Pozn.: ověřujeme jen body (iii), (iv) u Věty 18.12.

(C1) $F(a) = \int_0^\infty \underbrace{\frac{1-\exp(-ax^2)}{x^2 \exp(x^2)}}_{f(a,x)}\, dx$;

$$\frac{\partial f}{\partial a} = -(-x^2)\exp(-ax^2)\cdot\frac{1}{x^2 \exp x^2} = \exp\left(-(a+1)x^2\right).$$

majoranta: $a \in I = (-1+\delta, +\infty)$; $\delta > 0$ pevné.

$$\sup_{a\in I}\left|\frac{\partial f}{\partial a}(a,x)\right| = \sup_{a>-1+\delta} \exp\left(-(a+1)x^2\right) = \exp(-\delta x^2) \in L(0,+\infty).$$

Pomocný výpočet: $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$;

subst.: $x = \sqrt{a}\,y$ $\quad \int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

(iv): $a_0 = 0$; $f(0,x) = 0 \in L(0,+\infty)$.

V. 18.12. $\longrightarrow$ $F'(a) = \int_0^\infty \exp\left(-(a+1)x^2\right)dx = \frac{1}{2}\sqrt{\frac{\pi}{a+1}}$;

$\forall a > -1+\delta$; $\delta > 0$ libovolné $\Rightarrow$ zřejmě pro $\forall a > -1$.

dopočtem $F(a) = \int \frac{1}{2}\sqrt{\frac{\pi}{a+1}}\,da + C$

$$F(a) = \sqrt{\pi(a+1)} + C$$

$F(0) = 0$: $\quad C = -\sqrt{\pi}$.

+

(C2) $F(a) = \int_0^1 \underbrace{\frac{x^a - 1}{\ln x}}_{f(a,x)} dx$

$$\frac{\partial f}{\partial a} = \frac{\partial}{\partial a}\left(\frac{\exp(a\cdot \ln x) - 1}{\ln x}\right) = \exp(a \cdot \ln x) = x^a .$$

majoranta: $a \in I = (-1+\delta, +\infty)$; $\delta > 0$ pevné...

$$\sup_{a \in I} \left|\frac{\partial f}{\partial a}(a,x)\right| = \sup_{a > -1+\delta} x^a = x^{-1+\delta} \in L(0,1)$$

nebot' $\underline{-1+\delta > -1}$.

x^a <u>klesá</u> v a pro $x \in (0,1)$

$= \exp(a \cdot \underbrace{\ln x}_{<0})$.

(iv): $a_0 = 0$: $f(0,x) = 1 \in L(0,1)$.

V. 18.12.: $F'(a) = \int_0^1 x^a dx = \frac{1}{a+1}$; $a \in I$

$\delta > 0$ libovolné: platí pro $\forall\, a > -1$.

dopočtení: $F(a) = \int \frac{da}{a+1} + C$

$F(a) = \ln(a+1) + C$

$F(0) = 0$: $C = 0$.

(C3)+ $F(a) = \int_0^{\infty} \underbrace{\exp(-bx) \frac{\sin ax}{x}}_{f(a,x)} dx$; $b>0$ je pevné.

$$\frac{\partial f}{\partial a} = \exp(-bx) \cos(ax); \quad a \in \mathbb{R} = I$$

majoranta: $\sup_{a \in I} \left| \frac{\partial f}{\partial a}(a,x) \right| = \sup_{a \in \mathbb{R}} \exp(-bx) |\cos ax| \leq \exp(-bx) \in L(0,+\infty)$.

(iv): $a_0 = 0$; $f(0,x) = 0 \in L(0,+\infty)$.

V. 18.12. $\longrightarrow$ $F'(a) = \int_0^{\infty} \exp(-bx) \cos ax \, dx$; $\forall a \in \mathbb{R}$

dopočet: $F'(a) = \frac{b}{a^2+b^2}$

$$F(a) = \int \frac{b}{a^2+b^2} da + C = \operatorname{arctg}\left(\frac{a}{b}\right) + C.$$

$$F(0) = 0 \longrightarrow C = 0.$$

(C5) $F(a) = \int_0^{\infty} f(a,x) dx$; $f(a,x) = \frac{\operatorname{arctg}(ax)}{x(1+x^2)}$

$$\frac{\partial f}{\partial a} = \frac{1}{x(1+x^2)} \frac{\partial}{\partial a} \operatorname{arctg}(ax) = \frac{1}{(1+x^2)(1+a^2x^2)}$$

majoranta: $\sup_{a \in \mathbb{R}} \left| \frac{\partial f}{\partial a} \right| \leq \frac{1}{1+x^2} \in L(0,+\infty)$.

mezi výpočet: $\int_0^{\infty} \frac{dx}{1+x^2} = [\operatorname{arctg} x]_0^{\infty} = \frac{\pi}{2}$.

$a_0 = 0$: $f(0,x) = 0 \in L(0,+\infty)$.

V. 18.12. $\longrightarrow$ $F'(a) = \int_0^{\infty} \frac{dx}{(1+x^2)(1+a^2x^2)}$

pomocný výpočet $(a>0;\ a \neq 1)$ $\frac{1}{(1+x^2)(1+a^2x^2)} = \frac{1}{a^2-1} \left(\frac{a^2}{1+a^2x^2} - \frac{1}{1+x^2} \right)$

C5 - dokončení: $\displaystyle\int_0^{\infty} \frac{a^2}{1+a^2x^2}\,dx \ \left|\ \begin{array}{l} \text{subs.} \\ ax=y \\ dx=\frac{dy}{a} \end{array}\right. = \int_0^{\infty} \frac{a}{1+y^2}\,dy = a\cdot\frac{\pi}{2}.$

celkem: $\displaystyle F'(a) = \frac{1}{a^2-1}\left(a\frac{\pi}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2(a+1)}.$

Pozn.: $\displaystyle F'(a) = \frac{\pi}{2(a+1)}$ odvozeno pro $a>0;\ a\neq 1.$

Dle Lemmatu 6.3 (NMAF051; ZS 2008/09)

to platí pro $\forall a>0$.

$F(a)$ liché; $F'(a)$ sudé; tedy $\displaystyle F'(a) = \frac{\pi}{2(|a|+1)}$; $\forall a\in\mathbb{R}$.

integrací: $F(a) = \begin{cases} \frac{\pi}{2}\ln(|a|+1)\cdot \operatorname{sgn} a; & a\neq 0 \\ 0; & a=0. \end{cases}$

(C7) $\displaystyle F(a) = \int_0^{\frac{\pi}{2}} \underbrace{\frac{\ln(1+a\cdot\sin x)}{\sin x}}_{f(a,x)}\,dx \quad ; \quad a\in(-1,1).$

? integrand má smysl: $\sin x \neq 0;\ x\in(0,\frac{\pi}{2}).$

$1+a\cdot\sin x \geq 1-\underbrace{|a\cdot\sin x|}_{<1} > 0.$

$\displaystyle\frac{\partial f}{\partial a} = \frac{1}{1+a\cdot\sin x};$

majoranta: $a\in(-\Delta,\Delta);\ \Delta\in(0,1)$ pevné.

$\displaystyle\left|\frac{\partial f}{\partial a}(a,x)\right| = \frac{1}{|1+a\cdot\sin x|} \leq \frac{1}{1-\underbrace{|a\cdot\sin x|}_{\leq\Delta}} \leq \frac{1}{1-\Delta} =: g(x)$

$g\in L(0,\frac{\pi}{2}).$

$a_0=0:\ f(0,x)=0\in L(0,\frac{\pi}{2}).$

V. 18.12. $\rightarrow$ $\displaystyle F'(a) = \int_0^{\frac{\pi}{2}} \frac{dx}{1+a\cdot\sin x}.$

C7 – dopočet

pomocný výpočet: $\int_0^{\frac{\pi}{2}} \frac{dx}{1+a\cdot\sin x}$.

subs.: $t = \operatorname{tg}\frac{x}{2}$

$\sin x = \frac{2t}{1+t^2}$

$dx = \frac{2}{1+t^2}dt$

$t \in (0,1)$.

$$= \int_0^1 \frac{2\,dt}{1+a\cdot\frac{2t}{1+t^2}}\cdot\frac{1}{1+t^2} = \int_0^1 \frac{2\,dt}{t^2+2at+1}$$

$$= \left[\frac{2}{\sqrt{1-a^2}}\cdot\operatorname{arctg}\left(\frac{t+a}{\sqrt{1-a^2}}\right)\right]_{t=0}^{t=1} = ?\,..$$

OPRAVA ZADÁNÍ: $\int_{-\pi}^{\pi} \frac{\ln(1+a\cdot\sin x)}{\sin x}\,dx$.

~ vše analogické; jen: $F'(a) = \int_{-\pi}^{\pi}\frac{dx}{1+a\cdot\sin x} = \int_{-\infty}^{\infty}\frac{2\,dt}{t^2+2at+1}$

$x \in (-\pi,\pi) \longleftrightarrow t \in \mathbb{R}$

$$= \left[\frac{2}{\sqrt{1-a^2}}\operatorname{arctg}\left(\frac{t+a}{\sqrt{1-a^2}}\right)\right]_{t=-\infty}^{t=+\infty} = \frac{2}{\sqrt{1-a^2}}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{2\pi}{\sqrt{1-a^2}}.$$

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celkem: $F'(a) = \frac{2\pi}{\sqrt{1-a^2}}$; $a \in (-1,1)$

$\rightsquigarrow F(a) = 2\pi \arcsin a$.

C8: velmi podobně se ukáže: $F(a) = \int_0^{\pi}\frac{\ln(1+a\cdot\cos x)}{\cos x}$; $a \in (-1,1)$

$$F'(a) = \int_0^{\pi}\frac{dx}{1+a\cdot\cos x}.$$

subs.: $t = \operatorname{tg}\frac{x}{2}$

$t \in (0,+\infty)$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2}{1+t^2}dt$

$$= \int_0^{\infty}\frac{1}{1+a\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int_0^{\infty}\frac{2\,dt}{t^2\underbrace{(1-a)}_{>0}+\underbrace{(1+a)}_{>0}}$$

$= \frac{\pi}{\sqrt{1-a^2}}$ · -- dopočet:

$F(a) = \pi\arcsin a$; $a \in (-1,1)$.

Pomocný vzoreček. (d.ú.)

$$\int_0^{\infty}\frac{dx}{Ax^2+B} = \frac{\pi}{2\sqrt{AB}};\quad A,B>0.$$