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INERTIAL MANIFOLDS AND THE CONE CONDITION

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ABSTRACT: The "cone condition", used in passing in many proofs of the existence of inertial manifolds, is examined in more detail. Invariant manifolds for dissipative flows can be obtained directly using no other dynamical information. After finding a condition for the exponential attraction of trajectories to such a manifold, a cone invariance property is used to show the existence of orbits on the manifold which track a given orbit of the flow. This leads to a concise proof which guarantees the existence of inertial manifolds with the asymptotic completeness property. Furthermore it is shown that the "strong squeezing property" implies directly the existence of such an inertial manifold. There follows a brief discussion of the rôle of the cone condition in the Lyapunov-Perron fixed point method of proof, and a comparison with previous results.

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1. INTRODUCTION

In this paper a new and concise proof of the existence of inertial manifolds is given for the abstract evolution equation on a separable Hilbert space H,

$$\dot{u} + Au + f(u) = 0. \tag{1}$$

The following class of parabolic problems is considered : it is assumed that A is an unbounded positive self-adjoint linear operator with compact inverse, and that the nonlinear term $R(u)$ maps $D(A^\alpha)$ into $D(A^\beta)$ and satisfies, for some constants C_0, C_1, and ρ

$$|A^\beta R(u)| \leq C_0 \ \ \forall\ u \in D(A^\alpha),$$

$$|A^\beta (R(u_1) - R(u_2))| \leq C_1 |A^\alpha (u_1 - u_2)| \ \ \forall\ u_1, u_2 \in D(A^\alpha), \text{ and}$$

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$$\mathrm{supp}(R) \subset \Omega_\rho \equiv \{u \in D(A^\alpha) : |A^\alpha u| \leq \rho\}. \tag{2}$$

For technical reasons attention is restricted to the case $0 \leq \alpha - \beta \leq \frac{1}{2}$. This problem is considered in Foias et al. [3], who have $\alpha = 1, \beta = \frac{1}{2}$; Temam [12], who deals explicitly only with the case $\beta = \alpha - \frac{1}{2}$; and Mallet-Paret & Sell [7], who work with reaction-diffusion equations for which $\alpha = \beta = 0$. Chow et al. [1] cover the more general case $0 \leq \alpha - \beta < 1$ and Rodriguez Bernal [11] considers the same problem for flows on a Banach space.

The conditions above in fact pose little restriction, and many of the well-known dissipative partial differential equations can be reduced to this case after a careful prepartion of the equation, for example the 1D Kuramoto Sivashinsky equation, the 2D Navier Stokes equation, and many reaction-diffusion equations (Foias et al. [3], Mallet-Paret & Sell [7]).

Applying the results of Henry [6], equation (1) generates a semigroup $S(t)$ on $D(A^\alpha)$, such that the solution at time t through an initial condition $u_0 \in D(A^\alpha)$ is given by $u(t) = S(t)u_0$. For $t > 0$, $u(t)$ is more regular than the initial condition, with $u(t) \in D(A^{1+\beta})$ and $\dot{u} \in D(A^\beta)$. Furthermore, the solutions are unique in both forwards and backwards time (forwards uniqueness is standard, backwards uniqueness follows from Temam [12], pp 168–170, replacing his $|A^{\frac{1}{2}}w|^2/|w|^2$ with $|A^{\beta+\frac{1}{2}}w|^2/|A^\beta w|^2$). These regularity results make it possible to work with the equation itself and take inner products rather than have to use the variation of constants formula; in particular expressions such as

$$\frac{1}{2}\frac{d}{dt}|A^\alpha u| = (A^\beta \dot{u}, A^{2\alpha-\beta}u)$$

make sense since $\dot{u} \in D(A^\beta)$ and $u \in D(A^{1+\beta})$, $1 + \beta > 2\alpha - \beta$ since $\alpha - \beta \leq \frac{1}{2}$.

The eigenvalues of A are denoted by λ_j, counted according to their multiplicity and ordered so that $\lambda_{j+1} \geq \lambda_j$. The corresponding eigenfunctions w_j are assumed to be orthonormal in H.

Now choose $n > 0$ such that $\lambda_{n+1} \neq \lambda_n$, define the projection operator P_n to be the orthogonal projector onto the space spanned by the eigenfunctions $\{w_j : 1 \leq j \leq n\}$, and set $Q_n = I - P_n$. In what follows P_n, Q_n will often be abbreviated to P, Q when there is no ambiguity in the value of n. Also, when n is clearly fixed denote λ_n by λ and λ_{n+1} by Λ.

An inertial manifold $\mathcal{M}$ for (1) is a finite dimensional positively invariant Lipschitz manifold which attracts all orbits at an exponential rate,

$$\mathrm{dist}(S(t)u_0, \mathcal{M}) \leq Ce^{-kt},$$

where $C = C(X)$ is uniform for $u_0 \in X$, a bounded set in $D(A^\alpha)$. By restricting attention to trajectories on this manifold, which are governed by a finite-dimensional dynamical system, a good understanding of the asymptotic dynamics of the equation should be obtained. If every solution of (1) approaches some trajectory on $\mathcal{M}$, then the "asymptotic completeness property" is said to hold.

Invariance properties for the cone

$$\mathcal{C}_l^n = \{w(t) \equiv u_1(t) - u_2(t) : |A^\alpha Q_n w(t)| \leq l |A^\alpha P_n w(t)|\}$$

are used in many of the current existence proofs of inertial manifolds (here, $u_i(t)$ are two solutions of equation (1)).

Often this "cone condition" is supplemented by the "squeezing" property, which states that solutions outside the interior of the cone decay exponentially to zero, i.e. if $w(t_0) \notin \text{int } \mathcal{C}_l^n$ then

$$|A^\alpha Q_n w(t)| \leq e^{-kt} |A^\alpha Q_n w(0)| \tag{3}$$

for some $k > 0$ and all $0 \leq t \leq t_0$. Usually the squeezing property is only stated to hold for $w(t_0)$ strictly outside the cone, but all standard proofs give decay for $w(t_0) \in \partial\mathcal{C}$ also (see [2],[4],[11],[12]).

Together these two are termed the "strong squeezing property", which was first used in the analysis of the Kuramoto-Sivashinsky equation presented by Foias et al. [2]; it is usually employed to prove the exponential convergence of trajectories to an invariant manifold, once one has been shown to exist.

A different approach is adopted here; starting with the cone invariance property it is possible to prove the existence of an invariant manifold with no additional assumptions. A new proof of exponential convergence towards the invariant manifold is then given, which is simpler than those currently to be found in the literature. Finally, the cone property can be used to give an essentially geometric proof that the inertial manifold is asymptotically complete. When the strong squeezing propery is assumed, these last two steps can be combined in a very simple way.

Once these results have been put together to give a concise existence proof, conditions for the strong squeezing (and hence cone invariance) property to hold will be derived, and it will be shown how this analysis relates to many of the existence proofs to be found in the literature.

2. AN EXISTENCE PROOF BASED ON THE CONE CONDITION

The following three propositions build into the main theorem of this paper. The first proposition shows that the cone condition is enough to ensure the existence of an invariant manifold. The proof uses a theorem of Hale [5] which is a generalisation to infinite dimensional spaces of the result that, for the flow generated by a set of ordinary differential equations, an absorbing region in $\mathbb{R}^n$ will contain a fixed point.

Define the space $\mathcal{F}_l^n$ of Lipschitz functions $\phi : P_nH \to Q_nH \cap D(A^\alpha)$ which satisfy

$$|A^\alpha(\phi(p_1) - \phi(p_2))| \le l|A^\alpha(p_1 - p_2)|,$$

$$\|\phi\| \equiv \sup_{p \in P_nH} |A^\alpha \phi(p)| < \infty, \text{ and}$$

$$\operatorname{supp}(\phi) \subset P_n\Omega_\rho.$$

Clearly $\mathcal{F}_l^n$ can be considered as a closed convex subset of the Banach space consisting of all C^0 functions from $P_n\Omega_\rho$ into $Q_nH \cap D(A^\alpha)$.

Proposition 1 *Provided that for some l and n the cone $\mathcal{C}_l^n$ is invariant under the flow $S(t)$ induced by (1), there is an invariant manifold given as the graph of a function $\phi \in \mathcal{F}_l^n$.*

Proof The following theorem of Hale [5] (Theorem 3.4.8) is used:

Consider a semigroup $T(t)$ defined on a Banach space or a closed convex subset of a Banach space. If the semigroup $T(t)$ is dissipative, C^0, and completely continuous then $T(t)$ has a fixed point. *Completely continuous* means that, for any bounded set B and for any $t > 0$, the closure of $T(t)B$ is compact.

Because of the cone condition, the flow $S(t)$ on $D(A^\alpha)$ induces a flow on $\mathcal{F}_l$ which will be denoted $T(t)$ and is given by

$$S(t)\mathcal{G}[\phi(\tau)] = \mathcal{G}[\phi(t+\tau)] = \mathcal{G}[T(t)\phi(\tau)],$$

where the notation $\mathcal{G}[\psi]$ denotes the manifold given as the graph of ψ.

Indeed, any two points on $\mathcal{M}_t = S(t)\mathcal{G}[\phi(0)]$ must satisfy $|A^\alpha(q_1 - q_2)| \le |A^\alpha(p_1 - p_2)|$, and so for each value of p in $P\mathcal{M}_t$ there exists a unique value of q, denoted $\psi(p)$, for which $p + \psi(p) \in \mathcal{M}_t$. Over its domain of definition, ψ satisfies the Lipschitz bound necessary for functions in $\mathcal{F}_l$. From (2) it is clear that $\psi(p) = 0$ for $|A^\alpha p| > \rho$. It remains to show that there are "no holes" in $\mathcal{M}_t$, i.e. that $P\mathcal{M}_t = PH$.

Following Foias et al. [4], $\mathcal{M}_0$ is clearly homeomorphic to PH; due to backwards uniqueness and continuous dependence on initial conditions $\mathcal{M}_t$ is homeomorphic to $\mathcal{M}_0$. By virtue of the cone condition $P\mathcal{M}_t$ is homeomorphic to $\mathcal{M}_t$ and hence to PH. Now, using condition (2), for every $p \notin P\Omega_\rho$ there exists a unique $\tilde{p} \notin P\Omega_\rho$ with $S(t)\tilde{p} = p$, and which is given by $\tilde{p} = e^{At}p$. Thus $P\mathcal{M}_t$ must be equal to PH, and consequently $S(t)\mathcal{G}[\phi(0)] = \mathcal{G}[\phi(t)]$ with $\phi(t) = \psi$. The operator $T(t)$ defined by $T(t)\phi(0) = \phi(t)$ clearly inherits the semigroup properties of $S(t)$.

If $T(t)$ has a fixed point ϕ, then the manifold $\mathcal{G}[\phi]$ must be invariant under the flow $S(t)$. The theorem above is now shown to apply and ensure the existence of such a fixed point.

The evolution equation for the Q component of u is

$$\dot{q} + Aq + QR(u) = 0,$$

which using the variation of constants formula for $q(t)$ gives

$$q(t) = e^{-At}q(0) - \int_0^t e^{-A(t-s)}QR(u(s))\, ds.$$

Hence

$$|A^\alpha q(t)| \le |A^\alpha q(0)|e^{-\Lambda t} + C_0 \int_0^t \|A^{\alpha-\beta}e^{-AQ(t-s)}\|\, ds.$$

The integral can be bounded by using expressions found in Foias *et al.* [3] for example, by I_0, say, so that

$$|A^\alpha q(t)| \le |A^\alpha q(0)|e^{-\Lambda t} + I_0C_0,$$

and so

$$\|\phi(t)\| \le \|\phi(0)\|e^{-\Lambda t} + I_0C_0 \tag{4}$$

which ensures that the flow $T(t)$ is dissipative.

It further follows from (4) that $T(t)$ maps bounded sets into bounded sets.

Now, due to the Lipschitz property of functions in $\mathcal{F}_l$, any subset is equicontinuous. A bounded subset of $\mathcal{F}_l$ is certainly bounded in C^0, and thus by the Arzelà-Ascoli theorem for Banach spaces (Naylor & Sell [8]), any closed bounded subset of $\mathcal{F}_l$ is a compact subset of $C^0(P\Omega_\rho, QH \cap D(A^\alpha))$. But the topology on $\mathcal{F}_l$ generated by its norm coincides with that of $\mathcal{F}_l$ as a subset of C^0, and $\mathcal{F}_l$ is complete with respect to this norm, so a closed bounded subset of $\mathcal{F}_l$ is also compact in $\mathcal{F}_l$. Therefore $T(t)$ is completely continuous.

Thus $T(t)$ satisfies the conditions of the theorem and there exists an invariant manifold for the flow $S(t)$ given as the graph of a function $\phi \in \mathcal{F}_l$.

∎

Note that although the cone condition ensures that manifolds given as the graphs of C^1 functions stay C^1 provided R is smooth, the space C^1_l of functions whose derivatives satisfy $\|A^\alpha D\phi\| \le l$ is not complete. Its completion is the space $\mathcal{F}_l$ used above and so the simple method used for proposition 1 cannot be applied to yield a C^1 invariant manifold, and more complicated arguments are necessary (see Mallet-Paret & Sell [7], Raugel [9]).

For the invariant manifold to be an inertial manifold, trajectories must converge towards it exponentially. A condition for this is given in proposition 2.

Proposition 2 *Suppose equation (1) possesses an invariant manifold $\mathcal{M}$ given as the graph of a function $\phi \in \mathcal{F}_l^n$, and define μ by*

$$\mu = \Lambda - C_1(\Lambda^{\alpha-\beta} + l\lambda^{\alpha-\beta}). \tag{5}$$

Then provided $\mu > 0$, trajectories of (1) converge to $\mathcal{M}$ at an exponential rate, that is

$$|A^\alpha(q(t) - \phi(p(t)))| \le |A^\alpha(q_0 - \phi(p_0))|e^{-\mu t}.$$

Proof The time evolution of the quantity $\delta(t) \equiv q(t) - \phi(p(t))$ is given by the equation

$$\dot{\delta} + Aq + QR(p+q) + \dot{\phi}(p)_{(p,q)} = 0,$$

whenever $\dot{\phi}$ exists. (Compare this approach with that of Chow et al. [1] who use the variation of constants formula).

Now, ϕ is a Lipschitz function defined on a finite dimensional space, and thus it is differentiable almost everywhere. However, this does not preclude the possibility that $\phi(p(t))$ is nowhere differentiable along a trajectory, and so it is not possible to replace $\dot{\phi}$ by $D\phi \cdot \dot{p}$ and obtain an equation valid for almost all t. However, it is clear that $A^\alpha\phi(p(t))$ is Lipschitz in t, since it is Lipschitz in $A^\alpha p$ and

$$|A^\alpha \dot{p}| \le -\lambda_1|A^\alpha p| + \lambda^{\alpha-\beta}C_0,$$

which implies that $A^\alpha p(t)$ is Lipschitz in t. Thus $\dot{\phi}(t)$ exists for almost all t (Young & Young [13]).

Since the manifold $\mathcal{M}$ is invariant, $\dot{\delta} = 0$ when $q = \phi(p)$, so the following expression, valid for almost all t :

$$A\phi(p) + QR(p + \phi(p)) + \dot{\phi}(p)_{(p,\phi(p))} = 0,$$

can be subtracted from the equation for $\dot{\delta}$ to yield

$$\dot{\delta} + A\delta + QR(p + q) - QR(p + \phi(p)) + (\dot{\phi}_{(p,q)} - \dot{\phi}_{(p,\phi(p))}) = 0.$$

Taking the inner product with $A^{2\alpha}\delta$ produces

$$\frac{1}{2}\frac{d}{dt}|A^\alpha\delta|^2 \leq -|A^{\alpha+\frac{1}{2}}\delta|^2 + C_1|A^{2\alpha-\beta}\delta||A^\alpha\delta| + |A^\alpha(\dot{\phi}_{(p,q)} - \dot{\phi}_{(p,\phi(p))})||A^\alpha\delta|.$$

If ϕ were C^1 then $\dot{\phi} = D\phi \cdot \dot{p}$, and the final term could be bounded by

$$\begin{aligned}
|A^\alpha D\phi(p) \cdot (PR(p,q) - PR(p,\phi(p)))| &\leq l|A^\alpha(PR(p,q) - PR(p,\phi(p)))| \\
&\leq C_1 l\lambda^{\alpha-\beta}|A^\alpha(q - \phi(p))| \\
&= C_1 l\lambda^{\alpha-\beta}|A^\alpha\delta|.
\end{aligned}$$

The same bound can be obtained for Lipschitz ϕ by the following procedure: for each t where the two derivatives are defined, denote by $\bar{u}(\tau)$ the trajectory with initial condition $\bar{u}(t) = p(t) + \phi(p(t))$. Then

$$\begin{aligned}
&|A^\alpha(\dot{\phi}(t)_{(p,q)} - \dot{\phi}(t)_{(p,\phi(p))})| \\
&\quad = \left|\lim_{\delta t\to 0} \frac{A^\alpha(\phi(p(t+\delta t)) - \phi(p(t)) - \phi(\bar{p}(t+\delta t)) + \phi(\bar{p}(t)))}{\delta t}\right| \\
&\quad = \left|\lim_{\delta t\to 0} \frac{A^\alpha(\phi(p(t+\delta t)) - \phi(\bar{p}(t+\delta t)))}{\delta t}\right| \\
&\quad \leq \lim_{\delta t\to 0} l\left|\frac{A^\alpha(p(t+\delta t) - \bar{p}(t+\delta t))}{\delta t}\right| \\
&\quad = l\left|\lim_{\delta t\to 0} \frac{A^\alpha(p(t+\delta t) - p(t) - \bar{p}(t+\delta t) + \bar{p}(t))}{\delta t}\right| \\
&\quad = l|A^\alpha(\dot{p}(t) - \dot{\bar{p}}(t))| \\
&\quad = l|A^\alpha(PR(p+q) - PR(p+\phi(p)))| \\
&\quad \leq C_1 l\lambda^{\alpha-\beta}|A^\alpha\delta|.
\end{aligned}$$

With this bound,

$$\frac{1}{2}\frac{d}{dt}|A^\alpha\delta|^2 \leq -|A^{\alpha+\frac{1}{2}}\delta|^2 + C_1|A^{2\alpha-\beta}\delta||A^\alpha\delta| + lC_1\lambda^{\alpha-\beta}|A^\alpha\delta|^2.$$

Now consider the other two terms

$$-|A^{\alpha+\frac{1}{2}}\delta|^2 + C_1|A^{2\alpha-\beta}\delta||A^\alpha\delta|$$

which occur on the right-hand side above. Since $\alpha + \frac{1}{2} > 2\alpha - \beta$ this expression is bounded by

$$-|A^{\alpha+\frac{1}{2}}\delta|^2 + C_1\Lambda^{\alpha-\beta-\frac{1}{2}}|A^{\alpha+\frac{1}{2}}\delta||A^\alpha\delta|.$$

Observe that $-X^2 + cX$ is decreasing for $X > c/2$, and so provided that

$$|A^{\alpha+\frac{1}{2}}\delta| > \Lambda^{\frac{1}{2}}|A^\alpha\delta| > C_1\Lambda^{\alpha-\beta-\frac{1}{2}}|A^\alpha\delta|/2$$

which reduces to

$$\Lambda^{1-\alpha+\beta} > C_1/2, \tag{6}$$

the following expression is valid :

$$\frac{d}{dt}|A^\alpha\delta| \leq -(\Lambda - C_1(\Lambda^{\alpha-\beta} + l\lambda^{\alpha-\beta}))|A^\alpha\delta|.$$

The result follows at once, noting that $\mu > 0$ ensures condition (6). ∎

It remains to prove the asymptotic completeness property of the inertial manifold, which ensures that any trajectory can be approximated, to within an exponentially decaying transient, by a trajectory lying within the manifold. This is stated more precisely in the following proposition.

Proposition 3 *If trajectories of (1) converge exponentially towards an invariant manifold given as the graph of a function $\phi \in \mathcal{F}_l^n$ and the cone $\mathcal{C}_m^n$ is invariant for some $m > l$, then for any trajectory $u(t)$ of (1) there exists a point $u_0 \in \mathcal{M}$ such that*

$$|A^\alpha(u(t) - S(t)u_0)| \leq \frac{1+m}{m-l}|A^\alpha(q(0) - \phi(p(0)))|e^{-\mu t}.$$

Proof The proof is based on that in Foias et al. [4] with considerable simplifications. Consider the complement of $\mathcal{C}_m$ centred at the point $u(t)$ of a trajectory, and denote this by $\mathcal{K}(u(t))$, so that

$$\mathcal{K}(u) \equiv \{v \in D(A^\alpha) : |A^\alpha Qw| > m|A^\alpha Pw|, w = v - u\}.$$

In the language of Foias et al., this cone casts a "shadow" $\mathcal{V}(u(t))$ on the inertial manifold,

$$\mathcal{V}(u) = \mathcal{K}(u) \cap \mathcal{M}.$$

FIGURE 1

This constuction is illustrated in figure 1.

Denote by $V(u)$ the closure of $\mathcal{V}(u)$ so that

$$V(u) = \{v \in \mathcal{M} : |A^\alpha Qw| \geq m|A^\alpha Pw|, w = v - u\}.$$

As the distance of $u(t)$ to the manifold decreases so does the distance between u and all points in $V(u)$; a bound on this distance implies that $V(u)$ is compact. This is the content of lemma 1.

Lemma 1 *Consider $u \in D(A^\alpha)$ with $u \notin \mathcal{M}$. Then $V(u)$ is compact, and for all $v \in V(u)$, the distance in $D(A^\alpha)$ between u and v is bounded according to*

$$|A^\alpha(u - v)| \leq \frac{1+m}{m-l}|A^\alpha(q - \phi(p))|.$$

Proof For $u = p + q$ and $v \in V(u)$ with $v = \pi + \phi(\pi)$, $\pi \in PH$,

$$\begin{aligned} |A^\alpha(u - v)| &= |A^\alpha(p + q - \phi(p) + \phi(p) - \pi - \phi(\pi))| \\ &\leq |A^\alpha(q - \phi(p))| + (1 + l)|A^\alpha(p - \pi)|. \end{aligned}$$

Now suppose that $|A^\alpha Q(u - v)| = \gamma|A^\alpha P(u - v)|$ with $\gamma \geq m$. Then if $\pi = p + r\omega$ for some $r > 0$ and $\omega \in PD(A^\alpha)$ with $|A^\alpha \omega| = 1$,

$$|A^\alpha(\phi(p + r\omega) - q)| = \gamma r.$$

Therefore

$$|A^\alpha(\phi(p + r\omega) - \phi(p) + \phi(p) - q)| = \gamma r$$

and using the Lipschitz property of ϕ

$$lr + |A^\alpha(q - \phi(p))| \geq mr$$

so that

$$|A^\alpha(p - \pi)| = r \leq \frac{|A^\alpha(q - \phi(p))|}{m - l}.$$

This ensures that $V(u)$ is compact (it is a closed bounded subset of the finite dimensional manifold $\mathcal{M}$), and provides the bound

$$|A^\alpha(u - v)| \leq \frac{1 + m}{m - l}|A^\alpha(q - \phi(p))|.$$

∎

Write $\mathcal{K}_t, \mathcal{V}_t, V_t$ for $\mathcal{K}(u(t)), \mathcal{V}(u(t)), V(u(t))$. It is now necessary to show that there exists a point within $\mathcal{V}_0$ which remains within the shadow for all time. This will provide the trajectory on $\mathcal{M}$ that tracks $u(t)$.

Since both $\mathcal{M}$ and the complement of $\mathcal{K}_t$ are positively invariant, trajectories can only leave the shadows $\mathcal{V}_t$. Taking the limit of a sequence of points $S(-t_n)v(t_n)$ with $v(t_n) \in \mathcal{V}_t$ will give the required point in V_0. This is lemma 2.

Lemma 2 *There exists a point $u_0 \in V_0$ such that $S(t)u_0 \in V_t$ for all $t \geq 0$.*

Proof Consider a sequence of times $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, and choose for each t_n a $v_{t_n} \in \mathcal{V}_{t_n}$. Set $u(n) = S(-t_n)v_{t_n}$; since points can only leave $\mathcal{V}_t$ this sequence will be contained in $\mathcal{V}_0$. Using lemma 1, V_0 is compact, so there exists a subsequence of $u(n)$ which converges to a point $u_0 \in V_0$. This point must be contained in V_t for all $t \geq 0$, for if not there exists a neighbourhood N of u_0 in $\mathcal{M}$ and a time t_0 such that for all $t \geq t_0$, $S(t)N$ is disjoint from V_t, contradicting the definition of u_0. ∎

The trajectory of u_0 tracks $u(t)$ exponentially, since using lemma 1,

$$\begin{aligned}|A^\alpha(u(t) - S(t)u_0)| &\leq \frac{1 + m}{m - l}|A^\alpha(q(t) - \phi(p(t)))| \\ &\leq \frac{1 + m}{m - l}|A^\alpha(q(0) - \phi(p(0)))|e^{-\mu t}.\end{aligned}$$

∎

Combining propositions 1-3 yields the first theorem of this paper.

Theorem 1 *If for some n the flow induced by equation (1) leaves the cone $\mathcal{C}_l^n$ invariant, and the eigenvalues $\lambda = \lambda_n$ and $\Lambda = \lambda_{n+1}$ satisfy*

$$\Lambda > C_1(\Lambda^{\alpha-\beta} + l\lambda^{\alpha-\beta}) \tag{7}$$

then there exists an inertial manifold given as the graph of a function $\phi \in \mathcal{F}_l^n$. If in addition the cone $\mathcal{C}_m^n$, for some $m > l$, is invariant, the inertial manifold is asymptotically complete.

3. AN EXISTENCE PROOF BASED ON THE STRONG SQUEEZING PROPERTY

The aim in section 2 was to produce an existence proof given only the existence of certain invariant cones. However, if the strong squeezing property is assumed, propositions 2 and 3 can be combined in a concise way.

First note that the strong squeezing property includes the cone invariance necessary for proposition 1 and hence ensures the existence of an invariant manifold given as the graph of some $\phi \in \mathcal{F}_l$, which will be denoted by $\mathcal{M}$. The following argument can then be applied to provide a tracking trajectory within this manifold.

Proposition 4 *If the strong squeezing property holds, then for every trajectory $u(t)$ there exists a point $u_0 \in \mathcal{M}$, such that*

$$|A^\alpha(u(t) - S(t)u_0)| \leq (1 + l^{-1})(|A^\alpha Qu(0)| + \|\phi\|)e^{-kt}. \tag{8}$$

Proof Define $\mathcal{V}(u)$ as in proposition 3, except set $m = l$. The first stage is to show that $V(u)$ is compact for $u \notin \mathcal{M}$. Since $V(u)$ is by definition closed, it suffices to show that $V(u)$ is bounded. Because $V(u) \subset \mathcal{M}$, this is equivalent to $PV(u)$ bounded in $PD(A^\alpha)$. Indeed, for $v \in V(u)$ with $v = \pi + \phi(\pi)$,

$$|A^\alpha(p - \pi)| \leq l|A^\alpha(q - \phi(\pi))|,$$

and as $|A^\alpha(p - \pi)| \geq |A^\alpha\pi| - |A^\alpha p|$,

$$|A^\alpha\pi| - |A^\alpha p| \leq l(|A^\alpha q| + \|\phi\|),$$

which gives the bound on $|A^\alpha\pi|$,

$$|A^\alpha\pi| \leq |A^\alpha p| + l(|A^\alpha q| + \|\phi\|),$$

ensuring compactness of $V(u)$. Therefore lemma 2 can be applied to give an initial condition $u_0 \in V_0$ which satisfies $S(t)u_0 \in V_t$ for all $t \geq 0$.

Now the squeezing property is used: since $S(t)u_0 \in V_t$ for all t, the difference

$w(t) = S(t)u_0 - u(t)$ lies either outside $\mathcal{C}_l$ or on $\partial\mathcal{C}_l$ for all $t \geq 0$, and so

$$\begin{aligned} |A^\alpha(u(t) - S(t)u_0)| &= |A^\alpha w(t)| \\ &\leq (1 + l^{-1})|A^\alpha Qw(t)| \\ &\leq (1 + l^{-1})|A^\alpha Qw(0)|e^{-kt} \\ &\leq (1 + l^{-1})|A^\alpha Q(u(0) - u_0)|e^{-kt} \\ &\leq (1 + l^{-1})(|A^\alpha Qu(0)| + \|\phi\|)e^{-kt}, \end{aligned}$$

which is "exponential tracking". In particular the manifold is exponentially attractive,

$$\mathrm{dist}(S(t)u(0), \mathcal{M}) \leq C(X)e^{-kt} \qquad C(X) = (1 + l^{-1})(\varrho + \|\phi\|), \tag{9}$$

uniform for $u(0) \in \Omega_\varrho$. ∎

Therefore propositions 1 and 4 provide an extremely concise existence proof, summarised as

Theorem 2 *The Strong Squeezing Property ensures the existence of an asymptotically complete inertial manifold, given as the graph of a function $\phi \in \mathcal{F}_l^n$.*

4. VERIFYING THE CONE CONDITION

The natural question to ask now is "when does the cone invariance property hold?". In the following proposition a condition is given which will in fact ensure that $\mathcal{C}_l$ satisfies the strong squeezing property.

Proposition 5 *If there exists an n such that the eigenvalues $\lambda = \lambda_n$ and $\Lambda = \lambda_{n+1}$ satisfy*

$$\Lambda - \lambda > C_1\{(1 + l)\lambda^{\alpha-\beta} + (1 + l^{-1})\Lambda^{\alpha-\beta}\} \tag{10}$$

then the cone $\mathcal{C}_l^n$ is positively invariant, and if $w(t_0) \notin \mathrm{int}\, \mathcal{C}_l$ then

$$|A^\alpha Q_n w(t)| \leq |A^\alpha Q_n w(0)|e^{-kt},$$

with $k = \Lambda - C_1\Lambda^{\alpha-\beta}(1 + l^{-1})$, for all $0 \leq t \leq t_0$.

Proof The proof is simple (see also Rodriguez Bernal [11] and Temam [12]). The difference w of two solutions satisfies the equation

$$\dot{w} + Aw + R(u_1) - R(u_2) = 0. \tag{11}$$

Set $p = Pw$ and $q = Qw$. Then the P component of (11) reads

$$\dot{p} + Ap + PR(u_1) - PR(u_2) = 0,$$

and taking the scalar product with $A^{2\alpha}p$ yields

$$\frac{1}{2}\frac{d}{dt}|A^\alpha p|^2 \geq -\lambda|A^\alpha p|^2 - C_1\lambda^{\alpha-\beta}|A^\alpha w||A^\alpha p|,$$

or when $|A^\alpha q(0)| = l|A^\alpha p(0)|$,

$$\frac{d}{dt}|A^\alpha p|_{t=0} \geq -(\lambda + C_1\lambda^{\alpha-\beta}(1+l))|A^\alpha q|/l.$$

For the Q equation,

$$\dot{q} + Aq + QR(u_1) - QR(u_2) = 0$$

the same device as was used in the exponential convergence proof must be employed. Take the scalar product with $A^{2\alpha}q$ to obtain

$$\frac{1}{2}\frac{d}{dt}|A^\alpha q|^2 \leq -|A^{\alpha+\frac{1}{2}}q|^2 + C_1|A^\alpha w||A^{2\alpha-\beta}q|,$$

and consider the derivative when $|A^\alpha q(0)| = l|A^\alpha p(0)|$. The equation becomes

$$\frac{1}{2}\frac{d}{dt}|A^\alpha q|^2_{t=0} \leq -|A^{\alpha+\frac{1}{2}}q|^2 + C_1(1+l^{-1})|A^\alpha q||A^{2\alpha-\beta}q|.$$

Apply the argument of proposition 2 to give

$$\frac{d}{dt}|A^\alpha q|_{t=0} \leq -(\Lambda - C_1\Lambda^{\alpha-\beta}(1+l^{-1}))|A^\alpha q|$$

provided that

$$\Lambda^{1-\alpha+\beta} > C_1(1+l^{-1})/2. \tag{12}$$

Then at $t = 0$

$$\frac{d}{dt}(|A^\alpha q| - l|A^\alpha p|)_{t=0} \leq -(\Lambda - \lambda - C_1(1+l)\lambda^{\alpha-\beta} - C_1(1+l^{-1})\Lambda^{\alpha-\beta})|A^\alpha q(0)|,$$

which is negative provided that condition (10) holds, and so $\mathcal{C}_l^n$ is invariant. Note also that (10) implies (12).

The squeezing property follows directly from the Q component of the equation with $|A^\alpha q| \geq l|A^\alpha p|$. ∎

5. AN INERTIAL MANIFOLD THEOREM

An immediate corollary of proposition 5 is the following

Theorem 3 *If for some n the eigenvalues $\lambda = \lambda_n$ and $\Lambda = \lambda_{n+1}$ satisfy*

$$\Lambda - \lambda > C_1(\lambda^{\frac{(\alpha-\beta)}{2}} + \Lambda^{\frac{(\alpha-\beta)}{2}})^2 \tag{13}$$

then there exists an inertial manifold with the asymptotic completeness property, given as the graph of a function $\phi \in \mathcal{F}_l^n$, where $l = (\Lambda/\lambda)^{(\alpha-\beta)/2}$.

Proof Condition (13) is obtained from

$$\Lambda - \lambda > C_1\{(1+l)\lambda^{\alpha-\beta} + (1+l^{-1})\Lambda^{\alpha-\beta}\}$$

by minimising the right hand side with respect to l. This minimum occurs at $l_{min} = (\Lambda/\lambda)^{(\alpha-\beta)/2}$. Condition (13) is therefore the least restrictive version of (10) to ensure the strong squeezing property and hence an asymptotically complete inertial manifold. ∎

6. COMPARISON WITH OTHER METHODS OF PROOF

It has been shown that the cone condition, supplemented by the attraction of trajectories to the invariant manifold, or the strong squeezing property alone, is sufficient to provide a concise existence proof. It is interesting how the method above is related to those employed in the papers mentioned in the introduction, where the cone condition plays an often unacknowledged rôle.

The most explicit use of the cone condition is in the paper Mallet-Paret & Sell [7] on reaction-diffusion equations. They consider the linear variational equation along a trajectory, which is

$$\begin{aligned} \dot{\rho} &= DF(u)(\rho, \sigma) \\ \dot{\sigma} &= -A\sigma + DG(u)(\rho, \sigma) \end{aligned}$$

in their notation, where $F(u) = -Ap + PR(u)$ and $G(u) = QR(u)$. The cone condition is expressed as $\dot{V} < 0$ whenever $|\rho| = |\sigma|$, where $V \equiv \frac{1}{2}|\sigma|^2 - \frac{1}{2}|\rho|^2$. This implies invariance of the cone $\mathcal{C}_1$ (see [7]). For reaction-diffusion equations, $\alpha = \beta = 0$, and smoothness results can be obtained under the stronger assumption that $\dot{V} \leq -\xi|\sigma|^2$. Generalising this condition gives

Definition *The V-condition : with $V \equiv \frac{1}{2}(|A^\alpha\sigma|^2 - l^2|A^\alpha\rho|^2)$, one has $\dot{V} \leq -\xi|A^{2\alpha-\beta}\sigma|^2$ whenever $|A^\alpha\sigma| = l|A^\alpha\rho|$.*

The invariance of the cone $\mathcal{C}_l$ is ensured by $\dot{W} < 0$ on the boundary of $\mathcal{C}_l$, where $W \equiv \frac{1}{2}(|A^\alpha q|^2 - l^2|A^\alpha p|^2)$ with $p = Pw, q = Qw$, and $w = u_1 - u_2$. Strengthening this to a uniform condition yields

Definition *The W-condition : with W as above one has $\dot{W} \leq -\xi|A^{2\alpha-\beta}q|^2$ whenever $|A^\alpha q| = l|A^\alpha p|$.*

This is equivalent to the V-condition :

Proposition 6 *If R is C^1, the W-condition holds if and only if the V-condition holds.*

Proof That the V-condition implies the W-condition can be shown using the following argument from Mallet-Paret & Sell [7]. Setting $p = p_1 - p_2$ and $q = q_1 - q_2$ for two solutions u_1, u_2 of (1), p and q satisfy

$$\begin{aligned}\dot{p} &= F(u_1) - F(u_2)\\ \dot{q} &= -Aq + G(u_1) - G(u_2).\end{aligned}$$

Using the mean value formula,

$$\begin{aligned}\dot{p} &= \int_1^2 DF(u_\theta)u \; d\theta\\ \dot{q} &= -Aq + \int_1^2 DG(u_\theta)u \; d\theta,\end{aligned}$$

where $u_\theta = u_1 + (\theta - 1)(u_2 - u_1)$. For each value of θ the V condition states that when $|A^\alpha q| = l|A^\alpha p|$,

$$(A^{2\alpha}q, -Aq + DG(u_\theta)u) - l^2(A^{2\alpha}p, DF(u_\theta)u) \leq -\xi|A^{2\alpha-\beta}q|^2.$$

Integrating this over θ yields $\dot{W} \leq -\xi|A^{2\alpha-\beta}q|^2$ as required.

For the converse, set $p = \epsilon\rho$ and $q = \epsilon\sigma$ with $|A^\alpha\rho| = l|A^\alpha\sigma|$, and then

$$\begin{aligned}&(A^{2\alpha}\epsilon\sigma, -A\epsilon\sigma + F(u_1) - F(u_1 + \epsilon(\rho + \sigma)))\\ &\qquad - (A^{2\alpha}\epsilon\rho, G(u_1) - G(u_1 + \epsilon(\rho + \sigma))) \leq -\xi\epsilon^2|A^{2\alpha-\beta}\sigma|^2.\end{aligned}$$

Letting $\epsilon \to 0$ gives

$$(A^{2\alpha}\sigma, -A\sigma + DF(u_1)(\rho, \sigma)) - (A^{2\alpha}\rho, DG(u_1)(\rho, \sigma)) \leq -\xi|A^{2\alpha-\beta}\sigma|^2$$

which is the V-condition. ∎

Note that although $\dot{V} < 0$ implies that $\dot{W} < 0$, the action of taking limits in the proof of proposition 6 would yield only $\dot{V} \leq 0$ from $\dot{W} < 0$. Thus requiring $\dot{W} < 0$ is slightly stronger than the simple cone invariance condition for $\mathcal{C}_l$.

To show the existence of an invariant manifold in [7] the Hadamard or "graph transform" method is used. The flat manifold PH evolves with the flow, and the cone condition implies that $S(t)PH = \mathcal{G}[\phi_t]$ as in proposition 1. It is then shown that ϕ_t converges to some ϕ^* as $t \to \infty$, the graph of which is invariant for $S(t)$. A condition similar to (5) ensures this convergence. From proposition 2 it is clear that given (5)

$$\|\phi_t - \phi^*\| \leq \|\phi_0 - \phi^*\| e^{-\mu t}$$

and hence a similar construction will work in this case. It should also be possible to adapt their smoothness proof to the more general case considered here, assuming that the V condition holds (see Raugel [9]).

The method used in all the other papers cited above ([1], [2], [3], [4], [11], [12]) is to reduce the search for an invariant manifold to a fixed point problem. For a given $\phi \in \mathcal{F}_l^n$, consider the finite-dimensional equation for $p \equiv P_n u$

$$\dot{p} + Ap + PR(p + \phi(p)) = 0, \tag{14a}$$

and denote by $p(t)$ the solution for which $p(0) = p_0$.

Suppose that $\mathcal{G}[\phi]$ were an invariant manifold. Then the evolution of $q \equiv Q_n u$ on the manifold would be given by

$$\dot{q} + Aq + QR(p + \phi(p)) = 0, \tag{14b}$$

with $|A^\alpha q(t)| < \|\phi\|$ and so bounded for all time. From Temam [12] such a solution is unique and can be written, using the variation of constants formula, as

$$q(t) = -e^{-At} \int_{-\infty}^{t} e^{\tau AQ} QR(p(\tau) + \phi(p(\tau)))\, d\tau, \tag{15}$$

so that in particular

$$q(0) = -\int_{-\infty}^{0} e^{\tau AQ} QR(p + \phi(p))\, d\tau.$$

If $\mathcal{G}[\phi]$ is in fact invariant, then it must be the case that $q(0) = \phi(p_0)$. Therefore define an operator T acting on the space $\mathcal{F}_l$ by

$$T\phi(p_0) = -\int_{-\infty}^{0} e^{\tau AQ} QR(p + \phi(p))\, d\tau, \tag{16}$$

and solutions of the fixed point problem $T\phi = \phi$ will be invariant manifolds for (1).

In proving the existence of a fixed point for T, two methods can be employed; either the Schauder fixed point theorem (since $\mathcal{F}_l$ is convex) or the contraction mapping theorem. In both cases it is important to show that $T\phi \in \mathcal{F}_l$ whenever $\phi \in \mathcal{F}_l$. This is where the cone condition can be employed.

The properties of the operator T are clearly related to the dynamical properties of the family of equations

$$\dot{u} + Au + R(p + \phi(p)) = 0 \tag{17}$$

for $\phi \in \mathcal{F}_l$ (this is equations (14a) and (14b) combined). Indeed,

Proposition 7 *If all members of the family of equations (17) leave the cone $\mathcal{C}_l^n$ invariant, then T maps $\mathcal{F}_l^n$ into itself.*

Proof Using proposition 1 equation (17) possesses an invariant manifold given as the graph of a function $\psi \in \mathcal{F}_l$. The trajectory on this manifold with $u(0) = (p_0, \psi(p_0))$ satisfies equation (14a) and $|A^\alpha q(t)| \leq \|\psi\|$ for all time. Thus from the uniqueness of such a solution, $q(t)$ must be given by (15), and so $T\phi = \psi \in \mathcal{F}_l$. ∎

The action of T is thus to pick out the invariant manifold for equation (17); indeed, if the nonlinear term R depends only on p :

Proposition 8 *The equation*

$$\dot{u} + Au + R(p) = 0$$

possesses an invariant manifold given as the graph of the function ϕ, where

$$\phi(p_0) = -\int_{-\infty}^{0} e^{\tau AQ} R(p(\tau))\, d\tau$$

and $p(t)$ is the solution of

$$\dot{p} + Ap + PR(p) = 0 \qquad p(0) = p_0.$$

Proof $T\phi = T0$ for all ϕ. In particular, $T^2 0 = T0$, so $T0$ is a fixed point for T. ∎

It would be satisfying if the cone invariance property for (1) ensured that the condition of proposition 7 were fulfilled, but generally this need not be the case. Indeed, the simple two-dimensional system

$$\begin{cases} \dot{x} = -\lambda x + \mu y \\ \dot{y} = -\Lambda y + \mu x \end{cases}$$

leaves $\mathcal{C}_1^1$ invariant for any $\Lambda > \lambda$ and for all μ. Here, $R(x,y) = \mu(y,x)$; setting $\phi(x) \equiv 0$ gives an equation in the family (17)

$$\begin{cases} \dot{x} = -\lambda x \\ \dot{y} = -\Lambda y + \mu x \end{cases}$$

which does not satisfy the cone condition if $\Lambda - \lambda < \mu$.

However, results like proposition 5 which rely only on the Lipschitz bound of $R(u)$ will carry over, replacing $R(u)$ by $R(p + \phi(p))$ throughout.

It is worthy of note that T is always continuous (Robinson [10]), and so Proposition 7 ensures a fixed point for T by the Schauder fixed point theorem, in other words an invariant manifold for (1). This comes as no surprise, since if the assumption of proposition 7 holds the cone $\mathcal{C}_l$ is invariant for (1); for any u_1, u_2 whose difference lies on $\partial\mathcal{C}_l$ use the W-condition for (17) with $p_i = Pu_i$ $(i = 1, 2)$, and a ϕ with $\phi(p_i) = Qu_i$ (which can be found since $|A^\alpha Q(u_1 - u_2)| = l|A^\alpha P(u_1 - u_2)|$) to ensure that $\frac{d}{dt}(|A^\alpha q|^2 - l^2|A^\alpha p|^2) < 0$.

It can therefore easily be seen why these two different methods of proof furnish the same constraints on the eigenvalues of A. For comparison, the conditions from the papers mentioned in the introduction are now given (re-worked into a more explicit form) and condition (13) is best re-expressed (and slightly coarsened) as

$$\Lambda - \lambda > 2C_1(\Lambda^{\alpha-\beta} + \lambda^{\alpha-\beta}). \tag{18}$$

Unless otherwise stated, the conditions are shown to ensure the existence of Lipschitz manifolds with no asymptotic completeness.

Chow et al. [1] have the tightest condition

$$\Lambda - \lambda > 4C_1(\Lambda^{\alpha-\beta} + \lambda^{\alpha-\beta})$$

which they show also ensures that the manifold is C^1 if the nonlinear term R is C^1, and asymptotic completeness.

Foias et al. [3] obtain, for the case $\alpha = 1$ and $\beta = \frac{1}{2}$,

$$\Lambda - \lambda > 18C_1(\Lambda^{\frac{1}{2}} + \lambda^{\frac{1}{2}}),$$

the same in form as (18) but with a larger constant.

The results of Mallet-Paret & Sell [7] are not directly comparable, as the aim of their paper was to find conditions to ensure cone invariance when a spectral gap like (18) could not be verified. However, using the "principle of spatial averaging", they show that for reaction-diffusion equations

$$\dot{u} + \nu \nabla^2 u + f(x,u) = 0$$

there exist C^1 inertial manifolds for all ν and various boundary conditions in : all domains in $\mathbb{R}$, rectangular domains in $\mathbb{R}^2$ of the form $(0, 2\pi/a_1) \times (0, 2\pi/a_2)$, and the cubic domain $(0, 2\pi)^3$ in $\mathbb{R}^3$, the last result requiring f to be C^3.

The conditions of Rodriguez Bernal [11] become equation (7) above and the extraordinary two inequalities

$$\delta_2 - \lambda \geq C_1(1+l)\lambda^{\alpha-\beta} + (K_{\alpha-\beta} C_1 (1+l^{-1})\Gamma(1+\beta-\alpha))^{\frac{1}{1+\beta-\alpha}}$$

$$\lambda^{\alpha-\beta-1} C_1 l + \delta_2^{\alpha-\beta-1} K_{\alpha-\beta} C_1 \Gamma(1+\beta-\alpha) < 1,$$

where $\delta_2 = \Lambda(1-\epsilon)$ and $K_\theta = \theta^\theta \epsilon^{-\theta} e^{-\theta}$. Λ has to be replaced by $\delta_2 < \Lambda$ since remark (iv) on page 100 is incorrect; indeed, K_θ arises in the bound $\|A^\theta e^{-AQt}\| \leq K_\theta t^{-\theta} e^{-\delta_2 t}$, which holds for no K_θ when $\delta_2 = \Lambda$.

Temam [12] allows $0 \leq \alpha - \beta \leq \frac{1}{2}$ yielding two conditions

$$\Lambda - \lambda > 56 C_1 (\lambda^{\alpha-\beta} + \Lambda^{\alpha-\beta}) \qquad \Lambda > C_1^2 (20 + 8e^{-\frac{1}{2}})$$

which reduce to

$$\Lambda - \lambda > 56 C_1 (\lambda^{\frac{1}{2}} + \Lambda^{\frac{1}{2}})$$

when $\beta = \alpha - \frac{1}{2}$. This is (18) again with a very inflated constant.

Thus equation (13) is less restrictive than all the above cases, although the asymptotic conditions necessary to ensure the existence of inertial manifolds for all $\nu > 0$ in the equations

$$\dot{u} + \nu A u + R(u) = 0$$

are identical.

The two existence proofs presented here highlight the importance of the cone invariance and squeezing properties, and in their argument simplify those in the literature. That the condition obtained to ensure existence is essentially identical to those found previously is explained by the relationship of the cone condition to the properties of the operator T used in the fixed point method.

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