Řešte Bernoulliho rovnice:

1. $y' + 4xy + 2e^{-x^2}\sqrt{y} = 0$, $y(0) = 0$.

2.

$$3y' + \frac{y}{(1+x^2)(\pi/2 - \operatorname{arccotg} x)} + \frac{1}{(1+x^2)y^2} = 0, \qquad y(1) = -\sqrt[3]{\pi}/2.$$

3. $2y' - 6y \operatorname{tg} x + 3\sqrt[3]{y}\sin x = 0$, $y(0) = (1/3)^{2/3}$.

4. $y' - xy + xy^2 = 0$

5. $y' + 2y/x^2 + 2\sqrt{y}/x^2 = 0$

6. $y' + y \operatorname{cotg} x + \cos x/2y = 0$, $y(\pi/2) = 2/\sqrt{3}$.

7. $y' - y - y^2 = 0$

8.

$$3y' + \frac{2y}{x \ln x} + \frac{2}{x\sqrt{y}} = 0$$

9.

$$4y' + \frac{3y}{\sqrt{x}} + \frac{3}{\sqrt{x}\sqrt[3]{y}} = 0$$

10. $4y' + 3y + 3(\cos^2 x - 1/2)e^{-x}/\sqrt[3]{y} = 0$

Výsledky:

1) $y = 0$, $x \in \mathbb{R}$, nebo $y = (x-c)^2 e^{-2x^2}$, $x < c$ a nula jinde. *Pozn.* $z = y^{1/2}$, nutně $y \geq 0$, větvení v bodě $y = 0$.

2) $y = -\sqrt[3]{\operatorname{arctg} x/2}$, $x \in (0, \infty)$. *Pozn.* $z = y^3$, nutně $y \neq 0$, $x \neq 0$; všimnu si, že $\pi/2 - \operatorname{arccotg} x = \operatorname{arctg} x$. Obecné řešení má tvar $y = \sqrt[3]{C/\operatorname{arctg} x - \operatorname{arctg} x/2}$.

3) $y = (\cos x/3)^{3/2}$, $x \in (-\pi/2, \pi/2)$. *Pozn.* y je řešení $\implies$ $-y$ je řešení; $y = 0$ je řešení. $z = y^{2/3}$ za předpokladu $y > 0$ dává $y = (C/\cos^2 x + \cos x/3)^{2/3}$, je-li výraz uvnitř závorky kladný.

4) $y = 0$, $x \in \mathbb{R}$. Pro $y \neq 0$ kladu $z = 1/y$, odtud $y = 1/(Ce^{-x^2/2} + 1)$, $x \in I_C$, kde $I_C = \mathbb{R}$ pro $C > -1$; $I_C = (-\infty, 0)$ nebo $(0, \infty)$ pro $C = -1$ a konečně $I_C = (-\infty, -\sqrt{2\ln(-C)})$ nebo $(-\sqrt{2\ln(-C)}, \sqrt{2\ln(-C)})$ nebo $(\sqrt{2\ln(-C)}, \infty)$ pro $C < -1$.

(1) $y' + 4xy + 2e^{-x^2}\sqrt{y} = 0; \quad x \in \mathbb{R}$

$y \geq 0$

$a = \frac{1}{2}: \quad z = y^{1/2}$

$y = 0;\ x \in \mathbb{R}$ je řešení

$z' = \frac{1}{2} y^{-\frac{1}{2}} y' \quad \Big| \cdot \frac{1}{2} y^{-\frac{1}{2}}$

(čehém nětření pro $y = 0$.)

$$z' + 2xz + e^{-x^2} = 0 \quad \Big/ \, e^{x^2}$$

$$(z e^{x^2})' + 1 = 0$$

$$z e^{x^2} = C - x$$

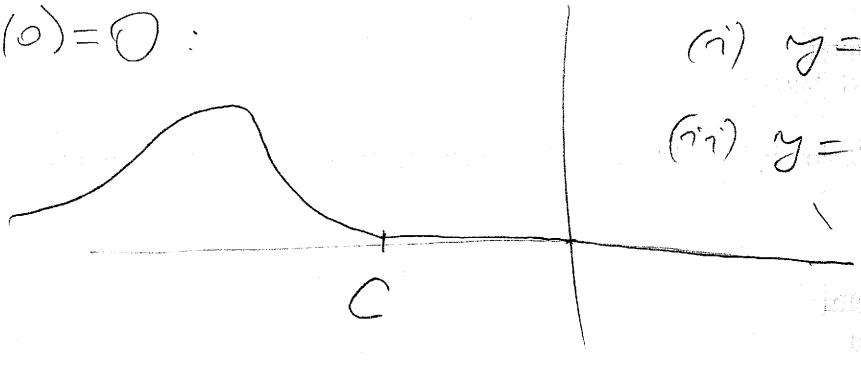
$$z = (C - x) e^{-x^2}.$$

$z > 0: \quad C - x > 0: \quad I_C = (-\infty, C)$

$$y = (C - x)^2 e^{-2x^2}; \quad x \in I_C$$

lze napojit na nulové řešení.

$y(0) = 0$:



(i) $y = 0; \quad x \in \mathbb{R}$

(ii) $y = 0; \quad x \geq C$

$(C - x)^2 e^{-2x^2}; \quad x < C.$

② $z' + \frac{z}{(1+x^2)(\text{arctg}\, x)} + \frac{1}{1+x^2} = 0 \quad ; \ x \neq 0.$

$z = y^3 \neq 0.$ $\qquad \frac{\pi}{2} - \text{arctg}\, x \equiv \text{arccotg}\, x \ ; \ \forall x \in \mathbb{R}.$

n.f.: $\text{arctg}\, x$:

$$(z \cdot \text{arctg}\, x)' + \frac{\text{arctg}\, x}{1+x^2} = 0$$

$$z \cdot \text{arctg}\, x + \frac{1}{2} \text{arctg}^2 x = C$$

$z(1) = -\frac{\pi}{8}$: $\qquad -\frac{\pi^2}{32} + \frac{1}{2} \cdot \frac{\pi^2}{16} = C$

$\text{arctg}\, 1 = \frac{\pi}{4}$ $\qquad 0 = C$

$$z = -\frac{\text{arctg}\, x}{2}$$

$$y = -\sqrt[3]{\frac{\text{arctg}\, x}{2}} \ ; \ x \in (0, \infty).$$

(3) $2y' - 6y\,\mathrm{tg}\,x + 3\sqrt[3]{y}\sin x = 0$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$y = 0$ řešení; $\tilde{y}$ řešení $\Rightarrow -\tilde{y}$ řešení.

BÚNO: $y > 0$: $z = y^{2/3} > 0$

$z' = \frac{2}{3} y^{-\frac{1}{3}} y' \quad \Big/ \cdot \frac{1}{3} y^{-\frac{1}{3}}$

$z' - 2\,\mathrm{tg}\,x \cdot z + \sin x = 0$

$2\int \frac{-\sin x}{\cos x}\,dx = 2\ln|\cos x| = \ln\cos^2 x\,; \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

i.f.: $\cos^2 x$

$z' \cdot \cos^2 x - 2\sin x\cos x \cdot z = -\sin x \cdot \cos^2 x$

$(z\cos^2 x) = C + \frac{1}{3}\cos^3 x$

$z = \cos^{-2} x \left(C + \frac{1}{3}\cos^3 x\right)$

$y(0) = \left(\frac{1}{3}\right)^{3/2}$

$z = \frac{1}{3}\cos x > 0$ v $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$z(0) = \frac{1}{3} \Rightarrow C = 0$:

$y = \left(\frac{\cos x}{3}\right)^{3/2}.$

④ $y' - xy + xy^2 = 0$; $z = \frac{1}{y} \neq 0$.

$y \equiv 0$.

$z' + xz = x$; i.f. $e^{\frac{x^2}{2}}$ $\quad z' = -\frac{y'}{y^2} \Big/ \cdot -\frac{1}{y^2}$

$$\left(z e^{\frac{x^2}{2}}\right)' = x e^{\frac{x^2}{2}} = \left(e^{\frac{x^2}{2}}\right)'$$

$$z e^{\frac{x^2}{2}} = C + e^{\frac{x^2}{2}}$$

$$z = C e^{-\frac{x^2}{2}} + 1 \;; \quad y = \frac{1}{C e^{-\frac{x^2}{2}} + 1}$$

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⑤ $y' + \frac{2y}{x^2} + \frac{2\sqrt{y}}{x^2} = 0$; $y \equiv 0$; $y \geq 0$

dodefinovat $y \equiv 0$.

$z = \sqrt{y}$

$z' = \frac{1}{2} y' y^{-\frac{1}{2}}$

$z' + \frac{z}{x^2} + \frac{1}{x^2} = 0$;

i.f. $e^{-\frac{1}{x}}$ $\quad \Big/ \cdot \frac{1}{2} y^{-\frac{1}{2}}$

$$\left(z e^{-\frac{1}{x}}\right)' + \frac{1}{x^2} e^{-\frac{1}{x}} = 0 \;;$$

$C \leq 0$ nikdy!!

$$z e^{-\frac{1}{x}} + e^{-\frac{1}{x}} = C$$

$C e^{\frac{1}{x}} \geq 1$

$$z = C e^{\frac{1}{x}} - 1 \geq 0.$$

$e^{\frac{1}{x}} \geq \frac{1}{C}$

$$y = \left(C e^{\frac{1}{x}} - 1\right)^2.$$

$\Leftrightarrow \frac{1}{x} > -\ln C$

$$Ce^{-\frac{x^2}{2}}+1\geq 0$$

$$Ce^{-\frac{x^2}{2}} \geq -1 \quad \Big/ e^{+\frac{x^2}{2}}$$

$$C \geq -e^{+\frac{x^2}{2}}$$

$$e^{\frac{x^2}{2}} \geq -C \quad : \quad C > -1$$

$$\frac{x^2}{2} = \ln(-C) \qquad -C < 1 \text{ vždy}$$

$$x^2 = 2\ln(-C) \qquad C=0$$

$$x^2 = \ln(-C)^2 = \ln C^2$$

$$x = \pm\sqrt{2\ln(C^{2})}$$

$$C>0:$$

$$Ce^{\frac{1}{x}} \geq 1 \qquad C \ngeq 1: \; x\in(0,\infty)$$

$$e^{\frac{1}{x}} \geq \frac{1}{C}$$

$$\frac{1}{x} \geq \ln\frac{1}{C} = -\ln C$$

$$x>0: \quad x > \frac{-1}{\ln C} \qquad I_C = (0,\infty)$$

$$\boxed{\frac{1}{x} > -\ln(C)} \qquad x>$$

$$x>0: \quad x < \frac{-1}{\ln C}$$

$Ce^{\frac{1}{x}} > 1 \qquad C > 0$ nutně

$e^{\frac{1}{x}} > \frac{1}{C}$

$\frac{1}{x} > \ln \frac{1}{C} = -\ln C$

(i) $C \geq 1$: PS ≤ 0: $x \in (0, \infty)$ o.k.

~~(ii)~~ $C = 1$: $\frac{1}{x} > 0$: $x \in (0, \infty)$

$C \in (0,1)$: $\frac{1}{x} > \ln \frac{1}{C} \to$ kladné.

$\frac{1}{x} > -\ln C \qquad / \frac{x}{-\ln C} > 0$

$\frac{-1}{\ln C} > x$

$x \in \left(-\infty, -\frac{1}{\ln C}\right)$. $C > 1$.

$C \in (0,1)$: $\frac{1}{x} > -\ln C$ min. $x < 0$

$1 > -x \ln C$

$\frac{-1}{\ln C} > x \qquad x \in \left(0, -\frac{1}{\ln C}\right)$.

(6) $y' + y \cot g\, x + \frac{\cos x}{2y} = 0$; $y\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{3}}$

$x \in (0, \pi)$

$y \neq 0$

$z = y^2 \quad / \cdot 2y$

$z' + 2 \cot g\, x \, z + \cos x = 0$;

$\int 2 \frac{\cos x}{\sin x} \, dx = \ln(\sin^2 x)$.

i.f.: $\sin^2 x$

$(z \sin^2 x)' + \cos x \sin^2 x = 0$

$z \sin^2 x + \frac{1}{3} \sin^3 x = C$.

$$\boxed{z = \frac{C}{\sin^2 x} - \frac{\sin x}{3}} = \frac{3C - \sin^3 x}{3 \sin^2 x}$$

dosadit: $z\left(\frac{\pi}{2}\right) = y^2\left(\frac{\pi}{2}\right) = 0$: $\quad 0 = C - \frac{1}{3}$

obecně: $z = 0$: $\quad \sin^3 x = 3C$

$\sin x = \sqrt[3]{3C}$.

(i) $3C > 1$ nebo $3C < 0$: nemá řešení $x \in (0, \pi)$.

(ii) $3C \in [0, 1]$: $x = \arcsin \sqrt[3]{3C}$

nebo

$x = \pi - \arcsin \sqrt[3]{3C}$.

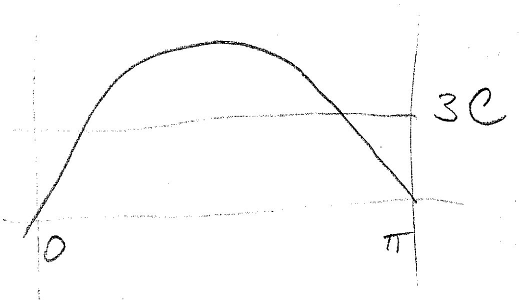
Zj: $y = \pm\sqrt{3C - \sin^3 x}$

$3C - \sin^3 x > 0$: (i) $3C > 1$

$C > \frac{1}{3}$ .. vždy $x \in (0, \pi)$

(ii) $3C \leq 0$ .. nikde na $(0, \pi)$.

(iii) $3C \in (0, 1]$:



$\sin^3 x < 3C$

$\sin x < \sqrt[3]{3C}$.

$x \in (0, A)$ nebo $x \in (\pi - A, \pi)$.

$A = \arcsin \sqrt[3]{3C}$

Obecné řešení: $y = \pm\sqrt{\frac{3C - \sin^3 x}{3\sin^2 x}}$, na výše uvedených intervalech.

Jediné se maximální řešení

($y \to 0$ nebo $y \to +\infty$ v krajních bodech).

$y(\frac{\pi}{2}) = \frac{2}{\sqrt{3}}$: $\frac{3C - 1}{1} = 4$,

$C = 5/3$

$y = \sqrt{\frac{5 - \sin^3 x}{3\sin^2 x}}$; $x \in (0, \pi)$.

(7) $y' - y - y^2 = 0$; $a = 2$

$z = y^{-1} \neq 0$

$y \equiv 0$; $x \in \mathbb{R}$ řešení.

$/ \cdot \left(-\frac{1}{y^2}\right)$

$z' = \frac{-y'}{y^2}$

$z' + z + 1 = 0$; F.S. $\{e^{-x}\}$

$z_p = -1$

obecné $z = Ce^{-x} - 1$

? $z = 0$: $Ce^{-x} = 1$ (i) $C \leq 0$ nikdy

(ii) $C > 0$: $e^{-x} = C^{-1}$ $/ \cdot \ln$

$-x = -\ln C$

$x = \ln C.$

$y = \frac{1}{Ce^{-x} - 1}$; $x \in I_C$

$I_C = \mathbb{R}$; $C \leq 0$

$= (-\infty, \ln C)$

nebo $(\ln C, +\infty)$; $C > 0$.

(8) $3y' + \frac{2y}{x\ln x} + \frac{2}{x\sqrt{y}} = 0$; máme $x, y > 0$
$x \neq 1$.

$\alpha = -\frac{1}{2}$

$z = y^{3/2}$; $z' = \frac{3}{2} y^{1/2} y'$ / $\cdot \frac{1}{2} y^{1/2}$

$$z' + \frac{z}{x\ln x} + \frac{1}{x} = 0 \quad ;$$

$\int \frac{dx}{x\ln x} = \ln|\ln x|$; $x \in (0,1)$ nebo $(1, +\infty)$.

tj.: $|\ln x|$ nebo $\ln x$ (nezáleží, jde i násobek)

$$(z\ln x)' + \frac{\ln x}{x} = 0$$

$$z\ln x + \frac{1}{2}\ln^2 x = C$$

$$z = \frac{C}{\ln x} - \frac{\ln x}{2} = \frac{2C - \ln^2 x}{2\ln x}.$$

$z > 0$: (i) $x \in (0,1)$: $2C - \ln^2 x < 0$

$2C < \ln^2 x$

$C \leq 0$: $I_C = (0,1)$

$C > 0$: $|\ln x| > \sqrt{2C}$: $I_C = (0, e^{-\sqrt{2C}})$

$\ln x < -\sqrt{2C}$

$z < 0$:

(iii) $x \in (1, \infty)$: $\quad 2c - \ln^2 x > 0$

$$2c > \ln^2 x$$

$$c \le 0 : \quad I_c = \emptyset$$

$$c > 0 : \quad \sqrt{2c} > |\ln x| = \ln x$$

$$I_c = (1, \sqrt{2c}).$$

Obecné řešení: $\quad y = \left(\dfrac{2c - \ln^2 x}{2 \ln x} \right)^{2/3}; \quad x \in I_c.$

– maximální: $y \to +\infty$;

nebo hraniční body $0, 1, +\infty$.

(9) $4y' + \frac{3y}{\sqrt{x}} + \frac{3}{\sqrt{x}\sqrt[3]{y}} = 0.$ $\quad y \neq 0;\ x > 0$

y .. řešení $\rightarrow -y$ je řešení.

BÚNO: $y > 0$.

$a = -\frac{1}{3}$

$z = y^{4/3}$

$z' = \frac{4}{3} y^{\frac{1}{3}} y' \Big/ \cdot \frac{y^{\frac{1}{3}}}{3}$

$z' + \frac{z}{\sqrt{x}} + \frac{1}{\sqrt{x}} = 0$

$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x};\ x \in (0, \infty).$ $\quad$ if: $\exp(+2\sqrt{x})$.

$\left(z \exp(+2\sqrt{x})\right)' + \frac{1}{\sqrt{x}} \exp(+2\sqrt{x}) = 0$

$z \exp(+2\sqrt{x}) + \exp(+2\sqrt{x}) = C$

$z = C \exp(-2\sqrt{x}) + 1$

$z > 0$: (i) $C \geq 0$: $I_C = (0, +\infty)$.

(ii) $C < 0$: $C \exp(-2\sqrt{x}) > -1$

$C > -\exp(2\sqrt{x})$

$\exp(2\sqrt{x}) > -C$

$2\sqrt{x} > \ln(-C)$

$y = \left(C \exp(-2\sqrt{x}) + 1\right)^{3/4}.$

$x > \frac{\ln^2(-C)}{4}$. $I_C = \left(\frac{\ln^2(-C)}{4}, +\infty\right)$

(10) $4y' + 3y + 3\frac{(\cos^2 x - \frac{1}{2})e^{-x}}{\sqrt[3]{y}} = 0$; $x \in \mathbb{R}$

$y \neq 0$; BGNO $y > 0$

$z = y^{4/3}$; $z' = \frac{4}{3}y^{\frac{1}{3}}y'$

$\cdot / \frac{1}{3}y^{\frac{1}{3}}$

$z' + z + e^{-x}(\cos^2 x - \frac{1}{2}) = 0 \quad / \cdot e^x$

$(ze^x)' + \cos^2 x - \frac{1}{2} = 0$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}(x + \frac{1}{2}\sin 2x)'$.

$ze^x + \frac{1}{2}(x + \frac{1}{2}\sin 2x) = C$

$z = C - \cancel{\frac{1}{2}x} - \frac{1}{4}\sin 2x$.

?? $z > 0$:

Riprova: $\cos^2 x \leftarrow \cos^2 x - \frac{1}{2}$

$\cos^2 x - \frac{1}{2} = \frac{1}{2}\cos 2x$

$C - \frac{1}{4}\sin 2x > 0$

$\sin 2x < 4C$.

$4C \geq 1 \quad I_c = \mathbb{R}$.

$4C \leq -1 \quad I_c = \cancel{\mathbb{R}}$

$4C \in (-1, 1)$: $I_c = \left(\frac{A}{2}, \pi - \frac{A}{2}\right)$

$A = \arcsin 4C$.

$y = \pm\left(C - \frac{1}{4}\sin 2x\right)^{3/4} e^{-3/4x}$.

$x \in I_c + k\pi$.