A1. $f(x) = \sin^4 x = [\sin^2 x]^2$, $\sin^2 y = \frac{1}{2}(1-\cos 2y)$ $\quad \cos^2 y = \frac{1}{2}(1+\cos 2y)$

$$= \left[\tfrac{1}{2}(1-\cos 2y)\right]^2 = \tfrac{1}{4}\left(1 - 2\cos 2y + \cos^2 2y\right)$$

$$= \tfrac{3}{8} - \tfrac{1}{2}\cos 2y + \tfrac{1}{8}\cos 4y \; ;$$

toto je (triviálně) absolutně konvergující trig. řada a tedy také $F_{f(x)}$ – Věta 21.1

A2: $f(x) = \cos^4 x = [\cos^2 x]^2 = \frac{1}{4}\left[1 + 2\cos 2y + \cos^2 2y\right]$

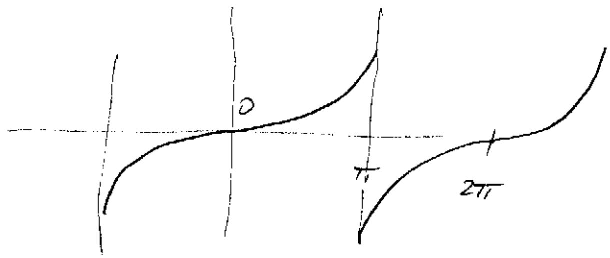
$a_0 = \frac{3}{2}$ $\qquad = \frac{3}{8} + \frac{1}{2}\cos 2y + \frac{1}{8}\cos 4y$

Parseval: $$\frac{1}{\pi}\int_0^{2\pi} (f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 .$$

$$\frac{1}{\pi}\underbrace{\int_0^{2\pi} \sin^8 x\,dx}_{I} = \left(\frac{3}{4}\right)^2 \cdot \frac{1}{2} + \frac{1}{4} + \frac{1}{16}$$

$$I = \frac{35}{64}\pi$$

A7.: $f(x) = \sinh(ax)$; $a \neq 0$; $x \in (-\pi, \pi\rangle$.



$a_k = 0 \quad \forall k \geq 0$

$$\frac{\pi}{2} b_k = \underbrace{\int_0^{\pi} \sinh(ax)\sin kx\,dx}_{I} = \quad \left| \begin{array}{ll} u = \sinh ax & u' = a\cosh ax \\ v' = \sin kx & v = -\frac{1}{k}\cos kx \end{array} \right.$$

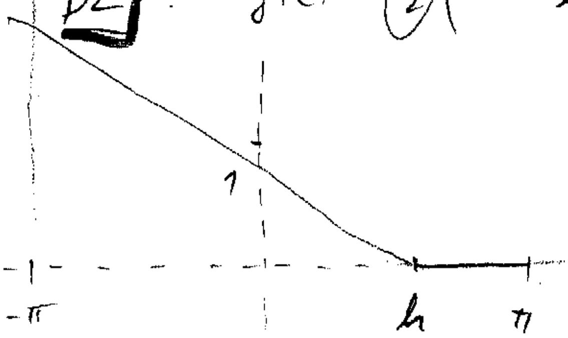
$$I = \left[\sinh ax \cdot \left(-\tfrac{1}{k}\cos kx\right)\right]_0^{\pi} + \frac{a}{k}\int_0^{\pi} \cosh ax \cdot \cos kx$$

$$= -\frac{1}{k}\sinh a\pi\,(-1)^k + \frac{a}{k}\left\{\left[\cosh ax \cdot \tfrac{1}{k}\sin kx\right]_0^{\pi} - \frac{a}{k} I\right\}$$

$$I\left(1 + \frac{a^2}{k^2}\right) = \frac{(-1)^{k+1}}{k}\sinh a\pi \; ; \quad b_k = (-1)^{k+1}\frac{2k}{\pi(k^2+a^2)}\sinh a\pi \quad ✓$$

$$\frac{k^2+a^2}{k^2}$$

D2: $f(x) = \frac{1}{2}\left(1 - \frac{x}{h}\right)\underbrace{\left(1 + \operatorname{sgn}\left(1 - \frac{x}{h}\right)\right)}_{\substack{0 \quad x > h \\ 2 \quad x < h}}$; $h \in (0, \pi)$



$a \neq k \in \mathbb{Z}$

D4 $f(x) = \sin(ax)$ .. lichá $\Rightarrow a_k = 0$

$$\frac{\pi}{2} b_k = \int_0^\pi \sin(ax) \sin kx = \frac{1}{2} \int_0^\pi \cos(a-k)x - \cos(a+k)x$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha+\beta)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(a-k)x}{a-k} - \frac{\sin(a+k)x}{a+k}\right]_0^\pi$$

$$= \frac{1}{2}\left(\frac{\sin(a-k)\pi}{a-k} - \frac{\sin(a+k)\pi}{a+k}\right) = -$$

$$F_f(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{\sin(a-k)\pi}{a-k} - \frac{\sin(a+k)\pi}{a+k}\right]\sin kx$$

A1 $f(x) = \sin^4 x = \left[\frac{1}{2}(1-\cos 2x)\right]^2 = \frac{1}{4} - \cos 2x + \cos^2 2x$

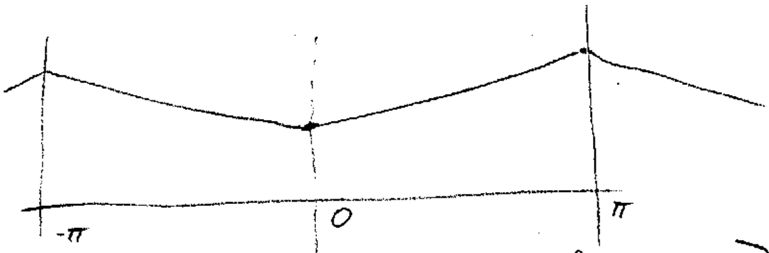
$$= \frac{1}{4} - \cos 2x + \frac{1}{2}(1+\cos 4x)$$

$$= \frac{3}{4} - \cos 2x + \frac{1}{2}\cos 4x.$$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

C5 $f(x) = \exp(ax)$



$$a_0 = \frac{2}{\pi}\int_0^\pi \exp(ax) = \frac{2}{\pi}\left[\frac{1}{a}\exp(ax)\right]_0^\pi = \frac{2}{\pi a}\left[e^{a\pi}-1\right]. \checkmark$$

$$\frac{\pi}{2}a_k = \int_0^\pi \exp(ax)\cos(kx) = Re\left\{\int_0^\pi \exp(x(a+ik))\,dx\right\}$$

$$= Re\left\{\left[\frac{1}{a+ik}\exp(x(a+ik))\right]_0^\pi\right\}$$

$$= Re\left\{\frac{a-ik}{a^2+k^2}\cdot\left[e^{a\pi}(-1)^k - 1\right]\right\}$$

$$= \frac{a}{a^2+k^2}\cdot\left[e^{a\pi}(-1)^k - 1\right].$$

$$F_f(x) = \frac{1}{\pi a}\left[e^{a\pi}-1\right] + \sum_{k=1}^{\infty}\frac{a}{a^2+k^2}\cdot\left[(-1)^k e^{a\pi}-1\right]\cdot\cos kx$$

$\frac{2}{\pi}$

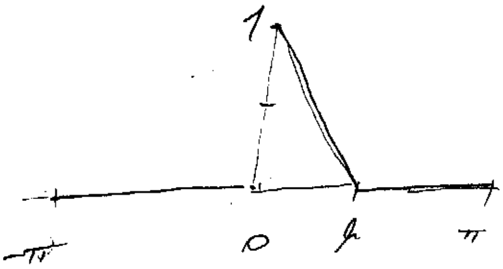
zderivováno: $\frac{2}{\pi} \ldots \sin(k\underbrace{(\pi+x)}_{y})$

⋮

D1. $f(x) = \frac{1}{4}\left(1-\frac{x}{h}\right)\left(1+\mathrm{sgn}\left(1-\frac{x}{h}\right)\right)\cdot\left(1+\mathrm{sign}\,x\right).$

$x < h$ $\qquad x \in (0,\pi)$

$$f(x) = \begin{cases} 1-\frac{x}{h}\ ; & x\in(0,h) \\ 0 & \text{jinde} \end{cases}$$



$$a_0 = \frac{1}{\pi}\int_0^h \left(1-\frac{x}{h}\right)dx = \frac{1}{\pi}\cdot\frac{h}{2}$$

$$\pi a_k = \int_0^h \left(1-\frac{x}{h}\right)\cos kx = \underbrace{\left[\left(1-\frac{x}{h}\right)\frac{\sin kx}{k}\right]_0^h}_{=0} + \frac{1}{h}\cdot\frac{1}{k}\int_0^h \sin kx\,dx$$

$$-\frac{1}{hk^2}\left[\cos kx\right]_0^h = +\frac{1}{hk^2}\left(1-\cos kh\right).$$

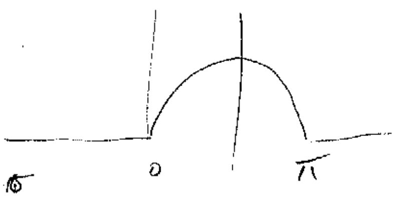
$$\pi b_k = \frac{1}{k^2 h}\left[kh - \sin kh\right].$$

$$F_f(x) = \frac{h}{4\pi} + \sum_{k=1}^{\infty} a_k\cos kx + b_k \sin kx\,;$$

$$x=0:\ \text{l.s.}:\ \frac{1}{2} = \frac{h}{4\pi} + \sum_{k=1}^{\infty}\frac{1}{hk^2}\left(1-\cos kh\right)$$

$$\boxed{\sum_{k=1}^{\infty}\frac{\cos kh}{k^2} = \frac{h^2}{4\pi} - \frac{h}{2} + \sum_{k=1}^{\infty}\frac{1}{k^2}}$$

D6: $f(x) = \max\{0, \sin x\}$; $\qquad a_0 = \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}$;



$$\pi a_k = \int_0^\pi \sin x \cos kx\,dx =$$

$$= \frac{1}{2}\int_0^\pi \sin(k+1)x + \sin(1-k)x\,dx = \begin{cases} 0 & ; k=1 \\ -\frac{(-1)^k+1}{k^2-1} & ; k\geq 2\end{cases}$$

$$\pi b_k = \int_0^\pi \sin x\cdot\sin kx\,dx = \begin{cases}\frac{\pi}{2}; & k=1 \\ 0; & k\geq 2\end{cases}$$

(D11)* $f(x) = \ln|\operatorname{tg}\frac{x}{2}|$; sudá ... $b_k = 0$

$$a_0 = \int_{-\pi}^{\pi} \ln|\operatorname{tg}\tfrac{x}{2}|\,dx = \,? \quad ; \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cdots$$

$$\frac{\pi}{2}a_k = \int_0^{\pi} f(x)\underbrace{\cos kx}_{g'(x)}\,dx = \underbrace{\left[f(x)\,\frac{\sin kx}{k}\right]_0^{\pi}}_{0} - \frac{1}{k}\underbrace{\int_0^{\pi}\frac{\sin kx}{\sin x}\,dx}_{I_k}$$

$$f'(x) = \frac{1}{\operatorname{tg}\frac{x}{2}}\cdot\left(\operatorname{tg}\frac{x}{2}\right)' = \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\cdot\frac{1}{2}\cdot\frac{1}{\cos^2\frac{x}{2}} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{1}{\sin x}$$

---

spočtěl I_k: $\quad I_1 = \int_0^{\pi}\frac{\sin x}{\sin x} = \pi$;

$$I_2 = \int_0^{\pi}\frac{\sin 2x}{\sin x} = 2\int_0^{\pi}\frac{\sin x\cos x}{\sin x}\,dx = 2\int_0^{\pi}\cos x = 0.$$

---

indukcí: $$I_{k+2} = \int_0^{\pi}\frac{\sin(kx+2x)}{\sin x} = \int_0^{\pi}\frac{\sin kx\cdot\cos 2x + \cos kx\cdot\sin 2x}{\sin x}$$

$$= I_k - 2\int_0^{\pi}\sin kx\,\sin x\,dx$$

$$+ 2\int_0^{\pi}\cos kx\,\cos x\,dx$$

$\cos 2x = 1 - 2\sin^2 x$

$\sin 2x = 2\sin x\cos x$

pro $k \geq 2$ je to 0; viz Lemma 21.1

(D11) – pokr.:

$$I_k = \int_0^\pi \frac{\sin kx}{\sin x}\,dx = \begin{cases} 0 ; & k \text{ sudé} \\ \pi , & \text{liché}. \end{cases}$$

$$\mathcal{F}_f(t) = \frac{a_0}{2} - \sum_{n=0}^{\infty} \frac{2}{2n+1} \cos(2n+1)x .$$

---

diskuse: $f(t)$ je po částech C^1 v $(+\delta, \pi-\delta)$; $\delta > 0$

a tedy $f(t) = \mathcal{F}_f(t)$ v $(0,\pi)$.

zec: $0 = f\left(\frac{\pi}{2}\right) = \frac{a_0}{2}$; neboť $\cos(2n+1)\frac{\pi}{2} = 0$

tedy $a_0 = 0$.

(D9)* analogicky: $f(t) = \ln\left|\sin\frac{x}{2}\right|$

$$f'(t) = \frac{1}{2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \frac{\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1+\cos x}{2\sin x}$$

-- ${}^{\text{ln.}}$ posloup ředě na integrál

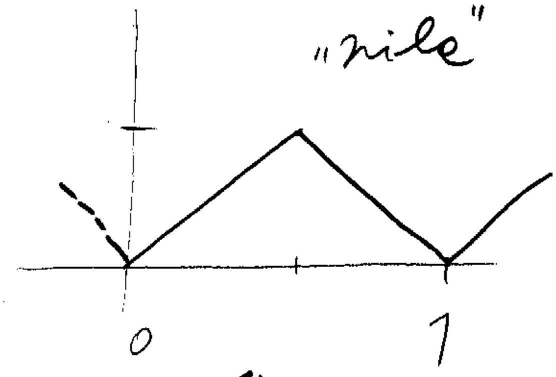
$$J_k = \frac{1}{2}\int_0^\pi \frac{1+\cos x}{\sin x}\cdot \sin kx\,dx = \frac{1}{2}\left(I_k + \int_0^\pi \frac{\cos x}{\sin x}\sin kx\right)$$

rekurentní vzorec ??

Pozn: D9–D11 jsou příklady (*) – TĚŽŠÍ (na písemku)

(B4) $f(x) = \mathrm{dist}(x, \mathbb{Z}) = \inf\{|x-n|;\ n \in \mathbb{Z}\}$.

„pila"



sudá, spojitá, po částech C^1

1-periodická.

$$a_0 = 2\int_0^1 f(x)\,dx = \frac{1}{2}.$$

$$b_k = 0\ ; \quad a_k = 2\int_0^1 f(x)\cos(2k\pi x)\,dx = 4\int_0^{1/2} x\cos(2k\pi x)\,dx$$

$$= 4\left[x\,\frac{\sin(2k\pi x)}{2k\pi}\right]_0^{1/2} - \frac{4}{2k\pi}\int_0^{\frac{1}{2}} \sin(2k\pi x)\,dx\ ;$$

$$\int_0^{\frac{1}{2}} \sin(2k\pi x)\,dx = \frac{1}{2k\pi}\Big[-\cos(2k\pi x)\Big]_0^{\frac{1}{2}} = \frac{1}{2k\pi}\Big(\underbrace{-\cos(k\pi)}_{(-1)^k} + 1\Big)$$

---

celkem: $a_k = \frac{1}{k^2\pi^2}\left((-1)^k - 1\right) = \begin{cases} 0, & k \text{ sudé} \\ \frac{-2}{k^2\pi^2}, & \text{liché.} \end{cases}$

$$F_f(x) = \frac{1}{4} + \sum_{n=0}^{\infty} \frac{-2}{\pi^2(2n+1)^2}\cos(2n+1)2\pi x\ .$$

V.21.2: $f(x) = F_f(x)$ pro $\forall x$.

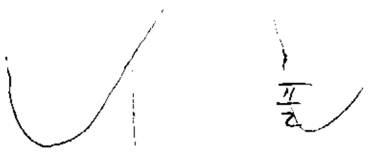
$x=0$: $\to \displaystyle\sum_{k=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

Parseval: $\displaystyle 2\int_0^1 |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2$

$$\frac{1}{6} = \frac{1}{8} + \sum_{n=0}^{\infty} \frac{4}{\pi^4(2n+1)^4}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

A8. $f(x) = x\cos x;\quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



$f'(x) = \cos x - x\sin x$

$f'\left(\frac{\pi}{2}-\right) = -\frac{\pi}{2};\quad f'\left(-\frac{\pi}{2}+\right) = -\frac{\pi}{2}$. $f(x)$ je C^1

$f''(x) = -\sin x - \sin x - x\cos x$

$= -2\sin x - x\cos x;\quad f''\left(\frac{\pi}{2}-\right) = -2$

$f''\left(-\frac{\pi}{2}+\right) = 2$ není C^2.

$\Rightarrow |a_k| + |b_k| \sim \frac{1}{k^3}$.

$$b_k = \frac{2}{l}\int_a^{a+l} f(x)\sin\left(\frac{2\pi}{l}kx\right)dx \qquad \boxed{l=\pi}.$$

liché: $a_k = 0$

$$b_k = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{f(x)\sin 2kx}_{\text{sudé fce}}\,dx; = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}$$

$$\frac{\pi}{4}b_k = \int_0^{\frac{\pi}{2}} x\cos x\sin 2kx\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cdot\left[\sin(2k+1)x + \sin(2k-1)x\right]dx$$

$$\cos a\sin b = \frac{1}{2}\left[\sin(a+b) + \sin(b-a)\right]$$

$$\int_0^{\frac{\pi}{2}} x\sin mx = \underbrace{\left[-x\frac{\cos mx}{m}\right]_0^{\frac{\pi}{2}}}_{=\cancel{\frac{\pi}{2}}\,0} + \frac{1}{m}\int_0^{\frac{\pi}{2}}\cos mx = \frac{1}{m^2}\left[\sin mx\right]_0^{\frac{\pi}{2}}$$

liché.

$$= \frac{1}{m^2}\sin\left(m\frac{\pi}{2}\right).$$

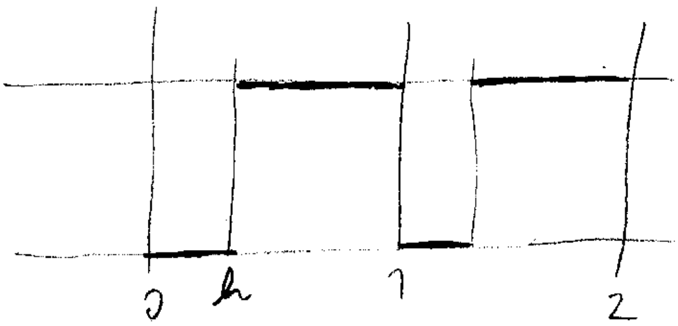
$$\frac{1}{2m^2}\left[\frac{\sin(2k+1)\frac{\pi}{2}}{(2k+1)^2} + \frac{\sin(2k-1)\frac{\pi}{2}}{(2k-1)^2}\right]. \qquad \frac{\pm(-1)^k}{(2k\pm1)^2}$$

$$(-1)^k\left[\frac{1}{(2k+1)^2} - \frac{1}{(2k-1)^2}\right] = \frac{(-1)^k}{(2k+1)^2(2k-1)^2}\cdot\left[-4k\right].\quad \underline{\text{o.k.}}$$

$$b_k = \frac{16}{\pi}(-1)^{k-1}\frac{k}{(4k^2-1)^2}.$$

A3 $\quad f(x) = \begin{cases} 0; & x \in (0,h) \\ 1; & x \in (h,1) \end{cases}$ $\quad$ perióda 1

$h \in (0,1)$ zadané



$$a_0 = 2\int_0^1 f(x)\,dx = 2(1-h)$$

$$a_k = 2\int_0^1 f(x)\cos(2k\pi x)\,dx$$

$$= 2\int_h^1 \cos(2k\pi x)\,dx = \frac{-\sin(2k\pi h)}{k\pi}$$

$$b_k = 2\int_0^1 f(x)\sin(2k\pi x)\,dx$$

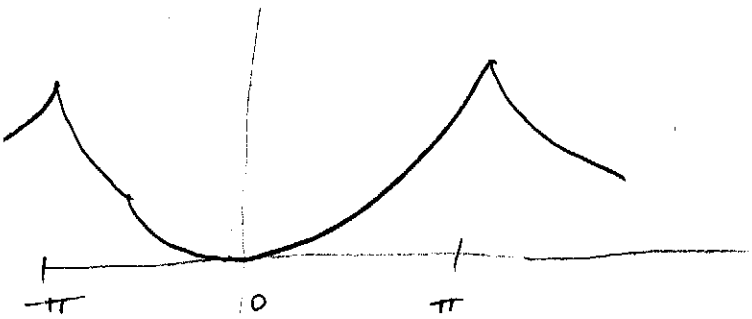
$$= 2\int_h^1 \sin(2k\pi x)\,dx = \frac{1}{k\pi}\Big[-\cos(2k\pi x)\Big]_h^1 = \frac{\cos(2k\pi h)-1}{k\pi}$$

$$F_f(x) = 1-h + \sum_{k=1}^{\infty}\Big[\frac{-1}{k\pi}\sin(2k\pi h)\cos(2k\pi x) + \frac{1}{k\pi}\big(\cos(2k\pi h)-1\big)\sin 2k\pi x\Big].$$

$$x=0: \quad F_f(x) = \frac{1}{2} = 1-h + \sum_{k=1}^{\infty}\left(\frac{-1}{k\pi}\right)\sin(2k\pi h)$$

$$\to \sum_{k=1}^{\infty}\frac{\sin(2k\pi h)}{k} = \pi\left(\frac{1}{2}-h\right) \; ; \; \forall h \in (0,1)$$

A51 $\quad f(x) = x^2 \; ; \; x \in (-\pi, \pi) \; ; \; 2\pi$-per.



spojitá, po částech C^1

sudá: $b_k = 0$;

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2}{3} \pi^3 .$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos kx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos kx \, dx = (2 \times \text{per-partes})$$

$$= \frac{4}{k^2} \cos k\pi = (-1)^k \frac{4}{k^2} .$$

$$F_f(x) = \frac{\pi^3}{3} + \sum_{k=1}^{\infty} (-1)^k \frac{4}{k^2} \cos kx$$

Věta 21.2 $\to \quad f(x) = F_f(x)$ pro $\forall x \in \mathbb{R}$

$$x = 0 : \to \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$$

Parseval:
$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

$$\frac{4}{9} \pi^4 \cdot \frac{1}{2} + \sum_{k=1}^{\infty} \frac{16}{k^4} = \frac{1}{\pi} \cdot \frac{2}{5} \pi^5$$

$$\to \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$