Výpočet rezidua

Pravidla: ① $\operatorname{res}_{z_0} \dfrac{g(z)}{(z-z_0)^k} = \dfrac{1}{(k-1)!}\, g^{(k-1)}(z_0)$

speciálně: $\operatorname{res}_{z_0} \dfrac{g(z)}{z-z_0} = g(z_0)$

$\operatorname{res}_{z_0} \dfrac{g(z)}{(z-z_0)^2} = g'(z_0).$

② $h(z_0)=0$, $h'(z_0)\neq 0$ $\Rightarrow$ $\operatorname{res}_{z_0} \dfrac{g(z)}{h(z)} = \dfrac{g(z_0)}{h'(z_0)}$

Lemma: $e^z = e^w \iff \operatorname{Re} z = \operatorname{Re} w$, $\operatorname{Im} z = \operatorname{Im} w + 2k\pi$

$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{i}\sinh(iz)$

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cosh(iz)$

$\sin iz = i\sinh(z)$

$\cos iz = \cosh(z)$

① $z_0 = 0$: $\quad \operatorname{rez}_0 \frac{1}{z}\cdot\frac{1}{z^2+1} = \frac{1}{z^2+1}\Big|_{z=0} = 1$

$z_0 = i$: $\quad \operatorname{rez}_i \frac{1}{(z-i)}\cdot\frac{1}{z(z+i)} = \frac{1}{z(z+i)}\Big|_{z=i} = -\frac{1}{2}$

② $\frac{z^2}{z^4+1} = f(z)$; singularity $\quad z^4 = -1$

$z^2 = \pm i$;

$z = \frac{1}{\sqrt{2}}(\pm 1 \pm i)$

jde o čtyři jednoduché póly:

např. pro $z_1 = \frac{1}{\sqrt{2}}(1+i)$

$$\operatorname{res}_{z_1} \frac{z^2}{z^4+1} = \frac{z^2}{(z^4+1)'}\Big|_{z=z_1} = \frac{z^2}{4z^3}\Big|_{z=z_1}$$

$$= \frac{1}{4z_1} = \frac{1-i}{4\sqrt{2}}$$

③ $\frac{z^2}{(z+1)^3}$; $\quad \operatorname{res}_{-1} = \frac{1}{2!}\left(z^2\right)''\Big|_{z=-1} = 1$

④ $\frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3}\cdot\frac{1}{(z-i)^3}$.

$$\operatorname{res}_i = \left(\frac{1}{(z+i)^3}\right)''\cdot\frac{1}{2!}\Big|_{z=i} = \frac{12}{(z+i)^5}\cdot\frac{1}{2}\Big|_{z=i} = -\frac{3i}{16}$$

5 $f = \frac{1}{(z^2+1)(z-1)}$; $\operatorname{res}_1 = -\frac{1}{2}$; $\operatorname{res}_{\pm i} = \frac{1}{4}$

6 $f = \frac{z^{2n}}{(z-1)^n}$; $\operatorname{res}_1 = \frac{\left(z^{2n}\right)^{(n-1)}}{(n-1)!}\Bigg|_{z=1} = \frac{2n\cdot(2n-1)\cdots n}{(n-1)!}$

$$\left(z^{2n}\right)' = 2n\cdot z^{2n-1} \qquad = \frac{(2n)!}{n!\,(n-1)!}$$

$$\left(z^{2n}\right)'' = 2n\cdot(2n-1)\cdot z^{2n-2}$$

$$\left(z^{2n}\right)^{(n-1)} = 2n\cdot(2n-1)\cdots(2n-(n-1+1))\cdot z^{2n-(n-1)}$$

7 $\frac{1}{\sin \pi z}$ .. ? singularity $\quad \sin \pi z = 0$

$$\frac{1}{2i}\left(e^{i\pi z} - e^{-i\pi z}\right) = 0$$

$$e^{2\pi i z} = 1 = e^0$$

$$2\pi i z = 2k\pi i \; ; \; k \in \mathbb{Z}$$

$$\boxed{z \in \mathbb{Z}}$$

$(\sin \pi z)' = \pi \cos \pi z \neq 0$ pro $z \in \mathbb{Z}$

→ jednoduché póly!!

$$\operatorname{res}_k \frac{1}{\sin \pi z} = \frac{1}{\pi \cos \pi z}\Bigg|_{z=k} = \frac{1}{\pi \cos k\pi} = \frac{(-1)^k}{\pi}.$$

(8) $f = \frac{\cos \pi z}{\sin \pi z}$; singularity $z \in \mathbb{Z}$ (viz př. 7)

$$\operatorname{res}_k f = \frac{\cos \pi z}{(\sin \pi z)'}\Big|_{z=k} = \frac{(-1)^k}{\pi (-1)^k} = \frac{1}{\pi}.$$

(9) $f = \frac{1}{\sinh z}$; ? singularity $\frac{1}{2}(e^{z} - e^{-z}) = 0$

$$e^{2z} = 1 = e^{0}$$

$$2z = 2k\pi i \; ; \; k \in \mathbb{Z}$$

$$\boxed{z = k\pi i \; ; \; k \in \mathbb{Z}}$$

$$\operatorname{res}_{k\pi i} f = \frac{1}{(\sinh z)'}\Big|_{z=k\pi i}$$

$$= \frac{1}{\cosh(k\pi i)} = \frac{1}{\cos k\pi} = (-1)^k$$

(10) $f = \frac{1}{\cosh z}$? singularity $\frac{1}{2}(e^{z} + e^{-z}) = 0$

$$e^{2z} = -1 = e^{i\pi}$$

$$2z = i\pi + 2k\pi i$$

$$\boxed{z_k = i\left(\frac{\pi}{2} + k\pi\right); k \in \mathbb{Z}}$$

$$\operatorname{res}_{z_k} = \frac{1}{(\cosh z)'}\Big|_{z=z_k}$$

$$= \frac{1}{\sinh i\left(\frac{\pi}{2} + k\pi\right)} = \frac{1}{i \sin\left(\frac{\pi}{2} + k\pi\right)}$$

$$= \frac{(-1)^k}{i} = (-1)^{k+1} i$$

(11) $f = \operatorname{tgh} z = \frac{\sinh z}{\cosh z}$; jednoduché póly $z_k = i\left(\frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}$

... viz př. 10

$$\operatorname{res}_{z_k} = \frac{\sinh z}{(\cosh z)'}\Big|_{z=z_k} = \frac{\sinh z}{\sinh z}\Big|_{z=z_k} = 1$$

(12) $\frac{\cos z}{(z-1)^2} = f \quad \operatorname{res}_1 f = (\cos z)'\Big|_{z=1} = \sin 1$

(13) $f = \frac{1}{e^z+1}$; ? singularity $\quad e^z = -1 = e^{i\pi}$

$$\boxed{z_k = i\pi + 2k\pi i;\ k \in \mathbb{Z}}$$

$$\operatorname{res}_{z_k} f = \frac{1}{(e^z+1)'}\Big|_{z=z_k} = e^{-z_k} = -1$$

(14) $f = \frac{\sin \pi z}{(z-1)^3}$; $\operatorname{res}_1 f = \frac{1}{2}\cdot(\sin \pi z)''\Big|_{z=1} = \frac{-\pi^2}{2}\sin \pi z\Big|_{z=1}$

$= 0$

(15) $\frac{1}{\sin z^2} = f = \frac{1}{h(z)}$; $\quad h'(z) = 2z\cdot\cos z^2$

$h''(z) = 2\cos z^2 - 4z^2 \sin z^2$

$h(0) = h'(0) = 0$

$h''(0) \neq 0.$ $\quad \to$ 2. nebo ? řád.

$$\operatorname{res}_0 f = \lim_{z\to 0}\left(\frac{z^2}{\sin z^2}\right)' ;$$

$$\left(\frac{z^2}{\sin z^2}\right)' = \frac{2z\cdot\sin z^2 - 2z^3\cos z^2}{\sin^2(z^2)} \to 0;\ z \to 0;\ \text{protože}$$

čitatel: $2z\left(\sin z^2 - z^2\cos z^2\right) = 2z\left(z^2 - z^2\left(1+o(z^2)\right) + o(z^4)\right)$

$= o(z^5)$

jmenovatel $\sim z^4$

celkem: $\operatorname{res}_0 f = 0$ (je to vidět ze „sudosti" fce!!)

(16) $\frac{1}{z^6(z-2)}$; $\operatorname{res}_0 = \frac{1}{5!}\left((z-2)^{-1}\right)^{(5)}\Big|_{z=0} = \frac{(-1)^5\, 5!}{5!\,(-2)^6}$

$\left((z-2)^{-1}\right)^{(5)} = (-1)(-2)\cdots(-5)(z-2)^{-6}$ $\qquad = \frac{-1}{64}$

$\operatorname{res}_2 = \frac{1}{2^6} = \frac{1}{64}$.

(17) $f = \frac{z^8+1}{z^6(z+2)}$; $\operatorname{res}_{-2} = \frac{z^8+1}{z^6}\Big|_{z=-2} = \frac{257}{64}$

$\operatorname{res}_0 f = ?$ $\qquad f = \underbrace{\frac{z^2}{z+2}}_{\in \mathcal{H}(U(0))} + \tilde{f}$; $\tilde{f} = \frac{1}{z^6(z+2)}$

$\to \operatorname{res}_0 f = \operatorname{res}_0 \tilde{f} = -\frac{1}{64}$ (viz výše př. 16)

(18) $f = \frac{z^{10}+1}{z^6(z^2+4)}$; singularity $0, \pm 2i$

neboť $z^2+4 = (z+2i)(z-2i)$

$\operatorname{res}_{2i} = \frac{z^{10}+1}{z^6(z+2i)}\Big|_{z=2i} = \frac{-1023i}{256}$; $\operatorname{res}_{-2i} = \frac{1023i}{256}$.

$\operatorname{res}_0 f = 0$; (funkce je sudá $\Rightarrow$ Laurentův rozvoj obsahuje jenom sudé mocniny: $a_{2k+1} = 0$; $k \in \mathbb{Z}$)

(19) $f = \dfrac{\cos z}{(z^2+1)^2}$; singularity $z = \pm i$ -- 2násobné póly.

$$\operatorname{res}_i f = \left(\frac{\cos z}{(z+i)^2}\right)'\Bigg|_{z=i} = \frac{-\left((z+i)\sin z + 2\cos z\right)}{(z+i)^3}\Bigg|_{z=i}$$

$= \dfrac{i}{4}\left(\sinh 1 - \cosh 1\right)$; dle mocnin: $\sin iz = i\sinh z$

$\cos iz = \cosh(z)$

(20) $f = \dfrac{\sin z}{(z^2+1)^2}$; $\operatorname{res}_i = \dfrac{7}{4}\left(\sinh 1 - \cosh 1\right)$