Stochastic Processes 2

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Literature

Basic Lecture Notes:

- Z. Prášková: Stochastic Processes 2, on-line version
- J. Dvořák, M. Prokešová: Stochastic Processes 2, Collection of solved exercises, on-line

Supplementary texts:

Definitions and basic characteristics
**Definition:**
Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(S, \mathcal{E})$ a measurable space, and $T \subseteq \mathbb{R}$. A family of (real-valued) random variables $\{X_t, t \in T\}$ defined on $(\Omega, \mathcal{A}, P)$ with values in $S$ is called a **stochastic (random) process**.

$T = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ or $T \subseteq \mathbb{Z} - \{X_t, t \in T\}$ is **discrete time stochastic process**, **time series**

$T = [a, b], -\infty \leq a < b \leq \infty$ - $\{X_t, t \in T\}$ is a **continuous time stochastic process**.

For any $\omega \in \Omega$ fixed, $X_t(\omega)$ is a function on $T$ with values in $S$ which is called a **trajectory** of the process.

**Definition:**
A pair $(S, \mathcal{E})$, where $S$ is a set of values of random variables $X_t$ and $\mathcal{E}$ is a $\sigma$—algebra of subsets of $S$, is called the **state space** of the process $\{X_t, t \in T\}$. 
Finite-dimensional distributions:

∀n ∈ ℕ and any finite subset \( \{t_1, \ldots, t_n\} \subset T \) there is a system of random variables \( X_{t_1}, \ldots, X_{t_n} \), with the joint distribution function

\[
P [X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n] = F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)
\]

for all real \( x_1, \ldots, x_n \).

A system of distribution functions is said to be consistent, if

\[F_{t_{i_1},\ldots,t_{i_n}}(x_{i_1}, \ldots, x_{i_n}) = F_{t_1,\ldots,t_n}(x_1, \ldots, x_n)\]

for any permutation \((i_1, \ldots, i_n)\) of \((1, \ldots, n)\) (symmetry)

\[\lim_{x_n \to \infty} F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{t_1,\ldots,t_{n-1}}(x_1, \ldots, x_{n-1})\]

(consistency)
A system of characteristic functions
The characteristic function of a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is

$$\varphi_{\mathbf{X}}(\mathbf{u}) := \mathbb{E} e^{i \mathbf{u}^\top \mathbf{X}} = \mathbb{E} e^{i \sum_{j=1}^n u_j X_j}, \quad \mathbf{u} = (u_1, \ldots, u_n) \top \in \mathbb{R}_n$$

$$F_{t_1, \ldots, t_n}(x_1, \ldots, x_n) \leftrightarrow \varphi_{X_{t_1}, \ldots, X_{t_n}}(u_1, \ldots, u_n) := \varphi(u_1, \ldots, u_n)$$

Consistent system of characteristic functions:

- symmetry:

$$\varphi(u_{i_1}, \ldots, u_{i_n}) = \varphi(u_1, \ldots, u_n)$$

for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$,

- consistency:

$$\lim_{u_n \to 0} \varphi_{X_{t_1}, \ldots, X_{t_n}}(u_1, \ldots, u_n) = \varphi_{X_{t_1}, \ldots, X_{t_{n-1}}}(u_1, \ldots, u_{n-1})$$
Daniell-Kolmogorov theorem

For any stochastic process there exists a consistent system of distribution functions and,

**Theorem 1:**
Let \( \{F_{t_1,\ldots,t_n}(x_1,\ldots,x_n)\}\) be a consistent system of distribution functions. Then there exists a stochastic process \( \{X_t, t \in T\} \) such that for any \( n \in \mathbb{N}, \) any \( t_1, \ldots, t_n \in T \) and any real \( x_1, \ldots, x_n \) it holds

\[
P [X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n] = F_{t_1,\ldots,t_n}(x_1,\ldots,x_n).
\]

**Definition:**
A complex-valued random variable $X$ is defined by $X = Y + iZ$, where $Y$ and $Z$ are real random variables, $i = \sqrt{-1}$.

The mean value of a complex-valued random variable $X = Y + iZ$ is defined by

$$E_X = E_Y + iE_Z$$

provided the mean values $E_Y$ and $E_Z$ exist.

The variance of a complex-valued random variable $X = Y + iZ$ is defined by

$$\text{var } X := E \left[ (X - E_X)(\overline{X} - \overline{E_X}) \right] = E|X - E_X|^2 \geq 0$$

provided the second moments of random variables $Y$ and $Z$ exist.

**Definition:**
A complex-valued stochastic process is a family of complex-valued random variables on $(\Omega, \mathcal{A}, P)$. 
Definition:
Let \( \{X_t, t \in T\} \) be a stochastic process such that \( \mathbb{E}X_t \) exists for all \( t \in T \). Then the function \( \mu_t = \mathbb{E}X_t \) defined on \( T \) is called the mean value of the process \( \{X_t, t \in T\} \). We say that the process is centred if its mean value is zero for all \( t \in T \).

Definition:
Let \( \{X_t, t \in T\} \) be a process with finite second moments, i.e., \( \mathbb{E}|X_t|^2 < \infty, \forall t \in T \). Then a (complex-valued) function defined on \( T \times T \) by

\[
R(s, t) = \mathbb{E} [(X_s - \mu_s)(\overline{X}_t - \overline{\mu}_t)]
\]

is called the autocovariance function of the process \( \{X_t, t \in T\} \). The value \( R(t, t) \) is the variance of the process at time \( t \).
Definition:

**Autocorrelation function** of the process \( \{X_t, t \in T\} \) with positive variances is defined by

\[
r(s, t) = \frac{R(s, t)}{\sqrt{R(s, s)}\sqrt{R(t, t)}}, \quad s, t \in T.
\]

Definition:

Stochastic process \( \{X_t, t \in T\} \) is called **Gaussian**, if for any \( n \in \mathbb{N} \) and \( t_1, \ldots, t_n \in T \), the vector \((X_{t_1}, \ldots, X_{t_n})^\top\) is normally distributed \( \mathcal{N}_n(m_t, V_t) \), where \( m_t = (E X_{t_1}, \ldots, E X_{t_n})^\top \) and

\[
V_t = \begin{pmatrix}
\text{var} X_{t_1} & \text{cov}(X_{t_1}, X_{t_2}) & \ldots & \text{cov}(X_{t_1}, X_{t_n}) \\
\text{cov}(X_{t_2}, X_{t_1}) & \text{var} X_{t_2} & \ldots & \text{cov}(X_{t_2}, X_{t_n}) \\
\ldots & \ldots & \ldots & \ldots \\
\text{cov}(X_{t_n}, X_{t_1}) & \text{cov}(X_{t_n}, X_{t_2}) & \ldots & \text{var} X_{t_n}
\end{pmatrix}.
\]
Definition:
Stochastic process \( \{X_t, \ t \in T\} \) is said to be **strictly stationary**, if for any \( n \in \mathbb{N} \), for any \( x_1, \ldots, x_n \) real and for any \( t_1, \ldots, t_n \) a \( h \) such that \( t_k \in T, t_k + h \in T, 1 \leq k \leq n \),

\[
F_{t_1,\ldots,t_n}(x_1,\ldots,x_n) = F_{t_1+h,\ldots,t_n+h}(x_1,\ldots,x_n).
\]

Definition:
Stochastic process \( \{X_t, \ t \in T\} \) with finite second moments is said to be **weakly stationary** or **second order stationary**, if its mean value is constant, \( \mu_t = \mu, \ \forall t \in T \) and if its autocovariance function \( R(s, t) \) is a function of \( s - t \), only. If only the latter condition is satisfied, the process is called **covariance stationary**.
Autocovariance function of weakly stationary process:

\[ R(t) := R(t, 0), \ t \in T, \]

(function of one variable).

Autocorrelation function in such case:

\[ r(t) = \frac{R(t)}{R(0)}. \]

**Theorem 2:**

Strictly stationary stochastic process \( \{X_t, \ t \in T\} \) with finite second moments is also weakly stationary.

**Proof:**

\( \{X_t, \ t \in T\} \) strictly stationary \( \Rightarrow \) \( X_t \) are equally distributed for all \( t \in T \) and thus with the mean value

\[ EX_t = EX_{t+h}, \ \forall t \in T, \ \forall h : t + h \in T \]

especially, for \( h = -t \):

\[ EX_t = EX_0 = \text{const} \]
Proof of Theorem 2, continued

Similarly, \((X_t, X_s)\) are equally distributed and

\[
E[X_t X_s] = E[X_{t+h} X_{s+h}] \ \forall s, t \in T, \forall h : s + h \in T, t + h \in T
\]
especially, for \(h = -t\):

\[
E[X_t X_s] = E[X_0 X_{s-t}]
\]
is a function of \(s - t\). \[\Box\]
Example: 
\{X_t, \ t \in T\} - a sequence of iid random variables with a distribution function \(F\)

\[F_{t_1,\ldots,t_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \Pr[X_{t_i} \leq x_i] = \prod_{i=1}^{n} F(x_i),\]

\[F_{t_1+h,\ldots,t_n+h}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \Pr[X_{t_i+h} \leq x_i] = \prod_{i=1}^{n} F(x_i),\]

\[\Rightarrow \{X_t, \ t \in T\} \text{ is strictly stationary.}\]
**Example:**

\{X_t, t \in \mathbb{Z}\} - a sequence defined by

\[ X_t = (-1)^t X, \]

where \( X \) is a random variable:

\[ X = \begin{cases} 
-\frac{1}{4} & \text{with probability } \frac{3}{4}, \\
\frac{3}{4} & \text{with probability } \frac{1}{4}. 
\end{cases} \]

Then \( \{X_t, t \in \mathbb{Z}\} \) is weakly stationary, since

\[ E[X_t] = 0, \]

\[ \text{var } X_t = \sigma^2 = \frac{3}{16}, \]

\[ R(s, t) = \sigma^2(-1)^{s+t} = \sigma^2(-1)^{s-t}, \]

but it is not strictly stationary (variables \( X \) and \( -X \) are not equally distributed).
Theorem 3:
A weakly stationary Gaussian process \(\{X_t, t \in T\}\) is also strictly stationary.

Proof:
Weak stationarity of the process \(\{X_t, t \in T\}\) implies
\[
E(X_t) = \mu, \quad \text{cov}(X_t, X_s) = R(t - s) = \text{cov}(X_{t+h}, X_{s+h}), t, s \in T,
\]
thus
\[
E(X_{t_1}, \ldots, X_{t_n}) = E(X_{t_1+h}, \ldots, X_{t_n+h}) = (\mu, \ldots, \mu) := \mu
\]
\[
\text{var}(X_{t_1}, \ldots, X_{t_n}) = \text{var}(X_{t_1+h}, \ldots, X_{t_n+h}) := \Sigma
\]
\[ \Sigma = \begin{pmatrix} R(0) & R(t_2 - t_1) & \ldots & R(t_n - t_1) \\ R(t_2 - t_1) & R(0) & \ldots & R(t_n - t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & R(0) \end{pmatrix}. \]

Since the normal distribution is uniquely defined by the mean value vector and the variance matrix, \((X_{t_1}, \ldots, X_{t_n}) \sim \mathcal{N}(\mu, \Sigma)\), and \((X_{t_1+h}, \ldots, X_{t_n+h}) \sim \mathcal{N}(\mu, \Sigma)\) \(\Rightarrow\) \(\{X_t, t \in T\}\) is strictly stationary. \(\square\)
Properties of autocovariance function

**Theorem 4:**

Let \( \{X_t, t \in T\} \) be a process with finite second moments. Then its autocovariance function satisfies

\[
R(t, t) \geq 0, \\
|R(s, t)| \leq \sqrt{R(s, s)} \sqrt{R(t, t)}.
\]

**Proof:**

The first assertion follows from the definition of the variance. The second one follows from the Schwarz inequality, since

\[
|R(s, t)| = |E(X_s - EX_s)(\overline{X_t} - \overline{EX_t})| \leq E|(X_s - EX_s)(\overline{X_t} - \overline{EX_t})| \\
\leq (E|X_s - EX_s|^2)^{\frac{1}{2}}(E|X_t - EX_t|^2)^{\frac{1}{2}} = \sqrt{R(s, s)} \sqrt{R(t, t)}
\]

Thus, for weakly stationary process \( R(0) \geq 0 \) a \( |R(t)| \leq R(0) \).
**Definition:**

Let $f(s, t)$ be a complex-valued function defined on $T \times T$, $T \subset \mathbb{R}$. We say that $f$ is **positive semidefinite**, sometimes: **non-negative definite**, if $\forall n \in \mathbb{N}$, any complex numbers $c_1, \ldots, c_n$ and any $t_1, \ldots, t_n \in T$ it holds

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} f(t_j, t_k) \geq 0.
$$

We say that a complex-valued function $g$ on $T$ is **positive semidefinite**, if $\forall n \in \mathbb{N}$, any complex numbers $c_1, \ldots, c_n$ and any $t_1, \ldots, t_n \in T$, such that $t_j - t_k \in T$, it holds

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} g(t_j - t_k) \geq 0.
$$
**Definition:**
We say that a complex-valued function $f$ on $T \times T$ is **Hermitian**, if $f(s, t) = \overline{f(t, s)} \ \forall s, t \in T$. A complex-valued function $g$ of one variable is called **Hermitian**, if $g(-t) = \overline{g(t)} \ \forall t \in T$.

**Theorem 5:**
Any positive semidefinite function is also Hermitian.
Proof:
Use the definition of positive semidefiniteness and for $n = 1$ choose $c_1 = 1$; for $n = 2$ choose $c_1 = 1, c_2 = 1$ and $c_1 = 1, c_2 = i (= \sqrt{-1})$.

**Remark:**
A positive semidefinite real-valued function $f$ on $T \times T$, is symmetric, i.e., $f(s, t) = f(t, s)$ for all $s, t \in T$. A positive semidefinite real-valued function $g$ on $T$ is symmetric, i.e, $g(t) = g(-t)$ for all $t \in T$. 


**Theorem 6:**

Let \( \{X_t, t \in T\} \) be a process with finite second moments. Then its autocovariance function is positive semidefinite on \( T \times T \).

**Proof:**

Suppose wlog that the process is centred. Then for any \( n \in \mathbb{N} \), complex constants \( c_1, \ldots, c_n \) and \( t_1, \ldots, t_n \in T \)

\[
0 \leq \mathbb{E} \left| \sum_{j=1}^{n} c_j X_{t_j} \right|^2 = \mathbb{E} \left[ \sum_{j=1}^{n} c_j X_{t_j} \sum_{k=1}^{n} c_k X_{t_k} \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} \mathbb{E}(X_{t_j} X_{t_k}) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} R(t_j, t_k).
\]
Theorem 7:
To any positive semidefinite function $R$ on $T \times T$ there exists a stochastic process $\{X_t, t \in T\}$ with finite second moments such that its autocovariance function is $R$.

Proof:
The proof will be given for real-valued function $R$, only. For the proof in a complex case see, e.g., Loève (1955), Chap. X, Par. 34.

Since $R$ is positive semidefinite, then for any $n \in \mathbb{N}$ and any real $t_1, \ldots, t_n \in T$ the matrix

$$
V_t = \begin{pmatrix}
R(t_1, t_1) & R(t_1, t_2) & \cdots & R(t_1, t_n) \\
R(t_2, t_1) & R(t_2, t_2) & \cdots & R(t_2, t_n) \\
\vdots & \vdots & \ddots & \vdots \\
R(t_n, t_1) & R(t_n, t_2) & \cdots & R(t_n, t_n)
\end{pmatrix}
$$

is positive semidefinite.
Proof of Theorem 7, continued

Function

\[ \varphi(u) = \exp \left\{-\frac{1}{2} u^\top V_t u \right\}, \quad u \in \mathbb{R}^n \]

is the characteristic function of the normal distribution \( \mathcal{N}_n(0, V_t) \). In this way, \( \forall n \in \mathbb{N} \) and any real \( t_1, \ldots, t_n \in T \) we get a consistent system of characteristic functions. The corresponding system of the distribution functions is also consistent. Thus according to Daniell-Kolmogorov theorem (Theorem 1) there exists a Gaussian stochastic process, covariances of which are given the values of the function \( R(s, t) \); hence, function \( R \) is the autocovariance function of this process.

\[ \square \]
**Example:**

Decide whether function $\cos t$, $t \in T = (-\infty, \infty)$ is an autocovariance function of a stochastic process.

**Solution:**

It suffices to show, that $\cos t$ is a positive semidefinite function. Consider $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ a $t_1, \ldots, t_n \in \mathbb{R}$. Then we have

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k \cos(t_j - t_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k (\cos t_j \cos t_k + \sin t_j \sin t_k)
$$

$$
= \left| \sum_{j=1}^{n} c_j \cos t_j \right|^2 + \left| \sum_{k=1}^{n} c_k \sin t_k \right|^2 \geq 0.
$$

Function $\cos t$ is positive semidefinite, and according to Theorem 6 there exists a (Gaussian) stochastic process $\{X_t, t \in T\}$, autocovariance function of which is $R(s, t) = \cos(s - t)$. 

Theorem 8:
The sum of two positive semidefinite functions is a positive semidefinite function.

Proof:
It follows from the definition of the positive semidefinite function. If \( f \) and \( g \) are positive semidefinite and \( h = f + g \), then for any \( n \in \mathbb{N} \), complex \( c_1, \ldots, c_n \) and \( t_1, \ldots, t_n \in T \)

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k h(t_j, t_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k [f(t_j, t_k) + g(t_j, t_k)]
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k f(t_j, t_k) + \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k g(t_j, t_k) \geq 0.
\]
**Corollary:**
Sum of two autocovariance functions is an autocovariance function of a stochastic process with finite second moments.

**Proof:**
It follows from Theorems 6 - 8.

**Theorem 9:**
The real part of an autocovariance function is an autocovariance function. The imaginary part is an autocovariance function (ACF) if and only if it is zero.

**Proof:**
Wlog, we prove the assertion for centred processes only. If $X_t = Y_t + iZ_t$ is complex with zero mean, then $EY_t = EZ_t = 0$ and $R(s, t) = EX_s X_t = E[(Y_s + iZ_s)(Y_t - iZ_t)] = EY_s Y_t + EZ_s Z_t + i(EZ_s Y_t - EY_s Z_t)$. The real part is an autocovariance function according to the previous Corollary. If imaginary part is zero, it is an ACF. If the imaginary part is ACF, then, since for $s = t$ (the variance in $s$) it is zero, the imaginary part must be zero.