I. Černý: (7.45) $f(x) = 2|\sin x| + |\cos 2x|$;

$\uparrow$ π- periodické $\quad$ $\uparrow$ $\frac{\pi}{2}$ -- periodické.

π- periodické, spojité v $\mathbb{R}$; $\lim_{x\to\pm\infty} f(x)$ neexistuje.

$\boxed{\text{Omezíme se na } x \in [0,\pi].}$ symetrie vůči $x = \frac{\pi}{2}$.

$f(x) = 0$ .... $\sin x = 0$ & $\cos 2x = 0$

$x = 0, \pi$ $\quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$ min

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$$f'(x) = 2\,\mathrm{sgn}(\sin x)\cdot\cos x + \mathrm{sgn}(\cos 2x)\cdot(-2\sin 2x)$$

$x \neq 0, \pi, \frac{\pi}{4}, \frac{3\pi}{4}$.

$$2\cos x\left(\mathrm{sgn}(\sin x) - 2\,\mathrm{sgn}(\cos 2x)\cdot\sin x\right)$$

$$f'_+(0) = \lim_{x\to 0+} f'(x) = \lim_{x\to 0+} \underbrace{2\cos x}_{\to 2}\left(1 - 2\underbrace{\sin x}_{\to 0}\cdot\mathrm{sgn}(\cos 2x)\right) = +2$$

$f'_-(0) = -2$; $\quad 2\pi$-per. $\quad f'_\pm(\pi) = \pm 2$.

$$f'_+\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}+}\left[\underbrace{2\cos x}_{\to 2\cdot\frac{1}{\sqrt{2}}}\underbrace{\left(\underbrace{\mathrm{sgn}(\sin x)}_{\equiv 1} - 2\underbrace{\mathrm{sgn}(\cos 2x)}_{\equiv -1}\underbrace{\sin x}_{\to\frac{1}{\sqrt{2}}}\right)}_{\text{na } P_+\left(\frac{\pi}{4}\right)}\right] =$$

$$= \sqrt{2}\left(1 + \sqrt{2}\right) = 2 + \sqrt{2} \doteq \underline{3.4}.$$

$$f'_-\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}-}\left[\underbrace{2\cos x}_{2\cdot\frac{1}{\sqrt{2}}}\left(\underbrace{\mathrm{sgn}(\sin x)}_{\equiv 1} - 2\underbrace{\mathrm{sgn}(\cos 2x)}_{\equiv 1}\underbrace{\sin x}_{\to\frac{1}{\sqrt{2}}}\right)\right] =$$

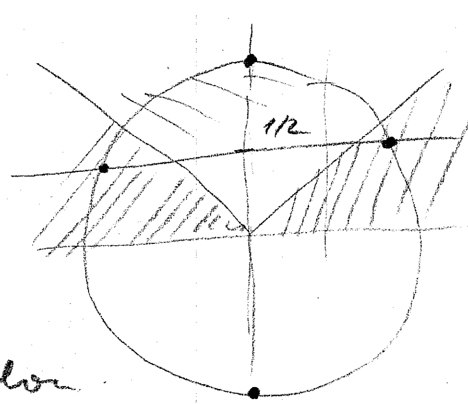
$$= \sqrt{2}\left(1 - \sqrt{2}\right) = \sqrt{2} - 2 \doteq 0.6$$

---

analogicky: $f'_+\left(\frac{3\pi}{4}\right) = \sqrt{2} - 2$; $\quad f'_-\left(\frac{3\pi}{4}\right) = 2 + \sqrt{2}$.

? $f'(x)=0$: $x\in(0,\pi)$; řeš.

$$f'(x) = 2\cos x\left(1 - 2\,\mathrm{sgn}(\cos 2x)\cdot \sin x\right)$$



A. $x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{3\pi}{4},\pi\right)$: $\cos 2x>0$:

$$0 = 2\cos x\left(1-2\sin x\right)$$

(a) $\cos x = 0$: $\frac{\pi}{2}+k\pi$ — neleží v daném oboru

(b) $\sin x = \frac{1}{2}$: $\boxed{x=\frac{\pi}{6},\ \frac{5\pi}{6}}$

B. $x\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$: $0=2\cos x\left(1+2\sin x\right)$

$\boxed{x=\frac{\pi}{2}}$ $\sin x = -\frac{1}{2}$ neleží v daném oboru.

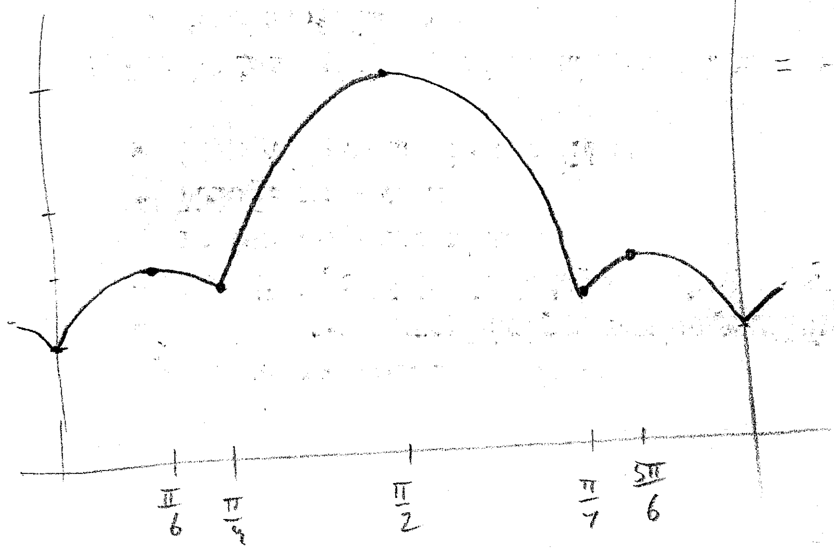
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$f(x)$	1	1.5	1.4	3	1.4	1.5	1

$$f\left(\frac{\pi}{6}\right) = 2\cdot\sin\frac{\pi}{6} + \left|\cos\frac{\pi}{3}\right| = 2\cdot\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$

$$f\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} = 1.4$$

$$f\left(\frac{\pi}{2}\right) = 2+1 = 3$$

pozn.: $f''(x) = \left[2\cos x\left(\mathrm{sgn}(\sin x) - 2\,\mathrm{sgn}(\cos 2x)\cdot\sin x\right)\right]$



Pozn.: $f''(x) \neq 0$ nikde $\Rightarrow$ nemění znaménko

A. $x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$:

$$f'(x) = 2\cos x\,(1 - 2\sin x)$$

$$f''(x) = -2\sin x\,(1 - 2\sin x) + 2\cos x\,(-2\cos x) = 8\sin^2 x - 2\sin x - 4$$

$$z = \sin x: \quad z_{1,2} = \frac{1 \pm \sqrt{33}}{8} \doteq \begin{cases} 0.84 \\ -0.59 \end{cases}$$

≤ 2

leč $\sin x \in \left[0, \frac{1}{\sqrt{2}}\right]$

$\frac{1}{\sqrt{2}} = 0.71$

B. $x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$:

$$f'(x) = 2\cos x\,(1 + 2\sin x)$$

$$f''(x) = -2\sin x\,(1 + 2\sin x) + 2\cos x\,(2\cos x)$$

$$= -2\sin x - 4\sin^2 x + 4\cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$= -2\sin x - 8\sin^2 x + 4$$

$$f''(x) = 0 \Leftrightarrow 8z^2 + 2z - 4 = 0; \quad (z = \sin x)$$

$$4z^2 + z - 2 = 0; \quad D = 1 + 32 = 33$$

$$z_{1,2} = \frac{-1 \pm \sqrt{33}}{8} = \begin{cases} \underline{0.6} < \frac{1}{\sqrt{2}} \\ \underline{-0.8} \end{cases}$$

Průběh: $f(x) = \sqrt[3]{x^2(x-6)}$; množné $D(f)=\mathbb{R}$ ($\sqrt[3]{y}$ spojitá s polynomem)

1 (bez f'')
2 (bez f'')

$f(\pm\infty\mp) = \pm\infty$.

nulové body: $x=0$, $x=6$

$f(0) \doteq -1.8\cancel{77}$

$$(\sqrt[3]{y})' = \frac{1}{3}\frac{1}{\sqrt[3]{y^2}}$$

$\neq y^{-\frac{2}{3}}\cdot\frac{1}{3}$ pouze pro $y>0$.

$\sqrt[3]{-1} = -1$

$(-1)^{2/3} := \exp(\ln(-1)\cdot\frac{2}{3})$ není definováno.

$$f'(x) = \frac{3x^2 - \cancel{6}\,12x}{3\sqrt[3]{x^4(x-6)^2}} \quad ; x \neq 0, 6$$

$$(x^3-6x^2)' = 3x^2 - \cancel{6}\,12x \qquad \boxed{= \frac{x-4}{\sqrt[3]{x(x-6)^2}}}$$

$$= \frac{x(x-2)}{\sqrt[3]{x^4(x-6)^2}} \quad ; \quad \text{nulový bod: } \underline{x=2}.$$

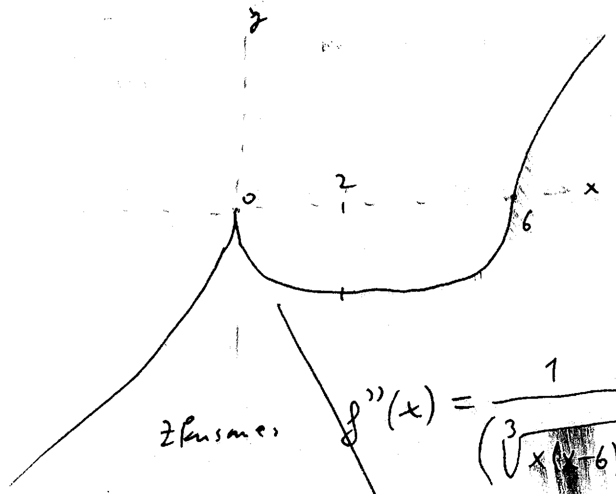
$$f'_{\pm}(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sqrt[3]{x^2(x-6)}}{\sqrt[3]{x^3}} = \lim_{x\to 0}\sqrt[3]{\frac{x-6}{x}}$$

$\to \pm\infty$ pro $x\to 0\mp$

i.e. $f'_{\pm}(0) = \mp\infty$.

$$f'(6) = \lim_{x\to 6} f'(x) = \lim_{x\to 6}\frac{x(x-2)}{\underbrace{\sqrt[3]{x^4(x-6)^2}}_{\to 0\,;\ >0 \text{ vždy}\ldots}} = +\infty$$

$\to 6\cdot 4 = 24$



$$f'(x) = \frac{(x-2)}{\sqrt[3]{x(x-6)^2}}$$

$\cancel{x^2-6x}$ $(x(x-6))^2 =$

Zkusme: $$f''(x) = \frac{1}{\left(\sqrt[3]{x(x-6)^2}\right)^2}\cdot\left\{1\cdot\sqrt[3]{x(x-6)^2} - (x-2)\cdot\frac{1}{3}\cdot\frac{3(x-2)(x-6)}{\sqrt[3]{x^2(x-6)^4}}\right\}$$

$\cancel{2x-6}$

$x \neq 0, 6$.

$f''(x)=0$: $\sqrt[3]{x^3(x-6)^6} = \frac{2}{3}(x-2)(x-3)$

$x(x-6)^2 = \frac{2}{3}(x-2)(x-3)$

$x(x-6)^2 = x(x^2-12x+36) = x^3-12x^2+36x$

$(\ )' = 3x^2-24x+36 = 3\cdot(x^2-8x+12) = 3(x-2)(x-6)$

$$f'(x) = \frac{x-4}{\sqrt[3]{x(x-6)^2}} \quad ; \quad x \neq 0, 6$$

$$f''(x) = \frac{1}{(\quad)^2} \left\{ \sqrt[3]{x(x-6)^2} - (x-4) \cdot \frac{3(x-2)(x-6)}{3\sqrt[3]{x^2(x-6)^4}} \right\}$$

$$= \frac{1}{(\quad)^2} \cdot \frac{1}{\sqrt[3]{x^2(x-6)^4}} \left\{ \sqrt[3]{x^3(x-6)^6} - (x-4)(x-2)(x-6) \right\}$$

$$h(x) = x(x-6)^2 - (x-4)(x-2)(x-6)$$

$$= (x-6) \cdot \left\{ x(x-6) - (x-2)(x-4) \right\}$$

$$= (x-6) \left\{ x^2 - 6x - (x^2 - 6x + 8) \right\}$$

$$= \frac{-8(x-6)}{(\quad)} .$$

$x<0$: $f''(x) > 0$: $f(x)$ ryze konvexní v $(-\infty, 0]$

$x \in (0,6)$: $f''(x) > 0$: $f(x)$ ryze konvexní v $[0, 6]$

$x > 6$: $f''(x) < 0$: $f(x)$

pozn.: $f(x)$ není konvexní na $(-\infty, 6]$

$x=6$ je inflexní bod ($f'(6)$ existuje!!)

Př.: $f(x) = 5 \sin x \cos^3 x$ — liché, π - periodické

zvolíme $I = \left[0, \frac{\pi}{2}\right]$.

$$f'(x) = 5 \cos x \cos^3 x - 35 \sin^2 \cos^2 x =$$
$$= 5 \cos^2 x \left(\cos^2 x - 3 \sin^2 x\right).$$

$$\frac{\sin^2 x}{\cos^2 x} = \frac{1}{3} \qquad x = \pm \frac{\pi}{6}$$

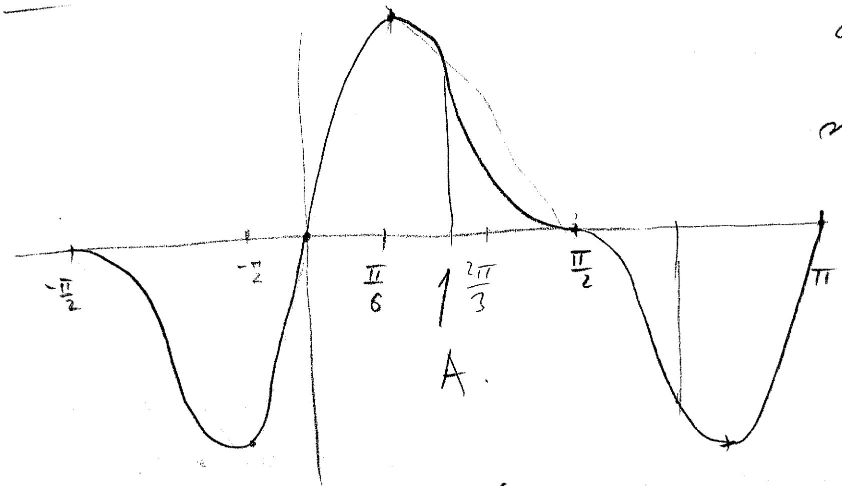
$\sin\frac{\pi}{6} = \frac{1}{2}$

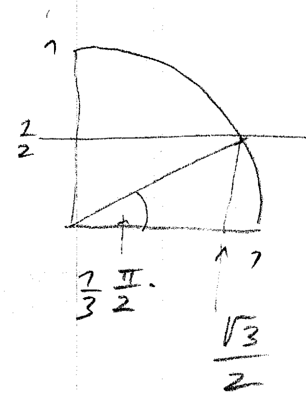
$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$$f\left(\pm\frac{\pi}{6}\right) = \pm 5 \cdot \frac{1}{2} \cdot \frac{3\sqrt{3}}{8} = \frac{15}{16}\sqrt{3} \doteq 1,628$$

$\cos\frac{\pi}{3} = \sin\frac{\pi}{6} = \frac{1}{2}$

$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$



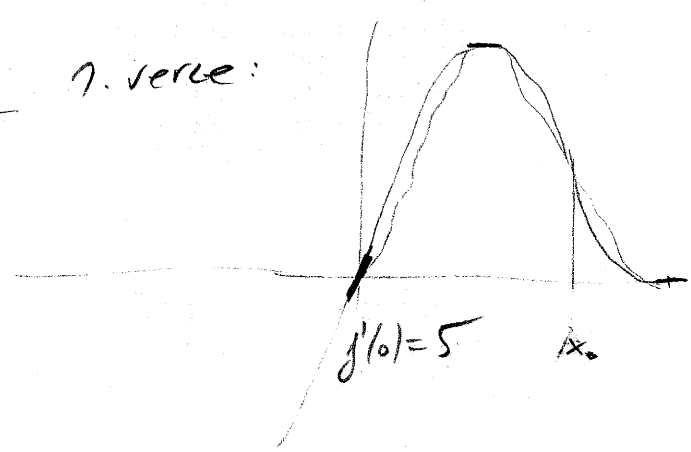


$$f''(x) = \left(5 \cos^2 x \left(\cos^2 x - 3\left(1 - \cos^2 x\right)\right)\right)'$$
$$= 10 \sin x \cos x \left(3 \sin^2 x - 5 \cos^2 x\right)$$

$x = k\frac{\pi}{2}$; $x = \pm \operatorname{arctg}\sqrt{\frac{5}{3}} = \pm 0.9 = \pm A$

$$H(f) = [-a, a].$$

graf: 1. verze:



$f(0) = f(\pm\frac{\pi}{2}) = 0$. ; lichá, π-periodická: $x \in [0, \frac{\pi}{2}]$.

$$f'(x) = 5\cos^2 x\,(\cos^2 x - 3\sin^2 x)$$
$$= 5\cos^4 x\left(1 - 3\cdot\frac{\sin^2 x}{\cos^2 x}\right)$$

$\mathrm{tg}^2 x = \frac{1}{3}$; $x = \pm\frac{\pi}{6}$

$|\mathrm{tg}\, x| = \frac{1}{\sqrt{3}}$ $\quad x = \frac{\pi}{6}$.

$x \in (-\frac{\pi}{6}, \frac{\pi}{6})$:

$I_1 = [0, \frac{\pi}{6}]$.. $f(x)$ spojitá v I; $f'(x) \neq 0$ uvnitř I $\Rightarrow$ $f(x)$ ryze monotónní v I

$f(0) = 0$; $f(\frac{\pi}{6}) = 5\cdot\sin\frac{\pi}{6}\cdot\cos^3(\frac{\pi}{6}) = a$

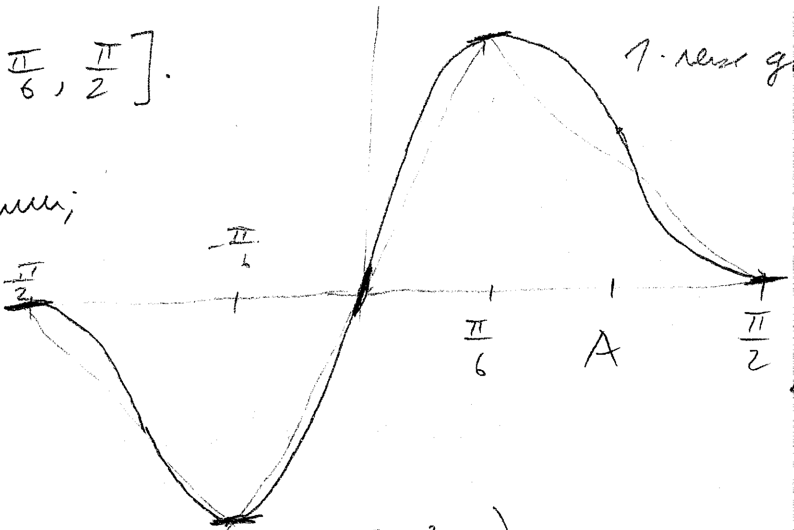
$\sin\frac{\pi}{6} = \frac{1}{2}$

$a = 5\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{3}{4} = \frac{\sqrt{3}\cdot 15}{16} \doteq 1.6$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$;

$f(x)$ klesá v $[\frac{\pi}{6}, \frac{\pi}{2}]$.

a lokální maximum;



1. náčrt grafu.

$f'(\pm\frac{\pi}{2}) = 0$

$f'(0) = 5$

$f'(\pm\frac{\pi}{6}) = 0$

$$f''(x) = 10\sin x\cos x\,\underbrace{(3\sin^2 x - 5\cos^2 x)}$$;

$= 0 \;\Leftrightarrow\; \mathrm{tg}^2 x = \frac{3}{5}$

$x = \mathrm{arctg}\sqrt{\frac{3}{5}} = A \doteq 0{,}9$

$f''(x) < 0 \quad x \in (0, A)$

$f''(x) > 0 \quad x \in (A, \frac{\pi}{2})$

Př. 4 $\quad f(x) = (x-1)\, e^{-|x-1|}$

spojité v $\mathbb{R}$; symetrické vůči $[1,0]$.

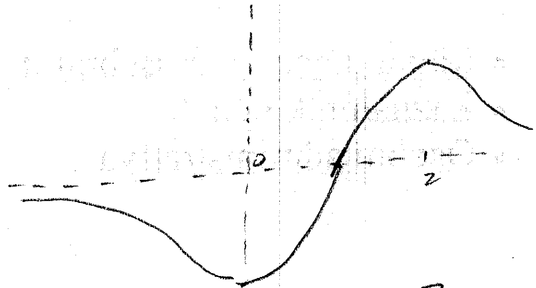
$\lim\limits_{x \to \pm\infty} f(x) = 0.$

$x \neq 1: \quad f'(x) = e^{-|x-1|} \left\{ 1 + (x-1) \cdot [-\operatorname{sgn}(x-1)] \right\}$

$\qquad\qquad\quad = e^{-|x-1|} \left\{ 1 - |x-1| \right\}$

$f'(1) = \lim\limits_{x \to 1} f'(x) = 1.$

$f'(x) = 0: \quad |x-1| = 1; \quad$ i.e.: $\quad x = 0, 2.$



$f(0) = -\frac{1}{e} \doteq 0.4$

$f(2) = +\frac{1}{e} \doteq 0.4$

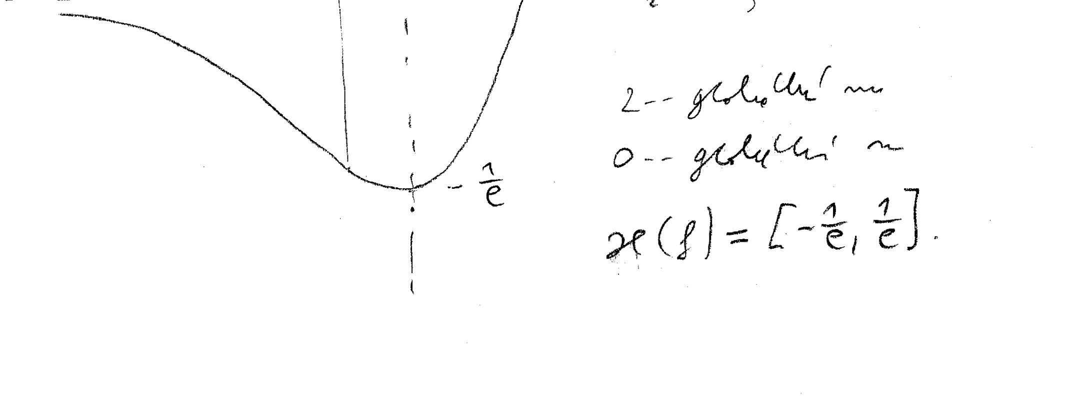
$f''(x) = \left[e^{-(x-1)} \{2-x\} \right]' = e^{-(x-1)} \left[-(2-x) - 1 \right] = e^{-(x-1)} \left[x - 3 \right].$

nechť $x > 1$

... dostaneme: $\quad x = 3$: inflexní bod

leč: $\quad x = 1$: též,

$\qquad x = -1$ též.



2 -- globální max

0 -- globální min

$\mathcal{H}(f) = \left[-\frac{1}{e}, \frac{1}{e} \right].$

Průběh: $f(x) = x + 2 + \dfrac{3}{|x+1|}$;

$D(f) = \mathbb{R} \setminus \{-1\}$; ani sudá, ani lichá ; spojitá na $D(f)$

$f(+\infty-) = +\infty$; $f(-\infty+) = -\infty$; $f(-1\pm) = +\infty$

$f'(x) = 1 + (-1)\cdot\dfrac{3}{|x+1|^2}\cdot \operatorname{sgn}(x+1)$; $x \neq -1$

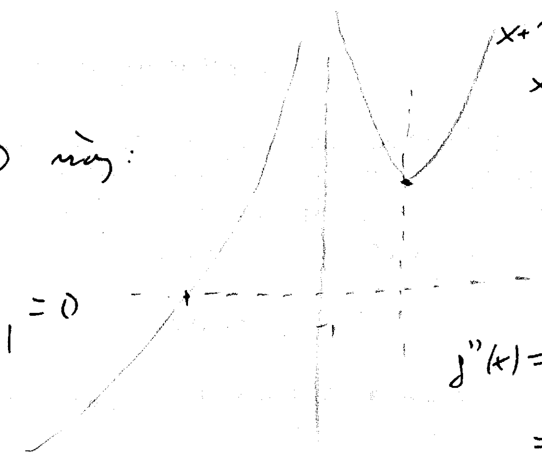
$= 1 - \dfrac{3\operatorname{sgn}(x+1)}{|x+1|^2}$; $f'(x) = 0$: $x > -1$:

$$1 - \frac{3\cancel{(x+1)}}{(x+1)^2} = 0$$

$$1 = \frac{3}{x+1} \quad x+1 = 3 \quad \underline{x = 2}.$$

$x+1 = \sqrt{3}$

$x = \sqrt{3} - 1 > -1$ ✓.



$x < -1$:

$f'(x) = 1 + \dfrac{3}{|x+1|^2} > 0$ vždy:

$f(x) = 0$: $x + 2 + \dfrac{3}{|x+1|} = 0$

$x+2 = \dfrac{+3}{x+1}$

$(x+2)(x+1)+3 = 0$

$x^2 + 3x + 5 = 0$; $D = 9 + 8 = 17$

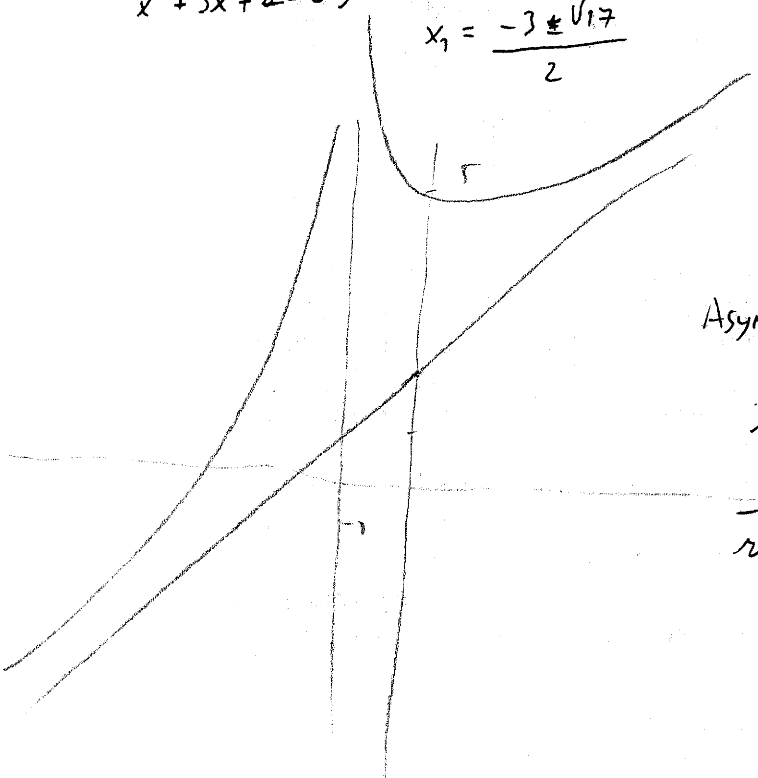
$x_1 = \dfrac{-3 \pm \sqrt{17}}{2}$

$f''(x) = 6\operatorname{sgn}(x+1)\cdot|x+1|^{-3}\cdot\operatorname{sgn}(x+1)$

$= 6\underbrace{\operatorname{sgn}^2(x+1)}_{=1 \text{ na } D(f)}\cdot|x+1|^{-3} > 0$ na

pozor: není pravda: f je konvexní na $D(f)$.

správně: f je konvexní na $(-\infty, -1)$ a na $(-1, +\infty)$



Asymptota: $p(x) = ax + b$ takové, že

$$\lim_{x\to+\infty} f(x) - p(x) = 0$$

vidět: že $p(x) = x+2$ je hledaná asymptota pro $x \to +\infty$ a taky $x \to -\infty$.

Průběh fce: $\boxed{f(x) = \arcsin\left(\frac{2x}{x^2+1}\right)}$

$\mathcal{D}(\arcsin) = [-1,1]$. tedy: $\frac{|2x|}{x^2+1} \le 1$

$2|x| \le x^2+1 = |x|^2+1$

$0 \le |x|^2 - 2|x| + 1 = (|x|-1)^2$

platí vždy; rovnost pro $|x| = 1$

i.e. $x = \pm 1$.

$f(x)$ je liché ($\arcsin$; $\frac{2x}{x^2+1}$ jsou liché)

omezené: $|\arcsin| \le \frac{\pi}{2}$; mezitě:

$f(+\infty-) = \arcsin 0 = 0$

$f(-\infty+) = 0$.

$(\arcsin y)' = \frac{1}{\sqrt{1-y^2}}$ $y \in (-1,1)$

$\left(\frac{2x}{x^2+1}\right)' = \frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$

$= \frac{2(1-\cancel{2}x^2)}{(x^2+1)^2}$

derivace: $f'(x) = \frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \cdot \left(\frac{2x}{x^2+1}\right)' =$

$$= \sqrt{\frac{(x^2+1)^2}{(x^2+1)^2-4x^2}} \cdot \frac{2(1-x^2)}{(x^2+1)^2} = \sqrt{\frac{|x^2+1|}{(x^2-1)^2}} \cdot \frac{2(1-x^2)}{(x^2+1)^2} =$$

$$= \frac{-2\,\mathrm{sgn}(x^2-1)}{(x^2+1)^2} \quad \text{pro } x \neq \pm 1$$

$f' < 0$ pro $x > 1$

$f' > 0$ pro $x \in (-1,1)$

$f' < 0$ pro $x < -1$.

body: $x = \underline{1}$: $f(1) = \arcsin 1 = \frac{\pi}{2}$;

$$f'_+(1) = \lim_{x\to 1+} f'(x) = \lim_{x\to 1+} \frac{-2\,\mathrm{sgn}(x^2-1)}{(x^2+1)^2} = \lim_{x\to 1+} \frac{-2}{(x^2+1)^2} = -\frac{1}{2}$$

$\mathrm{sgn}(x^2-1) = +1$ na $P_+(1)$

analogicky: $f'_-(1) = \frac{1}{2}$.

$$f''(x) = -2\,\mathrm{sgn}(x^2-1)\cdot\frac{-2\cdot(x^2+1)\,2x}{(x^2+1)^4} \qquad x \neq \pm 1$$

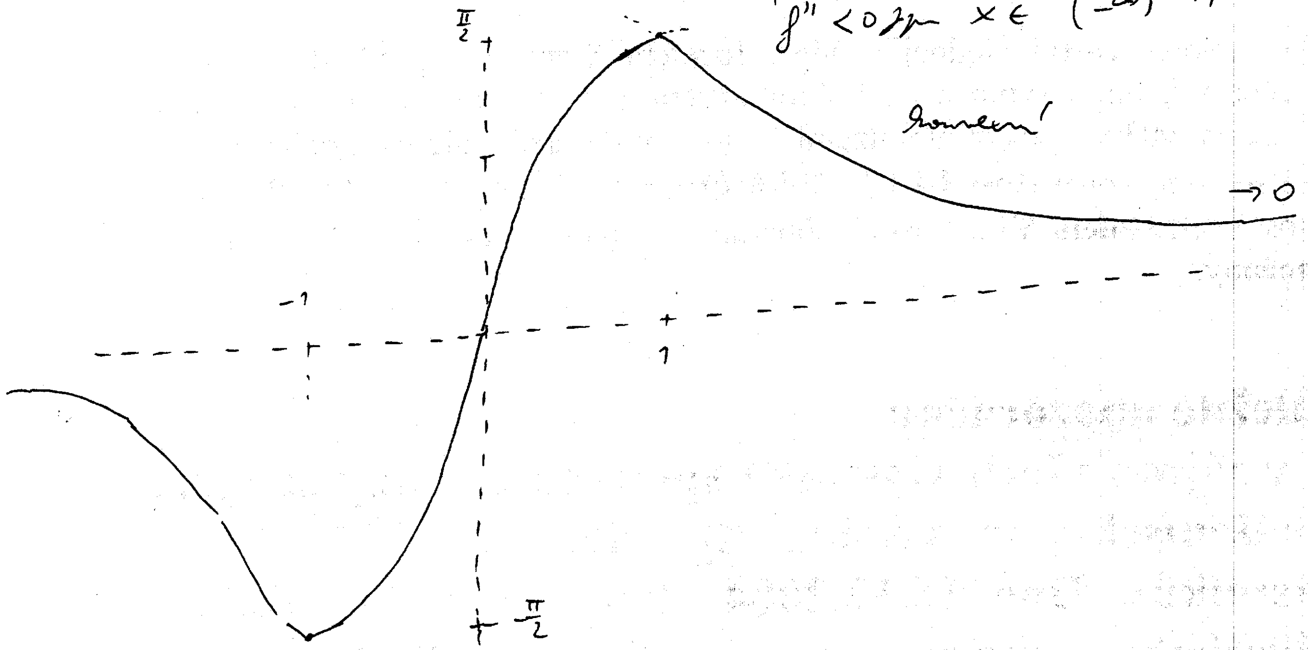
$$= 8\,\mathrm{sgn}(x^2-1)\cdot\frac{x}{(x^2+1)^3}\ ;$$

$f''>0$ pro $x>0$ -- ~~konkávní~~, konvexní

$f''>0$ pro $x\in(0,1)$ -- konkávní...

$f''<0$ pro $x\in(-1,0)$

$f''<0$ pro $x\in(-\infty,-1)$.



pozn.: $f(x)$ je konvexní v $[1,+\infty)$ leč 1 není inflexní bod.
konkávní v $[0,1]$ (neb $f'(1)$ neexistuje)