25. Laplaceova transformace

Def.

$$L^1_+ := \{ f : (0,\infty) \to \mathbb{C} \text{ měř.}, \exists c \in \mathbb{R} \mid |f(t)| e^{-ct} \in L^1(0,\infty) \}$$

Poznámka:
① $L^1(0,\infty) \subset L^1_+$ ... volme $c = 0$
$\not\supset$: $e^t \in L^1_+ \setminus L^1(0,\infty)$

② $f \in L^1_+ \Rightarrow f \subset L^1(0,K) \ \forall K < \infty$
„lokálně integrovatelné"

$$f(t) = \underbrace{e^{ct}}_{\text{omezená}} \underbrace{f(t) e^{-ct}}_{\text{integrovatelná}}$$

③ $e^{t^2} \notin L^1_+$

Značení $c_f := \inf\{ c \in \mathbb{R}, f(t) e^{-ct} \in L^1(0,\infty) \}$
obecně: $f(t) e^{-c_f t} \notin L^1(0,\infty)$
$f(t) e^{-ct} \in L^1(0,\infty) \ \forall c > c_f$

Def. [Laplaceova transformace funkce]
$f \in L^1_+$ $\quad \mathcal{L}[f](p) = F(p) = \int_0^\infty g(t) e^{-pt} \, dt \ \ \forall p \in \mathbb{C}$, $\operatorname{Re} p > C$

Poznámka: ① korektní $|f(t) e^{-pt}| = |f(t)| e^{-(\operatorname{Re} p) t} \in L^1(0,\infty)$
② souvislost s Four. transf. (kap. 24)

$$\int_0^\infty f(t) e^{-pt} = \int_{\mathbb{R}} f(t) e^{-(\operatorname{Re} p) t} e^{-2\pi i \left(\frac{\operatorname{Im} p}{2\pi}\right) t} \, dt =$$

$$= \left[f(t) e^{-(\operatorname{Re} p) t} \right]^\wedge \left(\frac{\operatorname{Im} p}{2\pi} \right)$$

Úmluva: $f \equiv 0 \ \forall t < 0, \ \forall f \in L^1_+$
$[f(t)]^\wedge(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} \, dt$

Věta 25.1. [Základní vlastnosti L. t.]
$f \in L^1_+$
(1) $F(p) \in \mathcal{H}(\{ p \in \mathbb{C}, \operatorname{Re} p > c_f \})$
(2) $\left(\frac{d}{dp}\right)^k F(p) = \mathcal{L}[(-t)^k f(t)](p)$
(3) $F(p) \to 0$, $\operatorname{Re} p \to \infty$, $\operatorname{Im} p$ pevné
(4) $F(p) \to 0$, $\operatorname{Im} p \to \infty$, $\operatorname{Re} p > c_f$ pevné

DK (1) $F(p)=\int_0^\infty f(t)e^{-pt}\,dt$, $F'(p)=\int_0^\infty (-t)f(t)e^{-pt}\,dt$

? záměna $\frac{d}{dp}$ a $\int$: majoranta, volme p_0,
$\mathrm{Re}\,p_0 > c_f$ ... pevné

$\delta>0$, $d>c_f$; $\mathrm{Re}\,p>d$ $\forall p\in U_\delta(p_0)$

$t\in(0,\infty)$, $|(-t)f(t)e^{-pt}| \le |t||f(t)e^{-dt}|$
$= \underbrace{|t|e^{-\varepsilon t}}_{\text{omezená}}\underbrace{|f(t)|e^{-(d-\varepsilon)t}}_{\in L^1(0,\infty)}$

$\Rightarrow$ (2) platí pro $l=1$, $\forall p\in U_\delta(p_0)$

(3) $F(p)=\int_0^\infty f(t)\exp[-(\mathrm{Re}\,p)t - i(\mathrm{Im}\,p)t]\,dt \xrightarrow{?} 0$, $\mathrm{Re}\,p\to\infty$

$|\cdot| = |f(t)|e^{-(\mathrm{Re}\,p)t}\cdot 1 \to 0$

? záměna lim a $\int$: $\le |f(t)|e^{-dt}$; $\mathrm{Re}\,p>d>c_f$
$\in L^1(0,\infty)$

(4) $F(p)=\Big[\underbrace{f(t)e^{-(\mathrm{Re}\,p)t}}_{\in L^1(\mathbb{R}),\ \mathrm{Re}\,p>c_f}\Big]^\wedge\left(\frac{\mathrm{Im}\,p}{2\pi}\right)$ Věta 25.? :
$f(t)\in L^1(\mathbb{R}) \Rightarrow \hat f(\xi)\to 0$, $|\xi|\to\infty$

Příklad: ① $f(t)\equiv 1$: $F(p)=\int_0^\infty 1\cdot e^{-pt}\,dt=\frac{1}{p}$, $\mathrm{Re}\,p>0=c_1$

② $f(t)=e^{at}$, $a\in\mathbb{R}$, $c_f=a$: $F(p)=\int_0^\infty e^{-(p-a)t}\,dt=\frac{1}{p-a}$

③ $f(t)=t^\alpha$, $\alpha\in\mathbb{R}$? $s^\alpha e^{-cs}\in L^1(0,\infty)$

(i) $(0,\delta)$ (ii) (K,∞)

(i) $t^\alpha e^{-ct}\sim t^\alpha$; $t\to 0+$
$\int_0^\delta < \infty \iff \alpha>-1$
(ii) stačí $c>0$

$\Rightarrow t^\alpha e^{-ct}\in L^1(0,\infty)$ pro $\alpha>-1$, $c>0$, $(c_f=0)$

$\mathcal{L}[t^\alpha](p)=\int_0^\infty t^\alpha e^{-pt}\,dt = \int_0^\infty \left(\frac{x}{p}\right)^\alpha e^{-x}\frac{dx}{p} = \frac{1}{p^{\alpha+1}}\int_0^\infty x^\alpha e^{-x}\,dx$

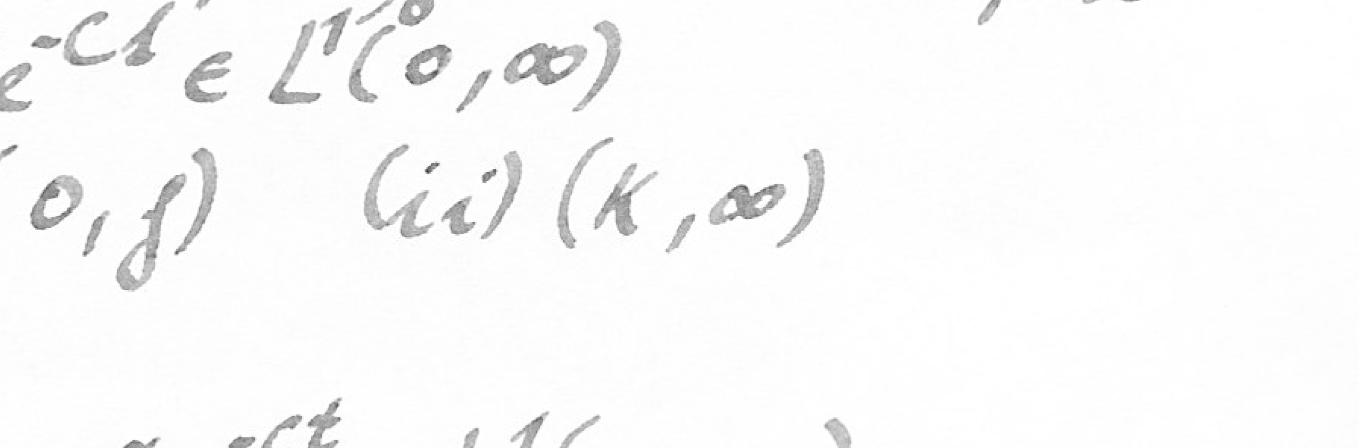
$\uparrow$ $pt=x$ $p\in(0,\infty)\subset\mathbb{R}$

$=\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$

? holomorfní rozšíření fce $\frac{1}{p^{\alpha+1}}$ do $\Omega=\{p\in\mathbb{C}, \mathrm{Re}\,p>0\}$

$\frac{1}{p^{\alpha+1}}=\exp(-(\alpha+1)\ln p)$, $p>0$: $\tilde F(p)=\exp(-(\alpha+1)\mathrm{Log}\,p)$, $\mathrm{Re}\,p>0$

$\mathrm{Log} := \left(\exp|_{-\pi<\mathrm{Im}<\pi}\right)^{-1}$



$\mathcal{L}[t^\alpha](p)=:F_1(p)$

$F_2(p):=\tilde F(p)\,\Gamma(\alpha+1)$

$F_1, F_2\in\mathcal{H}(\Omega)$

$F_1=F_2$ pro $p\in(0,\infty)$: má hromadný bod

Věta o jednoznačnosti: $F_1=F_2$ v Ω

Věta 25.2. [Další vlastnosti L. t.]

Nechť $f(t) \in L^1_+$

(1) $\mathcal{L}\{f(\alpha t)\}[p] = \frac{1}{\alpha}\mathcal{L}\{f(t)\}[p/\alpha]$; $\alpha > 0$; $\mathrm{Re}\, p > c_f \cdot \alpha$

(2) $\mathcal{L}\{f(t-a)\} = e^{-pa}\mathcal{L}\{f(t)\}[p]$; $a > 0$; $\mathrm{Re}\, p > c_f$

s úmluvou $f \equiv 0$; $t < 0$; $f \in L^1_+$

(3) $\mathcal{L}\{e^{at} f(t)\}[p] = \mathcal{L}\{f(t)\}[p-a]$

$\forall a \in \mathbb{C}$, $\mathrm{Re}\, p > c_f + \mathrm{Re}\, a$

DK (1) $\mathcal{L}\{f(\alpha t)\}[p] = \int_0^\infty f(\alpha t) e^{-pt}\, dt \;\Big|\; \begin{matrix} \alpha t = u \\ \in (0,\infty) \\ dt = \frac{1}{\alpha} du \end{matrix}$

$$= \frac{1}{\alpha}\int_0^\infty f(u) e^{-\frac{p}{\alpha} \cdot u}\, du = \frac{1}{\alpha}\mathcal{L}\{f(t)\}\left[\frac{p}{\alpha}\right]$$

$$\mathrm{Re}\left(\frac{p}{\alpha}\right) > c_f \Leftrightarrow \mathrm{Re}\, p > c_f \cdot \alpha$$

(2) $\mathcal{L}\{f(t-a)\}[p] = \int_0^\infty f(t-a) e^{-pt}\, dt \;\Big|\; \begin{matrix} t-a = u \\ \in (-a,\infty) \\ dt = du \end{matrix}$

$$= \underbrace{\int_{-a}^\infty f(u) \cdot e^{-p(u+a)}\, du}_{\text{úmluva !!}} = \int_0^\infty f(u) \cdot e^{-pu}\, du \cdot e^{-pa}$$

Pozn.: (2) neplatí pro $a < 0$

(3) $\int_0^\infty e^{at} f(t) e^{-pt}\, dt = \int_0^\infty f(t) e^{-(p-a)t}\, dt = \mathcal{L}\{f(t)\}[p-a]$

$\mathrm{Re}\,(p-a) > c_f$

$\mathrm{Re}\, p > \mathrm{Re}\, a + c_f$

Věta 25.3. [L. t. a derivace]

Nechť $f(t) \in C^\ell([0,\infty))$ a $f^{(j)} \in L^1_+$; $j = 0, \dots, \ell$; $\ell \in \mathbb{N}$

Potom

(1) $\mathcal{L}\{f^{(\ell)}(t)\}[p] = p^\ell \mathcal{L}\{f(t)\}[p] - \sum_{j=0}^{\ell-1} p^j f^{(\ell-j-1)}(0)$

Speciálně $\mathcal{L}\{f'(t)\}[p] = pF(p) - f(0)$

$\mathcal{L}\{f''(t)\}[p] = p^2 F(p) - \{f'(0) + p f(0)\}$

Opak.: $f \in C^\ell([0,\infty]) \Leftrightarrow \exists \tilde{f} \in C^\ell(\mathbb{R})$; $f = \tilde{f}$ v $[0,\infty)$

DK $\quad k=1 \quad \mathcal{L}\{f'(t)\}[p] = \int_0^\infty f'(t)\, e^{-pt}\, dt \overset{p.p.}{=} \underbrace{\left[f(t)e^{-pt}\right]_0^\infty}_{-f(0)} - \underbrace{\int_0^\infty f(t)(-p)e^{-pt}\,dt}_{p\,F(p)} =$

$\operatorname{Re} p > \max\{c_f, c_{f'}\}$

$k \to k+1 \quad \mathcal{L}\{f^{(k+1)}(t)\}[p] = \mathcal{L}\{[f^{(k)}(t)]'\}[p] \overset{\text{vzorec } k=1}{=}$

$$= \underbrace{p\,\mathcal{L}\{f^{(k)}(t)\}[p]}_{\text{předchozí krok}} - f^{(k)}(0) = p\left\{p^k F(p) - \sum_{j=0}^{k-1} p^j f^{(k-j-1)}(0)\right\} - f^{(k)}(0) =$$

$$= p^{k+1} F(p) - \left\{\sum_{j=0}^{k-1} p^{j+1} f^{(k-j-1)}(0) + f^{(k)}(0)\right\} =$$

$$= p^{k+1} F(p) - \left\{\sum_{j=1}^{k} p^j f^{(k-j)}(0) + f^{(k)}(0)\right\}$$

tj. (1) pro $k+1$

Příklad: $\quad u'(t) + u(t) = 1$, $\quad u(0) = 0$, $\quad t > 0 \quad \Big/ \mathcal{L} \quad U(p) := \mathcal{L}\{u(t)\}$

$$p\,U(p) - u(0) + U(p) = \frac{1}{p}$$

$$(p+1)\,U(p) = \frac{1}{p}$$

$$U(p) = \frac{1}{p(p+1)} = \overbrace{\frac{1}{p} - \frac{1}{p+1}}^{\text{parc. zlomky}} \quad \text{pak} \quad u(t) = 1 - e^{-t}$$

$$\frac{1}{p} \downarrow 1 \qquad \frac{1}{p+1} \downarrow e^{t}$$

Def.: Pro $f(t), g(t) \in L^1_+$ definujeme KONVOLUCI

$$[f * g](t) = \begin{cases} \int_0^t f(s)\, g(t-s)\, ds & ; \ t > 0 \\ 0 & ; \ t \le 0 \end{cases}$$

Pozn.: $\displaystyle \int_0^t \underbrace{f(s)}_{=0;\, s<0} \underbrace{g(t-s)}_{=0;\, t-s<0} ds = \underbrace{\int_{-\infty}^{\infty} f(s)\, g(t-s)\, ds}_{\text{definice konvoluce kap. 24}}$

úmluva ↑

Lemma 25.1. [Konvoluce v L^1_+]
Nechť $f(t), g(t) \in L^1_+$. Potom $[f*g](t)$ má smysl pro s.v. $t > 0$; náleží L^1_+ a navíc $c_{f*g} \le \max\{c_f, c_g\}$

DK pomocné funkce $\varphi(t) := |f(t)| \cdot e^{-ct}$

$\psi(t) := |g(t)| \cdot e^{-ct}$; $c > \max\{c_f, c_g\}$

$\Rightarrow \varphi, \psi \in L^1(\mathbb{R})$: V.24.? : $\|\varphi * \psi\|_{L^1} \le \|\varphi\|_{L^1} \|\psi\|_{L^1}$

spec. $[\varphi * \psi](t) \in \mathbb{R}$ pro s.v. t

$$|(f*g)(t)| = \left|\int_0^t f(s)\, g(t-s)\, ds\right| \le \int_{0\,(-\infty)}^{t\,(\infty)} \underbrace{|f(s)| e^{-cs}}_{\varphi(s)} \cdot \underbrace{|g(t-s)| e^{-c(t-s)}}_{\psi(t-s)} \cdot e^{ct}\, ds$$

$$= [\varphi * \psi](t) \cdot e^{ct}$$

$|(f*g)(t)| \cdot e^{-ct} \le |(\varphi * \psi)(t)|$

$\Rightarrow$ PS $\in L^1(0,\infty)$

spec. konečná pro s.v. t

$c > \max\{c_f, c_g\} \Rightarrow c_{f*g} \le \max\{c_f, c_g\}$

Věta 25.4. [L. t. a konvoluce]

Nechť $f(t), g(t) \in L^1_+$

Pak $\mathcal{L}\{(f*g)(t)\}[p] = \mathcal{L}\{f(t)\}[p] \cdot \mathcal{L}\{g(t)\}[p]$

$\operatorname{Re} p > \max\{c_f, c_g\}$

DK $LS = \int_0^\infty [f*g](t) e^{-pt}\, dt = \int_0^\infty \left(\int_0^t f(s)\, g(t-s)\, ds \right) e^{-pt}\, dt =$

$$\overset{(*)}{\underset{\text{Fubini}}{=}} \int_0^\infty \left(\int_s^\infty f(s)\, g(t-s)\, e^{-pt}\, dt \right) ds = \int_0^\infty f(s) \cdot e^{-ps} \Big(\underbrace{\int_s^\infty g(t-s) e^{-p(t-s)}}_{G(p)} \Big) ds =$$

$= G(p) F(p)$

(*) ? Fubini : $\iint_M |f(s)\, g(t-s)| \cdot e^{-\operatorname{Re} p\, t}\, ds\, dt < \infty$

nezáp. integrand : Fubini lze vždy :

$$\le \int_0^\infty |f(s)| e^{-\operatorname{Re} p \cdot s}\, ds \cdot \int_0^\infty |g(t)| \cdot e^{-\operatorname{Re} p\, t}\, dt < \infty$$

díky volbě p

Důsl. Vztah L. t. a primitivní funkce

$f(t) \in L^1_+ \ \dots\ h(t) := \int_0^t f(s)\, ds = \int_0^t f(s) \cdot 1(t-s)\, ds = (f * 1)(t)$

kde $1(t) := \begin{cases} 1 ; & t > 0 \\ 0 ; & t \le 0 \end{cases}$; platí $1(t) \in L^1_+$

Heaviside $\Rightarrow h(t) \in L^1_+$

$$\boxed{H(p) = F(p) \cdot \frac{1}{p}}$$

Opakování: V.24.11. $f \in \mathcal{S} \Rightarrow [\hat{f}]^\vee(x) = f(x) \quad \forall x \in \mathbb{R}^n$

v důkazu použito pouze: $f \in L^1 \cap C_b$, $\hat{f} \in L^1$

↳ spojitá a omezená

platí $\boxed{V.24.11.'}$ $f, \hat{f} \in L^1 \Rightarrow [\hat{f}]^\vee(x) = f(x)$ pro s.v. x

Věta 25.5. [Prostota $\mathcal{L}$. t.]

(1) Nechť $f(t) \in L^1_+$, nechť $\exists\, c^* \in \mathbb{R}$ tak, že

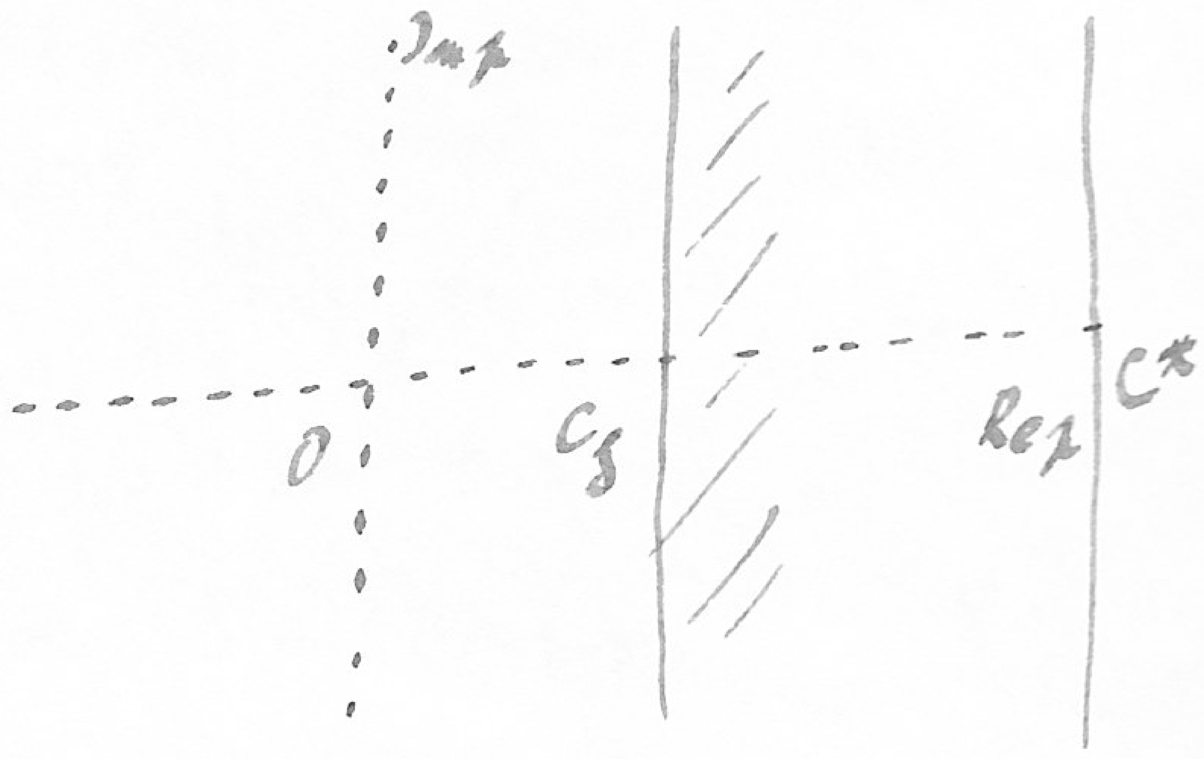
$F(p) = 0$ pro $\forall p \in \mathbb{C}, \operatorname{Re} p > c^*$

Potom $f(t) = 0$ s.v.

(2) Nechť $f(t), g(t) \in L^1_+$, nechť $N = \{p \in \mathbb{C}; F(p) = G(p)\}$

má hromadný bod v $\Omega := \{p \in \mathbb{C}; \operatorname{Re} p > \max\{c_f, c_g\}\}$

Potom $f(t) = g(t)$ s.v.



DK (1) $F(p) = \left[f(x)e^{-\operatorname{Re}p\, x}\right]^\wedge\left(\frac{\operatorname{Im} p}{2\pi}\right)$

volme $\cancel{p \in}$ $\cancel{c >}\, c^*$; $\operatorname{Re} p = c$

$c > \max\{c^*, c_f\}$; $\operatorname{Im} p \in \mathbb{R}$

$\varphi(x) \in L^1(\mathbb{R})$

$\hat{\varphi}(\xi) = 0 \ \forall \xi \in \mathbb{R}$; tedy $\hat{\varphi} \in L^1$

V.24.11.' $\Rightarrow [\hat{\varphi}]^\vee = [0]^\vee = 0 = \varphi(x)$ s.v. $\Rightarrow$

$\Rightarrow f(x) = 0$ s.v. $x > 0$; $f = 0$ pro $x < 0$ dle úmluvy

(2) $h(t) := f(t) - g(t) \in L^1_+$

$H(p) = F(p) - G(p)$ ... linearita $\mathcal{L}$

$N = \{p \in \mathbb{C}; \operatorname{Re} p > \max\{c_f, c_g\}; H(p) = 0\}$

má hromadný bod v Ω, $H \in \mathcal{H}(\Omega)$ $\Big\}$ $\Rightarrow H \equiv 0$ v $\Omega \overset{(1)}{\Rightarrow} h = 0$ s.v.

V.22.17. (o jednoznačnosti)

Věta 25.6. [Inverzní L. t.]

Nechť $F(p) \in \mathcal{H}(\mathbb{C} \setminus \{z_1, \dots, z_n\})$

nechť $|F(p)| \le \dfrac{K}{|p|^2}$; $p \to \infty$

Potom $\exists f(t) \in L^1_+$ tak, že $\mathcal{L}\{f(t)\}(p) = F(p)$

Navíc platí

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(z) e^{tz}\, dz = \sum_{\ell=1}^{n} \operatorname*{res}_{z=z_\ell} \{F(z) e^{tz}\}$$

pro s.v. t, všechna dost velká $a \in \mathbb{R}$ ($a > \operatorname{Re} z_\ell$; $\ell = 1, \dots, n$)

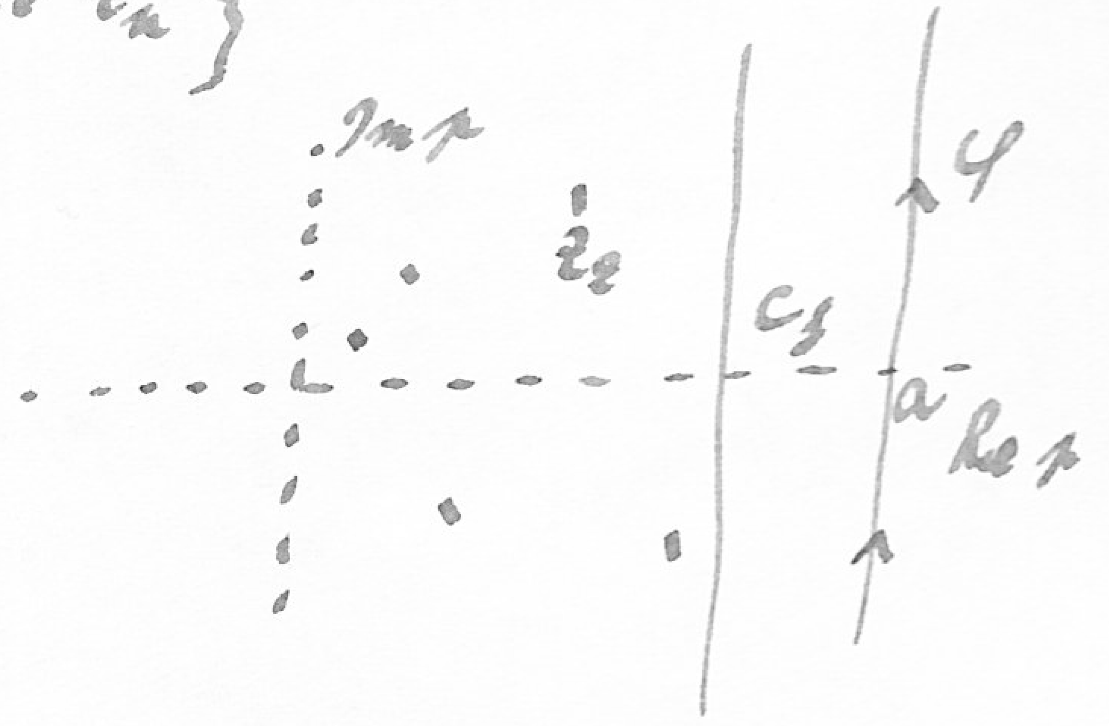
Značení $\displaystyle\int_{a-i\infty}^{a+i\infty} dz$ ... integrál dle $\varphi(t) = a + it$; $t \in (-\infty, \infty)$

DK (za silnějšího předpokladu: nechť $\exists f \in L^1_+$: $\mathcal{L}\{f\} = F \to$
$\to$ platí (1), (2))

$$F(p) = F(\operatorname{Re} p + i \operatorname{Im} p) = \mathcal{L}\{f(t)\}[p] = [\underbrace{f(x) e^{-\operatorname{Re} p\, x}}_{h(x)}]^{\wedge}\Big(\underbrace{\frac{\operatorname{Im} p}{2\pi}}_{\xi}\Big)$$

$$\hat{h}(\xi) = F(\operatorname{Re} p + 2\pi i \xi)$$

fixuji $\operatorname{Re} p = a$; $a > \max\{c_f, \operatorname{Re} z_1, \dots, \operatorname{Re} z_n\}$



$\Rightarrow h(x) \in L^1(\mathbb{R})$

$\hat{h}(\xi) \in L^1(\mathbb{R})$ spojitá v $\mathbb{R}$

$$|\hat{h}(\xi)| \le \frac{K}{|\xi|^2} \ ; \ |\xi| \to \infty$$

V.24.11.' : $h(x) = [\hat{h}]^{\vee}(x)$ s.v. x

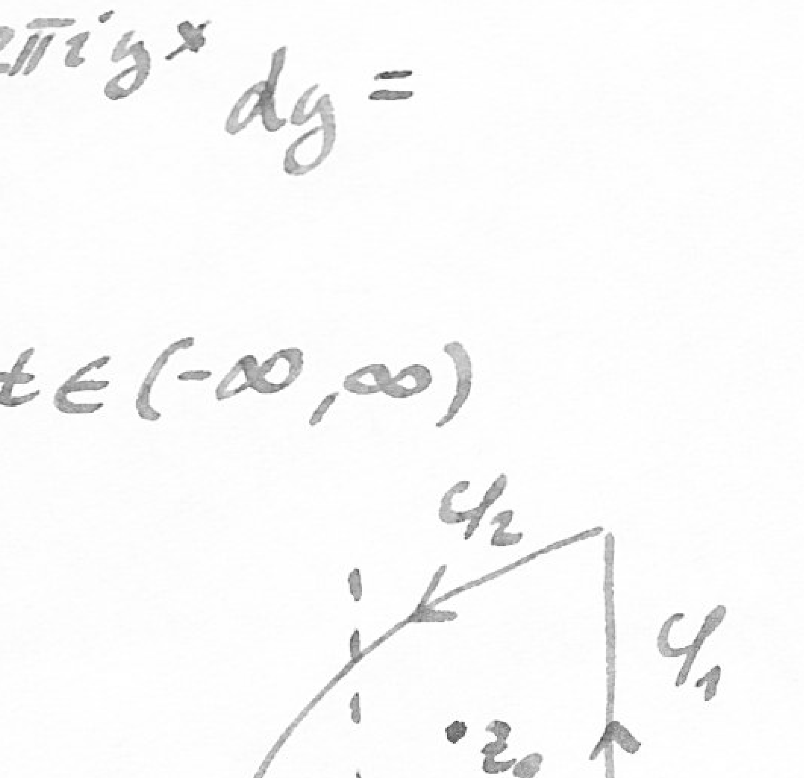
$$h(x) = e^{-ax} f(x) = \int_{-\infty}^{\infty} \hat{h}(y) \overbrace{e^{2\pi i x y}}^{e^{-ax}} dy = \int_{-\infty}^{\infty} F(\underbrace{a + 2\pi i y}_{z}) \overbrace{e^{2\pi i y x}}^{e^{-ax}} dy =$$

$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(z) e^{x(z-a)}\, dz$$

$z = a + 2\pi i t$; $t \in (-\infty, \infty)$

$dz = 2\pi i$

druhá část: aplikace reziduové věty:



$$\frac{1}{2\pi i} \int_{\varphi_1 \dotplus \varphi_2} F(z) e^{tz}\, dz = \sum_{\ell} \operatorname*{res}_{z_\ell} F(z) e^{tz}$$

$$\underbrace{\frac{1}{2\pi i} \int}_{\varphi_1} + \underbrace{\frac{1}{2\pi i} \int}_{\varphi_2} \ ; \ R \to \infty$$

$\to \int_\varphi$ $\quad \to 0$: Lemma o velké půlkružnici

$$|F(z)e^{tz}| \le \underbrace{|F(z)|}_{\le \frac{K}{|z|^2}} \cdot \underbrace{e^{\operatorname{Re} tz}}_{\le e^{\operatorname{Re} tc}} \qquad t \ge 0$$

26. Speciální funkce

Def. Gamma funkce (Eulerův integrál 2. druhu) je definována jakožto $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt$

$z \in \mathbb{C},\ \operatorname{Re} z > 0$

Pozn. definice korektní:

$$\int_0^\infty |t^{z-1}e^{-t}|\,dt = \int_0^\infty t^{\operatorname{Re} z-1} e^{-t}\,dt < \infty \Leftrightarrow \operatorname{Re} z > 0$$

$$t^{z-1} = \exp((z-1)\ln t)$$

$$t \in (0,\infty)$$

$$|t^{z-1}| = \exp(\operatorname{Re}(z-1)\ln t) = t^{\operatorname{Re}\{z-1\}}$$

- $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$
- $\Gamma(z+1) = z\,\Gamma(z) \quad \forall\, z \in \mathbb{C};\ \operatorname{Re} z > 0$ // d.cv. per-partes a indukce

  $\Gamma(z+n+1) = z(z+1)\cdots(z+n)\,\Gamma(z)$
- $\Gamma(n) = (n-1)! \quad \forall\, n \in \mathbb{N}$
- jiné vyjádření: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt = \left| \begin{matrix} t = u^2 & u = \sqrt{t} \\ dt = 2u\,du & \end{matrix} \right| =$

  $= 2\int_0^\infty u^{2z-1} e^{-u^2}\,du$

  spec. $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2} = \int_{-\infty}^\infty e^{-u^2}\,du = \sqrt{\pi}$

  $\Gamma(n+\frac{1}{2}) = \frac{1}{2}(\frac{1}{2}+1)\cdots(\frac{1}{2}+n-1)\sqrt{\pi}$

$$\int_0^\infty e^{-x^n}\,dx = \left| \begin{matrix} x^n = t \\ x = t^{1/n} \\ dx = \frac{1}{n} t^{\frac{1}{n}-1} \end{matrix} \right| = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$$

Motivace: zobecnění faktoriálu; $1, x, \frac{x^2}{2!}, \dots \frac{x^n}{n!}$

$[Pf](t) = \int_0^t f(s)\,ds$ … primitivní funkce

$Pf = f * 1$

- výpočty integrálů: $\mathcal{L}\{t^\alpha\} = \int_0^\infty t^\alpha e^{-pt}\,dt = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$

$P^{(n)}f = P(P\cdots Pf) = f * \frac{x^{n-1}}{(n-1)!}$: kdyby bylo n záporné → dostanu derivaci