A. Napište jako součet mocninné řady o daném středu:

1. $f(z) = \cosh^2 z$, $z = 0$
2. $f(z) = \sin^2 z$, $z = 0$
3. $f(z) = \frac{1}{az+b}$, $z = 0$, $b \neq 0$
4. $f(z) = \frac{z}{z^2-2z+5}$, $z = 1$
5. $f(z) = \sin(2z - z^2)$, $z = 1$
6. * $f(z) = \exp z \sin z$, $z = 0$
7. * $f(z) = \exp\left(\frac{z}{1-z}\right)$, $z = 0$

B. Rozviňte do Laurentovy řady o daném středu:

1. $f(z) = \frac{1}{(z-a)(z-b)}$, $z = a$
2. $f(z) = \frac{1}{(z^2+1)^2}$, $z = i$
3. $f(z) = (z+1)^2 \exp(1/z)$, $z = 0$
4. $f(z) = z^2 \sin\left(\frac{1}{z-1}\right)$, $z = 1$
5. * $f(z) = \sin z \sin(1/z)$, $z = 0$
6. * $f(z) = \exp(z + 1/z)$, $z = 0$

U příkladů * stačí nalézt část řady, jinak se žádá tvar celé sumy.

(A1) $f(z) = \cosh^2 z = \left(\frac{1}{2}(e^{z} + e^{-z})\right)^2$

$$= \frac{1}{4}\left(e^{2z} + e^{-2z} + 2\right) = \frac{1}{2}\left(1 + \cosh 2z\right)$$

víme: $\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad \forall x \in \mathbb{C}$.

$$\longrightarrow \quad f(z) = 1 + \sum_{k=1}^{\infty} \frac{2^{2k-1}}{(2k)!} z^{2k} \; ; \quad z \in \mathbb{C}$$

(A2) $f(z) = \sin^2 z = \frac{1}{2}\left(1 - \cos 2z\right)$;

víme: $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

$$\longrightarrow \quad f(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} z^{2k} \; ; \quad z \in \mathbb{C}$$

(A3) $f(z) = \frac{1}{az+b} = \frac{1}{b} \cdot \frac{1}{1 - \underbrace{\left(\frac{-a}{b} z\right)}_{Q}} = \frac{1}{b} \cdot \sum_{k=0}^{\infty} Q^k$ $\qquad |Q| < 1$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{a^k}{b^{k+1}} z^k \qquad \boxed{|z| < \frac{b}{a}}$$

(A4) $f(z) = \frac{z}{z^2 - 2z + 5} = \frac{z}{(z-1)^2 + 4}$; značme $z - 1 = w$ střed v $\underline{0}$.

$= \frac{w+1}{w^2+4}$; pomocný výpočet: $\frac{1}{w^2+4} = \frac{1}{4\left(1 - \frac{(-w^2)}{4}\right)} = \sum_{k=0}^{\infty} (-1)^k \frac{w^{2k}}{4^{k+1}}$

a tedy: $(w+1) \cdot \sum_{k=0}^{\infty} (-1)^k \frac{w^{2k}}{4^{k+1}} = \sum_{k=0}^{\infty} (-1)^k \left[\frac{w^{2k}}{4^{k+1}} + \frac{w^{2k+1}}{4^{k+1}}\right]$

$= \sum_{l=0}^{\infty} a_l w^l$; kde $a_{2k} = \frac{(-1)^k}{4^{k+1}}$; $k = 0, 1$

$$a_{2k+1} = (-1)^k / 4^{k+1}.$$

A5 $f(z) = \sin(2z - z^2) = \sin(\ldots)$; $z - 1 = w$
1 střed O.

$= \sin(1 - w^2)$; víme: $\sin(a-b) = \sin a \cos b - \cos a \sin b$.

$$= \sin 1 \cos w^2 - \cos 1 \sin w^2 = \sin 1 \cdot \sum_{k=0}^{\infty} (-1)^k \frac{w^{4k}}{(2k)!} - \cos 1 \cdot \sum_{k=0}^{\infty} (-1)^k \cdot \frac{w^{4k+2}}{(2k+1)!}$$

A6 $f(z) = \exp z \cdot \sin z$

$$= \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5)\right)\left(z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)\right)$$

$$= z + z^2 + z^3\left(-\frac{1}{6} + \frac{1}{2}\right) + z^4\left(\frac{1}{6} - \frac{1}{6}\right) + z^5\left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)$$

$$= z + z^2 + \frac{z^3}{3} - \frac{z^5}{30} \cdots$$

A7 $f(z) = \exp\left(\frac{z}{1-z}\right)$;

pomocný výpočet: $\underbrace{\frac{z}{1-z}}_{w} = z \cdot (1 + z + z^2 + \cdots) = z + z^2 + z^3 + \cdots$

$\exp w = 1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots$ do řádu z^3

$$w^2 = (z + z^2 + z^3 \ldots) \cdot (z + z^2 + z^3 \ldots) = z^2 + 2z^3 + O(z^4)$$

$$w^3 = (z + O(z^2)) \cdot (z^2 + O(z^3)) = z^3 + O(z^4)$$

$$f(z) = 1 + z + z^2\left(1 + \frac{1}{2}\right) + z^3\left(1 + \frac{2}{2} + \frac{1}{6}\right)$$

$$= 1 + z + \frac{3}{2} z^2 + \frac{13}{6} z^3 + O(z^3)$$

B1 $f(z)=\frac{1}{(z-a)(z-b)}$; $z_0=a$.

(i) $b=a$: $\frac{1}{(z-a)^2}$ (ii) $b\neq a$: $\frac{1}{z-b}=\frac{1}{(z-a)+a-b}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(b-a)^{k+1}}\cdot(z-a)^k$

$\rightarrow f(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(b-a)^{k+1}}(z-a)^{k-1}$. viz A3

$|z-a|<|a-b|$

B2 $f(z)=\frac{1}{(z^2+1)^2}=\frac{1}{(z-i)^2(z+i)^2}$; $\frac{1}{(z+i)^2}=-\frac{d}{dz}\frac{1}{z+i}$;

$\frac{1}{z+i}=\frac{1}{(z-i)+2i}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2i)^{k+1}}(z-i)^k$; $|z-i|<2$.

$\rightarrow f(z)=\frac{1}{(z-i)^2}\left(-\frac{d}{dz}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2i)^{k+1}}(z-i)^k=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\underbrace{(2i)^{k+1}}_{\left(\frac{i}{2}\right)^k}}\cdot(z-i)^{k-3}$.

B3 $f(z)=(z+1)^2\exp\left(\frac{1}{z}\right)$; $z_0=0$.

$=(z^2+2z+1)\cdot\underbrace{\sum_{k=0}^{\infty}\frac{1}{k!}z^{-k}}_{1+z^{-1}+\frac{z^{-2}}{2!}+\dots}=z^2+3z+\sum_{l=0}^{\infty}\underbrace{\left[\frac{1}{l!}+\frac{2}{(l+1)!}+\frac{1}{(l+2)!}\right]}z^{-l}$

jeden součin

dva součiny

tři součiny:

$z^{-l}\cdot 1$

$z^{-(l+1)}\cdot 2z$

$z^{-(l+2)}\cdot z^2$

B4 $f(z) = \left((z-1)+1\right)^2 \sin\left(\frac{1}{z-1}\right) = (w^2+2w+1)\sin\frac{1}{w}$

$w = z-1.$

$$= (w^2+2w+1)\cdot \underbrace{\sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(2k+1)!}\, w^{-(2k+1)}}_{w^{-1} - \frac{1}{3!}w^{-3} + \frac{1}{5!}w^{-5} + \dots};$$

$$= \underset{\uparrow}{w} + \cancel{2} + \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2}{(2k+1)!}\, w^{-2k}} + \sum_{k=0}^{\infty}\left[\underset{\uparrow}{\frac{(-1)^k}{(2k+1)!}} + \underset{\uparrow}{\frac{(-1)^{k+1}}{(2k+3)!}}\right] w^{-2k-1}$$

$w^2 \cdot w^{-1}$ sudé mocniny $2w\cdot\sin\frac{1}{w}$ $1\cdot\sin\frac{1}{w}$ $w^2\left(\sin\frac{1}{w} - \frac{1}{w}\right)$

B5 $f(z) = \sin z \cdot \sin\frac{1}{z} = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)\cdot\left(z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} + \dots\right)$

$$= z^0 \cdot \left(1 + \left(\frac{1}{3!}\right)^2 + \left(\frac{1}{5!}\right)^2 + \dots\right.$$

$$\left. + z^2 \cdot \left(-\frac{1}{3!}\cdot\frac{1}{1!} - \frac{1}{5!}\cdot\frac{1}{3!} - \frac{1}{7!}\cdot\frac{1}{5!} + \dots\right)\right)$$

B6 $f(z) = e^z \cdot e^{\frac{1}{z}} = \left(1 + z + \frac{z^2}{2!} + \dots \frac{z^k}{k!} + \dots\right)\left(1 + z^{-1} + \frac{z^{-2}}{2!} + \dots \frac{z^{-k}}{k!} + \dots\right)$

$$= z^0 \cdot \left(1 + \left(\frac{1}{1!}\right)^2 + \left(\frac{1}{2!}\right)^2 + \dots\right)$$

$$+ z\left(1 + \frac{1}{1!\,2!} + \frac{1}{2!\,3!} + \dots\right)$$

$$+ z^k\left(\frac{1}{0!\,k!} + \frac{1}{1!\,(k+1)!} + \dots \frac{1}{l!\,(k+l)!} + \dots\right)$$