the sake of completeness we gave also results for the kernels not used in this work. Another problem, coming in fact from the weighted estimates, was the necessity of investigations of certain perturbation to the Oseen problem, called here the modified Oseen problem. The most crucial problem were the asymptotic properties of its fundamental solution. Again, for the sake of completeness, we then presented complete theory of this linear problem in both two and three space dimensions (the extension to higher space dimension is straightforward), combining the approaches from [Ga1], [Ga2] with [No3].

The other problem, treated in this work, was the axially symmetric flow of both linearly viscous and ideal fluid in the whole $\mathbb{R}^3$. The proof presented here has several advantages in comparison with the original proofs presented in [Lad2] or [UcYu]. Unlike the above mentioned papers we essentially use the fact that we study the problem in the whole space and therefore we do not have to construct complicated basis in weighted spaces on the balls with growing diameters.

Finally, let us mention several problems which were either not attached or, in spite of certain attempts, remained unsolved. To the former belong study of asymptotic structure of problems in higher space dimensions or study of compressible viscoelastic fluids. Both problems seem to be only a little bit more technical but solvable in a similar way.

The problem of study of asymptotic structure to the second grade fluid seems to be much more difficult. It is essentially connected with the optimality of L^p–weighted estimates for kernels representing second gradients of the fundamental Oseen tensor. In order to exclude the disturbing logarithmic factors completely another technique must be used; the fact that the second gradient of the fundamental Oseen tensor represents L^p–L^p Lizorkin multiplier should be employed. This problem may serve as a starting point for further investigations.

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