[B]

① $f(x) = \sqrt{1+x} - \sqrt{x} = \dfrac{1}{\sqrt{1+x}+\sqrt{x}} \to \dfrac{1}{\infty} = 0; \quad x \to +\infty.$

neboť $\sqrt{x} \to +\infty; \quad x \to +\infty$

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② $f(x) = x\left(\sqrt{x^2+1} - \sqrt{x^2-1}\right) = \dfrac{2x}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \dfrac{2\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}\right)}$

neboť $\sqrt{1+\frac{1}{x^2}} \to \sqrt{1}$ $\quad \to 1; \quad x \to +\infty$

díky aritm. limit a spojitosti $\sqrt{y}$.

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③ $f(x) = x^{4/3}\left(\sqrt[3]{x^2+1} - \sqrt[3]{x^2-1}\right); \quad \sqrt[3]{a} - \sqrt[3]{b} = \dfrac{a-b}{(\sqrt[3]{a})^2 + \sqrt[3]{a}\sqrt[3]{b} + (\sqrt[3]{b})^2}$

$= \dfrac{2x^{4/3}}{(\sqrt[3]{x^2+1})^2 + \sqrt[3]{x^2+1}\cdot\sqrt[3]{x^2-1} + (\sqrt[3]{x^2-1})^2};$

protože $\sqrt[3]{x^2 \pm 1} = x^{2/3}\sqrt[3]{1 \pm \frac{1}{x^2}}$, je

$f(x) = \dfrac{2}{\left(\sqrt[3]{1+\frac{1}{x^2}}\right)^2 + \sqrt[3]{1+\frac{1}{x^2}}\sqrt[3]{1-\frac{1}{x^2}} + \left(\sqrt[3]{1-\frac{1}{x^2}}\right)^2} \to \dfrac{2}{3}$ [m.t.] $x \to +\infty$

neboť $1+\frac{1}{x^2} \to 1; \quad x \to +\infty$

$g(y) = \sqrt[3]{y}$ spojitá v bodě $y_0 = 1$. $\quad$ V. 2.6. (a)

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④ $f(x) = \dfrac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}} = \dfrac{x^{20}\left(2-\frac{3}{x}\right)^{20} x^{30}\left(3+\frac{2}{x}\right)^{30}}{x^{50}\left(2+\frac{1}{x}\right)^{50}}$

$= \dfrac{\left(2-\frac{3}{x}\right)^{20}\left(3+\frac{2}{x}\right)^{30}}{\left(2+\frac{1}{x}\right)^{50}} \to \dfrac{2^{20}\,3^{30}}{2^{50}} = \left(\dfrac{3}{2}\right)^{30}; \quad x \to +\infty.$

[m.t.]

(5) $f(x) = \dfrac{2x^2+1}{\sqrt{3x^4-6x^2+5}} = \dfrac{\cancel{x^2}\cdot\left(2+\frac{1}{x^2}\right)}{\cancel{x^2}\sqrt{3-\frac{6}{x^2}+\frac{5}{x^4}}} \to \dfrac{2}{\sqrt{3}}, \ x\to+\infty$

jmenovatel: $3-\frac{6}{x^2}-\frac{5}{x^4} \to 3$

V. 2.6. (a)

$g(y) = \sqrt{y}$ spojitá v $y_0 = 3$.

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(6) $f(x) = \dfrac{\ln(1+3^x)}{\ln(1+2^x)} = \dfrac{x\cdot\ln 3 + \ln(1+3^{-x})}{x\cdot\ln 2 + \ln(1+2^{-x})}$

viz: $\ln(1+3^x) = \ln\left\{3^x\left(1+\frac{1}{3^x}\right)\right\} = \ln 3^x + \ln\left(1+\frac{1}{3^x}\right)$

$= x\cdot\ln 3 + \ln\left(1+\frac{1}{3^x}\right)$

$= \dfrac{\ln 3 + \frac{1}{x}\ln\left(1+\frac{1}{3^x}\right)}{\ln 2 + \frac{1}{x}\ln\left(1+\frac{1}{2^x}\right)} \to \dfrac{\ln 3}{\ln 2}; \ x\to+\infty$

neboť $\frac{1}{x}\to 0$; $\ln\left(1+\frac{1}{\underbrace{3^x}_{\to+\infty}}\right)\to\ln(1)=0$; $x\to+\infty$.

podobně jmenovatel.

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(7) $f(x) = \sin\sqrt{x+1} - \sin\sqrt{x}$; vzoreček: $\sin a - \sin b$

$= 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$

$= 2\underbrace{\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}}_{\to 0;}\cdot\underbrace{\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}}_{\text{omezené}}$; tj. $f(x)\to 0, \ x\to+\infty$

díky V. 2.4.

neboť $\sqrt{x+1}-\sqrt{x}\to 0$

(viz B1)

(8) $f(x) = \exp\left(x \ln \frac{x+a}{x-a}\right) \longrightarrow e^{2a}; \quad x \to +\infty$

$a \neq 0$

$$x \cdot \ln \frac{x+a}{x-a} = \frac{\ln \frac{x+a}{x-a}}{\frac{x+a}{x-a} - 1} \cdot x \cdot \left(\frac{x+a}{x-a} - 1\right) = P_1 \cdot P_2 \to 2a$$

$$\left.\begin{array}{l} \frac{\ln z}{z-1} \to 1; \; z \to 1 \\ \frac{x+a}{x-a} \to 1; \; x \to +\infty \\ \quad \frac{x+a}{x-a} = 1: \; x+a \neq x-a \text{ na } P(+\infty) \\ \qquad\qquad a \neq 0 \end{array}\right\} \Rightarrow P_1 \to 1; \; x \to +\infty$$

$$P_2 = \frac{x \cdot 2a}{x-a} = \frac{2a}{1-\frac{a}{x}} \to \frac{2a}{1-\frac{a}{+\infty}} = 2a.$$

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(9) $f(x) = \frac{x^a}{\ln^b x} = x^a (\ln x)^{-b}$

$$\text{(i) } a>0, b<0: \left.\begin{array}{l} x^a \to +\infty \\ \ln^{-b} x \to +\infty \end{array}\right\} \Rightarrow f(x) \to +\infty \cdot +\infty = +\infty$$

$$\text{(ii) } a<0; b>0: \left.\begin{array}{l} x^a \to 0 \\ \ln^{-b} x \to 0 \end{array}\right\} \Rightarrow f(x) \to 0 . 0 = 0.$$

$a, b > 0: \quad f(x) = \left(\frac{x^{a/b}}{\ln x}\right)^b$

$$\text{víme: } \left.\begin{array}{l} \frac{y}{\ln y} \to \infty; \; y \to +\infty \\ x^\gamma \to +\infty; \; x \to +\infty \\ (\gamma > 0) \text{ zjevně } \neq +\infty \end{array}\right\} \Rightarrow \frac{x^\gamma}{\ln x^\gamma} = \frac{1}{\gamma} \cdot \frac{x^\gamma}{\ln x} \to +\infty$$

užij pro $\gamma = a/b > 0$;

$y^b \to +\infty; \; y \to +\infty.$

(10) $f(x) = x^a |\ln x|^b \to 0;\ x \to 0+;\ a, b > 0.$

$$\lim_{x\to 0+} f(x) = \lim_{y\to+\infty} f\left(\frac{1}{y}\right) = \lim_{y\to+\infty} \frac{|\ln \frac{1}{y}|^b}{y^a} = \lim_{y\to+\infty} \frac{1}{\frac{y^a}{\ln^b y}}$$

příkl. č. 9 $\Rightarrow \frac{y^a}{\ln^b y} \to +\infty$; tj. výsledek: $\underline{0}$.

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(11) $f(x) = \frac{x^a}{e^{bx}} = \left(\frac{x}{e^{\frac{bx}{a}}}\right)^a \to +\infty$

vίme: $\left.\begin{array}{l} \frac{y}{e^y} \to 0;\ y \to +\infty \\ \gamma x \to +\infty;\ x \to +\infty \\ \gamma > 0 \text{ pevné} \end{array}\right\} \Rightarrow \frac{\gamma \cdot x}{e^{\gamma x}} \to +\infty;\ x \to +\infty$

tedy: $\frac{x}{e^{\gamma x}} \to +\infty$
$\forall \gamma > 0$.

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(12) $f(x) = \frac{2^x + 3^x + 2xe^x}{x^n \ln x + 3^x} = \frac{\left(\frac{2}{3}\right)^x + 1 + 2x\left(e/3\right)^x}{\frac{x^n \ln x}{3^x} + 1} \cdot \cancel{\frac{3^x}{3^x}} \to 1$
$x \to +\infty$

vytknu vedoucí člen

jednotlivé členy:

$\left(\frac{2}{3}\right)^x = e^{x \ln \frac{2}{3}} \to 0$, $\quad e^y \to 0;\ y \to -\infty$
$\ln 2/3 < 0:\ x \cdot \ln 2/3 \to -\infty;\ x \to +\infty.$

$x\left(\frac{e}{3}\right)^x = \frac{x}{\exp[\ln 3 - 1]x} \to 0;\ x \to +\infty$ (př. 11)
$\ln 3 > 1$ $(3 > e)$.

$$\frac{x^n \cdot \ln x}{e^{x \cdot \ln 3}} = \frac{x^n}{e^{x \frac{1}{2} \ln 3}} \cdot \frac{\ln x}{x} \cdot \frac{x}{e^{x \frac{1}{2}\ln 3}} \to 0 \cdot 0 \cdot 0 = 0$$

$\frac{1}{2} \ln 3 > 0$; viz př. 11, 9.

(13) $f(x) = \dfrac{x \cdot \ln x \cdot \operatorname{arctg} x}{\sqrt{x} \cdot \ln x^2} = \frac{1}{2}\sqrt{x} \cdot \operatorname{arctg} x \to +\infty \cdot \frac{\pi}{2} = +\infty$;

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(14) $f(x) = (1-\cos x) \cdot \ln x = \dfrac{1-\cos x}{x^2} \cdot x^2 \ln x \to \frac{1}{2} \cdot 0 = 0$

viz př. 10 & výsledek limity pro $\underline{\cos x}$:

$$\frac{1-\cos x}{x^2} \cdot \underbrace{\frac{1+\cos x}{1+\cos x}}_{\neq 0 \text{ na } P(0,\pi)} = \frac{1-\cos^2 x}{x^2} \cdot \frac{1}{1+\cos x} = \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1+\cos x} \to \frac{1}{2}$$

$\frac{\sin x}{x} \to 1;\ x \to 0$; $\quad \dfrac{1}{1+\cos x} \to \dfrac{1}{1+\cos 0} = \dfrac{1}{2}$

$g(y) = y^2$ .. spojité v 0 $\quad$ $\cos x$ spojité v 0.

---

(15) $\dfrac{\ln(1+e^x)}{x} = \dfrac{\ln(e^x(1+e^{-x}))}{x} = \dfrac{x + \ln(1+e^{-x})}{x}$

$= 1 + \frac{1}{x}\ln(1+e^{-x}) \to 1$; $x \to +\infty$

$\left.\begin{array}{l} e^{-x} \to 0;\ x \to +\infty \\ \ln y \text{ spojitá v } y_0 = 1 \end{array}\right\} \xRightarrow{V.2.6(a)} \ln(1+e^{-x}) \to \ln 1 = 0$, $x \to +\infty$.

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(16) $f(x) = \dfrac{\sqrt{x^6(3+x^{-4}+x^{-6}e^{-x})}}{\sqrt[4]{x^{12}(1-x^{-24})}} = \dfrac{\cancel{x^3}\sqrt{3+x^{-4}+x^{-6}e^{-x}}}{\cancel{x^3}\sqrt[4]{1-x^{-24}}} \to \sqrt{3}$, $x \to +\infty$.

$x^{-a} \to 0;\ x \to +\infty;\ a > 0$

$e^{-x} \to 0$

$\sqrt[4]{y}, \sqrt{y}$ spojité $\forall y_0 > 0$.

(17) $\ln\left(\frac{x^2-3}{x+1}\right) = \ln x\cdot\left(\frac{x^2-3}{x(x+1)}\right) = \ln x + \ln\frac{x^2-3}{x^2+x}$

$$f(x) = \frac{\ln x + \ln\left(\frac{x^2-3}{x^2+x}\right)}{2\ln x} = \frac{1+\frac{\ln\frac{x^2-3}{x^2+x}}{\ln x}}{2} \to \frac{1}{2};\ x\to+\infty$$

neboť: $\ln x \to +\infty$

$$\frac{x^2-3}{x^2+x} = \frac{1-3/x^2}{1+\frac{1}{x}} \to \frac{1-\frac{3}{+\infty}}{1-\frac{1}{+\infty}} = 1;$$

$\ln y$ .. spojité v $y_0 = 1$.

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(18) $$f(x) = \frac{e\cdot x}{\ln\left(e^{3x}(1+e^{-6x})\right)} = \frac{e\cdot x}{3x+\ln(1+e^{-6x})} =$$

$$= \frac{e}{3+\frac{1}{x}\ln(1+e^{-6x})} \to \frac{e}{3};\ x\to+\infty$$

$\frac{1}{x}\to 0$; $1+e^{-6x}\to 1$; $\ln y$ spojité v $y_0 = 1$.

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(19) $$f(x) = \sqrt{x^3}\left(\sqrt{x+1}-\sqrt{x}+\sqrt{x-1}-\sqrt{x}\right) = (\sqrt{x})^3\left(\frac{1}{\sqrt{x+1}-\sqrt{x}} - \frac{1}{\sqrt{x-1}+\sqrt{x}}\right)$$

$$= \frac{\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x-1}+\sqrt{x}}\cdot\sqrt{x}\,\underbrace{\left(\sqrt{x-1}-\sqrt{x+1}\right)}$$

$$= \frac{-2}{\sqrt{x-1}+\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{1+\frac{1}{x}}+1}\cdot\frac{1}{\sqrt{1-\frac{1}{x}}+1}\cdot\frac{-2}{\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{1}{x}}} \longrightarrow \frac{1}{2}\cdot\frac{1}{2}\cdot\left(\frac{-2}{2}\right) = -\frac{1}{4}$$

(20) $$f(x)=\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x+\sqrt{x-\sqrt{x}}}=\frac{\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x+\sqrt{x-\sqrt{x}}}}$$

$$=\frac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}\cdot\frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x+\sqrt{x-\sqrt{x}}}}=P_1\cdot P_2$$

$$P_1=\frac{2}{\sqrt{1+\sqrt{\frac{1}{x^3}}}+\sqrt{1-\sqrt{\frac{1}{x^3}}}}\to\frac{2}{1+1}=2\,;$$

nelok.: $\frac{1}{x^3}\to 0$; >0 $\quad g(y)=\sqrt{y}$ spojitá v $y_0=1$
a v bodě $y_0=1$ zprava.

$$P_2\to\frac{1}{+\infty}=0,$$ nelos $\sqrt{y}\to+\infty$; $y\to+\infty$

$$x-\sqrt{x}=x\left(1-\frac{1}{\sqrt{x}}\right)\to+\infty\left(1-\frac{1}{+\infty}\right)=+\infty$$