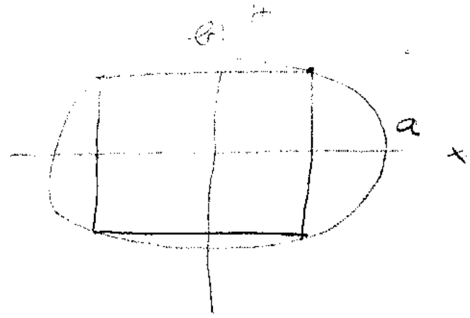
(1a) elipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $a,b>0$



$f = 2x\cdot 2y$; $g=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$;

$M=\{g=0\}\cap\{x\geq 0, y\geq 0\}$.

∃ globální extrémy (f spojitá; M omez. uzavř.)

zřejmě $f(a,0)=f(0,b)=0$ ... glob. minimum.

$A=(x,y)$ je podezřelý; $x,y>0$:

(a) f,g nehladké ... $\emptyset$

(b) $\nabla g=0$: $g_x=\frac{2x}{a^2}=0$ nikdy pro $x>0$

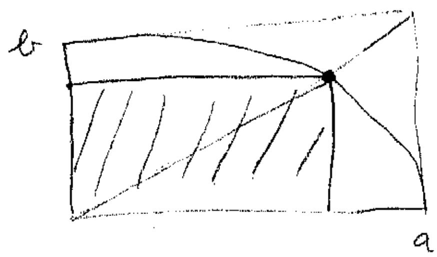
(c) $\nabla f=\lambda\nabla g$:

$$4y = 2\lambda\frac{x}{a^2}$$
$$4x = 2\lambda\frac{y}{b^2}$$

máme $x,y,\lambda>0$; podělením rovnic:

$$\frac{y}{x}=\frac{x}{y}\cdot\frac{b^2}{a^2}$$

$$\frac{x^2}{y^2}=\frac{a^2}{b^2};\ \text{tj.}\ \boxed{\frac{x}{a}=\frac{y}{b}}$$



(2) $f = xyz$; $M = \{g = 0\} \cap \{x, y, z > 0\}$

$$g = 2xy + 2yz + 2xz - 1.$$

podezřelé body:

(a) f, g nehladké … $\emptyset$

(b) $\nabla g = 0$: $g_x = 2(y+z) = 0$, $g_y = 2(x+z) = 0$, $g_z = 2(x+y) = 0$ — jediné pro $x = y = z = 0$ … není prvkem M.

(c) $\nabla f = \lambda \nabla g$:

$$yz = 2\lambda(y+z)$$
$$xz = 2\lambda(x+z)$$
$$xy = 2\lambda(x+y)$$

1. rce − 2. rce: $z(y-x) = 2\lambda(y-x)$

(α) $y \neq x \Rightarrow z = 2\lambda$; 2. rce: $xz = z(x+z)$

$x = x + z$: spor. (neboť $x, y, z > 0$).

tedy nutně:

(β): $y = x$; analogicky $y = z$; tj. $A = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ je jediný podezřelý bod.

$M_\varepsilon = M \cap \{x \geq \varepsilon, y \geq \varepsilon, z \geq \varepsilon\}$; $\varepsilon > 0$ malé.

$\exists$ max f při M_ε, neb M_ε je omez. uzavř.

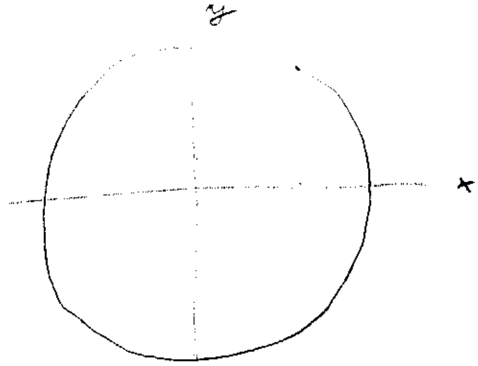
omezenost: $2xy \leq 1$; tedy $x \geq \varepsilon \to y \leq \frac{1}{2x} \leq \frac{1}{2\varepsilon}$ atd.

dále: (x, y, z) mimo M_ε; např.: $x < \varepsilon$; ale: $2yz \leq 1$;

odsud $f = xyz = \frac{1}{2}x \cdot 2yz < \frac{1}{2}\varepsilon \cdot 1 = \frac{\varepsilon}{2}$;

tj. f malé mimo $M_\varepsilon \Rightarrow \exists$ max. f při M.

(3) $M = \{x^2+y^2=1\} \cap \{x, y > 0\}$



$f = 2y \cdot \pi x^2$ (výška krát podstava)

$g = x^2+y^2-1$.

podezřelé body: nehladkost a $\nabla g = 0$ .. $\emptyset$

$\nabla f = \lambda \nabla g$: $\begin{array}{l} 2\pi \cdot 2xy = 2\lambda x \\ 2\pi x^2 = 2\lambda y \end{array}$; podělením rovnic (musí $x, y, \lambda > 0$)

$$\frac{2xy}{x^2} = \frac{x}{y} \; ; \; 2y^2 = x^2$$

$$\boxed{y = \frac{x}{\sqrt{2}}.}$$

$x^2 + \frac{x^2}{2} = 1$; $x = \sqrt{\frac{2}{3}}$ $\quad A = \left(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$.

Protože $f(0,1) = f(1,0) = 0$, je A zřejmě maximum.

(4) $f = \pi x^2 + \pi x\sqrt{x^2+y^2}$; $M = \{g=0\}$, $g = \frac{1}{3}\pi x^2 y - 1$.

x... poloměr podstavy, y... výška.

$\nabla g = 0$: $g_y = \frac{1}{3}\pi x^2 = 0$ .. jen pro $x = 0$. .. $\emptyset$

$\nabla f = \lambda \nabla g$:
$$2\pi x + \pi\sqrt{x^2+y^2} + \pi\frac{x^2}{\sqrt{x^2+y^2}} = \lambda \frac{2}{3}\pi xy \quad / \; x$$
$$\pi\frac{xy}{\sqrt{x^2+y^2}} = \lambda\frac{1}{3}\pi x^2 \quad / \; -2y$$

$$2\pi x^2 + \pi x\frac{2x^2+y^2}{\sqrt{x^2+y^2}} - \pi x\frac{2y^2}{\sqrt{x^2+y^2}} = 0 \quad / : \pi x$$

$$2x + \frac{2x^2-y^2}{\sqrt{x^2+y^2}} = 0 \; ; \quad 2 + \frac{(2-k^2)}{\sqrt{1+k^2}} = 0$$

volme $y = kx$ $\quad \to k = \sqrt{8}$