On the Uniqueness of the Solution and Finite-dimensional Attractors for the 3D Flow with Dynamic Slip Boundary Condition*

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Abstract

We consider 3D incompressible non-Newtonian fluid, subject to a dynamic boundary condition. Using an iteration scheme in Nikolski-Bochner spaces, we obtain additional fractional time regularity of arbitrary weak solution, provided the power-law exponent is above the critical value $r = 11/5$. This implies uniqueness of solutions. We also show existence of the global attractor and even a finite-dimensional exponential attractor.

Keywords: non-Newtonian fluid, uniqueness of solutions, dynamic boundary condition, exponential attractor

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1 Introduction

We consider a general class of incompressible, non-Newtonian fluids of power-law type (also called Ladyzhenskaya model), confined to a bounded 3D domain. Such problems have been extensively studied before; however, mostly in the setting of a homogenenous Dirichlet boundary condition, or still analytically simpler, but physically not very realistic periodic boundary condition. There is certainly a recent surge of interest in more exotic boundary setting, motivated by the study of fluid-structure interaction problem on the one hand, and more generally related to the systems whose boundary is open to exchange of the mass, energy, or forces with the exterior.

In the present paper, we focus on a rather general class of the so-called dynamic slip boundary conditions

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{n} &= 0 \\ \beta \partial_t \boldsymbol{u} + \boldsymbol{s} &= -\big[\boldsymbol{S}\boldsymbol{n}\big]_\tau \end{aligned}$$

where $\boldsymbol{n}$ is the outer normal, τ the tangential projection, $\boldsymbol{S}$ the Cauchy stress and β some positive constant. The quantity $\boldsymbol{s}$ has a non-linear, possibly even implicit relation to $\boldsymbol{u}$. Note that a number of well-known boundary conditions can be obtained as a special case. For example, $\beta = 0$ and $\boldsymbol{s} = \alpha \boldsymbol{u}$ is the Navier slip, and $\alpha \to +\infty$ leads to the Dirichlet boundary condition.

We remark that related problems have been studied before, as for example the so-called Tresca boundary condition (which includes a maximal monotone relation in the form of a convex subdifferential), see [14]. Systems involving the Cauchy stress on the boundary have been extensively studied by [2] and the references therein. The main novelty of the present paper is the explicit presence of the time derivative, and hence nonlinear dynamics taking place on the boundary. This in particular requires an extended mathematical setting, which has been

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recently developed in [1], see also [11]. Basic existence and uniqueness results in the class of weak solutions were obtained in the last two mentioned works.

Our aim here is to extend this analysis in two ways. Firstly, we use the approach of [7], to obtain additional time regularity of arbitrary weak solution, provided that the power-law exponent r stands above the critical value 11/5. Unlike most regularity techniques, this approach works with time differences only and is thus largely independent of the boundary setting. The consequence is well-posedness in the class of weak solutions in the supercritical range $r \geq 11/5$. The case $r = 11/5$ can also be included, using a delicate argument based on the reverse Hölder inequality, see [8]. Note also that for $r \geq 12/5$, the time derivative becomes an admissible test function, whence the improved regularity can be obtained in straightforward way, together with reasonable explicit estimates [5].

As a natural corollary, we establish existence of the global attractor. Assuming further that the boundary nonlinearity $\boldsymbol{s}$ is represented as a function of polynomial growth in $\boldsymbol{u}$, we show that the attractor is finite-dimensional. Exponential attractor can then also be constructed. Let us emphasize here that the mere existence of attractor can be obtained even if $\boldsymbol{s}$ and $\boldsymbol{u}$ are related via an maximal monotone graph; on the other hand, it seems that its finite-dimensionality requires explicit (though rather general) functional relation $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{u})$. Resolving the problem (perhaps in terms of an explicit counterexample) is a problem to be addressed in our future research.

The paper is organized as follows. Section 1 defines the studied system, and collects necessary mathematical preliminaries about the function spaces and their properties. Our main results: Theorem 1.1 (on time regularity and uniqueness) and Theorem 1.2 (on global and exponential attractors) are formulated. Concepts from the abstract theory of dynamical systems are also recalled.

In section 2, we introduce the concept of weak solution, and derive basic a priori estimates. Continuous dependency on the data (a weak-strong uniqueness type result) is also established here.

Section 3 is the main technical part of the paper. The improved time regularity of weak solution is obtained by iterative estimates in Nikolskii spaces, completing in particular the proof of Theorem 1.1.

The final Section 4 is devoted to the large time dynamics. We recall a general abstract scheme of the so-called method of ℓ-trajectories given in [12], and show how the previous analysis leads to Theorem 1.2.

1.1 Problem formulation

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We employ small boldfaced letters to denote vectors and bold capitals for tensors. The symbols "$\cdot$" and "$:$" stand for the scalar product of vectors and tensors, respectively and "$\otimes$" means the tensor product. Outward unit normal vector is denoted by $\boldsymbol{n}$ and for any vector-valued function $\boldsymbol{x} : \partial\Omega \to \mathbb{R}^3$, the symbol $\boldsymbol{x}_\tau$ stands for the projection to the tangent plane, i.e. $\boldsymbol{x}_\tau = \boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{n})\boldsymbol{n}$.

Standard differential operators, like gradient (∇), or divergence (div), are always related to the spatial variables only. By $\boldsymbol{D}\boldsymbol{u}$ we understand the symmetric gradient of the velocity field, i.e. $2\boldsymbol{D}\boldsymbol{u} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top$. We denote the trace of Sobolev functions as the original function, and if we want to emphasize it, we use the symbol "tr". Generic constants, that depend just on data, are denoted by c or C and may vary from line to line.

Our problem is the following. Let $\boldsymbol{f} : (0,T) \times \Omega \to \mathbb{R}^3$ is a given external force and $\boldsymbol{u}_0 : \overline{\Omega} \to \mathbb{R}^3$ is the initial velocity. We are looking for the velocity field $\boldsymbol{u} : (0,T) \times \overline{\Omega} \to \mathbb{R}^3$ and the pressure $p : (0,T) \times \Omega \to \mathbb{R}$ solutions to the generalized Navier-Stokes system

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } (0,T) \times \Omega, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } (0,T) \times \Omega, \tag{2}$$

completed by the boundary and initial conditions

$$\beta\partial_t \boldsymbol{u} + \boldsymbol{s} = -[\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u})\boldsymbol{n}]_\tau \quad \text{on } (0,T) \times \partial\Omega, \tag{3}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } (0,T) \times \partial\Omega, \tag{4}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \overline{\Omega}, \tag{5}$$

By $\boldsymbol{S} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ we understand the viscous part of the Cauchy stress such that $\boldsymbol{S}(\boldsymbol{0}) = \boldsymbol{0}$ which moreover satisfy for any $\boldsymbol{D}_1, \boldsymbol{D}_2 \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ the coercivity conditions

$$(\boldsymbol{S}(\boldsymbol{D}_1) - \boldsymbol{S}(\boldsymbol{D}_2)) : (\boldsymbol{D}_1 - \boldsymbol{D}_2) \geq \begin{cases} c_1|\boldsymbol{D}_1 - \boldsymbol{D}_2|^2 + c_1|\boldsymbol{D}_1 - \boldsymbol{D}_2|^r, \\ c_1(1 + |\boldsymbol{D}_1|^{r-2} + |\boldsymbol{D}_2|^{r-2})|\boldsymbol{D}_1 - \boldsymbol{D}_2|^2, \end{cases} \tag{6}$$

and the growth condition

$$|\boldsymbol{S}(\boldsymbol{D}_1) - \boldsymbol{S}(\boldsymbol{D}_2)| \leq c_2(1 + |\boldsymbol{D}_1|^{r-2} + |\boldsymbol{D}_2|^{r-2})|\boldsymbol{D}_1 - \boldsymbol{D}_2|. \tag{7}$$

The boundary term $\boldsymbol{s}$ is connected with $\boldsymbol{u}$ via the constitutive relation

$$(\boldsymbol{s}, \boldsymbol{u}) \in \mathcal{G}, \tag{8}$$

with $\mathcal{G}$ being a maximal monotone 2-graph. It means that $\mathcal{G} \subset \mathbb{R}^3 \times \mathbb{R}^3$ and there hold four conditions, namely

(G1) $(\boldsymbol{0}, \boldsymbol{0}) \in \mathcal{G}$.

(G2) For any $(\boldsymbol{s}_1, \boldsymbol{u}_1), (\boldsymbol{s}_2, \boldsymbol{u}_2) \in \mathcal{G}$:

$$(\boldsymbol{s}_1 - \boldsymbol{s}_2) : (\boldsymbol{u}_1 - \boldsymbol{u}_2) \geq 0.$$

(G3) If for some $(\boldsymbol{s}_1, \boldsymbol{u}_1) \in \mathbb{R}^3 \times \mathbb{R}^3$ and all $(\boldsymbol{s}_2, \boldsymbol{u}_2) \in \mathcal{G}$ there holds

$$(\boldsymbol{s}_1 - \boldsymbol{s}_2) : (\boldsymbol{u}_1 - \boldsymbol{u}_2) \geq 0,$$

then $(\boldsymbol{s}_1, \boldsymbol{u}_1) \in \mathcal{G}$.

(G4) There exists $c_3, c_4 \geq 0$ such that for all $(\boldsymbol{s}, \boldsymbol{u}) \in \mathcal{G}$ there holds

$$\boldsymbol{s} \cdot \boldsymbol{u} \geq c_3(|\boldsymbol{s}|^2 + |\boldsymbol{u}|^2) - c_4.$$

Moreover, to show that the attractor is finite-dimensional, we will introduce two other conditions. First of all, we require that

(G5) $\mathcal{G}$ is graph of a function $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{u})$ such that for some $q \geq 2$ and $c_4 > 0$

$$|\boldsymbol{s}(\boldsymbol{u}_1) - \boldsymbol{s}(\boldsymbol{u}_2)| \leq c_4 \left(1 + |\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}\right) |\boldsymbol{u}_1 - \boldsymbol{u}_2|. \tag{9}$$

In certain situations, we will also need an analogous lower-bound, i.e.

(G6) there exists $c_5 > 0$ such that

$$(\boldsymbol{s}(\boldsymbol{u}_1) - \boldsymbol{s}(\boldsymbol{u}_2)) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2) \geq c_5 \left(|\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}\right) |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2. \tag{10}$$

Note that these conditions allow for certain degeneracy of the function $\boldsymbol{s}(\boldsymbol{u})$ so that the derivative can vanish at some points or even intervals. This corresponds to horizontal components of the graph $\mathcal{G}$. On the other hand, vertical components of $\mathcal{G}$ are excluded if (G5) holds.

Remark. *By* data *we henceforth understand the domain Ω, right-hand side $\boldsymbol{f}$, initial condition $\boldsymbol{u}_0$, as well as the constants and exponents describing the growth of $\boldsymbol{S}$ and $\boldsymbol{s}$; in particular the exponents r and q.*

Further generalizations of our results are possible, involving boundary with external forcing, or some mix of various boundary conditions on different parts of $\partial\Omega$. Such modifications can be rather straightforward, after the function spaces and the (abstract) weak formulation are modified accordingly.

1.2 Function spaces

For a Banach space X over $\mathbb{R}$, its dual is denoted by X^* and $\langle x^*, x\rangle_X$ is the duality pairing. For $r \in [1,\infty]$ we denote $(L^r(\Omega), ||\cdot||_{L^r(\Omega)})$ and $(W^{1,r}(\Omega), ||\cdot||_{W^{1,r}(\Omega)})$ the Lebesgue and Sobolev spaces with corresponding norms. We often write just $||\cdot||_r$ or $||\cdot||_{1,r}$. The space of functions $\boldsymbol{u} : [0,T] \to X$ which are L^r integrable or (weakly) continuous with respect to time is denoted by $L^r(0,T;X)$, $C([0,T];X)$ or $C_{\mathrm{w}}([0,T];X)$ respectively.

To properly introduce the notion of a weak solution we need to pay close attention to boundary terms. Thus, we need more refined function spaces. We will follow the notation of [1, Chapter 3].

For $r \in (1,\infty)$ we introduce the spaces

$$\begin{aligned}
\mathcal{V} &:= \{(\boldsymbol{u},\boldsymbol{g}) \in \mathcal{C}^{0,1}(\overline{\Omega}) \times \mathcal{C}^{0,1}(\partial\Omega); \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ and } \boldsymbol{u} = \boldsymbol{g} \text{ on } \partial\Omega\},\\
V_r &:= \overline{\mathcal{V}}^{||\cdot||_{V_r}}, \text{ where } ||(\boldsymbol{u},\boldsymbol{g})||_{V_r} := ||\boldsymbol{u}||_{W^{1,r}(\Omega)} + ||\boldsymbol{u}||_{L^2(\Omega)} + ||\boldsymbol{g}||_{L^2(\partial\Omega)},\\
H &:= \overline{V_r}^{||\cdot||_H}, \text{ where } ||(\boldsymbol{u},\boldsymbol{g})||_H^2 := ||\boldsymbol{u}||^2_{L^2(\Omega)} + \beta||\boldsymbol{g}||^2_{L^2(\partial\Omega)}.
\end{aligned}$$

Space V_r is both reflexive and separable. Observe that, thanks to the Trace theorem, for $r = 2$ the norm on V_2 is equivalent to $||\cdot||_{1,2}$. Also, H is Hilbert space identified with its own dual H^*, with the inner product

$$((\tilde{\boldsymbol{u}},\boldsymbol{u}),(\tilde{\boldsymbol{g}},\boldsymbol{g}))_H := \int_\Omega \tilde{\boldsymbol{u}}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}x + \beta\int_{\partial\Omega} \tilde{\boldsymbol{g}}\cdot\overline{\boldsymbol{g}}\,\mathrm{d}S.$$

Moreover, for $r > 6/5$, $W^{1,r}(\Omega)$ is compactly embedded into $L^2(\Omega)$ and for $r > 3/2$, the trace operator is compact from $W^{1,r}(\Omega)$ into $L^2(\partial\Omega)$. Therefore,

$$V_r \hookrightarrow\hookrightarrow H \text{ if } r > \frac{3}{2}\ .$$

The duality pairing between V_r and V_r^* is defined in a standard way as a continuous extension of the inner product $(\cdot,\cdot)_H$ on H. As usual, we see that there is a Gelfand triplet

$$V_r \hookrightarrow H \equiv H^* \hookrightarrow V_r^*,$$

where both embeddings are continuous and dense.

We also briefly recall main properties of the so-called Nikolski spaces; for a detailed treatment, see [3] or [13]. For $\boldsymbol{u} : I \to X$, where $I \subset \mathbb{R}$ is a time interval and $h > 0$, we set

$$\begin{aligned}
I_h &= \{t \in I; t + h \in I\},\\
\tau^h\boldsymbol{u}(t) &= \boldsymbol{u}(t+h), \quad t \in I_h,\\
d^h\boldsymbol{u}(t) &= \boldsymbol{u}(t+h) - \boldsymbol{u}(t), \quad t \in I_h.
\end{aligned}$$

For $r \in [1,\infty]$ and $s \in [0,1]$, the Nikolskii space $N^{s,r}(I;X)$ is defined by the norm

$$||\boldsymbol{u}||_{L^r(I;X)} + \sup_{h>0} h^{-s}||d^h\boldsymbol{u}||_{L^r(I_h;X)}.$$

It is clear that for $s = 0$ the above norm is equivalent to $L^r(I;X)$ and similarly for $s = 1$ the norm is equivalent to $W^{1,r}(I;X)$. The following embedding can be obtained, see e.g. [13],

$$N^{s,r}(I;X) \hookrightarrow L^q(I;X) \quad \text{if} \quad \frac{1}{q} > \frac{1}{r} - s \geq 0. \tag{11}$$

Nikolskii spaces are not the best choice in view of interpolation or embedding properties. On the other hand, their definition is fairly simple and as we will see, it is rather straightforward to obtain estimates of the $N^{s,r}$-norm.

1.3 Dynamical systems

Our main goal is to show existence of a finite-dimensional (exponential) attractor. We recall some basic notions from the theory of dynamical systems. Let $\mathcal{X}$ be (a closed subset to) a normed space. Family of mappings $\{\Sigma_t\}_{t\geq 0} : \mathcal{X} \to \mathcal{X}$ is called a semigroup provided that $\Sigma_0 = I$ and $\Sigma_{t+s} = \Sigma_t \Sigma_s$ for all s, $t \geq 0$. Requiring also continuity of the map $(t, x) \mapsto \Sigma_t x$, the couple $(\Sigma_t, \mathcal{X})$ is referred to as a dynamical system.

Set $\mathcal{A} \subset \mathcal{X}$ is called a global attractor to the dynamical system $(\Sigma_t, \mathcal{X})$ if

(i) $\mathcal{A}$ is compact in $\mathcal{X}$,

(ii) $\Sigma_t \mathcal{A} = \mathcal{A}$ for all $t \geq 0$ and

(iii) for any bounded $\mathcal{B} \subset \mathcal{X}$ there holds

$$\operatorname{dist}(\Sigma_t \mathcal{B}, \mathcal{A}) \to 0 \text{ as } t \to \infty,$$

where $\operatorname{dist}(\mathcal{B}, \mathcal{A})$ is the standard Hausdorff semi-distance of the set $\mathcal{B}$ from the set $\mathcal{A}$, defined as $\operatorname{dist}(\mathcal{B}, \mathcal{A}) = \sup_{a\in\mathcal{A}} \inf_{b\in\mathcal{B}} ||b - a||_{\mathcal{X}}$.

Let us note that a dynamical system can have at most one global attractor. The condition (ii) says that the global attractor is (fully) invariant with respect to Σ_t.

Fractal dimension of a compact set $\mathcal{K} \subset \mathcal{X}$ is defined by

$$d_f^{\mathcal{X}}(\mathcal{K}) := \limsup_{\varepsilon\to 0_+} \frac{\log N_\varepsilon^{\mathcal{X}}(\mathcal{K})}{\log \frac{1}{\varepsilon}} ,$$

where $N_\varepsilon^{\mathcal{X}}(\mathcal{K})$ denotes the minimal number of ε-balls needed to cover the set $\mathcal{K}$.

Finally, we say that the set $\mathcal{E} \subset \mathcal{X}$ is an exponential attractor to $(\Sigma_t, \mathcal{X})$ if

(i) $\mathcal{E}$ is compact,

(ii) $\Sigma_t \mathcal{E} \subset \mathcal{E}$ for all $t \geq 0$,

(iii) $d_f^{\mathcal{X}}(\mathcal{E})$ is finite and

(iv) there exist $\sigma, \omega > 0$ such that for any $\mathcal{B} \subset \mathcal{X}$ bounded there exist $t_0 > 0$ such that $\operatorname{dist}(\Sigma_t \mathcal{B}, \mathcal{E}) \leq \omega e^{-\sigma t}$ for all $t \geq t_0$.

We note that the exponential attractor is not uniquely defined and necessarily, it contains the global attractor.

1.4 Main results

Main results of this article are summarized in the following two theorems. Their proofs will be given in Sections 3 and 4, respectively.

We remark that

$$\overline{r} := \frac{2r}{2r - 3} \tag{12}$$

is the critical integrability exponent which implies uniqueness in the class of weak solutions, cf. Theorem 2.2 below.

Theorem 1.1 (Regularity and uniqueness)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, $\boldsymbol{u}_0 \in H$, $r \geq 11/5$ and $\boldsymbol{f} \in L^{r'}(0, T; V_r^*)$. If $r \in (11/5, 5/2)$, assume moreover $\boldsymbol{f} \in N^{\delta,r'}(0, T; V_r^*)$ with some $\delta > \overline{\delta}$, where*

$$\overline{\delta} := (r - 1)\left(\frac{5}{2r} - 1\right).$$

Moreover, if $r = 11/5$ assume that $\boldsymbol{f} \in L^{q_0}(0, T; V_r^*)$ for some $q_0 > r'$.

Fix $t_0 < T/2$. Then for an arbitrary weak solution $\boldsymbol{u}$ of (1)–(8) we have $\boldsymbol{u} \in L^{\overline{r}}(2t_0, T; V_r)$. Moreover,

$$||\boldsymbol{u}||_{L^{\overline{r}}(2t_0,T;V_r)} \leq C_{reg},$$

where C_{reg} depends only on $\Omega, r, T, t_0, ||\boldsymbol{f}||_{L^{r'}(0,T;V_r^*)}, ||\boldsymbol{u}||_{L^\infty(0,T,H)}$ and $||\boldsymbol{u}||_{L^r(0,T,V_r)}$.

In particular, any weak solution has a uniquely determined continuation after arbitrary positive time.

Finally, if $\boldsymbol{u}_0 \in V_r$, the conclusion holds even for $t_0 = 0$.

Theorem 1.2 (Attractor)**.** *Assume that for any $T > 0$, $\boldsymbol{f}$ satisfies the assumptions of the previous theorem. Then the system (1)–(8) has a global attractor.*

Moreover, suppose that (9) holds and if $q > r/2 + 1$, we further assume that (10) holds too. Then the global attractor has finite fractal dimension, and there exists an exponential attractor.

1.5 Auxiliary inequalities

For reader's convenience, we summarize below some estimates that will be used throughout the paper.

Theorem 1.3 (Trace theorem)**.** *Let $r \in [1, \infty)$, then there exists unique, continuous and linear operator $\operatorname{tr} : W^{1,r}(\Omega) \to L^r(\partial\Omega)$ such that for all $\boldsymbol{u} \in C^1(\overline{\Omega})$ there holds $\operatorname{tr} \boldsymbol{u} = \boldsymbol{u}$ on $\partial\Omega$. In particular, there exists a constant $C_T > 0$ depending only on r and Ω such that for all $\boldsymbol{u} \in W^{1,r}(\Omega)$ there holds*

$$||\operatorname{tr} \boldsymbol{u}||_{L^r(\partial\Omega)} \leq C_T ||\boldsymbol{u}||_{1,r}.$$

Proof. See Theorem 6.4.3 in [10]. ■

Theorem 1.4 (Korn's inequality)**.** *Let Ω be a bounded Lipschitz domain, let $r \in (1, \infty)$. Then there exists a constant $C_K > 0$, depending only on Ω and r, such that for all $\boldsymbol{u} \in W^{1,r}(\Omega)$ which has $\operatorname{tr} \boldsymbol{u} \in L^2(\partial\Omega)$, the following inequalities hold*

$$||\boldsymbol{u}||_{1,r} \leq \begin{cases} C_K(||\boldsymbol{D}\boldsymbol{u}||_r + ||\operatorname{tr} \boldsymbol{u}||_{L^2(\partial\Omega)}) \\ C_K(||\boldsymbol{D}\boldsymbol{u}||_r + ||\boldsymbol{u}||_{L^2(\Omega)}) \end{cases}.$$

Proof. See Lemma 1.11 in [9]. ■

Theorem 1.5 (Interpolations)**.** *The following estimates hold true:*

(i) If $r \in [9/5, 3)$, then

$$||\boldsymbol{u}||_{2r'} \leq C||\boldsymbol{u}||_2^{\frac{5r-9}{5r-6}} ||\boldsymbol{u}||_{V_r}^{\frac{3}{5r-6}}. \tag{13}$$

(ii) If $r > 3$, then

$$||\boldsymbol{u}||_{2r'} \leq C||\boldsymbol{u}||_2^{\frac{2r-3}{2r}} ||\boldsymbol{u}||_{V_r}^{\frac{3}{2r}}. \tag{14}$$

(iii) Finally, for any $r \geq 2$, one has

$$||\boldsymbol{u}||_{2r'} \leq C||\boldsymbol{u}||_2^{\frac{2r-3}{2r}} ||\boldsymbol{u}||_{V_2}^{\frac{3}{2r}}. \tag{15}$$

Proof. The proof is a straightforward consequence of Hölder's inequality

$$||\boldsymbol{u}||_p \leq ||\boldsymbol{u}||_{p_1}^{\alpha} ||\boldsymbol{u}||_{p_2}^{1-\alpha} \quad \text{with} \quad \frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}, \alpha \in [0,1] \tag{16}$$

and Sobolev embeddings: in case (i), we use $\alpha = \frac{5r-9}{5r-6}$ and $W^{1,r} \hookrightarrow L^{\frac{3r}{3-r}}$. For (ii), the embedding $W^{1,r} \hookrightarrow L^6$ is used and $\alpha = \frac{2r-3}{2r}$. Finally, to prove (iii), one takes $\alpha = \frac{2r-3}{2r}$ together with $W^{1,2} \hookrightarrow L^6$. ■

Theorem 1.6 (Aubin-Lions-Simon)**.** *Let $r \in [1, \infty)$ and X_1, X_2, X_3 be Banach spaces such that*

$$X_1 \hookrightarrow\hookrightarrow X_2 \hookrightarrow X_3,$$

then

$$\{\boldsymbol{u} \in L^r(0, T; X_1); \partial_t \boldsymbol{u} \in L^1(0, T; X_3)\} \hookrightarrow\hookrightarrow L^r(0, T; X_2).$$

Proof. See Theorem II. 5. 16 in [4]. ■

2 Weak solution, existence and uniqueness

To properly define a notion of a weak solution to the problem (1)–(8), we start with formal derivation of a priori estimates. We take scalar product of (1) with an arbitrary smooth function $\boldsymbol{\varphi} \in V_r$, integrate over Ω and use Gauss's theorem to get

$$\int_\Omega \partial_t \boldsymbol{u} \cdot \boldsymbol{\varphi} - \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi} + \int_{\partial\Omega} (\boldsymbol{u} \otimes \boldsymbol{u})\boldsymbol{n} \cdot \boldsymbol{\varphi} + \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \nabla \boldsymbol{\varphi} - \int_{\partial\Omega} [\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u})\boldsymbol{n}]_\tau \cdot \boldsymbol{\varphi}$$
$$= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} - \int_\Omega p \operatorname{div} \boldsymbol{\varphi} + \int_{\partial\Omega} p\, \boldsymbol{n} \cdot \boldsymbol{\varphi}.$$

We realize that $\boldsymbol{n} \cdot \boldsymbol{\varphi} = 0$ on $\partial\Omega$ and $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω and the fact that $\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u})$ is symmetric matrix to obtain

$$\int_\Omega \partial_t \boldsymbol{u} \cdot \boldsymbol{\varphi} + \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{\varphi} - \int_{\partial\Omega} [\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u})\boldsymbol{n}]_\tau \cdot \boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} + \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi}.$$

Now, we employ the boundary condition (3) to get

$$\int_\Omega \partial_t \boldsymbol{u} \cdot \boldsymbol{\varphi} + \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{\varphi} + \int_{\partial\Omega} \beta \partial_t \boldsymbol{u} \cdot \boldsymbol{\varphi} + \boldsymbol{s} \cdot \boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} + \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi}.$$

For $\boldsymbol{f} \in V_r^*$ we identify the first integral on the right hand side with $\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{V_r}$. Using of the definition of H we finally obtain (formally) the equality

$$(\partial_t \boldsymbol{u}, \boldsymbol{\varphi})_H + \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{\varphi} + \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{\varphi} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{V_r} + \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi}. \tag{17}$$

To find the energy equality we set $\boldsymbol{\varphi} := \boldsymbol{u}$, the above equation reads now as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{u}||_H^2 + \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{u} + \int_{\partial\Omega} \boldsymbol{s} \cdot \boldsymbol{u} = \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{V_r} + \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{u}.$$

The last integral vanishes, as usual, because of (2) and (4). For the second term we use the r-coercivity of $\boldsymbol{S}$, i.e. (6) and for the third one we have condition (8), which enables us to use property (G4) from the definition of 2-graph. We obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{u}||_H^2 + c_1 \int_\Omega |\boldsymbol{D}\boldsymbol{u}|^2 + |\boldsymbol{D}\boldsymbol{u}|^r + c_3 \int_{\partial\Omega} |\boldsymbol{s}|^2 + |\boldsymbol{u}|^2 \le c_4 + \langle \boldsymbol{f}, \boldsymbol{u} \rangle_{V_r}.$$

Duality on the right hand side can be estimated as follows

$$\begin{aligned}
\langle \boldsymbol{f}, \boldsymbol{u} \rangle_{V_r} &\le ||\boldsymbol{f}||_{V_r^*} ||\boldsymbol{u}||_{V_r} = ||\boldsymbol{f}||_{V_r^*} (||\boldsymbol{u}||_{W^{1,r}(\Omega)} + ||\boldsymbol{u}||_{L^2(\Omega)} + ||\operatorname{tr} \boldsymbol{u}||_{L^2(\partial\Omega)}) \\
&\le ||\boldsymbol{f}||_{V_r^*} (c(||\boldsymbol{D}\boldsymbol{u}||_r + ||\operatorname{tr} \boldsymbol{u}||_{L^2(\partial\Omega)}) + ||\boldsymbol{u}||_{L^2(\Omega)} + ||\operatorname{tr} \boldsymbol{u}||_{L^2(\partial\Omega)}) \\
&\le c||\boldsymbol{D}\boldsymbol{u}||_r ||\boldsymbol{f}||_{V_r^*} + c||\boldsymbol{f}||_{V_r^*} ||\boldsymbol{u}||_H \\
&\le \varepsilon ||\boldsymbol{D}\boldsymbol{u}||_r^r + c_\varepsilon ||\boldsymbol{f}||_{V_r^*}^{r'} + c(1 + ||\boldsymbol{u}||_H^2) ||\boldsymbol{f}||_{V_r^*},
\end{aligned}$$

where we successively used the Korn's inequality, the trivial inequality $a + b \le \sqrt{2}\sqrt{a^2 + b^2}$ and Young's inequality twice. On the left hand side we can estimate, using the Korn's inequality and the Trace theorem, that

$$c_1 \int_\Omega |\boldsymbol{D}\boldsymbol{u}|^2 + c_3 \int_{\partial\Omega} |\boldsymbol{u}|^2 \ge c(||\boldsymbol{u}||_{1,2} + ||\operatorname{tr} \boldsymbol{u}||_{L^2(\partial\Omega)}) \ge c||\boldsymbol{u}||_H^2.$$

Altogether we get (for $\varepsilon > 0$ small enough)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{u}||_H^2 + \frac{c_1}{2}\int\limits_\Omega |\boldsymbol{Du}|^r + c_3\int\limits_{\partial\Omega} |\boldsymbol{s}|^2 + c||\boldsymbol{u}||_H^2 \le c_4 + c_\varepsilon ||\boldsymbol{f}||_{V_r^*}^{r'} + c(1+||\boldsymbol{u}||_H^2)||\boldsymbol{f}||_{V_r^*}.$$

This implies the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{u}||_H^2 + c||\boldsymbol{u}||_H^2 \le c_4 + c_\varepsilon ||\boldsymbol{f}||_{V_r^*}^{r'} + c||\boldsymbol{f}||_{V_r^*} + c||\boldsymbol{f}||_{V_r^*}||\boldsymbol{u}||_H^2,$$

from which follows, using of Grönwall's inequality, the uniform estimate

$$\sup_{t\in(0,T)} ||\boldsymbol{u}(t)||_H^2 \le C.$$

Integrating the preceding inequality over time and using assumption on data (in particular $\boldsymbol{f} \in L^{r'}(0,T;V_r^*), r' > 1$) and the uniform estimate above we obtain

$$\frac{c_1}{2}\int\limits_0^t\int\limits_\Omega |\boldsymbol{Du}|^r + c_3\int\limits_0^t\int\limits_{\partial\Omega} |\boldsymbol{s}|^2 + c\int\limits_0^t ||\boldsymbol{u}||_H^2 \le C.$$

By the Korn's inequality we obtain that

$$\boldsymbol{u} \in L^\infty(0,T;H) \cap L^r(0,T;V_r) \text{ and } \boldsymbol{s} \in L^2(0,T;L^2(\partial\Omega)).$$

Concerning the boundary force $\boldsymbol{s}$, we invoke (G4) and Young's inequality to obtain

$$c_3(|\boldsymbol{s}|^2 + |\boldsymbol{u}|^2) - c_4 \le \boldsymbol{s}\cdot\boldsymbol{u} \le \frac{c_3}{2}|\boldsymbol{s}|^2 + c|\boldsymbol{u}|^2.$$

This implies that

$$\boldsymbol{s} \in L^\infty(0,T;L^2(\partial\Omega)).$$

To estimate the time derivative we use duality argument and the relation (17)

$$\begin{aligned}
||\partial_t\boldsymbol{u}||_{V_r^*} &= \sup_{\boldsymbol{\varphi}} \langle\partial_t\boldsymbol{u},\boldsymbol{\varphi}\rangle_{V_r} \\
&= \sup_{\boldsymbol{\varphi}}\left[-\int\limits_\Omega \boldsymbol{S}(\boldsymbol{Du}):\boldsymbol{D\varphi} - \int\limits_{\partial\Omega}\boldsymbol{s}\cdot\boldsymbol{\varphi} + \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_{V_r} + \int\limits_\Omega(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\boldsymbol{\varphi}\right] \\
&\le \sup_{\boldsymbol{\varphi}}\Big[||\boldsymbol{S}(\boldsymbol{Du})||_{r'}||\boldsymbol{D\varphi}||_r + ||\boldsymbol{s}||_{L^2(\partial\Omega)}||\boldsymbol{\varphi}||_{L^2(\partial\Omega)} + ||\boldsymbol{f}||_{V_r^*}||\boldsymbol{\varphi}||_{V_r} + ||\boldsymbol{u}||_{2r'}^2||\nabla\boldsymbol{\varphi}||_r\Big],
\end{aligned}$$

where suprema are over all $\boldsymbol{\varphi} \in V_r$ from the unit ball. Therefore, by Korn's and Trace inequalities,

$$||\partial_t\boldsymbol{u}||_{V_r^*} \le ||\boldsymbol{S}(\boldsymbol{Du})||_{r'} + ||\boldsymbol{s}||_{L^2(\partial\Omega)} + ||\boldsymbol{f}||_{V_r^*} + ||\boldsymbol{u}||_{2r'}^2\ .$$

For the first term we use the growth condition and obtain

$$||\boldsymbol{S}(\boldsymbol{Du})||_{r'} \le c(1 + ||\boldsymbol{Du}||_r^{r-1}).$$

Suppose that $r \in [11/5, 3)$. We then interpolate the last term by (13) , i.e.

$$||\boldsymbol{u}||_{2r'}^2 \le ||\boldsymbol{u}||_2^{2(1-a)}||\boldsymbol{u}||_{1,r}^{2a}, \qquad \text{where } a = \frac{3}{5r-6}\ .$$

Because $\boldsymbol{u} \in L^\infty(0,T;H)$, $\boldsymbol{s} \in L^\infty(0,T;L^2(\partial\Omega))$ and $2a \le r-1$ (which is equivalent to $r \ge 11/5$), we obtain

$$||\partial_t\boldsymbol{u}||_{V_r^*} \le c(1 + ||\boldsymbol{u}||_{V_r}^{r-1} + ||\boldsymbol{f}||_{V_r^*}).$$

If $r = 3$, then $2r' = 3$ and so $||\boldsymbol{u}||_{2r'}^2 = ||\boldsymbol{u}||_r^{r-1} \le ||\boldsymbol{u}||_{V_r}^{r-1}$. Finally, assuming that $r > 3$, then we use (14), i.e.

$$||\boldsymbol{u}||_{2r'}^2 \le C||\boldsymbol{u}||_2^{\frac{2r-3}{r}}||\boldsymbol{u}||_{V_r}^{\frac{3}{r}}\ .$$

Observe that $\frac{3}{r} \le r-1$ and so, by the same argument as in the case $r < 3$, we conclude that

$$||\partial_t \boldsymbol{u}||_{V_r^*} \le c(1 + ||\boldsymbol{u}||_{V_r}^{r-1} + ||\boldsymbol{f}||_{V_r^*}) \quad \text{for} \quad r \ge \frac{11}{5}\ . \tag{18}$$

This in particular gives us that

$$\partial_t \boldsymbol{u} \in L^{r'}(0,T;V_r^*)$$

and together with $\boldsymbol{u} \in L^r(0,T;V_r)$ we obtain

$$\boldsymbol{u} \in \mathcal{C}([0,T];H).$$

Definition 1. *(Weak solution) We say that the couple $(\boldsymbol{u}, \boldsymbol{s})$ is a* ***weak solution*** *to the problem* (1)–(8) *if*

$$\begin{aligned} \boldsymbol{u} &\in L^r(0,T;V_r) \cap \mathcal{C}([0,T];H), \\ \partial_t \boldsymbol{u} &\in L^{r'}(0,T;V_r^*), \\ \boldsymbol{s} &\in L^\infty(0,T;L^2(\partial\Omega)) \end{aligned}$$

and for a. e. $t \in (0,T)$ satisfies relations

$$\partial_t \boldsymbol{u} + L(\boldsymbol{u}) = K_0(\boldsymbol{u}) + \boldsymbol{f} \quad \text{in } V_r^*, \tag{19}$$

$$(\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{G} \text{ a. e. on } \partial\Omega, \tag{20}$$

where

$$\begin{aligned} \langle L(\boldsymbol{u}), \boldsymbol{\varphi}\rangle_{V_r} &:= \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{\varphi} + \int_{\partial\Omega} \boldsymbol{s}\cdot\boldsymbol{\varphi}, \\ \langle K_0(\boldsymbol{u}), \boldsymbol{\varphi}\rangle_{V_r} &:= \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{\varphi}. \end{aligned}$$

The initial condition is attained strongly, i.e.

$$\lim_{t\to 0_+} ||\boldsymbol{u}(t) - \boldsymbol{u}_0||_H = 0.$$

We say that the solution satisfies the ***energy equality*** *if for all $t \in (0,T)$*

$$\frac{1}{2}||\boldsymbol{u}(t)||_H^2 + \int_0^t \int_\Omega \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}) : \boldsymbol{D}\boldsymbol{u} + \int_0^t \int_{\partial\Omega} \boldsymbol{s}\cdot\boldsymbol{u} = \frac{1}{2}||\boldsymbol{u}_0||_H^2 + \int_0^t \langle \boldsymbol{f}, \boldsymbol{u}\rangle_{V_r}\ . \tag{21}$$

Theorem 2.1 (Existence of weak solution)**.** *Let $\boldsymbol{u}_0 \in H$, $r \ge 11/5$ and $\boldsymbol{f} \in L^{r'}(0,T,V_r^*)$. Then there exists at least one weak solution satisfying the energy equality.*

Proof. We use formal estimates above on the level of Galerkin approximation. Usual compactness and monotonocity argument are used. For further details we refer e.g. to [1]. ■

Let us emphasize that the condition $r \ge 11/5$ guarantees that any weak solutions is an admissible test function, in particular, one always has the energy equality. This will be crucial for our regularity estimates in Section 3. On the other hand, the mere existence of a weak solution can be shown for $r > 6/5$, see [1].

Theorem 2.2 (Relative energy inequality)**.** *Let $(\boldsymbol{u}_1, \boldsymbol{s}_1), (\boldsymbol{u}_2, \boldsymbol{s}_2)$ be weak solutions to the problem* (1)–(8) *and let $r \ge \frac{11}{5}$. Then*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{w}||_H^2 + \frac{c_1}{2}(||\boldsymbol{D}\boldsymbol{w}||_2^2 + ||\boldsymbol{D}\boldsymbol{w}||_r^r) + c_1 \int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2})|\boldsymbol{D}\boldsymbol{w}|^2 \\ \le c\big(1 + ||\boldsymbol{u}_2||_{V_r}^{\overline{r}}\big)||\boldsymbol{w}||_H^2, \end{aligned} \tag{22}$$

where $\boldsymbol{w} := \boldsymbol{u}_1 - \boldsymbol{u}_2$, the constant c depends only on Ω, r and $\overline{r} = \frac{2r}{2r-3}$.

In particular, if $\boldsymbol{u}_2 \in L^{\overline{r}}(0,T;V_r)$, then $\boldsymbol{u}_2$ is unique in the class of weak solutions.

Proof. We take the difference of (19) for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and use $\boldsymbol{\varphi} := \boldsymbol{w}$ as a test function, which can be done, because $r \geq 11/5$. We get

$$\langle \partial_t \boldsymbol{w}, \boldsymbol{w}\rangle_{V_r} + \int\limits_{\Omega} (\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_1) - \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_2)) : \boldsymbol{D}\boldsymbol{w} + \int\limits_{\partial\Omega} (\boldsymbol{s}_1 - \boldsymbol{s}_2)\cdot \boldsymbol{w} = \int\limits_{\Omega} (\boldsymbol{u}_1 \otimes \boldsymbol{u}_1 - \boldsymbol{u}_2 \otimes \boldsymbol{u}_2) : \nabla \boldsymbol{w}.$$

The third term is non-negative thanks to the monotonicity, i.e. property (G2) of our 2-graph. We know that the first term is equal to $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{w}||_H^2$, for the second one we use r-coercivity of $\boldsymbol{S}$, i.e.

$$\int\limits_{\Omega} (\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_1) - \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_2)) : \boldsymbol{D}\boldsymbol{w} \geq \frac{c_1}{2}(||\boldsymbol{D}\boldsymbol{w}||_2^2 + ||\boldsymbol{D}\boldsymbol{w}||_r^r) + \frac{c_1}{2}\int\limits_{\Omega}(1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2})|\boldsymbol{D}\boldsymbol{w}|^2.$$

We need to deal with the convective term:

$$\begin{aligned}\left|\int\limits_{\Omega} (\boldsymbol{u}_1 \otimes \boldsymbol{u}_1 - \boldsymbol{u}_2 \otimes \boldsymbol{u}_2) : \nabla \boldsymbol{w}\right| &= \left| - \int\limits_{\Omega} \nabla \boldsymbol{u}_2 : (\boldsymbol{w}\otimes\boldsymbol{w})\right| \leq ||\nabla \boldsymbol{u}_2||_r ||\boldsymbol{w}||_{2r'}^2 \\ &\leq c||\boldsymbol{u}_2||_{V_r} ||\boldsymbol{w}||_2^{\frac{2r-3}{r}} ||\boldsymbol{w}||_{1,2}^{\frac{3}{r}} \\ &\leq c||\boldsymbol{u}_2||_{V_r} ||\boldsymbol{w}||_2^{\frac{2r-3}{r}} (||\boldsymbol{D}\boldsymbol{w}||_2 + ||\boldsymbol{w}||_H)^{\frac{3}{r}} \\ &\leq c||\boldsymbol{u}_2||_{V_r} ||\boldsymbol{w}||_2^{\frac{2r-3}{r}} (||\boldsymbol{D}\boldsymbol{w}||_2^{\frac{3}{r}} + ||\boldsymbol{w}||_H^{\frac{3}{r}}) \\ &\leq \varepsilon ||\boldsymbol{D}\boldsymbol{w}||_2^2 + c_\varepsilon\big(1 + ||\boldsymbol{u}_2||_{V_r}^{\overline{r}}\big)||\boldsymbol{w}||_H^2 .\end{aligned}$$

where we used the interpolation (15), the definition of V_2, Jensen's inequality (if $3/r > 1$, otherwise it is a trivial estimate) and Young's inequality. Hence (22) follows.

Concerning the second part, we apply Grönwall's lemma to obtain

$$||\boldsymbol{w}(t)||_H^2 \leq K||\boldsymbol{w}(s)||\,, \qquad 0 \leq s \leq t \leq T, \tag{23}$$

where K depends on the norm of $\boldsymbol{u}_2$ in $L^{\overline{r}}(0,T;V_r)$. ∎

3 Time regularity

The aim of this Section is to prove Theorem 1.1, i.e. improved local time $\overline{r}$-integrability of an arbitrary weak solution. Note that for $r \geq 5/2$, one has $r \geq \overline{r}$. Furthermore, for $r > 12/5$, one can test the equation by the time derivative (or time difference).

Hence, the main interest of the following approach lies in the low regularity regime $r \in [11/5, 12/5]$.

3.1 Auxiliary estimates

Lemma 3.1. *Let $(\boldsymbol{u}, \boldsymbol{s})$ be a weak solution of (1)–(8), $r \geq 11/5$, $\boldsymbol{f} \in L^{r'}(0,T;V_r^*)$. Assume that $\tau \in (0,T)$ is a semi-Lebesgue point of $\boldsymbol{f}$, i.e.*

$$\sup_{h\in(0,T-\tau)} \frac{1}{h}\int\limits_{\tau}^{\tau+h} ||\boldsymbol{f}(t)||_{V_r^*}^{r'}\,\mathrm{d}t < \infty$$

and $\boldsymbol{u}(\tau) \in V_r$. Then there exists a constant $C > 0$ such that for all $h \in (0, T-\tau)$:

$$||\boldsymbol{u}(\tau+h) - \boldsymbol{u}(\tau)||_H^2 \leq Ch\left(1 + ||\boldsymbol{u}(\tau)||_{V_r}^r + \frac{1}{h}\int\limits_{\tau}^{\tau+h} ||\boldsymbol{f}||_{V_r^*}^{r'}\right).$$

Proof. There holds

$$\begin{aligned}||\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau)||_H^2 &= (\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau),\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau))_H\\ &= (\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau),\boldsymbol{u}(\tau+h)+\boldsymbol{u}(\tau)-2\boldsymbol{u}(\tau))_H\\ &= ||\boldsymbol{u}(\tau+h)||_H^2-||\boldsymbol{u}(\tau)||_H^2-2(\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau),\boldsymbol{u}(\tau))_H.\end{aligned}$$

Due to the energy equality we have

$$||\boldsymbol{u}(\tau+h)||_H^2-||\boldsymbol{u}(\tau)||_H^2=\int_\tau^{\tau+h}\left(\langle\boldsymbol{f},\boldsymbol{u}\rangle_{V_r}-\int_\Omega\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}):\boldsymbol{D}\boldsymbol{u}-\int_{\partial\Omega}\boldsymbol{s}\cdot\boldsymbol{u}\right)$$

and we will apply again Hölder's, Korn's and Young's inequalities together with (6) and (8) to obtain

$$\begin{aligned}||\boldsymbol{u}(\tau+h)||_H^2-||\boldsymbol{u}(\tau)||_H^2 &\le \int_\tau^{\tau+h}(c_\varepsilon||\boldsymbol{f}||_{V_r^*}^{r'}+\varepsilon||\boldsymbol{D}\boldsymbol{u}||_r^r+\varepsilon||\boldsymbol{u}||_H^r)-c_1\int_\tau^{\tau+h}(||\boldsymbol{D}\boldsymbol{u}||_r^r+||\boldsymbol{D}\boldsymbol{u}||_2^2)\\ &\quad -c_3\int_\tau^{\tau+h}(||\boldsymbol{s}||_{L^2(\partial\Omega)}^2+||\boldsymbol{u}||_{L^2(\partial\Omega)}^2-c_4)\\ &\le c_3c_4h+c\int_\tau^{\tau+h}||\boldsymbol{f}||_{V_r^*}^{r'}-\frac{c_1}{2}\int_\tau^{\tau+h}||\boldsymbol{D}\boldsymbol{u}||_r^r\\ &\quad +\varepsilon c\int_\tau^{\tau+h}||\boldsymbol{u}||_H^2-C\int_\tau^{\tau+h}(||\boldsymbol{D}\boldsymbol{u}||_2^2+||\boldsymbol{u}||_{L^2(\partial\Omega)}^2).\end{aligned}$$

Now, because of Korn's inequality we have

$$\varepsilon c\int_\tau^{\tau+h}||\boldsymbol{u}||_H^2-C\int_\tau^{\tau+h}(||\boldsymbol{D}\boldsymbol{u}||_2^2+||\boldsymbol{u}||_{L^2(\partial\Omega)}^2)\le\varepsilon c\int_\tau^{\tau+h}||\boldsymbol{u}||_H^2-C\int_\tau^{\tau+h}||\boldsymbol{D}\boldsymbol{u}||_{V_2}^2,$$

and therefore, for $\varepsilon>0$ sufficiently small,

$$||\boldsymbol{u}(\tau+h)||_H^2-||\boldsymbol{u}(\tau)||_H^2\le C\left(h+\int_\tau^{\tau+h}||\boldsymbol{f}||_{V_r^*}^{r'}\right)-c\int_\tau^{\tau+h}(||\boldsymbol{D}\boldsymbol{u}||_r^r+||\boldsymbol{D}\boldsymbol{u}||_{V_2}^2).$$

Furthermore, observe that

$$-2(\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau),\boldsymbol{u}(\tau))_H=-2\int_\tau^{\tau+h}\langle\partial_t\boldsymbol{u}(s),\boldsymbol{u}(\tau)\rangle_{V_r}\,\mathrm{d}s$$

and thanks to (18) we get

$$|-2(\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau),\boldsymbol{u}(\tau))_H|\le 2c\int_\tau^{\tau+h}(1+||\boldsymbol{u}(s)||_{V_r}^{r-1}+||\boldsymbol{f}(s)||_{V_r^*})||\boldsymbol{u}(\tau)||_{V_r}\,\mathrm{d}s.$$

By Young's and Korn's inequalities, we have

$$
\begin{aligned}
&\big| -2(\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau),\boldsymbol{u}(\tau))_H\big| \\
&\qquad \le C\int_\tau^{\tau+h} (1+||\boldsymbol{u}(\tau)||_{V_r}^r+||\boldsymbol{f}(s)||_{V_r^*}^{r'})\,\mathrm{d}s+\varepsilon\int_\tau^{\tau+h} ||\boldsymbol{u}(s)||_{V_r}^{r'(r-1)}\,\mathrm{d}s \\
&\qquad \le Ch(1+||\boldsymbol{u}(\tau)||_{V_r}^r)+C\int_\tau^{\tau+h} ||\boldsymbol{f}||_{V_r^*}^{r'}+\varepsilon\int_\tau^{\tau+h} ||\boldsymbol{u}||_{V_r}^r \\
&\qquad \le Ch(1+||\boldsymbol{u}(\tau)||_{V_r}^r)+C\int_\tau^{\tau+h} ||\boldsymbol{f}||_{V_r^*}^{r'}+\varepsilon\int_\tau^{\tau+h} (||\boldsymbol{D}\boldsymbol{u}||_r^r+||\boldsymbol{u}||_H^r) \\
&\qquad \le Ch(1+||\boldsymbol{u}(\tau)||_{V_r}^r)+C\int_\tau^{\tau+h} ||\boldsymbol{f}||_{V_r^*}^{r'}+\varepsilon\int_\tau^{\tau+h} ||\boldsymbol{D}\boldsymbol{u}||_r^r+\varepsilon c\int_\tau^{\tau+h} ||\boldsymbol{u}||_H^2,
\end{aligned}
$$

where we used in the last inequality that $\boldsymbol{u}\in L^\infty(0,T;H)$. Altogether

$$
\begin{aligned}
||\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau)||_H^2 &\le ||\boldsymbol{u}(\tau+h)||_H^2-||\boldsymbol{u}(\tau)||_H^2+|2(\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau),\boldsymbol{u}(\tau))_H| \\
&\le C\left(h+h||\boldsymbol{u}(\tau)||_{V_r}^r+\int_\tau^{\tau+h} ||\boldsymbol{f}||_{V_r^*}^{r'}\right)-c\int_\tau^{\tau+h} (||\boldsymbol{D}\boldsymbol{u}||_r^r+||\boldsymbol{D}\boldsymbol{u}||_{V_2}^2) \\
&\le Ch\left(1+||\boldsymbol{u}(\tau)||_{V_r}^r+\frac{1}{h}\int_\tau^{\tau+h} ||\boldsymbol{f}||_{V_r^*}^{r'}\right).
\end{aligned}
$$

■

In the following lemma we show, in an elementary way, that $\boldsymbol{u}\in N^{\frac12,2}(0,T;H)$.

Lemma 3.2. *Let $(\boldsymbol{u},\boldsymbol{s})$ be a weak solution of (1)–(8). Then there exists $C>0$, depending only on the data, such that for all $h\in(0,T)$ there holds*

$$
\int_0^{T-h} ||\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau)||_H^2\,\mathrm{d}\tau\le Ch.
$$

Proof. Let us take $h\in(0,T)$ and $\tau\in(0,T-h)$. Observe that $h\in(0,T-\tau)$ for all $\tau\in(0,T-h)$. Because almost all $\tau\in(0,T)$ are semi-Lebesgue points of $\boldsymbol{f}$ and $\boldsymbol{u}(\tau)\in V_r$, the previous lemma says that

$$
\int_0^{T-h} ||\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau)||_H^2\mathrm{d}\tau\le Ch\int_0^{T-h}\left(1+||\boldsymbol{u}(\tau)||_{V_r}^r+\frac{1}{h}\int_\tau^{\tau+h}||\boldsymbol{f}||_{V_r^*}^{r'}\right)\mathrm{d}\tau.
$$

We assume that $\boldsymbol{u}\in L^r(0,T;V_r)$ and thus

$$
\begin{aligned}
\int_0^{T-h} ||\boldsymbol{u}(\tau+h)-\boldsymbol{u}(\tau)||_H^2\mathrm{d}\tau &\le Ch\left(T+||\boldsymbol{u}||_{L^r(0,T;V_r)}^r+\int_0^{T-h}\left(\frac{1}{h}\int_\tau^{\tau+h}||\boldsymbol{f}||_{V_r^*}^{r'}\right)\mathrm{d}\tau\right) \\
&\le Ch\left(T+||\boldsymbol{u}||_{L^r(0,T;V_r)}^r+||\boldsymbol{f}||_{L^{r'}(0,T;V_r^*)}^{r'}\right),
\end{aligned}
$$

where we used a simple boundedness of averaging operators on $L^p(\mathbb{R})$ for any $p\in[1,\infty]$. It completes the proof.

■

3.2 Regularity of solution

Our goal here will be to show, that any solution is slightly more regular in time, provided that right hand side is so. We will start in a similar way as in [7].

Lemma 3.3. *Let* $(\boldsymbol{u}, \boldsymbol{s})$ *be a weak solution of* (1)–(8)*, let moreover* $\boldsymbol{f} \in L^{q_0}(0,T;V_r^*)$ *for some* $q_0 > r'$*. Then there exists* $q > r$ *such that* $\boldsymbol{u} \in L^q_{loc}((0,T];V_r)$*, with estimate depending only on* t_0*, the above-mentioned norm of* $\boldsymbol{f}$ *and the data.*

Proof. The proof is based on the reverse Hölder inequality, and is a step by step analogue of Lemma 4.3 from [7]. Note that one can also show global version of the above lemma, provided that the initial condition belongs to V_r. ■

Lemma 3.4 (Regularity of non-linear Stokes)**.** *Let* $t_0 < T/2$*,* $N \in \mathbb{N}$ *and denote* $t_N := t_0 \sum_{n=0}^{N} 2^{-n}$*. Assume that* $(\boldsymbol{u}, \boldsymbol{s})$ *satisfy*

$$\partial_t \boldsymbol{u} + L(\boldsymbol{u}) = K(s) \quad \text{in } V_r^* \tag{24}$$

for a. e. $s \in (t_{N-1}, T)$*. Suppose that*

$$\boldsymbol{u} \in L^r(0,T;V_r) \quad \text{and} \quad \partial_t \boldsymbol{u} \in L^{r'}(0,T;V_r^*)$$

and for some $\delta > 0$ *there holds*

$$K \in N^{\delta, r'}(t_{N-1}, T; V_r^*).$$

Let us define

$$\tau := \frac{\delta r}{2(r-1)} \quad \text{and} \quad \sigma := \frac{\delta}{r-1} \,. \tag{25}$$

Then $\boldsymbol{u} \in N^{\tau,\infty}(t_N, T; H) \cap N^{\sigma,r}(t_N, T; V_r)$*.*

Moreover, the norm of $\boldsymbol{u}$ *in both* $N^{\tau,\infty}(t_N,T;H)$ *and* $N^{\sigma,r}(t_N,T;V_r)$ *is uniformly bounded by a constant which depends only on* $\delta, T, t_0, r, \Omega, ||\boldsymbol{f}||_{L^{r'}(0,T;V_r^*)}, ||\boldsymbol{u}||_{L^\infty(0,T;H)}$ *and* $||\boldsymbol{u}||_{L^r(0,T;V_r)}$*.*

Proof. Thanks to the assumption on $\boldsymbol{u}$ we can use Lemma 3.2 to obtain

$$\boldsymbol{u} \in N^{\frac{1}{2},2}(0,T,H)$$

and thus

$$\int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_H^2 \le ch.$$

Let $h \in (0, T - t_N)$. We apply d^h to (24) and use $\boldsymbol{\varphi} := d^h \boldsymbol{u} \in V_r$ as a test function, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} ||d^h \boldsymbol{u}||_H^2 + \left\langle d^h L(\boldsymbol{u}), d^h \boldsymbol{u} \right\rangle_{V_r} = \left\langle d^h K(s), d^h \boldsymbol{u} \right\rangle_{V_r}.$$

Because of (6) and the fact that $(\boldsymbol{u}, \boldsymbol{s}) \in \mathcal{G}$ we derive the estimate

$$\left\langle d^h L(\boldsymbol{u}), d^h \boldsymbol{u} \right\rangle_{V_r} \ge c_1 \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^2 + |\boldsymbol{D}(d^h \boldsymbol{u})|^r.$$

The right hand side is estimated as follows

$$\begin{aligned} \left\langle d^h K(s), d^h \boldsymbol{u} \right\rangle_{V_r} &\le ||d^h K(s)||_{V_r^*} ||d^h \boldsymbol{u}||_{V_r} \le ||d^h K(s)||_{V_r^*} (||\boldsymbol{D}(d^h \boldsymbol{u})||_r + ||d^h \boldsymbol{u}||_H) \\ &\le \varepsilon ||\boldsymbol{D}(d^h \boldsymbol{u})||_r^r + c_\varepsilon ||d^h K(s)||_{V_r^*}^{r'} + C ||d^h \boldsymbol{u}||_H^2 ||\boldsymbol{u}||_H^{r-2} \\ &\le \varepsilon ||\boldsymbol{D}(d^h \boldsymbol{u})||_r^r + c_\varepsilon ||d^h K(s)||_{V_r^*}^{r'} + C ||d^h \boldsymbol{u}||_H^2. \end{aligned}$$

Here we used Korn's and Young's inequalities and also the fact that $||\boldsymbol{u}(s)||_H$ is uniformly bounded. Altogether

$$\frac{\mathrm{d}}{\mathrm{d}t} ||d^h \boldsymbol{u}||_H^2 + c_1 \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^2 + |\boldsymbol{D}(d^h \boldsymbol{u})|^r \le C ||d^h \boldsymbol{u}||_H^2 + C ||d^h K(s)||_{V_r^*}^{r'},$$

which implies that for $t \in [t_{N-1}, T-h]$

$$||d^h \boldsymbol{u}(t)||_H^2 + c_1 \int_{t_{N-1}}^{t} \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^r \le ||d^h \boldsymbol{u}(t_{N-1})||_H^2 + C \int_{t_{N-1}}^{t} ||d^h \boldsymbol{u}||_H^2 + C \int_{t_{N-1}}^{t} ||d^h K(s)||_{V_r^*}^{r'}$$
$$\le ||d^h \boldsymbol{u}(t_{N-1})||_H^2 + C(h + h^{\delta r'})$$

and thus

$$||d^h \boldsymbol{u}(t)||_H^2 + c_1 \int_{t_{N-1}}^{t} \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^r \le ||d^h \boldsymbol{u}(t_{N-1})||_H^2 + Ch^{\delta r'}.$$

We integrate with respect to t_{N-1} over $(0, t_N)$, we get, for $t \in [t_N, T-h]$,

$$t_N ||d^h \boldsymbol{u}(t)||_H^2 + c_1 t_N \int_{t_N}^{t} \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^r \le \int_0^{t_N} ||d^h \boldsymbol{u}(t_{N-1})||_H^2 \mathrm{d}t_{N-1} + C t_N h^{\delta r'}$$

and by the Lemma 3.2 we obtain

$$t_N ||d^h \boldsymbol{u}(t)||_H^2 + c_1 t_N \int_{t_N}^{t} \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^r \le ch + C t_N h^{\delta r'} \le Ch^{\delta r'}$$

and finally

$$\sup_{t \in [t_N, T-h]} ||d^h \boldsymbol{u}(t)||_H^2 + c_1 \int_{t_N}^{T-h} \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^r \le \frac{C}{t_N} h^{\delta r'}.$$

It is enough to set $\tau = \delta r'/2$, which is exactly (25), to obtain that $\boldsymbol{u} \in N^{\tau,\infty}(t_N, T; H)$. From the last inequality we now get

$$\int_{t_N}^{T-h} \int_\Omega |\boldsymbol{D}(d^h \boldsymbol{u})|^r \le \frac{C}{t_N} h^{2\tau}.$$

Because $\boldsymbol{u} \in N^{\tau,\infty}(t_N, T; H)$, $||\boldsymbol{u}(t)||_H$ is uniformly bounded and $r > 2$ we also have that

$$\int_{t_N}^{T-h} ||d^h \boldsymbol{u}(t)||_H^r \le \frac{C}{t_N} h^{2\tau}.$$

We obtain

$$\int_{t_N}^{T-h} (||d^h \boldsymbol{u}(t)||_H + ||\boldsymbol{D}(d^h \boldsymbol{u})||_r)^r \le c \int_{t_N}^{T-h} (||d^h \boldsymbol{u}(t)||_H^r + ||\boldsymbol{D}(d^h \boldsymbol{u})||_r^r) \le \frac{C}{t_N} h^{2\tau}$$

and thanks to $||d^h \boldsymbol{u}||_{V_r} \le c(||\boldsymbol{D}(d^h \boldsymbol{u})||_r + ||d^h \boldsymbol{u}||_H)$ we get that $\boldsymbol{u} \in N^{\sigma,r}(t_N, T; V_r)$ for $\sigma = \frac{2\tau}{r}$. ■

Remark. *The previous lemma is a minor modification of Lemma 5.1 from [7]. The main difference is in the presence of numbers t_N, which enables us to obtain a uniform bound of the norm of $\boldsymbol{u}$.*

3.3 General scheme

Proceeding similarly as in Theorem 3.1 and Lemma 5.6 from [7], we can now prove our first main result.

Proof of the Theorem 1.1. Let us take any $2t_0 \in (0,T)$ and choose an arbitrary weak solution $(\boldsymbol{u}, \boldsymbol{s})$ of (1)–(8). Due to apriori estimates we have

$$\boldsymbol{u} \in N^{0,r}(0,T;V_r) \cap N^{0,\infty}(0,T;H)$$

and because of Lemma 3.2

$$\boldsymbol{u} \in N^{\frac{1}{2},2}(0,T;H).$$

In the case of $r = 11/5$ we invoke Lemma 3.3 to find $q > \frac{11}{5}$ such that $\boldsymbol{u} \in L^q(t_0,T;V_r)$, moreover, its norm is uniformly bounded. Therefore

$$\boldsymbol{u} \in N^{0,q}(t_0,T;V_r).$$

Step I: Improving time regularity of the convective term. We will show that for $r = 11/5$ we get

$$K_0(\boldsymbol{u}) \in N^{\delta_0,r'}(t_0,T;V_r^*) \quad \text{with} \quad \delta_0 = \frac{6}{55} \cdot \frac{5q-11}{q} \; .$$

We take $\boldsymbol{\varphi} \in L^r(t_0,T;V_r)$ from the unit ball and fix $h \in (0, T-t_0)$. We estimate

$$\begin{aligned}\left| \int_{t_0}^{T-h} \left\langle d^h K_0(\boldsymbol{u}), \boldsymbol{\varphi} \right\rangle_{V_r} \right| &\leq \int_{t_0}^{T-h} \int_\Omega \left| d^h(\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi} \right| \leq \int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_{2r'} ||\boldsymbol{u}||_{2r'} ||\nabla \boldsymbol{\varphi}||_r \\ &\leq \int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_2^{\alpha} ||d^h \boldsymbol{u}||_{V_r}^{1-\alpha} ||\boldsymbol{u}||_2^{\alpha} ||\boldsymbol{u}||_{V_r}^{1-\alpha} ||\boldsymbol{\varphi}||_{V_r} \, ,\end{aligned}$$

where we used the Hölder's inequality and the interpolation (13) with $\alpha = \frac{5r-9}{5r-6}$. Now we use the Hölder's inequality again, this time with following exponents

$$\frac{1}{p} + \frac{1-\alpha}{q} + \frac{1}{\infty} + \frac{1-\alpha}{q} + \frac{1}{r} = 1, \quad \text{i.e.} \quad \frac{1}{p} = \frac{6}{11} - \frac{6}{5q} \; ,$$

where the last expression is positive because $q > 11/5$. We obtain

$$\begin{aligned}\left| \int_{t_0}^{T-h} \left\langle d^h K_0(\boldsymbol{u}), \boldsymbol{\varphi} \right\rangle_{V_r} \right| &\leq \left(\int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_2^{\alpha p} \right)^{\frac{1}{p}} \left(\int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_{V_r}^{q} \right)^{\frac{1-\alpha}{q}} \cdot \sup_{t \in (t_0, T-h)} ||\boldsymbol{u}||_2^{\alpha} \cdot \\ &\cdot \left(\int_{t_0}^{T-h} ||\boldsymbol{u}||_{V_r}^{q} \right)^{\frac{1-\alpha}{q}} \left(\int_{t_0}^{T-h} ||\boldsymbol{\varphi}||_{V_r}^{r} \right)^{\frac{1}{r}} .\end{aligned}$$

We know that $\boldsymbol{u}$ is uniformly bounded in $L^\infty(0,T;H) \cap L^q(t_0,T;V_r)$. Together with our assumption on $\boldsymbol{\varphi}$ and the fact that $\boldsymbol{u} \in N^{0,q}(t_0,T;V_r)$ we get

$$\left| \int_{t_0}^{T-h} \left\langle d^h K_0(\boldsymbol{u}), \boldsymbol{\varphi} \right\rangle_{V_r} \right| \leq C \left(\int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_H^{\alpha p} \right)^{\frac{1}{p}} .$$

Observe that $\alpha p > 2$ (it is equivalent to $66 > 19q$) and thus we will continue as follows

$$\left| \int_{t_0}^{T-h} \left\langle d^h K_0(\boldsymbol{u}), \boldsymbol{\varphi} \right\rangle_{V_r} \right| \leq C \left(\int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_H^2 ||d^h \boldsymbol{u}||_H^{\alpha p - 2} \right)^{\frac{1}{p}} \leq C \left(\int_{t_0}^{T-h} ||d^h \boldsymbol{u}||_H^2 \right)^{\frac{1}{2} \cdot \frac{2}{p}} \leq C h^{\frac{1}{p}},$$

where we used again the uniform boundedness of $\boldsymbol{u}$ and the fact that $\boldsymbol{u} \in N^{\frac{1}{2},2}(0,T;H)$. The duality argument gives as desired result about $K_0(\boldsymbol{u})$.

For $r > 11/5$ we would proceed in a similar way. Instead of q we use r and then obtain the same result with $\delta_0 = \frac{5r^2 - 11r + 6}{r(5r-6)}$.

Step II: Improving time regularity of the solution. Let us consider $r = 11/5$ and set

$$K(t) := \boldsymbol{f}(t) + K_0(\boldsymbol{u}(t)),$$

due to the previous step and the assumption $\boldsymbol{f} \in N^{\delta,r'}(0,T;V_r^*)$ with $\delta > \frac{6}{5}\left(\frac{25}{22} - 1\right) = \frac{9}{55} > \delta_0$, if q is sufficiently close to the threshold $11/5$, we get $K \in N^{\delta_0,r'}(t_0,T;V_r^*)$. We invoke Lemma 3.4 to get

$$\boldsymbol{u} \in N^{\tau_0,\infty}(t_1,T;H) \cap N^{\sigma,r}(t_1,T;V_r)$$

with $\sigma = \frac{\delta_0}{r-1}$ and τ_0 which is a small fixed number from an interval $\left(0, \frac{\delta_0 r}{2(r-1)}\right]$.

For $r > 11/5$ we use either the same δ_0 as in the previous step, or we take some smaller value (because $\delta < \delta_0$ for r close to $5/2$).

In any case we get

$$\boldsymbol{u} \in N^{\tau_0,\infty}(t_1,T;H) \cap N^{\sigma,r}(t_1,T;V_r)$$

for small σ and small fixed τ_0.

Step III: Iterative improving of the convective term and the solution. We will proceed in the same fashion as in the first step, just the second Hölder's inequality will be used with exponents

$$\frac{1}{p} + \frac{1-\alpha}{r_\sigma} + \frac{1}{\infty} + \frac{1-\alpha}{r} + \frac{1}{r} = 1,$$

where r_σ is such that $N^{\sigma,r}(t_1,T;V_r) \hookrightarrow L^{r_\sigma}(t_1,T;V_r)$, i.e. as seen from (11) it holds with

$$\frac{1}{r_\sigma} = \frac{1}{r} - \sigma + \varepsilon,$$

where $\varepsilon > 0$ is small enough. Therefore,

$$\frac{1}{p} = \frac{5r-11}{5r-6} + (1-\alpha)(\sigma - \varepsilon) > 0.$$

The estimate reads

$$\begin{aligned}
\left| \int_{t_1}^{T-h} \langle d^h K_0(\boldsymbol{u}), \boldsymbol{\varphi} \rangle_{V_r} \right| &\le c \left(\int_{t_1}^{T-h} ||d^h \boldsymbol{u}||_H^{\alpha p} \right)^{\frac{1}{p}} \left(\int_{t_1}^{T-h} ||d^h \boldsymbol{u}||_{V_r}^{r} \right)^{\frac{1-\alpha}{r}} \left(\int_{t_1}^{T-h} ||\boldsymbol{u}||_{V_r}^{r_\sigma} \right)^{\frac{1-\alpha}{r_\sigma}} \\
&\le c h^{(1-\alpha)\sigma} \left(\int_{t_1}^{T-h} ||d^h \boldsymbol{u}||_H^2 ||d^h \boldsymbol{u}||_H^{\alpha p - 2} \right)^{\frac{1}{p}} \\
&\le c h^{(1-\alpha)\sigma} ||d^h \boldsymbol{u}||_{L^\infty(t_1,T;H)}^{\alpha - \frac{2}{p}} \left(\int_{t_1}^{T-h} ||d^h \boldsymbol{u}||_H^2 \right)^{\frac{1}{2}\cdot\frac{2}{p}} \\
&\le c h^{(1-\alpha)\sigma} h^{(\alpha - \frac{2}{p})\tau_0} h^{\frac{1}{p}},
\end{aligned}$$

where we used that $\alpha p > 2$. For $r = 11/5$ this condition reduces to $\sigma < 1/3$, but it holds for all $r \in [11/5, 5/2)$. This gives us that $K_0(\boldsymbol{u}) \in N^{\delta_1,r'}(t_1,T;V_r^*)$ with $\delta_1 = (1-\alpha)\sigma + (\alpha - \frac{2}{p})\tau_0 + \frac{1}{p}$. We again invoke Lemma 3.4 to find

$$\boldsymbol{u} \in N^{\tau_0,\infty}(t_2,T;H) \cap N^{\sigma_1,r}(t_2,T;V_r)$$

with $\sigma_1 = \frac{\delta_1}{r-1} = \frac{1-\alpha}{r-1}\sigma + \frac{\alpha - \frac{2}{p}}{r-1}\tau_0 + \frac{1}{r-1} \cdot \frac{5r-11}{5r-6} + \frac{1-\alpha}{r-1}(\sigma - \varepsilon)$.

As in [7], we obtain the formula for improving σ. For $r = 11/5$ and given σ small we find the new $\tilde{\sigma}$ given by

$$\tilde{\sigma} = \sigma + \frac{\tau_0}{3} - (\sigma - \varepsilon)\tau_0 - \frac{\varepsilon}{2} = (1-\tau_0)\sigma + \frac{\tau_0}{3} - \varepsilon\left(\frac{1}{2} - \tau_0\right).$$

If we define function θ by $\theta(\sigma) = \tilde{\sigma}$, it is a contraction and $\theta : [0,1] \to [0,1]$. Banach fixed point theorem shows that θ has a unique fixed point

$$\sigma_{\text{fix}} = \frac{1}{3} - \varepsilon\left(\frac{1}{2\tau_0} - 1\right).$$

Since we can assume $\sigma \le \overline{\sigma} := \frac{5}{2r} - 1 = \frac{3}{22} < \sigma_{\text{fix}}$, for ε small enough, we really reach little beyond the value $\overline{\sigma}$ after finitely many iterations. And so $\boldsymbol{u} \in N^{\overline{\sigma}+\overline{\varepsilon},r}(t_N, T; V_r)$ for $\overline{\varepsilon} > 0$ small and $N \in \mathbb{N}$ big enough. Due to the definition of t_N we see that $\boldsymbol{u} \in N^{\overline{\sigma}+\overline{\varepsilon},r}(2t_0, T; V_r)$ and by (11) we finally obtain

$$\boldsymbol{u} \in L^q(2t_0, T; V_r) \quad \text{if} \quad \frac{1}{q} > \frac{1}{r} - \overline{\sigma} - \overline{\varepsilon},$$

in other words we can set $q = \frac{1}{r} - \overline{\sigma}$, which gives the critical value $\overline{r} = 22/7$. For $r > 11/5$ we can find a similar contraction mapping and conclude the same result.

Step IV: Unique continuation of solutions. Take any two weak solutions $\boldsymbol{u}_1, \boldsymbol{u}_2$ on $[0, T]$. By (22), cf. Theorem 2.2 above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}||\boldsymbol{w}||_H^2 \le c\big(1 + ||\boldsymbol{u}_2||_{V_r}^{\overline{r}}\big)||\boldsymbol{w}||_H^2$$

where $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. Recall that $\overline{r} = \frac{2r}{2r-3}$ and thus for $r \ge 5/2$ we have $\boldsymbol{u} \in L^{\overline{r}}(0, T; V_r)$ because $r \ge \overline{r}$. For $r \in (11/5, 5/2)$ we can proceed as in [6] to obtain $\boldsymbol{u} \in L^\infty(t_0, T; V_r)$ for any $t_0 \in (0, T)$.

Finally, for $r \in [11/5, 12/5]$ we employ steps I–III above to conclude that for an arbitrarily small $t_0 > 0$ we have $\boldsymbol{u} \in L^{\overline{r}}(t_0, T; V_r)$.

Now let $t_1 \in (0, T]$ be such that $\boldsymbol{u}_1(t_1) = \boldsymbol{u}_2(t_1)$. Since we can can assume $t_0 < t_1$, it follows by Theorem 2.2 that $\boldsymbol{u}_1(t) = \boldsymbol{u}_2(t)$ for all $t \in [t_1, T]$. This finishes the proof.

■

Remark. *Proceeding as in [7], one can also show that*

$$\boldsymbol{u} \in N^{\frac{1}{2},\infty}(t_0, T; H) \cap N^{\frac{1}{r},r}(t_0, T; V_r) \cap N^{\frac{1}{2},2}(t_0, T; V_2).$$

Similarly, we can obtain all the above conclusions globally, i.e. for $t_0 = 0$, provided that $\boldsymbol{u}_0 \in V_r$ and 0 is a semi-Lebesgue point of $\boldsymbol{f}$.

4 Attractor

4.1 General method

Here we briefly outline the general method of trajectories, following closely the exposition in [12]. Let X, Y, Z be Banach spaces, such that X is both reflexive and separable and we have embeddings

$$Y \hookrightarrow\hookrightarrow X \text{ and } X \hookrightarrow Z.$$

For $r \in [2, \infty)$ and $\tau > 0$ fixed we denote

$$X_\tau := L^2(0, \tau; X),$$
$$Y_\tau := \{\boldsymbol{u} \in L^r(0, \tau; Y), \partial_t \boldsymbol{u} \in L^1(0, \tau; Z)\}.$$

(A1) For any $\boldsymbol{u}_0 \in X$ and arbitrary $T > 0$ there exists $\boldsymbol{u} \in \mathcal{C}_w([0, T]; X) \cap Y_T$ a solution on $[0, T]$ with $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Moreover, for any solution the estimates of $||\boldsymbol{u}||_Y$ are uniform with respect to $||\boldsymbol{u}(0)||_X$.

(A2) There exists a bounded set $B^0 \subset X$ such that, if $\boldsymbol{u}$ is an arbitrary solution to (19) with initial condition $\boldsymbol{u}_0 \in X$ then

(i) there exists $t_0 = t_0(||\boldsymbol{u}(0)||_X)$ such that $\boldsymbol{u}(t) \in B^0$ for all $t \ge t_0$ and

(ii) if $\boldsymbol{u}_0 \in B^0$ then $\boldsymbol{u}(t) \in B^0$ for all $t \ge 0$.

Now, let $\ell > 0$ be an arbitrary fixed number. By the ℓ-trajectory we mean any solution on the interval $[0, \ell]$. The set of all such ℓ-trajectories is denoted by $\mathcal{X}_\ell$ and is equipped with the topology of X_ℓ. Instead of uniqueness of the solution, we will require that

(A3) Each ℓ-trajectory has among all solutions unique continuation. More precisely, if two solutions coincide on the interval $[0, \ell]$, they coincide for all subsequent times $t \ge \ell$.

We can thus define the semigroup L_t on $\mathcal{X}_\ell$ by

$$L_t \boldsymbol{\xi}(\tau) := \boldsymbol{u}(t + \tau), \tau \in [0, \ell],$$

where $\boldsymbol{u}$ is the unique solution on $[0, \ell+\tau]$ such that $\boldsymbol{u}|_{[0,\ell]} = \boldsymbol{\xi} \in \mathcal{X}_\ell$.

Moreover, we define

$$B^0_\ell := \{\boldsymbol{\xi} \in \mathcal{X}_\ell;\ \boldsymbol{\xi}(0) \in B^0\},$$

it is the set of all ℓ-trajectories starting at any point of B^0.

(A4) For all $t > 0$, $L_t : X_\ell \to X_\ell$ is continuous on B^0_ℓ.

(A5) For some $\tau > 0$, $\overline{L_\tau(B^0_\ell)}^{X_\ell} \subset B^0_\ell$.

Now, using (A5), we introduce the set

$$B^1_\ell := \overline{L_\tau(B^0_\ell)}^{X_\ell} \subset B^0_\ell\,.$$

The above assumptions are sufficient to obtain existence of the global attractor in the space of trajectories. The following one is a criterion of its finite-dimensionality.

(A6) There exists a space W_ℓ with $W_\ell \hookrightarrow\hookrightarrow X_\ell$ and $\tau > 0$ such that $L_\tau : X_\ell \to W_\ell$ is a Lipschitz continuous on B^1_ℓ.

To get the results from the space of trajectories to the original space of initial conditions, we introduce a mapping $e : \mathcal{X}_\ell \to X$ by the formula

$$e(\boldsymbol{\xi}) = \boldsymbol{\xi}(\ell).$$

Semigroup L_t worked on B^1_ℓ and the solution operator S_t is defined on the set B^1, which is defined using of the mapping e as

$$B^1 := e(B^1_\ell).$$

Note that S_t is well-defined in virtue of the assumption (A3).

(A7) The mapping $e : X_\ell \to X$ is continuous on B^1_ℓ.

To get the finiteness of the fractal dimension of $\mathcal{A}$ it is necessary to strengthen the previous assumption.

(A8) The mapping $e : X_\ell \to X$ is α-Hölder continuous on B^1_ℓ.

At last, to construct an exponential attractor we will need two additional assumptions.

(A9) For all $\tau > 0$ the operators $L_t : X_\ell \to X_\ell$ are uniformly Lipschitz continuous on B^1_ℓ with respect to $t \in [0, \tau]$.

(A10) For all $\tau > 0$ there exists $c > 0$ and $\gamma \in (0, 1]$ such that for all $\boldsymbol{\xi} \in B^1_\ell$ and all $t_1, t_2 \in [0, \tau]$ it holds that $||L_{t_1}\boldsymbol{\xi} - L_{t_2}\boldsymbol{\xi}||_{X_\ell} \le c|t_1 - t_2|^\gamma$.

Now, we mention one result, which is quite useful in verifying the assumptions (A4), (A7), (A8) and (A9).

Lemma 4.1. *Let $\mathcal{T}_\ell \subset \mathcal{X}_\ell$ be a set of trajectories and let $\mathcal{T} \subset X$ be defined by*

$$\mathcal{T} := \{\boldsymbol{\xi}(t);\ \boldsymbol{\xi} \in \mathcal{T}_\ell,\ t \in [\ell/2, \ell]\}\,.$$

Let the solution operators S_t be well-defined and moreover, uniformly (with respect to $t \in [0, \tau]$) Lipschitz continuous on the set $\mathcal{T} \subset X$. Then

(i) the operators $L_t : X_\ell \to X_\ell$ are uniformly (with respect to $t \in [0, \tau]$) Lipschitz continuous on $\mathcal{T}_\ell$ and

(ii) the operator $e : X_\ell \to X$ is Lipschitz continuous on $\mathcal{T}_\ell$.

Proof. See Lemma 2.1 in [12]. ■

In the following theorem we summarize results about existence of (exponential) attractor from [12].

Theorem 4.2.

(i) Let (A1)–(A5) hold. Then the dynamical system $(L_t, \mathcal{X}_\ell)$ possesses global attractor $\mathcal{A}_\ell$. Its fractal dimension in $\mathcal{X}_\ell$ is finite provided that (A6) holds too.

(ii) Let (A1)–(A5) and (A7) hold. Then the dynamical system (S_t, B^1) possesses global attractor $\mathcal{A}$, which is given by $\mathcal{A} = e(\mathcal{A}_\ell)$. Its fractal dimension in X is finite provided that both (A6) and (A8) hold.

(iii) Let assumptions (A1)–(A10) hold. Then the dynamical system (S_t, B^1) possesses an exponential attractor $\mathcal{E}$.

Proof. See Theorems 2.1–2.6 in [12]. ■

4.2 Existence and uniqueness of attractor

Most of the assumptions are easily obtained from results about existence and continuous dependence on initial condition. From now on, we will assume that the conditions of our main Theorem 1.1 hold true.

We set

$$Y := V_r, \quad X := H \quad \text{and} \quad Z := V_r^*,$$

which is a Gelfand triple. Now we will verify step by step all the desired properties.

Property (A1). The first property is just Theorem 2.1.

Property (A2). This follows from the energy equality (21), and the a priori estimates, which led to the very existence of solution.

From now on, let $\ell > 0$ be fixed.

Property (A3). This property was verified in the proof of Theorem 1.1. Let us recall that for $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, the difference of two ℓ-trajectories, $\ell < T$, we obtained

$$||\boldsymbol{w}(t)||_H^2 \leq K||\boldsymbol{w}(s)||_H^2 \quad \text{for} \quad t_0 \leq s \leq t \leq T, \tag{26}$$

where $K = \exp(C \int_{t_0}^{T} ||\boldsymbol{u}_2||_{V_r}^{\overline{r}} + ||\boldsymbol{u}_2||_{V_r})$ is uniformly bounded and $t_0 > 0$ can be arbitrarily small.

Property (A4). Here we use Lemma 4.1. We set $\mathcal{T}_\ell := B_\ell^0$ which is the set defined after establishing of (A3). In the proof of (A3) we took t_0 small, so it can be smaller than $\frac{\ell}{2}$, it means that the solutions operator S_t of (19) makes sense on

$$\mathcal{T} = \left\{\boldsymbol{\xi}(t); \boldsymbol{\xi} \in \mathcal{T}_\ell, t \in \left[\frac{\ell}{2}, \ell\right]\right\}.$$

Moreover, thanks to (26) we know that $S_t : H \to H$ is uniformly Lipschitz continuous on $\mathcal{T}$. Now we can invoke Lemma 4.1 to obtain (A4).

Property (A5). Observe that, due to (A2), the set B_ℓ^0 is positively invariant with respect to L_τ, i.e. $L_\tau B_\ell^0 \subset B_\ell^0$. Therefore, to establish the fifth assumption, it is enough to show that

$$\overline{B_\ell^0}^{X_\ell} \subset B_\ell^0.$$

To verify that we would consider an arbitrary sequence $\{\boldsymbol{\xi}_n\} \subset B_\ell^0$ such that $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$ in X_ℓ. Then we follow the proof of Theorem 2.1, where the main point is indeed compactness of bounded sequences of solutions, see [1] for details. By this we find that $\boldsymbol{\xi}$ is indeed a solution of (19) (together with some $\boldsymbol{s}$). The fact that $\boldsymbol{\xi}(0) \in B^0$ is a simple consequence of closedness of B^0 together with continuity of $\boldsymbol{\xi}$ and the fact that $\boldsymbol{\xi}_n(t) \in B^0, t \geq 0$, which holds due to by (A2). This shows (A5).

Now we fix $\tau > 0$.

Properties (A7), (A8) and (A9). Because of (A5) we have that $B_\ell^1 \subset B_\ell^0$. We can use Lemma 4.1 again, but now with $\mathcal{T}_\ell := B_\ell^1$. The rest is the same as before.

Property (A10). Here we want to find $c > 0$ and $\beta \in (0, 1]$ (possibly depending on τ) such that for all $\boldsymbol{\xi} \in B_\ell^1$ and all $t_1, t_2 \in [0, \tau]$ there holds

$$\int_0^\ell ||\boldsymbol{u}(t + t_1) - \boldsymbol{u}(t + t_2)||_H^2 \, \mathrm{d}t \leq c|t_1 - t_2|^{2\beta},$$

where $\boldsymbol{u}$ is a unique continuation of $\boldsymbol{\xi}$ on $[0, \ell + \tau]$.

This is just Lemma 3.2. Without loss of generality we can assume that $t_1 > t_2$ and set $\delta := t_1 - t_2$. The left hand side becomes

$$\int_{t_2}^{\ell + t_2} ||\boldsymbol{u}(s + \delta) - \boldsymbol{u}(s)||_H^2 \, \mathrm{d}s.$$

Almost all times $s \in (t_2, \ell+t_2)$ are semi-Lebesgue points of $\boldsymbol{f}$ and $\boldsymbol{u}(s) \in V_r$. By Lemma 3.1 we get $C > 0$ such that for any $h \in (0, \ell+\tau-s)$ there holds

$$||\boldsymbol{u}(s+h) - \boldsymbol{u}(s)||_H^2 \leq Ch \left(1 + ||\boldsymbol{u}(s)||_{V_r}^r + \frac{1}{h} \int_s^{s+h} ||\boldsymbol{f}||_{V_r^*}^{r'} \right).$$

We see that $\delta \in (0, \ell+\tau-s)$ for all $s \in (t_2, \ell+t_2)$ and thus we can use the estimate with $h := \delta$. Then we integrate over $s \in (t_2, \ell+t_2)$ and use boundedness of averaging operator on L^1, which finishes the proof of (A10) with $\beta = \frac{1}{2}$.

We have verified all the assumptions except (A6). Thus, we can use the general result of Theorem 4.2(i) to obtain the first part of Theorem 1.2. To obtain its second part, i.e. finite dimension of the attractor and existence of an exponential attractor, we need to verify the key condition (A6).

4.3 Finite dimension of attractor

We need to check out the sixth assumption. The procedure is similar to the case of Dirichlet boundary condition, treated in [6]. Note that here is the only place where we actually use the existence of a selection $\boldsymbol{s} = \boldsymbol{s}(\boldsymbol{u})$ with certain polynomial growth, i.e. (9) and (10).

We will verify (A6) with $\tau := \ell$ and

$$W_\ell := \{\boldsymbol{u} \in L^2(0, \ell; V_2), \partial_t \boldsymbol{u} \in L^1(0, l; V_r^*)\}.$$

By the Aubin-Lions-Simon theorem we get the embedding $W_\ell \hookrightarrow\hookrightarrow L^2(0, \ell; H) = X_\ell$. We wish to show that for any $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in B_\ell^1$ there hold estimates

$$\begin{aligned} ||L_\ell \boldsymbol{\xi}_1 - L_\ell \boldsymbol{\xi}_2||_{L^2(0,\ell;V_2)} &\leq C ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||_{L^2(0,\ell;H)}, \\ ||\partial_t L_\ell \boldsymbol{\xi}_1 - \partial_t L_\ell \boldsymbol{\xi}_2||_{L^1(0,\ell;V_r^*)} &\leq C ||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||_{L^2(0,\ell;H)}. \end{aligned}$$

By (A3) we find $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ unique extensions to $[0, 2\ell]$ of $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ respectively, i.e. $\boldsymbol{u}_1 = \boldsymbol{\xi}_1$ and $\boldsymbol{u}_2 = \boldsymbol{\xi}_2$ on $[0, \ell]$. Set $\boldsymbol{w} := \boldsymbol{u}_1 - \boldsymbol{u}_2$. We need to verify that

$$||\boldsymbol{u}_1(\ell + \cdot) - \boldsymbol{u}_2(\ell + \cdot)||_{L^2(0,\ell;V_2)} = \int_\ell^{2\ell} ||\boldsymbol{w}||_{V_2}^2 \leq C \int_0^\ell ||\boldsymbol{w}||_H^2, \tag{27}$$

$$||\partial_t \boldsymbol{u}_1(\ell + \cdot) - \partial_t \boldsymbol{u}_2(\ell + \cdot)||_{L^1(0,\ell;V_r^*)} = \int_\ell^{2\ell} ||\partial_t \boldsymbol{w}||_{V_r^*} \leq C \left(\int_0^\ell ||\boldsymbol{w}||_H^2 \right)^{\frac{1}{2}}. \tag{28}$$

As in the proof of (A3) we will start with the inequality for a difference of two solutions (see (22)), but we will also incorporate better estimate of the boundary term which is due to (10), provided that $q > r/2 + 1$, which is tantamount to $q - 2 > \frac{r-2}{2}$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} ||\boldsymbol{w}||_H^2 + \frac{c_1}{2} (||\boldsymbol{D}\boldsymbol{w}||_2^2 + ||\boldsymbol{D}\boldsymbol{w}||_r^r) + c_1 I^2 + c_5 J^2 \leq c\big(1 + ||\boldsymbol{u}_2||_{V_r}^{\overline{r}}\big) ||\boldsymbol{w}||_H^2, \tag{29}$$

where we write

$$I^2 = \int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2}) |\boldsymbol{D}\boldsymbol{w}|^2, \tag{30}$$

$$J^2 = \int_{\partial\Omega} (|\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}) |\boldsymbol{w}|^2. \tag{31}$$

As in the proof of (A3) we fix small t_0 for which we obtained $\boldsymbol{u}_2 \in L^{\overline{r}}(t_0, 2\ell; V_r)$. Let us take $s \in (t_0, \ell)$ a integrate (29) over $t \in (s, 2\ell)$ to get

$$||\boldsymbol{w}(2\ell)||_H^2 + \frac{c_1}{2} \int_s^{2\ell} (||\boldsymbol{D}\boldsymbol{w}||_2^2 + ||\boldsymbol{D}\boldsymbol{w}||_r^r) + c_1 \int_s^{2\ell} I^2 \leq ||\boldsymbol{w}(s)||_H^2 + C \int_s^{2\ell} (||\boldsymbol{u}_2||_{V_r}^{\overline{r}} + ||\boldsymbol{u}_2||_{V_r}) ||\boldsymbol{w}||_H^2,$$

because of (26) we can estimate

$$\frac{c_1}{2}\int_\ell^{2\ell}(||\boldsymbol{D}\boldsymbol{w}||_2^2+||\boldsymbol{D}\boldsymbol{w}||_r^r)+c_1\int_\ell^{2\ell} I^2 \le ||\boldsymbol{w}(s)||_H^2\left(1+C\int_{t_0}^{2\ell}||\boldsymbol{u}_2||_{V_r}^{\overline{r}}+||\boldsymbol{u}_2||_{V_r}\right).$$

The bracket on the right hand side can be (see Theorem 1.1) uniformly estimated by a constant which depends only on $r, t_0, \ell, \Omega, ||\boldsymbol{f}||_{L^{q_0}(0,T;V_r^*)}$ and uniformly bounded norms of solution of (1)–(8). We get the inequality

$$\int_\ell^{2\ell}(||\boldsymbol{D}\boldsymbol{w}||_2^2+||\boldsymbol{D}\boldsymbol{w}||_r^r)+\int_\ell^{2\ell} I^2 \le C||\boldsymbol{w}(s)||_H^2.$$

We integrate it over $s\in(t_0,\ell)$ to obtain

$$(\ell-t_0)\int_\ell^{2\ell}(||\boldsymbol{D}\boldsymbol{w}||_2^2+||\boldsymbol{D}\boldsymbol{w}||_r^r)+(\ell-t_0)\int_\ell^{2\ell} I^2 \le C\int_{t_0}^{\ell}||\boldsymbol{w}(s)||_H^2$$

$$\int_\ell^{2\ell}(||\boldsymbol{D}\boldsymbol{w}||_2^2+||\boldsymbol{D}\boldsymbol{w}||_r^r)+\int_\ell^{2\ell} I^2 \le \frac{2C}{\ell}\int_0^{\ell}||\boldsymbol{w}||_H^2,$$

where we used that $t_0<\frac{\ell}{2}$ from the proof of (A4). The inequality

$$\int_\ell^{2\ell}||\boldsymbol{D}\boldsymbol{w}||_2^2 \le \frac{C}{\ell}\int_0^{\ell}||\boldsymbol{w}||_H^2,$$

together with (26) implies (27). Let us emphasize that we obtained also inequalities

$$\int_\ell^{2\ell} I^2 \le \frac{2C}{\ell}\int_0^{\ell}||\boldsymbol{w}||_H^2, \tag{32}$$

$$\int_\ell^{2\ell} J^2 \le \frac{2C}{\ell}\int_0^{\ell}||\boldsymbol{w}||_H^2, \tag{33}$$

which will be useful later on.

To show (28) we start with the duality argument and use (19) to obtain

$$||\partial_t\boldsymbol{w}||_{L^1(\ell,2\ell;V_r^*)}=\sup_{\boldsymbol{\varphi}}\int_\ell^{2\ell}\langle\partial_t\boldsymbol{w},\boldsymbol{\varphi}\rangle_{V_r}$$

$$\int_\ell^{2\ell}\langle\partial_t\boldsymbol{w},\boldsymbol{\varphi}\rangle_{V_r}=\int_\ell^{2\ell}\langle K_0(\boldsymbol{u}_1)-K_0(\boldsymbol{u}_2),\boldsymbol{\varphi}\rangle_{V_r}-\int_\ell^{2\ell}\int_\Omega(\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_1)-\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_2)):\boldsymbol{D}\boldsymbol{\varphi}-\int_\ell^{2\ell}\int_{\partial\Omega}(\boldsymbol{s}_1-\boldsymbol{s}_2)\cdot\boldsymbol{\varphi}$$

$$=: I_1+I_2+I_3\,.$$

where supremum is taken over all $\boldsymbol{\varphi}\in L^\infty(\ell,2\ell;V_r)$ from the unit ball.

The first integral I_1 is estimated as follows:

$$I_1\le\int_\ell^{2\ell}\int_\Omega|\boldsymbol{u}_1\otimes\boldsymbol{u}_1-\boldsymbol{u}_2\otimes\boldsymbol{u}_2||\nabla\boldsymbol{\varphi}|\le\int_\ell^{2\ell}\int_\Omega|\boldsymbol{w}|(|\boldsymbol{u}_1|+|\boldsymbol{u}_2|)|\nabla\boldsymbol{\varphi}|$$

$$\le\int_\ell^{2\ell}||\boldsymbol{w}||_{2r'}(||\boldsymbol{u}_1||_{2r'}+||\boldsymbol{u}_2||_{2r'})||\nabla\boldsymbol{\varphi}||_r\le\int_\ell^{2\ell}||\boldsymbol{w}||_{2r'}(||\boldsymbol{u}_1||_{2r'}+||\boldsymbol{u}_2||_{2r'}).$$

Suppose that $r \in [11/5, 3)$. We interpolate the first term by (15) and others by (13). Because of uniform estimates of solutions in $L^\infty(0,T;H)$ and $L^r(0,T;V_r)$ we obtain

$$
\begin{aligned}
I_1 &\le C \int_\ell^{2\ell} ||\boldsymbol{w}||_H^{\frac{2r-3}{2r}} ||\boldsymbol{w}||_{V_2}^{\frac{3}{2r}} (||\boldsymbol{u}_1||_{V_r}^{\frac{3}{5r-6}} + ||\boldsymbol{u}_2||_{V_r}^{\frac{3}{5r-6}}) \cdot 1 \\
&\le C(\ell) \left(\int_\ell^{2\ell} ||\boldsymbol{w}||_H^2 \right)^{\frac{2r-3}{4r}} \cdot \left(\int_\ell^{2\ell} ||\boldsymbol{w}||_{V_2}^2 \right)^{\frac{3}{4r}} \cdot \left[\left(\int_\ell^{2\ell} ||\boldsymbol{u}_1||_{V_r}^r \right)^{\frac{3}{r(5r-6)}} + \left(\int_\ell^{2\ell} ||\boldsymbol{u}_2||_{V_r}^r \right)^{\frac{3}{r(5r-6)}} \right] \\
&\le C(\ell) \left(\int_\ell^{2\ell} ||\boldsymbol{w}||_{V_2}^2 \right)^{\frac{2r-3}{4r}} \cdot \left(\int_\ell^{2\ell} ||\boldsymbol{w}||_{V_2}^2 \right)^{\frac{3}{4r}} \le C(\ell) \left(\int_\ell^{2\ell} ||\boldsymbol{w}||_{V_2}^2 \right)^{\frac{1}{2}} \\
&\le \frac{C(\ell)}{\sqrt{\ell}} \left(\int_0^{\ell} ||\boldsymbol{w}||_H^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

where we used the fact that

$$
\frac{2r-3}{4r} + \frac{3}{4r} + \frac{3}{r(5r-6)} \le 1,
$$

the trivial embedding $V_2 \hookrightarrow H$ and the inequality (27).

For $r \ge 3$ we can either proceed as in [6] or we simply recall that any solution is in $L^\infty(t_0,T;V_r)$, and thus the rest follows immediately. It means that we got desired estimate of the convective term.

Now we estimate I_2. We start with (7) and the Hölder's inequality with exponents

$$
\frac{1}{2} + \frac{r-2}{2r} + \frac{1}{r} = 1
$$

to obtain

$$
\begin{aligned}
I_2 &\le \int_\ell^{2\ell} \int_\Omega \Big| (\boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_1) - \boldsymbol{S}(\boldsymbol{D}\boldsymbol{u}_2)) : \boldsymbol{D}\boldsymbol{\varphi} \Big| \le c_2 \int_\ell^{2\ell} \int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2})^{\frac{1}{2}+\frac{1}{2}} |\boldsymbol{D}\boldsymbol{w}||\boldsymbol{D}\boldsymbol{\varphi}| \\
&\le C \int_\ell^{2\ell} \Bigg[\left(\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2})^{2\cdot\frac{1}{2}} |\boldsymbol{D}\boldsymbol{w}|^2 \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2})^{\frac{2r}{r-2}\cdot\frac{1}{2}} \right)^{\frac{r-2}{2r}} ||\boldsymbol{D}\boldsymbol{\varphi}||_r \Bigg] \\
&\le C \int_\ell^{2\ell} \left(\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2}) |\boldsymbol{D}\boldsymbol{w}|^2 \right)^{\frac{1}{2}} \left(\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^{r-2} + |\boldsymbol{D}\boldsymbol{u}_2|^{r-2})^{\frac{r}{r-2}} \right)^{\frac{r-2}{2r}}.
\end{aligned}
$$

Now it is time to invoke quantity called I^2 from (30). Using Jensen's inequality we have

$$
\begin{aligned}
I_2 &\le C \int_\ell^{2\ell} I \cdot \left(\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{u}_1|^r + |\boldsymbol{D}\boldsymbol{u}_2|^r) \right)^{\frac{r-2}{2r}} \le C \int_\ell^{2\ell} I \cdot (1 + ||\boldsymbol{D}\boldsymbol{u}_1||_r^r + ||\boldsymbol{D}\boldsymbol{u}_2||_r^r)^{\frac{1}{2}} \\
&\le C \left(\int_\ell^{2\ell} I^2 \right)^{\frac{1}{2}} \cdot \left(\int_\ell^{2\ell} (1 + ||\boldsymbol{D}\boldsymbol{u}_1||_r^r + ||\boldsymbol{D}\boldsymbol{u}_2||_r^r) \right)^{\frac{1}{2}} \\
&\le \frac{C}{\sqrt{\ell}} \left(\int_0^{\ell} ||\boldsymbol{w}||_H^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

where the last estimate used (32) and uniform boundedness of solutions in $L^r(0,T;V_r)$.

The boundary term, i.e. I_3, is can be estimated from the above provided that (G5) holds. We get

$$I_3 \leq \int_\ell^{2\ell}\int_{\partial\Omega} \left|(\boldsymbol{s}_1-\boldsymbol{s}_2)\cdot\boldsymbol{\varphi}\right| \leq \int_\ell^{2\ell}\int_{\partial\Omega} |\boldsymbol{s}_1-\boldsymbol{s}_2||\boldsymbol{\varphi}|$$
$$\leq c_4 \int_\ell^{2\ell}\int_{\partial\Omega} \left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)|\boldsymbol{u}_1-\boldsymbol{u}_2||\boldsymbol{\varphi}|.$$

If $q-2 \leq \frac{r-2}{2}$ we use the Hölder's inequality with exponents

$$\frac{r-2}{2r}+\frac{1}{2}+\frac{1}{r}=1$$

to obtain

$$I_3 \leq C\int_\ell^{2\ell}\left(\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)^{\frac{2r}{r-2}}\right)^{\frac{r-2}{2r}} ||\boldsymbol{w}||_{L^2(\partial\Omega)}||\boldsymbol{\varphi}||_{L^r(\partial\Omega)}$$
$$\leq C\left(\int_\ell^{2\ell}||\boldsymbol{w}||^2_{L^2(\partial\Omega)}\right)^{\frac{1}{2}}\left[\int_\ell^{2\ell}\left(\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)^{\frac{2r}{r-2}}\right)^{\frac{r-2}{r}}\right]^{\frac{1}{2}}.$$

But the second integral is clearly bounded, as we can see using of Jensen's inequality, trivial estimate $(1+|x|)^{\frac{r-2}{r}} \leq 1+|x|$ and the fact that $2r\frac{q-2}{r-2}\leq r$ and so $W^{1,r}(\Omega)\hookrightarrow L^{2r\frac{q-2}{r-2}}(\partial\Omega)$:

$$\int_\ell^{2\ell}\left(\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)^{\frac{2r}{r-2}}\right)^{\frac{r-2}{r}} \leq C\int_\ell^{2\ell}\left(\int_{\partial\Omega}1+|\boldsymbol{u}_1|^{2r\frac{q-2}{r-2}}+|\boldsymbol{u}_2|^{2r\frac{q-2}{r-2}}\right)^{\frac{r-2}{r}}$$
$$\leq C\int_\ell^{2\ell}\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{2r\frac{q-2}{r-2}}+|\boldsymbol{u}_2|^{2r\frac{q-2}{r-2}}\right)$$
$$\leq C\int_\ell^{2\ell}\left(1+||\boldsymbol{u}_1||^{2r\frac{q-2}{r-2}}_{L^{2r\frac{q-2}{r-2}}(\partial\Omega)}+||\boldsymbol{u}_2||^{2r\frac{q-2}{r-2}}_{L^{2r\frac{q-2}{r-2}}(\partial\Omega)}\right)$$
$$\leq C\int_\ell^{2\ell}\left(1+||\boldsymbol{u}_1||^r_{1,r}+||\boldsymbol{u}_2||^r_{1,r}\right).$$

If $\frac{r-2}{2} < q-2 \leq r-2$ we use the Hölder's inequality in the same way as in the estimating of I_2, i.e.

$$I_3 \leq C\int_\ell^{2\ell}\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)^{\frac{1}{2}+\frac{1}{2}}|\boldsymbol{w}||\boldsymbol{\varphi}|$$
$$\leq C\int_\ell^{2\ell}\left[\left(\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)|\boldsymbol{w}|^2\right)^{\frac{1}{2}}\left(\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)^{\frac{r}{r-2}}\right)^{\frac{r-2}{2r}}\right]$$
$$\leq C\left(\int_\ell^{2\ell}||\boldsymbol{w}||^2_H+J^2\right)^{\frac{1}{2}}\left(\int_\ell^{2\ell}\left(\int_{\partial\Omega}\left(1+|\boldsymbol{u}_1|^{q-2}+|\boldsymbol{u}_2|^{q-2}\right)^{\frac{r}{r-2}}\right)^{\frac{r-2}{r}}\right)^{\frac{1}{2}},$$

where we invoked quantity J^2 from (31). At this point we need assumption (G6), because of (27) and (33) we obtain

$$\left(\int_{\ell}^{2\ell} ||\boldsymbol{w}||_H^2 + J^2\right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{\ell}} \left(\int_0^{\ell} ||\boldsymbol{w}||_H^2\right)^{\frac{1}{2}}.$$

And similarly as before we use Jensen's inequality and the embedding $W^{1,r}(\Omega) \hookrightarrow L^{r\frac{q-2}{r-2}}(\partial\Omega)$, which is valid provided that $q \leq r$, to estimate

$$\begin{aligned}
\int_{\ell}^{2\ell} \left(\int_{\partial\Omega} \left(1 + |\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}\right)^{\frac{r}{r-2}}\right)^{\frac{r-2}{r}} &\leq C \int_{\ell}^{2\ell} \left(1 + ||\boldsymbol{u}_1||^{r\frac{q-2}{r-2}}_{L^{r\frac{q-2}{r-2}}(\partial\Omega)} + ||\boldsymbol{u}_2||^{r\frac{q-2}{r-2}}_{L^{r\frac{q-2}{r-2}}(\partial\Omega)}\right)^{\frac{r-2}{r}} \\
&\leq C \int_{\ell}^{2\ell} \left(1 + ||\boldsymbol{u}_1||_{1,r}^r + ||\boldsymbol{u}_2||_{1,r}^r\right).
\end{aligned}$$

Together we obtain

$$I_3 \leq C \left(\int_0^{\ell} ||\boldsymbol{w}||_H^2\right)^{\frac{1}{2}}.$$

What remains is to get the same estimate also in the case $q > r$. We realize that we can use much "worse" W_ℓ than before and still obtain the desired result. Let us redefine

$$W_\ell := \{\boldsymbol{u} \in L^2(0,\ell;V_2), \partial_t \boldsymbol{u} \in L^1(0,l;\mathcal{V}^*)\}.$$

Now, the proof of (27) is the same as before, but the proof of (28) is much simpler. Reason is that our test function $\boldsymbol{\varphi}$ in the duality argument is essentially bounded in the time–space.

Therefore, we estimate I_3 using of (G5) and Hölder's inequality as follows

$$\begin{aligned}
I_3 &\leq \int_{\ell}^{2\ell}\int_{\partial\Omega} |\boldsymbol{s}_1 - \boldsymbol{s}_2||\boldsymbol{\varphi}| \leq C \int_{\ell}^{2\ell}\int_{\partial\Omega} \left(1 + |\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}\right)^{\frac{1}{2}+\frac{1}{2}} |\boldsymbol{w}| \\
&\leq C \left(\int_{\ell}^{2\ell}\int_{\partial\Omega} \left(1 + |\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}\right) |\boldsymbol{w}|^2\right)^{\frac{1}{2}} \left(\int_{\ell}^{2\ell}\int_{\partial\Omega} \left(1 + |\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

The first integral is again estimated thanks to (27) and (33) like

$$\left(\int_{\ell}^{2\ell}\int_{\partial\Omega} \left(1 + |\boldsymbol{u}_1|^{q-2} + |\boldsymbol{u}_2|^{q-2}\right) |\boldsymbol{w}|^2\right)^{\frac{1}{2}} \leq \frac{C}{\sqrt{\ell}} \left(\int_0^{\ell} ||\boldsymbol{w}||_H^2\right)^{\frac{1}{2}}$$

and the second integral is uniformly bounded. This is due to the fact that in the energy equality (21) we can estimate the boundary integral using of (G6) instead of (G4) as usual. It gives us that any weak solution satisfies for all $t \in (0,T)$ that

$$\int_0^t \int_{\partial\Omega} |\boldsymbol{u}|^q \leq C.$$

In summary, we were able to obtain the estimate

$$I_3 \leq C \left(\int_0^{\ell} ||\boldsymbol{w}||_H^2\right)^{\frac{1}{2}}$$

for any $q \geq 2$. Thanks to those three estimates of I_1, I_2 and I_3 we can conlude that (28) holds. The argument is complete.

Proof of the Theorem 1.2. We verified all the assumptions needed in the abstract method described in [12], which is here contained in Theorem 4.2. By parts (i) and (ii) we get existence and finite-dimensionality of the global attractor. Existence of an exponential attractor follows from part (iii). ■

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