All this activity based on an unproved conjecture (with sometimes paradoxical consequences) reminds me of the work of Lobachevskii, who constructed a beautiful theory of his geometry, undeterred by having an unproved hypothesis at the foundation. Now we know that there are two geometries, one where Lobachevskii's hypothesis is satisfied and one where it is not. They simply describe the geometry of different surfaces.

American universities boast about what famous Russian mathematicians they have rejected.

It seems doubtful that there can be a mathematics that contains exponentially hard problems (impossible to solve without combinatorial search) and another that does not. In any case, various aspects of deterministic, randomized, and derandomized algorithms provided many interesting lectures at Zurich (the section on computer science).

Most talks at the Congress, however, were like sermons. The lecturers plainly didn't expect that listeners would understand anything. Sometimes they went so far as to state obviously false theorems to the respectfully silent auditorium. The sermon mood was so pervasive that most of the introducers didn't even ask for questions at the end. And when some old-fashioned professors, like J. Moser (Director of the Mathematical Institute of ETH Zurich, the principal mathematical center in Switzerland), did urge people to ask questions, very few listeners overcame fear of exposing their ignorance sufficiently to do so.

The talks differed from sermons, however, in not being free. For those not registered as participants, the fee to attend a talk was considerable, as for a concert or a play.

I take pleasure in reporting that representatives of the Russian school generally were on the more comprehensible side. It is part of our tradition that a survey talk should emphasize new ideas and illuminating examples and not technical details.

I find rather worrisome the distinct shift in interests of our younger researchers (especially those working in the West) from directions long pursued by us to those fashionable in the USA. Such a shift of interests (doubtless related to the difficult conditions of job-hunting in American universities, some of which boast about what famous Russian mathematicians they have rejected) is inevitably negative. World leaders in one field leave it to race in a pack of jostling competitors following some other leader. Could this explain the distinct decrease in the proportion of our mathematicians among speakers at Congresses?

It is a pleasure to note also the large number of young Congress participants, including graduate students, from Russia and other countries of the former Soviet Union. Their attendance was made possible by generous support from the Swiss Organizing Committee of the Congress and the Soros Foundation.

Swiss mathematicians did everything possible to make our stay pleasant: participants were offered trips all over Switzerland (Lucerne, Interlaken, Bern, etc.), trips to the mountains (to Rigi Kulm overlooking the Vierwaldstätter See), to the Rhine waterfall (comparable to Niagara), concerts of classical and folk music. I was impressed by the small and little-known Bürlet art gallery in Zurich—Rembrandt and Franz Hals, El Greco and Goya, Canaletto and Tiepolo, Greuze and Ingres, Corot and Courbet, Cézanne, van Gogh, Matisse, Pissarro, Picasso.

After the tiring Congress, I spent a day at the home of my old friend A. Haefliger near Geneva. We climbed from 1500m to 3000m in the mountains near the Rhone valley, about halfway between the Jungfrau and the Matterhorn, and I got to swim in a glacial lake. On the return I picked mushrooms, sorrel, blueberries, and wild strawberries, and made my hosts a dinner from these gifts of nature (overcoming their doubts as to their edibility). The next day I returned to Moscow.

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A Nobel Prize for John Nash

John Milnor

John F. Nash created an impressive array of exciting mathematics during the 10 years of his mathematical activity. To some, the brief paper written at age 21, for which he has won a Nobel prize[1] in economics, may seem like the least of his achievements. Nevertheless, I applaud the wisdom of the selection committee in making this award. It is notoriously difficult to apply precise mathematical methods in the social sciences, yet the ideas in Nash's thesis are simple and rigorous, and provide a firm background, not only for economic theory but also for research in evolutionary biology, and more generally for the study of any situation in which human or nonhuman beings face competition or conflict. According to P. Ordeshook [O], p. 118,

> The concept of a Nash equilibrium n-tuple is perhaps *the* most important idea in noncooperative game theory. . . .Whether we are analyzing candidates' election strategies, the causes of war, agenda manipulation in legislatures, or the actions of interest groups, predictions about events reduce to a search for and description of equilibria. Put simply, equilibrium strategies are the things that we predict about people.

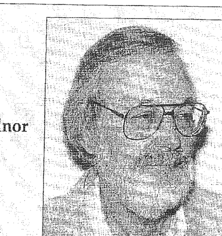
The first section of this article will describe this prize work. After a short digression, the third section will outline some of the work for which Nash is famous among mathematicians, and the last will briefly describe events since 1958.

[1] Compare [P]. This is the third Nobel prize to be awarded to a Princeton mathematics graduate. The first two were physics prizes, both to John Bardeen.

Game Theory

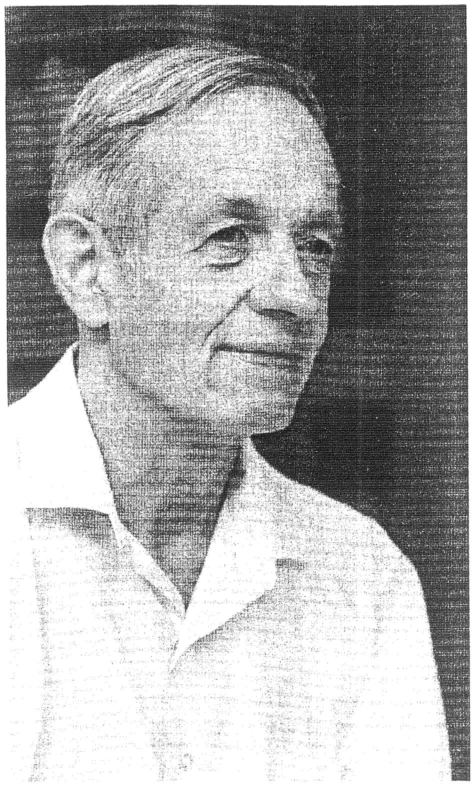
In the framework set down by von Neumann and Morgenstern [NM], an n-person game can be described as follows. There are n agents or players, numbered from 1 to n. For each i between 1 and n, the ith player has a set S_i of possible strategies, and chooses some element $s_i \in S_i$, where these choices are to be made simultaneously. The outcome of the game is then a function of the n choices $s_1, \ldots, s_n$. The ith player also has a preference ordering for the set of possible outcomes. This is conveniently described by a real-valued function

$$p_i: S_1 \times \cdots \times S_n \to \mathbf{R}.$$



John Milnor

John Milnor was educated at Princeton. He has worked in topology, geometry, algebra, dynamics, and (long ago) game theory. Since 1989, he has been Director of the Institute for Mathematical Sciences at SUNY Stony Brook.



John Nash in September 1994.

called his payoff function. Each player's objective is to make his own choice s_i in such a way as to maximize his payoff $p_i(s_1, \ldots, s_n)$, with the understanding that for each $j \neq i$, the jth player is simultaneously choosing s_j, trying to maximize the value of his own payoff p_j.

In interpreting this mathematical model, the various "players" can be individual persons. However, there are many other possibilities: The players can be nations, corporations, armies, teams, human-programmed computers, or animals. For the study of evolution, one considers competition between species or between genes. (Compare [MSP], [MS], and [D].) In the case of a game which extends over time, the "strategies" chosen by the players should be thought of, not as individual choices, but rather as comprehensive prescriptions of what to do in every conceivable situation which may arise during the course of the game. For example, a strategy for the game of chess might consist of a computer program which selects a move for every possible chess position. The "payoff" functions are usually not measured in something simple and objective, such as money, but rather are supposed to incorporate every relevant motivation which the players may have, whether selfish, altruistic, or whatever.

Although von Neumann and Morgenstern developed a beautiful theory of two-person games which are zero-sum, in the sense that $p_1 + p_2 = 0$, their theory for the more general case was complicated and unconvincing. Nash's theory, on the other hand, is direct and elegant:

Definition. An n-tuple of strategies $(s_1, \ldots, s_n)$ constitutes an *equilibrium point* for the game if no player can increase his payoff $p_i(s_1, \ldots, s_n)$ by changing only s_i while the other s_j remain fixed.

It is not claimed that an equilibrium point is a particularly desirable outcome for a game. Indeed, it may well be disastrous for all parties. (As an example, it is not difficult to describe a game of "atomic warfare" with only one equilibrium point, which requires each player to annihilate the others.) Rather, we must think of an equilibrium point as a description of what is likely to happen in a completely noncooperative situation, in which the players pursue their individual goals without any cooperation, either because they cannot communicate or because they have no mechanism or no will for cooperation. This contrasts with the von Neumann–Morgenstern work, which considered only cooperative games.

I am indebted to Hector Sussmann for two examples which show that equilibrium points are relevant even in everyday life:

Example 1. At a boring party, all of the guests want to go home early, but no one is willing to leave before midnight unless someone else leaves first. There is just one equilibrium point: everyone stays until midnight. (Compare [Sch].)

Example 2. A group of 20 is going to dinner, and each one has the choice of an adequate meal for 10 dollars or an excellent meal for 20 dollars. If paying individually, each one would choose the cheaper meal. However, they have decided to split the bill. Since the marginal cost of the more expensive meal for each person is only 50 cents, everyone chooses it.

Before stating Nash's basic existence theorem, we must introduce probabilities, via the von Neumann–Morgenstern theory of mixed strategies. To see why this is necessary, consider the following.

Example 3. A simple combination lock has 1000 possible combinations; the owner is free to choose any one of them. A would-be thief will have just one chance to guess the combination. Thus, we could take S_1 and S_2 to be finite sets, with 1000 elements each. However, with this mathematical model, no equilibrium point would exist.

What we must rather do, in order to obtain a reasonable theory, is to allow randomization in making choices. In fact, we take S_1 and S_2 to be 999-dimensional simplexes, with 1000 vertices each. A point of the simplex S_1 or of S_2 is to be construed as a probability distribution over the 1000 possible combinations. Now there is a unique equilibrium point (s_1, s_2), where each s_i is the probability distribution which assigns each choice of combination the probability 1/1000. The thief then has just 1 chance in 1000 of making the correct guess. (This is an example of a two-person zero-sum game, so a Nash equilibrium point in this case is the same thing as a pair of optimal strategies in the sense of von Neumann and Morgenstern.)

Following von Neumann and Morgenstern, such a weighted average of finitely many pure strategies, where the weighting coefficients are interpreted as probabilities, is called a *mixed strategy*. The set of all mixed strategies for a given player forms a finite-dimensional simplex.

EXISTENCE THEOREM. *If the space of strategies S_i for each player is a finite-dimensional simplex, and if each payoff function $p_i(s_1, \ldots, s_n)$ is continuous as a function of n variables, and is linear as a function of s_i when the other variables are kept fixed, then at least one equilibrium point exists.*

To prove this statement, we embed each S_i in a Euclidean space $\mathbf{R}^{d_i}$ of the same dimension, and consider the Cartesian product

$$K = S_1 \times \cdots \times S_n \subset \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_n}.$$

We can then construct a continuous vector field

$$(s_1, \ldots, s_n) \mapsto (v_1, \ldots, v_n) \in \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_n} \qquad (*)$$

as follows: The component v_i in the $\mathbf{R}^{d_i}$ direction is to be the gradient vector $\partial p_i / \partial s_i$ for the function p_i when considered as a linear function of s_i, with s_j kept fixed for every $j \neq i$. We will need the following:

LEMMA. *If $K \subset \mathbf{R}^d$ is a compact, convex set, and $v \colon K \to \mathbf{R}^d$ is any continuous function, then there exists at least one point $\hat{s} \in K$ where the vector field v either vanishes, or points out of K in the sense that every point of K lies in the half-space*

$$\{s \in \mathbf{R}^d : s \cdot v(\hat{s}) \leq \hat{s} \cdot v(\hat{s})\}.$$

Proof Outline. (See Figure 1.) Let $\rho : \mathbf{R}^d \to K$ be the canonical retraction, which carries every point of $\mathbf{R}^d$ to the closest point of K. Then the composition $s \mapsto \rho(s + v(s))$ maps K into itself and hence, by the Brouwer Fixed Point Theorem, has a fixed point

$$\hat{s} = \rho(\hat{s} + v(\hat{s})).$$

It is easy to check that $v(\hat{s})$ either vanishes or points out of K.

Applying this lemma to the vector field $(*)$, we obtain an n-tuple $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_n)$, which is the required equilibrium point. □

Commentary. As with any theory which constructs a mathematical model for some real-life problem, we must ask how realistic the model is. Does it help us to understand the real world? Does it make predictions which can be tested?

In the case of Nash's equilibrium-point theory, we might first ask whether this is intended as a descriptive theory which tells us how people actually act in a competitive situation, or a normative theory which tells us how rational people ought to act in order to achieve the best possible outcome. The answer is probably both, and neither. In fact, the two aspects can never really be separated, since a descriptive theory of how the other players are making their choices may be crucial for making one's own choice.

First let us ask about the realism of the underlying model. The hypothesis is that all of the players are rational, that they understand the precise rules of the game, and that they have complete information about the objectives of all of the other players. Clearly, this is seldom completely true.

One point which should particularly be noticed is the linearity hypothesis in Nash's theorem. This is a direct application of the von Neumann–Morgenstern theory of numerical utility: the claim that it is possible to measure the relative desirability of different possible outcomes by a real-valued function which is linear with respect to probabilities. In other words, if the strategy s yields a 50% chance of an outcome with utility a and a 50% chance of an outcome with utility b, whereas the strategy s' yields a 100% chance of an outcome with utility $(a+b)/2$, then the player should be indifferent between s and s'. This concept has been studied by many authors.

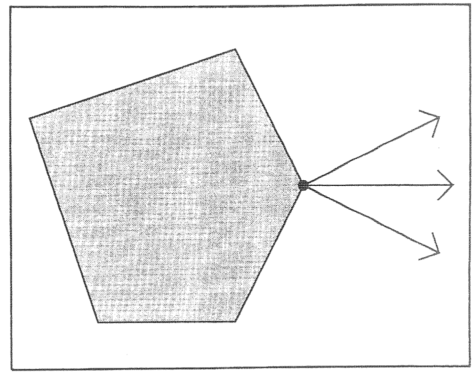


Figure 1. Examples of vectors pointing "out" at a boundary point of a compact convex set.

(See for example [HM].) My own belief is that this is quite reasonable as a normative theory, but that it may not be realistic as a descriptive theory.

Under what conditions does equilibrium-point theory really apply? A game may have many equilibrium points, some better than others for one player or even for all of the players. If the game is played only once, with no communication at all between the players, how will they know which of these equilibrium points is relevant? If the players do communicate, at what point does it cease being a "noncooperative" game? Another common scenario would be a game which is played over and over again, perhaps gradually settling toward some equilibrium. In this case, there is the added complication that the players' utility functions may not remain the same for each play.

Evidently, Nash's theory was not a finished answer to the problem of understanding competitive situations. Rather, it was a starting point, which has led to much further study during the intervening years. In fact it should be emphasized that no simple mathematical theory can provide a complete answer, since the psychology of the players and the mechanism of their interaction may be crucial to a more precise understanding.

Games

Nash entered Princeton as a graduate student in 1948, the same year that I entered as a freshman. I quickly got to know him, since we both spent a great deal of time in the common room. He was always full of mathematical ideas, not only on game theory, but in geometry[2] and topology as well. However, my most vivid memory of this time is of the many games which were played in the common room. I was introduced to Go and Kriegspiel, and also to an ingenious topological game which we called Nash, in honor of the inventor. In fact it was later discovered that the same game had been invented a few years earlier by Piet Hein in Denmark. Hein called it Hex, and it is now commonly known by that name. An $n \times n$ Nash or Hex board consists of a rhombus which is tiled by n^2 hexagons, as illustrated in Figure 2. (The recommended size for an enjoyable game is 14×14. However, a much smaller board is shown here for illustrative purposes.) Two opposite edges are colored black, and the remaining two are colored white. The players alternately place pieces on the hexagons, and once played, a piece is never moved. The black player tries to construct a connected chain of black pieces joining the two black boundaries, while the white player tries to form a connected chain of white pieces joining the white boundaries. The game continues until one player or the other succeeds.

[2] Here is one question asked by Nash. Let V_0 be a singular algebraic variety of dimension k, embedded in some smooth variety M_0, and let $M_1 = G_k(M_0)$ be the Grassmann variety of tangent k-planes to M_0. Then V_0 lifts naturally to a k-dimensional variety $V_1 \subset M_1$. Continuing inductively, we obtain a sequence of k-dimensional varieties $V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \cdots$. Do we eventually reach a variety V_q which is nonsingular? Even today, this has been proved only in special cases. (Compare [G-S], [H], and [Sp].)

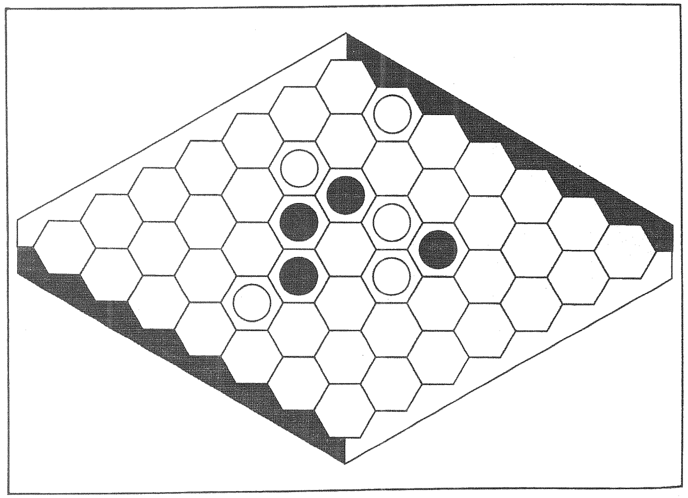


Figure 2. A typical situation in the game of Hex. Problem: Black to move and win. Alternate Problem: White to move and win. (Solution on page 56.)

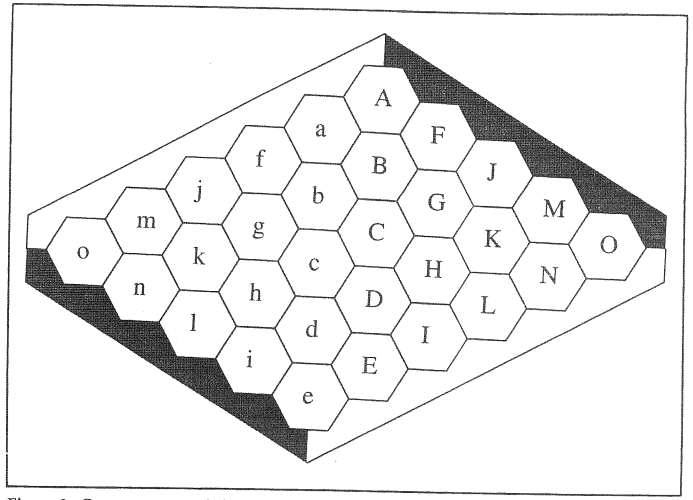


Figure 3. On an asymmetric board, as shown, white can win even if black moves first. The winning strategy can be explicitly described by "doubles": Whatever move black makes, white responds by playing in the hexagon which is marked with the corresponding symbol, where the correspondence (for example, $a \leftrightarrow A$) is a glide reflection which flips the left half of the board onto the right half.

THEOREM. *On an $n \times n$ Hex board, the first player can always win.*

Nash's proof is marvelously nonconstructive and can be outlined as follows.

First Step. A purely topological argument shows that, in any play of the game, one player or the other must win: If the board is covered by black and white pieces, then there exists either a black chain from black to black or a white chain from white to white, but never both.

Second Step. Since this game is finite, with only two possible outcomes, and since the players move alternately with complete information, a theorem of Zermelo, rediscovered by von Neumann and Morgenstern, asserts that one of the two players must have a winning strategy.

Third Step, by symmetry. If the second player had a winning strategy, the first player could just make an initial move at random, and then follow the strategy for the second player. Since his initial play can never hurt him, he must win. Thus, the hypothesis that the second player has a winning strategy leads to a contradiction. (This is a well-known argument which applies to some other symmetric games, such as Five-in-a-Row.) □

Note that this proof depends strongly on the symmetry of the board. On an $n \times (n+1)$ board, the player with the shorter distance to connect can always win, even if the other player has the first move. (Compare Fig. 3.)

Geometry and Analysis

After receiving his doctorate, Nash moved to M.I.T., where he produced a remarkable series of papers. The first was a basic contribution to the theory of real algebraic varieties.

THEOREM. *Given any smooth compact k-dimensional manifold M, there exists a real algebraic variety $V \subset \mathbf{R}^{2k+1}$ and a connected component V_0 of V so that V_0 is a smooth manifold diffeomorphic to M.*

He complemented this theorem by giving an abstract characterization of such manifolds V_0 by means of a suitable algebra of real-valued functions.

As one example of the power of this result, let me describe an important application. A basic problem in dynamics is to understand how the number of periodic points of period p for a smooth map can increase as a function of p.

THEOREM OF ARTIN AND MAZUR. *Any smooth map from a compact manifold to itself can be approximated by a smooth map such that the number of periodic points of period p grows at most exponentially with p.*

The only known proof of this result makes essential use of Nash's work, in order to translate the dynamic problem into an algebraic one of counting solutions to polynomial equations.

Two years later, he attacked one of the fundamental unsolved problems in Riemannian geometry, namely, the Isometric Embedding Problem for Riemannian manifolds. In other words, he considered the system of differential equations

$$\frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j} = g_{ij}(u_1, \ldots, u_k),$$

where the u_i are local coordinates for some given k-dimensional Riemannian manifold, where $g_{ij}(u_1, \ldots, u_k)$ is the prescribed Riemannian metric, and where $\mathbf{x}(u_1, \ldots, u_k)$ is the unknown isometric embedding into Euclidean n-space. This is a system of $k(k+1)/2$ nonlinear differential equations in n unknown functions, to be solved globally, over the entire manifold. He first tackled the C^1-case.

Every student of differential geometry knows that a compact surface without boundary in Euclidean 3-space must have points of positive curvature. (*Proof*: Enclose the surface in a spherical balloon, and then move the balloon until it first touches the surface. Then both principal curvatures at the point of contact must be nonzero, with the same sign. Hence, the Gaussian curvature at this point is strictly positive.) As an example, it follows that a flat torus $S^1 \times S^2 \subset \mathbf{R}^2 \times \mathbf{R}^2$ has no smooth isometric embedding into 3-space.

Nash ignored such difficulties. Incorporating a later improvement by Kuiper [K], we can state his result as follows.

THEOREM. *If a compact Riemannian manifold (M, g) can be smoothly embedded in the Euclidean space $\mathbf{R}^n$, then it can be C^1-isometrically embedded in $\mathbf{R}^n$.*

Here is a very rough outline of the proof. Start with any smooth embedding and shrink it uniformly until all distances in the induced metric are shorter than distances in the given metric g. Next introduce small sinusoidal ripples in the embedded manifold so as to increase the Euclidean lengths of curves in one coordinate patch after another. Now repeat this process, keeping careful control of first derivatives at every stage, so that the Riemannian metric induced from the embedding will increase monotonically toward the required metric g.

The catch in this construction is that there is no way of keeping control of second derivatives. Thus, the embedding which is constructed will never be C^2-smooth. Since the concept of curvature involves second derivatives in an essential way, any argument involving principal curvatures simply will not apply to the resulting embedding.

Next he attacked the much more serious C^r-embedding problem, for $r > 1$.

THEOREM. *If $n \geq k(k+1)(3k+11)/2$, then every k-dimensional Riemannian manifold of class C^r can be C^r-isometrically embedded into $\mathbf{R}^n$, for $3 \leq r \leq \infty$.*

To prove this result, he introduced an entirely new method into nonlinear analysis. As later generalized by Moser, this method can be roughly described as follows. We are trying to solve some system of equations in an infinite-dimensional space of functions f. Given an approximate solution f_0, we can apply some linear approximation procedure analogous to Newton's method, to produce a better approximation g_0. The difficulty is that this linear procedure typically involves differentiation, so that g_0 is less differentiable than f_0. The trick is then to apply a smoothing operator, approximating g_0 by a function f_1 which has better smoothness properties. We can then continue inductively, constructing a sequence of approximations $f_0, f_1, f_2, \ldots$, with extremely careful estimates at every stage. With appropriate hypotheses, these will converge to the required solution. (For further development of these ideas, see [Gr] and [Gü].)

At this time he began a deep study of parabolic and elliptic differential equations, proving basic local existence, uniqueness, and continuity theorems (and also speculating about relations with statistical mechanics, singularities, and turbulence). This work has been somewhat neglected. In fact, a 1957 paper by De Giorgi [DG] has tended to dominate the field. The methods were quite different, but both authors were strikingly original, and made real breakthroughs. De Giorgi considered only the elliptic case, whereas Nash rather assigned a primary role to parabolic equations. His methods, based on a moment inequality for the fundamental solution, are quite powerful. (Compare [FS].) Here are some quotations (abridged and mildly edited) from his paper on "continuity of solutions" (Nash [12]), which help to describe his vision and goals in 1958.

> The open problems in the area of non-linear partial differential equations are very relevant to applied mathematics and science as a whole, perhaps more so than the open problems in any other area of mathematics, and this field seems poised for rapid development. Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid. Also, the relationship between this continuum description of a fluid and the more physically valid statistical mechanical description is not well understood. Probably one should first try to prove existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain *gross* types of singularity, such as infinities of temperature or density. A result of this kind would clarify the turbulence problem.
>
> Successful treatment of non-linear partial differential equations generally depends on 'a priori' estimates, which are themselves theorems about linear equations. . . . The methods used here were inspired by physical intuition, of diffusion, Brownian movement, and flow of heat or electrical charges, but the ritual of mathematical exposition tends to hide this natural basis.

Epilogue

In 1958, at the age of 30, Nash suffered a devastating attack of mental illness. (Compare [N].) There followed many horrible years: periods of confinement to mental hospitals, usually involuntary and often accompanied by shock treatments, interspersed with periods of partial recovery. During a brief respite in 1966, he published one further paper, showing that his isometric embedding theorem, and more generally the Nash–Moser implicit function machinery, can be extended to the real-analytic case. There followed an extremely long fallow period. I lost touch with him during this time; however, I was very happy to hear that in recent years his illness has abated, and that he has regained interest in major unsolved problems. This year, Nash not only attended the award ceremonies in Stockholm but also gave a seminar in Uppsala on his recent work in mathematical physics.

I conclude by congratulating John Nash, not just for his prize, but for his many contributions to human knowledge, and offer him all best wishes for the future.

Acknowledgments

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[3] Note that an evolutionarily stable strategy in the sense of Maynard Smith and Price is nearly the same thing as a Nash equilibrium point.