① $f_n(x) = \dfrac{n^2x^2}{1+n^2x^2}$; $x \ge 0$.

bodová limita: $f_n(x) \to f(x) = \begin{cases} 0; & x=0 \\ 1; & x>0. \end{cases}$

$\left.\begin{array}{l} f_n(x) \text{ spojité} \\ f(x) \text{ nespojité v } 0 \text{ zprava} \end{array}\right\}$ $f_n \not\rightrightarrows 0$ v $[0,\delta)$ $\delta>0$

a tedy ani v $[0,+\infty)$.

?? $f_n \rightrightarrows f$ v $[\delta,+\infty)$; $\delta>0$ jeme.

$$\sigma_n = \sup_{x\ge\delta}|f_n(x)-f(x)| = \sup_{x\ge\delta} \underbrace{\frac{1}{1+n^2x^2}}_{\text{klesající v } x \text{ kladná}} = \frac{1}{1+n^2\delta^2}.$$

zjevně $\lim_{n\to\infty}\sigma_n = 0$.

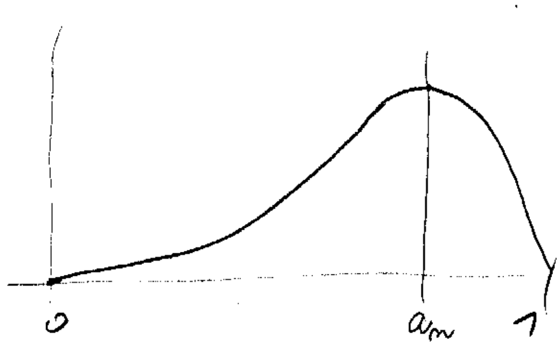
③ $f_n(x) = x^n - x^{n+1} = x^n(1-x)$.

$x^n \to 0$; $\forall |x|<1 \Rightarrow f_n(x)\to f(x)=0 \quad \forall x\in(-1,1]$

?? $f_n \rightrightarrows 0$ v $[0,1]$. ANO:

$$\sigma_n = \sup_{x\in[0,1]}|f_n(x)-0| = \sup_{x\in[0,1]} x^n(1-x)\ldots$$

pomocná úloha: průběh fce $x^n(1-x) = f_n(x)$



$$\sigma_n = f_n(a_n) = \left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1} \to e^{-1}\cdot 0 = 0$$

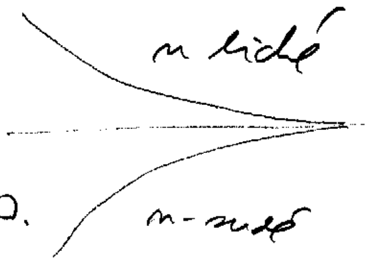
$a_n = \dfrac{n}{n+1}$ (dokažte si podrobně !!)

?? $f_n \rightrightarrows 0$ v $(-1,0]$ .. NE!

$$\sigma_n = \sup_{x\in(-1,0]} |f_n(x)-0| = \sup_{x\in(-1,0]} |f_n(x)|$$

$$f_n'(x) = \underbrace{x^{n-1}}_{\neq 0}\underbrace{\left(n-(n+1)x\right)}_{<0} \quad \text{pro } \forall x \in (-1,0).$$

$\Rightarrow$ f_n roste v $(-1,0]$ pro n liché.
 klesá $\qquad$ n sudé.

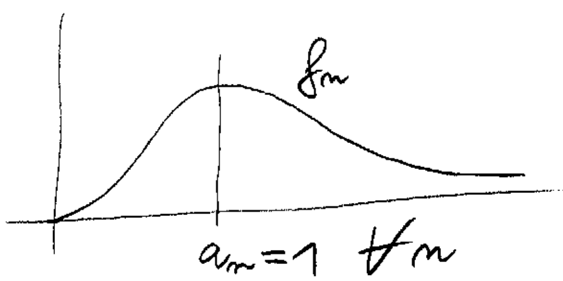


$f_n(0)=0$:

$$\sigma_n = |f_n(-1)| = |(-1)^n(1-(-1))| = 2 \not\to 0.$$

⑥ $f_n(x) \to 0 \quad \forall x \in [0,\infty)$

průběh $f_n(x)$:



$$\sigma_n = \sup_{x\geq 0} |f_n(x)-0| = \sup_{x\geq 0} f_n(x) = f_n(1) \to 0.$$

tedy $f_n \rightrightarrows 0$ v $[0,+\infty)$.

⑨ $f_n(x) = \frac{nx}{1+n+x} = \frac{x}{\frac{1}{n}+1+\frac{x}{n}} \to x \; ; \quad \forall x \in [0,\infty)$

?? stejn: NE $\quad \sigma_n = \sup\limits_{x\in[0,+\infty)} |f_n(x) - x| = \sup\limits_{x\geq 0} \underbrace{\frac{x^2+x}{1+n+x}}_{g_n(x)}$

zjevně $g_n(x) \to +\infty$; $x \to +\infty$

$\Rightarrow \sigma_n = +\infty \not\to 0$.

?? stejn. v $[0,M]$; $M>0$ pevné: ANO.

$$\sigma_n = \sup_{x\in[0,M]} |f_n(x) - x| = \sup_{x\in[0,M]} \left|\frac{x^2+x}{1+n+x}\right| \leq \frac{M^2+M}{1+n},$$

neboť: $x^2+x \leq M^2+M$

$1+n+x \geq 1+n$

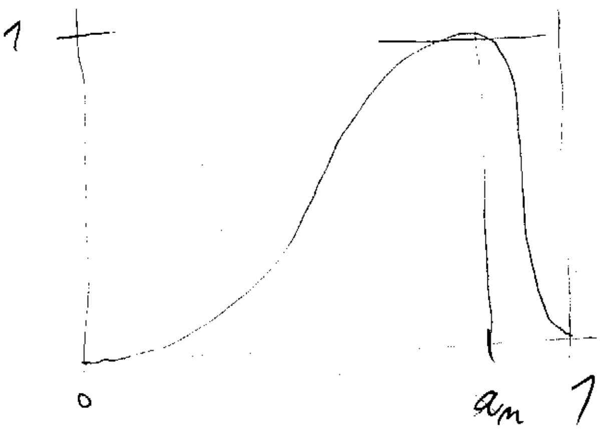
$0 \leq \sigma_n \leq \frac{M(M+1)}{1+n} \quad \forall n$; tedy $\sigma_n \to 0$.

---

⑩ $f_n(x) = \sin(\pi x^n)$; BÚNO $x \in [0,1]$.

$\left.\begin{array}{ll} x\in[0,1): & \pi x^n \to 0 \\ x = 1 & \pi x^n = \pi \end{array}\right\} \Rightarrow f_n(x) \to 0$ bodově v celém $[0,1]$.

$\sigma_n = \sup\limits_{x\in[0,1]} |f_n(x) - 0| = \sup\limits_{x\in[0,1]} f_n(x) = f_n(a_n) = 1 \not\to 0.$



$f_n'(x) = \underbrace{\cos(\pi x^n)}_{=0} \cdot \underbrace{\pi x^{n-1} \cdot n}_{>0 \text{ pro } x\in(0,1)}$

$\Leftrightarrow \pi x^n = \frac{\pi}{2}$

$x = a_n = \left(\frac{1}{2}\right)^{\frac{1}{n}}$

tedy; $f_n \not\rightrightarrows 0$ v $[0,1]$.

?? $f_n \rightrightarrows 0$ v $[0,\eta]$; $\eta \in (0,1)$ pevné: <u>ANO.</u>

$$\sigma_n = \sup_{x\in[0,\eta]} |f_n(x)-0| = \sup_{x\in[0,\eta]} f_n(x) ;$$

BÚNO: n velké $\Rightarrow$ $a_n > \eta$ (neboť $a_n \to 1$).

$\Rightarrow$ f_n <u>rostoucí</u> na $[0,\eta]$

$\Rightarrow$ $\sigma_n = f_n(\eta) \to 0$.

---

(4) $f_n(x) = \sqrt[n]{1+x^n}$ ...

$x\in[0,1]$: $\quad 1 \le f_n(x) \le \sqrt[n]{2} \to 1$

$x > 1$: $\quad f_n(x) = x\sqrt[n]{1+\frac{1}{x^n}}$

$\to 1$ viz argument výše.

bodová limita: $f_n(x) \to f(x) = \begin{cases} 1 ; & x\in[0,1] \\ x ; & x \ge 1. \end{cases}$

?? stejn. v $[0,1]$: ANO

$$\sigma_n = \sup_{x\in[0,1]} |f_n(x)-1| = \sup_{x\in[0,1]} \left(\sqrt[n]{1+x^n}-1\right) \le \sqrt[n]{2}-1 \ \forall n$$

tedy: $0 \le \sigma_n \le \sqrt[n]{2}-1 \to 0$.

?? stejn. v $[1,+\infty)$: ANO.

$$\sigma_n = \sup_{x\geq 1} |f_n(x) - x| = \sup_{x \geq 1} \Big(\underbrace{\sqrt[n]{1+x^n} - x}_{g_n(x)} \Big).$$

$$g_n'(x) = \left((1+x^n)^{\frac{1}{n}} - x \right)' = \frac{1}{\cancel{n}} (1+x^n)^{\frac{1}{n}-1} \cdot \cancel{n} x^{n-1} - 1$$

$$= \frac{x^{n-1}}{\left(\sqrt[n]{1+x^n}\right)^{n-1}} - 1 < 0$$

$\Rightarrow$ $g_n(x)$ klesá v $[1,+\infty)$

$$\sigma_n = g_n(1) = \sqrt[n]{2} - 1 \to 0.$$

(8) $f_n(x) = \left(1+\frac{x}{n}\right)^n = \exp\left(n\cdot\ln\left(1+\frac{x}{n}\right)\right) \to e^x$;

$$n\cdot\ln\left(1+\frac{x}{n}\right) = \underbrace{\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}}}_{\to 1}\cdot x$$

neboť $\frac{\ln(1+y)}{y} \to 1$; $y\to 0$

$\frac{x}{n}\to 0$; $\frac{x}{n}\neq 0$ $\forall n$;

$x\neq 0$

$$\sigma_n = \sup_{x\in\mathbb{R}} |f_n(x) - e^x| = \sup_{x\in\mathbb{R}} \Big|\underbrace{\left(1+\frac{x}{n}\right)^n - e^x}_{g_n(x)}\Big|.$$

$|g_n(x)| \to +\infty$ pro $x\to\pm\infty$

$\Rightarrow \sigma_n = +\infty$; tedy $f_n(x) \not\rightrightarrows e^x$ v $\mathbb{R}$

a z téhož důvodu ani $[a,+\infty)$

nebo $(-\infty, b]$.

?? $f_n \rightrightarrows e^x$ v $[-M, M]$; $M>0$ pevné!! : <u>ANO</u>.

podobně jako v (15).

(11) $f_n(x) = \dfrac{\operatorname{arctg}(nx)}{nx}$; $x > 0$.

nebot' $\lim\limits_{y\to+\infty} \operatorname{arctg} y = \dfrac{\pi}{2}$.

x pevné: $f_n(x) \longrightarrow \dfrac{\frac{\pi}{2}}{x \cdot \infty} = 0$;

$$\sigma_n = \sup_{x>0} |f_n(x) - 0| = \sup_{x>0} \underbrace{\frac{\operatorname{arctg}(nx)}{nx}}_{g_n(x)}.$$

$\lim\limits_{x\to 0+} g_n(x) = 1$; tedy $\sigma_n \geq 1 \not\to 0$.

dokonce $f_n \not\rightrightarrows 0$ na $(0,\delta)$ pro žádné $\delta > 0$.

?? $f_n \rightrightarrows 0$ v $(\delta, +\infty)$; $\delta > 0$ pevné: ANO

$$\sigma_n = \sup_{x\geq\delta} |f_n(x) - 0| = \sup_{x\geq\delta} \frac{\operatorname{arctg}(nx)}{nx} \leq \frac{\pi}{2} \cdot \frac{1}{n\delta}$$

nebot' $\operatorname{arctg} < \dfrac{\pi}{2}$

$nx \geq n\delta$

$$0 \leq \sigma_n \leq \frac{\pi}{2\delta} \cdot \frac{1}{n} \to 0.$$

(15) $f_n(x) = n \cdot \ln\left(1 + \frac{x}{n}\right)$; $x > -1$.

$$\left.\begin{array}{l} x=0: \ f_n(0) = n \cdot \ln 1 = 0 \\ x \neq 0: \ f_n(x) = x \cdot \underbrace{\dfrac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}}}_{\to 1} \to x. \end{array}\right\} \ \underline{f_n(x) \to x.}$$

?? stejn. v $(-1, +\infty)$: NE

$$\sigma_n = \sup_{x>-1} |f_n(x) - x| = \sup_{x>-1} \Big|\underbrace{n \ln\left(1+\frac{x}{n}\right) - x}_{g_n(x)}\Big|$$

$$g_n(x) = x\left(\underbrace{\frac{n}{x}}\cdot \ln\left(1+\frac{x}{n}\right) - 1\right) \to +\infty\,(0-1) = -\infty$$

pro $x \to +\infty$

$\Rightarrow \sigma_n = +\infty$; z téhož důvodu: $f_n \not\rightrightarrows x$ v $[K, +\infty)$

pro žádné $\underline{K}$.

?? $f_n \rightrightarrows x$ v $(-1, M)$; $M > 0$ pevné: <u>ANO</u>.

$x \in (-1, M)$: $\quad |f_n(x) - x| = \left|x\left(\dfrac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}} - 1\right)\right| < \varepsilon$

$x \neq 0$

pro $\forall n \geq n_0$??

$\varepsilon > 0$ dáno: $\quad \exists \delta > 0$; $\left|\dfrac{\ln(1+y)}{y} - 1\right| < \dfrac{\varepsilon}{M} \quad \forall y \in P(0, \delta)$

vol $n_0 > \dfrac{M}{\delta}$ ... $\forall n \geq n_0$; $x \in (-1, M)$: $\left|\dfrac{x}{n}\right| \leq \dfrac{M}{n_0} < \delta$

$$\Rightarrow \left|x\left(\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}} - 1\right)\right| \leq M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

(14) $f_n(x) = \sqrt[2n]{x^n + |\ln x|}$; $x > 0$.

? bodová limita :

$x \in (0,1)$ $\quad \sqrt[2n]{|\ln x|} \le f_n(x) \le \sqrt[2n]{1+|\ln x|}$

$\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow$

$\qquad\qquad\qquad 1 \qquad\qquad\qquad\qquad 1$

neboť $\sqrt[2n]{a} = \exp\left(\frac{1}{2n} \ln a\right) \to 1$ $\forall a>0$ pevné

$\frac{1}{2n} \to 0$

$f_n(1) = 1 \quad \forall n$

$x > 1$: $f_n(x) = \sqrt{x} \sqrt[2n]{1 + \frac{\ln x}{x^n}}$

$$1 \le \dots \le \sqrt[2n]{1 + \frac{\ln x}{x}} \to 1.$$

---

celkem: $f_n(x) \longrightarrow f(x) = \begin{cases} 1; & x \in (0,1] \\ \sqrt{x}; & x > 1. \end{cases}$

?? $f_n \rightrightarrows 1$ v $(0,1]$: NE

$\sigma_n = \sup\limits_{x \in (0,1]} |f_n(x) - 1| \ge \lim\limits_{x \to 0+} f_n(x) = +\infty.$

?? $f_n \rightrightarrows 1$ v $[\delta, 1]$; $\delta \in (0,1)$ pevné: ANO:

$\sigma_n = \sup\limits_{x \in [\delta,1]} |f_n(x) - 1| = \sup\limits_{x \in [\delta,1]} \left|\sqrt[2n]{x^n + |\ln x|} - 1\right|$

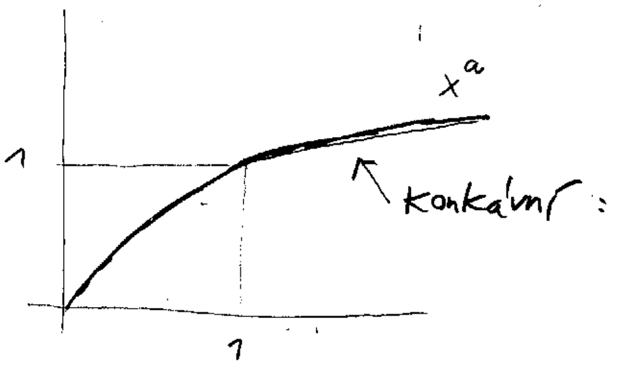
odtud $x \in [\delta, 1] \Rightarrow f_n(x) \le 1 + |\ln \delta|$

$$f_n(x) \ge \delta^{n}$$

$$\underbrace{\sqrt[2n]{\delta^n} - 1}_{\rightrightarrows 0} \le f_n(x) - 1 \le \underbrace{\sqrt[2n]{1 + |\ln \delta|} - 1}_{\rightrightarrows 0} .$$

$x \ge 1$: $\quad \sigma_n = \sup\limits_{x \ge 1} \left| f_n(x) - \sqrt{x} \right| = \sup\limits_{x \ge 1} \Big(\underbrace{\sqrt[2n]{x^n + \ln x} - \sqrt{x}}_{g_n(x).} \Big).$

$$g_n(x) = \sqrt{x} \left(\sqrt[2n]{1 + \frac{\ln x}{x^n}} - 1 \right)$$

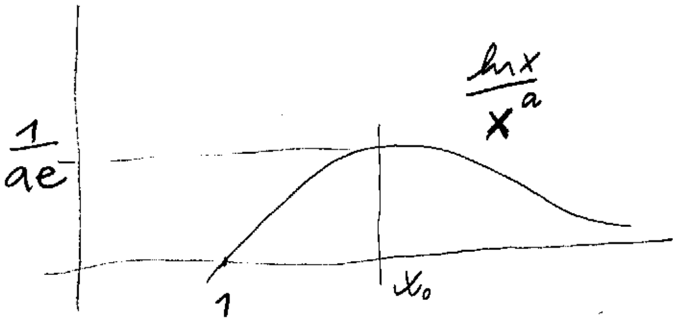


$$\boxed{(1+h)^a \le 1 + ah \quad \forall h > 0, \; a \in (0,1).}$$

$$g_n(x) = x^{\frac{1}{2}} \left(\left(1 + \frac{\ln x}{x^n} \right)^{\frac{1}{2n}} - 1 \right) \le x^{\frac{1}{2} - n} \cdot \ln x \cdot \frac{1}{2n} \le \frac{C}{2n}$$

$\rightarrow$ stačí $x^{\frac{1}{2} - n} \ln x \le C \quad x \in [1, \infty)$

konstanta nezávislá na $n \in \mathbb{N}$.



$$C = \sup_n \frac{1}{e(n - \frac{1}{2})} = \frac{2}{e} .$$