Def. na prostoru $\mathcal{S}(\mathbb{R})$ definuji operátory:

$$X : f(x) \longmapsto x\, f(x)$$

$$D : f(x) \longmapsto \frac{1}{2\pi i} f'(x).$$

$$\|Xf\|_2^2 = \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \quad \ldots \text{ míra „rozptýlenosti" funkce } f(x)$$

V. 23.4: $\widehat{Df} = \xi \hat{f}(\xi)$,

$$\|Df\|_2^2 \underset{\substack{\uparrow \\ \text{Plancherel}}}{=} \|\widehat{Df}\|_2^2 = \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \quad \ldots \text{ míra „rozptýlenosti" funkce } \hat{f}(\xi)$$

Věta 23.14. [Heisenbergův princip neurčitosti]

Nechť $f(x) \in \mathcal{S}(\mathbb{R})$; $\|f\|_{L^2} = 1$. Potom

$$\|Xf\|_2^2 \cdot \|Df\|_2^2 \geq \frac{1}{4\pi}.$$

Rovnost, právě když f je ([illegible]) gaussián.

Názorně: funkce f a $\hat{f}$ nemohou být zároveň příliš koncentrované.

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dk.: pomocné tvrzení: X, D jsou samoadjungované.

(1) $\langle Xf, g \rangle = \langle f, Xg \rangle$

(2) $\langle Df, g \rangle = \langle f, Dg \rangle$.

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ad (1): $\langle Xf, g \rangle = \int_{\mathbb{R}} x f(x) \cdot \overline{g(x)} \, dx = \int_{\mathbb{R}} f(x) \cdot \overline{x g(x)} \, dx = \langle f, Xg \rangle.$

ad(2): $\langle Df, g\rangle \underset{\text{Plancherel}}{=} \langle \widehat{Df}, \hat{g}\rangle = \int_{\mathbb{R}} \xi \hat{f}(\xi)\cdot \overline{\hat{g}(\xi)}\, d\xi$ $\qquad$ $\widehat{Df} = \xi \hat{f}$, $\widehat{Dg} = \xi \hat{g}$

$$= \int_{\mathbb{R}} \hat{f}(\xi)\cdot \overline{\xi \hat{g}(\xi)}\, d\xi = \langle \hat{f}, \widehat{Dg}\rangle = \langle f, Dg\rangle.$$

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(3) platí: $[DX - XD] f = \frac{1}{2\pi i} f$.

Dk: $DXf - XDf = \underbrace{\frac{1}{2\pi i}\left[x f(x)\right]'}_{= \frac{1}{2\pi i}\left(f(x) - \cancel{x f'(x)}\right)} - \cancel{x\cdot\left[\frac{1}{2\pi i} f'(x)\right]} = \frac{1}{2\pi i} f(x).$

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vlastní důkaz: buď $\lambda \in \mathbb{R}$ libovolné.

$$0 \leq \|\lambda\cdot Xf - i\cdot Df\|_2^2 = \langle \lambda\cdot Xf - i\cdot Df, \lambda\cdot Xf - i\cdot Df\rangle$$

$$= \underbrace{\langle \lambda\cdot Xf, \lambda\cdot Xf\rangle}_{(\alpha)} + \underbrace{\langle \lambda\cdot Xf, (-i)\cdot Df\rangle + \langle (-i)\cdot Df, \lambda\cdot Xf\rangle}_{(\beta)} + \underbrace{\langle (-i)\cdot Df, (-i)\cdot Df\rangle}_{(\gamma)}.$$

$$= (\alpha) + (\beta) + (\gamma).$$

$(\alpha) = \lambda^2 \|Xf\|_2^2$

$(\gamma) = (-i)\cdot\overline{(-i)}\,\|Df\|_2^2 = \|Df\|_2^2$; neboť $\langle \varphi, c\psi\rangle = \bar{c}\langle\varphi,\psi\rangle$

$(\beta) = \lambda i \left\{\langle Xf, Df\rangle - \langle Df, Xf\rangle\right\}$ samoadj.: (1,2)

$= \lambda i \left\{\langle DXf - XDf, f\rangle\right\} \underset{(3)}{=} \lambda i \left\langle \frac{1}{2\pi i} f, f\right\rangle = \frac{1}{2\pi}\|f\|_2^2 = \frac{1}{2\pi}$

ale: $g(\lambda) = \lambda^2 \|xf\|_2^2 + \lambda \cdot \frac{1}{2\pi} + \|Df\|_2^2 \geq 0$

polynom st. 2: $\Rightarrow \quad \Delta \leq 0$

$$\frac{1}{(2\pi)^2} - 4\,\|xf\|_2^{2} \cdot \|Df\|_2^{2} \leq 0$$

$$\|xf\|^2 \cdot \|Df\|^2 \geq \frac{1}{(4\pi)^2}$$

q.e.d.

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2. část: d.cv.: $f(x) = \sqrt{a} \cdot e^{-\frac{(a\pi x)^2}{2}}$ splní „$=$“; $a > 0$ libovolné.

$\boxed{„=“} \Rightarrow f$ má tento tvar:

$\Delta = 0$: $\exists \lambda \in \mathbb{R}$ kořen $g(\lambda)$.

$$0 = \|\lambda x f - i Df\|_{L^2};$$

$\Rightarrow \lambda x f = i Df$ ve smyslu $L^2(\mathbb{R})$

tj: $\lambda x f(x) = \frac{i}{2\pi i} f'(x)$ pro s.v. $x \in \mathbb{R}$

spojitost: pro $\forall x \in \mathbb{R}$.

$$f'(x) - 2\pi\lambda \cdot x f(x) = 0; \quad / \exp(-\pi\lambda x^2)$$

$$\left[f(x) \cdot \exp(-\pi\lambda x^2)\right]' = 0$$

$$f(x) \cdot \exp(-\pi\lambda x^2) = C$$

$$f(x) = C \cdot \exp(\pi\lambda x^2);$$

$f \in \mathcal{S} \Rightarrow$ nutně $\lambda < 0$;

např: $\lambda = -\frac{a^2}{2}$;

C .. volíme tak, že

$C = \sqrt{a}$.