COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **10**, Number **5**, September **2011**

doi:10.3934/cpaa.2011.10.1415

pp. **1415–1445**

ASYMPTOTIC DYNAMICS OF REVERSIBLE CUBIC AUTOCATALYTIC REACTION-DIFFUSION SYSTEMS

YUNCHENG YOU

Department of Mathematics and Statistics
University of South Florida, Tampa, FL 33620, USA

ABSTRACT. In this paper we prove the existence of a global attractor, an (H, E) global attractor, and an exponential attractor for the cubic autocatalytic reaction-diffusion systems represented by the reversible Gray-Scott equations. The two pairs of oppositely signed nonlinear terms feature the challenge in conducting various estimates. A new rescaling and grouping estimation method is introduced and combined with the other approaches to achieve the proof of dissipation, asymptotic compactness, and discrete squeezing property in all the stages.

1. **Introduction.** In recently published papers [51, 52, 54] by this author, it has been proved that for a class of nonlinear reaction-diffusion systems in the form

$$\frac{\partial u}{\partial t} = d_1 \Delta u + a_1 u + b_1 v + f(u, v) + g_1,$$
$$\frac{\partial v}{\partial t} = d_2 \Delta v + a_2 u + b_2 v - f(u, v) + g_2,$$

where the nonlinear reaction term $f(u, v) = u^2 v$ represents the type of cubic autocatalytic chemical or biochemical reactions, with homogeneous Dirichlet or Neumann boundary condition on a bounded, locally Lipschitz domain $\Omega \subset \Re^n$, $n \leq 3$, there exists a global attractor in the phase space $L^2(\Omega) \times L^2(\Omega)$, whose Hausdorff and fractal dimensions are finite.

This class of reaction-diffusion systems includes some significant pattern formation equations arising from modeling of kinetics of chemical or biochemical reactions and from biological and cellular pattern formation. The following four model equations are typical in this class:

Brusselator equations:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + a - (b + 1)u + u^2 v, \quad \frac{\partial v}{\partial t} = d_2 \Delta v + bu - u^2 v.$$

Gray-Scott equations:

$$\frac{\partial u}{\partial t} = d_1 \Delta u - (F + k)u + u^2 v, \quad \frac{\partial v}{\partial t} = d_2 \Delta v + F(1 - v) - u^2 v.$$

Selkov equations:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \rho - au + kv + u^2 v, \quad \frac{\partial v}{\partial t} = d_2 \Delta v + b - kv - u^2 v.$$

2000 *Mathematics Subject Classification.* Primary: 37L30; Secondary: 35B40, 35B41, 35K55, 35K57, 35Q80.

Key words and phrases. Reversible reaction-diffusion system, cubic autocatalysis, Gray-Scott equations, asymptotic dynamics, global attractor, exponential attractor.

Schnackenberg equations:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \rho - au + u^2 v, \quad \frac{\partial v}{\partial t} = d_2 \Delta v + b - u^2 v.$$

In these four systems, all the involved parameters are positive constants.

The Brusselator system is originally a system of ordinary differential equations as the reaction rate equations proposed by the scientists in the Brussels school [32]. The Gray–Scott system was originated from describing an isothermal, autocatalytic, continuously fed, unstirred reaction and diffusion of two chemicals [16, 17]. The Selkov system was proposed [38] as a simplified model of the phosphofructokinase reactions in glycolysis involving ATP, ADP, and AMP. The Schnackenberg system originally formulated in [36] serves as a simplified model of pattern formation in embryogenesis and in skin analysis [5, 40].

It is known that several examples of real autocatalytic reactions can be modeled by the aforementioned reaction-diffusion systems, such as acidic bromate oxidation of cerium, chlorite-iodide-malonic acid (CIMA) reaction, ferrocyanide-iodate-sulphite reaction, iodate oxidation of arsenite, and some enzyme catalytic reactions in biochemistry [1, 6, 15, 23, 24].

These mathematical models of the cubic autocatalytic reactions did not attract much attention until 1993 when J. E. Pearson [31] discovered self-replication of spot-like patterns and some other interesting patterns in numerical simulations of the 2D Gray-Scott equations. Since then, researches on these types of cubic autocatalytic reaction-diffusion systems on 1D and 2D domains have been reported in many aspects such as Turing patterns, spike patterns, stripe patterns, ring-like patterns, mesa-type patterns, and Hopf bifurcations by experiments [18, 22–24], numerical simulations [31, 33, 37], and mathematical analysis [8, 9, 20, 21, 26, 28, 29, 35, 45–48].

However, we have not seen substantial research results in the front of global dynamics for this type reaction-diffusion systems, especially in space dimension $n \leq 3$, until recently. The following proposition states the results in [51, 52, 54] on the global attractors for these nonreversible cubic autocatalytic reaction-diffusion systems.

Proposition 1.1. *For the solution semiflow of each of the Brusselator equations, Gray-Scott equations, Selkov equations and Schnackenberg equations on any bounded, locally Lipschitz domain of space dimension $n \leq 3$ with homogeneous Dirichlet or Neumann boundary conditions, and for any respectively involved positive parameters, there exists a global attractor $\mathscr{A}$ in the phase space $H = L^2(\Omega) \times L^2(\Omega)$ and the global attractor $\mathscr{A}$ has finite Hausdorff dimension and finite fractal dimension.*

In this paper, we shall study the asymptotic dynamics of the reversible cubic autocatalytic reaction-diffusion systems. These are more realistic and more precise model equations. In order to fix the idea, we take the reversible Gray–Scott equations as the representative. Let us briefly describe the derivation of the reversible Gray–Scott equations based on the simplified stoichiometry and the reduced scheme of chemical reactions:

$$2\mathrm{A} + \mathrm{B} \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 3\mathrm{A}, \quad \mathrm{A} \xrightarrow{k_2} \mathrm{P}.$$

Here A is an autocatalytic reactant which decays to form a product P in the irreversible second reactions shown above, while B is a reactant for which higher concentrations beyond a certain level increase the rate of its own removal, and the first reaction is the key autocatalytic reaction and is in general reversible, even is the generator of a linear analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on the Hilbert space H. By the fact that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous embedding for $n \leq 3$ and using the generalized Hölder inequality,

$$\|u^2 v\| \leq \|u\|_{L^6}^2 \|v\|_{L^6} \quad \text{and} \quad \|u^3\| = \|u\|_{L^6}^3, \quad \text{for } u, v \in L^6(\Omega),$$

one can verify that the nonlinear mapping

$$f(u,v) = \begin{pmatrix} -(F+k)u + u^2 v - Gu^3 \\ F(1-v) - u^2 v + Gu^3 \end{pmatrix} : E \longrightarrow H \tag{8}$$

is well defined on E and locally Lipschitz continuous. Then the initial-boundary value problem (3)–(6) is formulated into an initial value problem of the Gray–Scott evolutionary equation:

$$\begin{aligned} \frac{dw}{dt} &= Aw + f(w), \quad t > 0, \\ w(0) &= w_0 = \operatorname{col}(u_0, v_0) \in H, \end{aligned} \tag{9}$$

where $w(t) = \operatorname{col}(u(t,\cdot), v(t,\cdot))$, or simply written as $(u(t,\cdot), v(t,\cdot))$. Accordingly we shall write $w_0 = (u_0, v_0)$.

By conducting *a priori* estimates on the Galerkin approximate solutions of the IVP (9) and the weak convergence, we can prove the local existence and uniqueness of the weak solution $w(t)$ of (9) in the sense specified in [7, Section XV.3], which turns out to be a local strong solution for $t > 0$. Moreover, by taking the H-inner-product of (9) with this strong solution $w(t)$ itself and conducting *a priori* estimates, one can prove the continuous dependence of the solutions on the initial data and the following property satisfied by the strong solution,

$$w \in C([0, T_{\max}); H) \cap C^1((0, T_{\max}); H) \cap L^2(0, T_{\max}; E), \tag{10}$$

where $[0, T_{\max})$ is the maximal interval of existence.

For the Dirichlet boundary condition (5), the Poincaré inequality holds,

$$\|\nabla \varphi\|^2 \geq \eta \|\varphi\|^2, \quad \text{for any } \varphi \in H_0^1(\Omega), \tag{11}$$

where $\eta > 0$ is a uniform constant.

We refer to [39] and [43] for the concepts and basic facts in the theory of infinite dimensional dynamical systems. Below just mention few for clarity.

Definition 1.2. Let $\mathcal{M}$ be a complete metric space. A time-dependent family of mappings $\{S(t)\}_{t\geq 0}$ is called a *semiflow* (or *semigroup*) on $\mathcal{M}$, if the following conditions are satisfied:

(i) $S(0)w = w$, for all $w \in \mathcal{M}$.
(ii) $S(t+s) = S(t)S(s)$ on $\mathcal{M}$, for all $t, s \geq 0$.
(iii) The mapping $S(\cdot) : (t, w) \mapsto S(t)w$ is continuous from $[0, \infty) \times \mathcal{M}$ into $\mathcal{M}$.

Definition 1.3. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a complete metric space $\mathcal{M}$. A bounded subset B_0 of $\mathcal{M}$ is called an absorbing set in $\mathcal{M}$ if, for any bounded subset $B \subset \mathcal{M}$, there is some finite time $t_0 \geq 0$ depending on B such that $S(t)B \subset B_0$ for all $t \geq t_0$.

Definition 1.4. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a complete metric space $\mathcal{M}$ whose metric is denoted by $d(\cdot,\cdot)$. A subset $\mathscr{A}$ of $\mathcal{M}$ is called a global attractor for this semiflow, if $\mathscr{A}$ has the following properties:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in the sense that $S(t)\mathscr{A} = \mathscr{A}$ for any $t \geq 0$.

though the rate constant k_{-1} of the reverse reaction might be relatively small. Let $a = [A]$ and $b = [B]$ be the concentrations of the reactants A and B, respectively. The law of mass actions and the Fick's law of diffusion yield the following equations

$$\frac{\partial a}{\partial \tau} = D_1 \Delta a - \lambda a - k_2 a + k_1 a^2 b - k_{-1} a^3, \tag{1}$$

$$\frac{\partial b}{\partial \tau} = D_2 \Delta b + \lambda (b_0 - b) - k_1 a^2 b + k_{-1} a^3, \tag{2}$$

where D_1 and D_2 are the diffusive constants, k_1, k_{-1} and k_2 are the respective reaction rate constants, λ is the constant rate of feeding of the reactant B with reference to a constant concentration b_0 and also the rate of removing product P out.

By nondimensionalization, define

$$u = \frac{a}{b_0}, \quad v = \frac{b}{b_0}, \quad t = k_1 b_0^2 \tau, \quad d_1 = \frac{D_1}{k_1 b_0^2}, \quad d_2 = \frac{D_2}{k_1 b_0^2},$$

$$F = \frac{\lambda}{k_1 b_0^2}, \quad k = \frac{k_2}{k_1 b_o^2}, \quad G = \frac{k_{-1}}{k_1},$$

where k is called the effective production rate constant, $1/F$ is the mean residence time in dimensionless time units, and G is the ratio of the reverse versus forward reaction rates. Then the system of equations (1) and (2) is reduced to the dimensionless form of the reversible Gray-Scott equations (briefly called RGSE)

$$\frac{\partial u}{\partial t} = d_1 \Delta u - (F + k) u + u^2 v - G u^3, \quad t > 0, \ x \in \Omega, \tag{3}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + F(1 - v) - u^2 v + G u^3, \quad t > 0, \ x \in \Omega. \tag{4}$$

where d_1, d_2, F, k and G are positive constants. Consider the homogeneous Dirichlet (non-slip) boundary condition

$$u(t, x) = 0, \quad v(t, x) = 0, \quad t > 0, \ x \in \partial\Omega. \tag{5}$$

For the initial condition

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega, \tag{6}$$

we do not assume initial data (u_0, v_0) to be nonnegative or bounded.

To formulate this initial-boundary value problem of RGSE into an evolutionary equation, define the product Hilbert spaces

$$H = L^2(\Omega) \times L^2(\Omega),$$

$$E = H_0^1(\Omega) \times H_0^1(\Omega),$$

$$Z = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right).$$

The norm and inner-product of H or $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The norms of E and Z will be denoted by $\| \cdot \|_E$ and $\| \cdot \|_Z$, respectively. The norm of $L^p(\Omega)$ will be denoted by $\| \cdot \|_{L^p}$. We use $| \cdot |$ to denote an absolute value or a vector norm in a Euclidean space.

It is easy to check that, by the Lumer–Phillips theorem and the analytic semigroup generation theorem [39], the densely defined, sectorial operator

$$A = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix} : D(A) (= Z) \longrightarrow H \tag{7}$$

(ii) $\mathscr{A}$ attracts any bounded set B of $\mathcal{M}$ in the sense that, in terms of the Hausdorff semidistance,

$$\operatorname{dist}(S(t)B, \mathscr{A}) = \sup_{x \in B} \inf_{y \in \mathscr{A}} d(S(t)x, y) \to 0, \quad \text{as } t \to \infty.$$

Definition 1.5. A semiflow $\{S(t)\}_{t \geq 0}$ on a complete metric space $\mathcal{M}$ is asymptotically compact if for any sequences $\{u_n\}$ which is bounded in $\mathcal{M}$ and $\{t_n\} \subset (0, \infty)$ with $t_n \to \infty$, there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, such that $\lim_{k \to \infty} S(t_{n_k}) u_{n_k}$ exists in $\mathcal{M}$.

The next proposition [39, 43] states concisely the basic result on the existence of a global attractor for a semiflow.

Proposition 1.6. *Let $\{S(t)\}_{t \geq 0}$ be a semiflow on a Banach space $\mathcal{X}$, which has the following two properties:*

(i) *there exists a bounded absorbing set $B_0 \subset \mathcal{X}$ for $\{S(t)\}_{t \geq 0}$, and*
(ii) *$\{S(t)\}_{t \geq 0}$ is asymptotically compact on $\mathcal{X}$.*

Then there exists a global attractor $\mathscr{A}$ for $\{S(t)\}_{t \geq 0}$ in $\mathcal{X}$, which is the ω-limit set of B_0,

$$\mathscr{A} = \omega(B_0) \stackrel{\text{def}}{=} \bigcap_{\tau \geq 0} Cl_{\mathcal{X}} \bigcup_{t \geq \tau} (S(t)B_0).$$

In this work we shall investigate the asymptotic dynamics for the *reversible* Gray-Scott equations, as a representative of all the other reversible cubic autocatalytic reaction-diffusion systems. Specifically, we shall prove the following main results for the solution semiflow of the reversible Gray-Scott evolutionary equation (9), which are also valid for the reversible Brusselator equations, reversible Selkov equations, and reversible Schnackenberg equations with the homogeneous Dirichlet boundary conditions.

Main results:

1. There exists a global attractor $\mathscr{A}$ in H. It is shown in Theorem 4.2.
2. The global attractor $\mathscr{A}$ is an (H, E) global attractor shown in Theorem 5.6.
3. There exists an exponential attractor $\mathscr{E}$ in H. It is shown in Theorem 6.5.

The common feature shared by these reversible cubic autocatalytic reaction-diffusion systems and by the corresponding non-reversible reaction-diffusion systems (with $G = 0$) is that the asymptotic sign condition or called the dissipative condition on the nonlinear vector function $f(u, v)$,

$$\limsup_{|(u,v)| \to \infty} f(u, v) \cdot (u, v) \leq C$$

for some positive constant C is not satisfied due to the pair of oppositely signed, cubic terms $\pm u^2 v$.

Beside this common difficulty, there are more serious difficulties in analyzing the reversible cubic autocatalytic reaction-diffusion systems, which do not occur in treating Brusslater equations [51], Gray-Scott equations [52], and Selkov equations [54]. The new challenge is that due to the second pair of oppositely signed terms $\mp G u^3$ in the two equations we can no longer make dissipative and bounding *a priori* estimates on the v-component first by using the v-equation alone separately and then use the sum $y(t, x) = u(t, x) + v(t, x)$ to deal with the u-component in proving absorbing property and proving asymptotic compactness and so on for the solution semiflow as we did in [51, 52, 54]. The novel feature in this paper is to overcome this obstacle and to make the dissipative and bounding *a priori* estimates for u-component and v-component together by the new rescaling and grouping estimation method, that will be shown in Sections 2, 3 and 5.

We emphasize that the aforementioned Main Results on the existence of global attractor, (H, E) global attractor, and exponential attractor is unconditional, neither assuming that initial data or solutions are nonnegative, nor imposing any restriction on any positive parameters involved in these equations.

In Section 2, we shall prove the absorbing properties in the product spaces $L^p \times L^p$, $1 \le p \le 6$, for the solution semiflow of the reversible Gray-Scott evolutionary equation by introducing the rescaling and grouping estimation method. In Section 3 and Section 4, a decomposition approach combined with the rescaling and grouping estimation method will be exploited to show the asymptotic compactness via the uniform smallness with respect to a truncation of the u-component and through the κ-contracting property. In Section 5, it is shown that the global attractor is an (H, E) global attractor through the ultimate boundedness in transition from H to E. In Section 6, the existence of an exponential attractor in H is proved based on the existence of the (H, E) global attractor.

2. **Absorbing property.** In this section first we show that for any initial data $w_0 = (u_0, v_0) \in H$, the unique strong solution of (9) exists globally and defines a semiflow. Then we address the absorbing properties of this reversible Gray-Scott semiflow in the space H and in the spaces $L^p(\Omega) \times L^p(\Omega)$, for $1 \le p \le 6$.

Lemma 2.1. *For any initial data $w_0 = (u_0, v_0) \in H$, there exists a unique, global, weak solution $w(t) = (u(t), v(t))$, $t \in [0, \infty)$, of the reversible Gray–Scott evolutionary equation (9). Moreover, there exists a bounded absorbing set $\mathcal{B}_0$ in H for the solution semiflow of this reversible Gray–Scott evolutionary equation,*

$$\mathcal{B}_0 = \left\{ (\varphi, \psi) \in L^2(\Omega) \times L^2(\Omega) : \|\varphi(\cdot)\|^2 + \|\psi(\cdot)\|^2 \le K_0 \right\}, \tag{12}$$

where K_0 is a positive constant.

Proof. Taking the L^2 inner-products $\langle (3), Gu(t, \cdot) \rangle$ and $\langle (4), v(t, \cdot) \rangle$, then summing up the resulting equalities, by the boundary condition (5), we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(G\|u\|^2 + \|v\|^2 \right) + d_1 G \|\nabla u\|^2 + d_2 \|\nabla v\|^2 \\
&= \int_\Omega Fv \, dx - \left((F + k) G \|u\|^2 + F\|v\|^2 \right) - \int_\Omega \left(G^2 u^4 - 2Gu^3 v + u^2 v^2 \right) dx \\
&= \int_\Omega Fv \, dx - \left((F + k) G \|u\|^2 + F\|v\|^2 \right) - \int_\Omega \left(Gu^2 - uv \right)^2 dx \\
&\le \frac{F|\Omega|}{2} - \frac{1}{2} \left((F + k) G \|u\|^2 + F\|v\|^2 \right).
\end{aligned} \tag{13}$$

Then we have the following two differential inequalities,

$$\frac{d}{dt} \left(G\|u\|^2 + \|v\|^2 \right) + F \left(G\|u\|^2 + \|v\|^2 \right) \le F|\Omega|, \tag{14}$$

and

$$\frac{d}{dt} \left[G\|u\|^2 + \|v\|^2 \right] + 2 \left(d_1 G \|\nabla u\|^2 + d_2 \|\nabla v\|^2 \right) \le F|\Omega|. \tag{15}$$

The differential inequality (14) yields

$$G\|u(t, \cdot)\|^2 + \|v(t, \cdot)\|^2 \le e^{-Ft} \left(G \|u_0\|^2 + \|v_0\|^2 \right) + |\Omega|, \quad \text{for } t \in [0, T_{\max}). \tag{16}$$

This inequality (16) shows that the weak solution $w(t;w_0)=(u(t,\cdot),v(t,\cdot))$ of (9) will never blow up at any finite time, so that $T_{\max}=\infty$ for any initial data in H. Combined with (10) and the joint continuous dependence of the weak solution on time t and the initial data w_0, cf. [39], we see that the family of all the weak solutions defines a semiflow $\{S(t)\}_{t\geq 0}$ on the Hilbert space H,

$$S(t)w_0 = w(t;w_0), \quad t\geq 0,\ w_0\in H.$$

We shall call this semiflow as the solution semiflow of the reversible Gray-Scott equations, or simply the *reversible Gray–Scott semiflow.*

Moreover, the inequality (16) also shows that

$$\limsup_{t\to\infty}\left(\|u(t,\cdot)\|^2+\|v(t,\cdot)\|^2\right)\leq \frac{|\Omega|}{\min\{G,1\}}+1 \tag{17}$$

and there is a bounded absorbing set for the reversible Gray-Scott semiflow $\{S(t)\}_{t\geq 0}$ in the phase space $H=L^2(\Omega)\times L^2(\Omega)$,

$$\mathcal{B}_0=\left\{(\varphi,\psi)\in L^2(\Omega)\times L^2(\Omega):\|\varphi(\cdot)\|^2+\|\psi(\cdot)\|^2\leq K_0\right\}, \tag{18}$$

where the constant K_0 is given by

$$K_0=\frac{|\Omega|}{\min\{G,1\}}+1.$$

The proof is completed. □

Let $d_0=\min\{d_1,d_2\}$. Note that (15) implies

$$\int_0^t e^{d_2\eta s}\frac{d}{ds}\left(G\|u(s)\|^2+\|v(s)\|^2\right)ds+\int_0^t e^{d_2\eta s}d_1G\|\nabla u(s)\|^2ds\leq\frac{1}{d_2\eta}e^{d_2\eta t}F|\Omega|,$$

and from (15) we have

$$G\|u(t,\cdot)\|^2+\|v(t,\cdot)\|^2\leq e^{-d_0\eta t}\left(G\|u_0\|^2+\|v_0\|^2\right)+\frac{F}{d_0\eta}|\Omega|. \tag{19}$$

Therefore,

$$\begin{aligned}\int_0^t e^{d_2\eta s}\|\nabla u(s)\|^2ds &\leq\frac{1}{d_1G}\left(G\|u_0\|^2+\|v_0\|^2\right)+\frac{e^{d_2\eta t}F}{d_1d_2\eta G}|\Omega|\\ &\quad+\frac{d_2\eta}{d_1G}\int_0^t e^{d_2\eta s}\left[e^{-d_0\eta s}\left(G\|u_0\|^2+\|v_0\|^2\right)+\frac{F}{d_0\eta}|\Omega|\right]ds.\end{aligned} \tag{20}$$

This is for the later use.

In the next lemma we shall prove that the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$ possesses the absorbing property in the space $L^p(\Omega)\times L^p(\Omega)$ for $2\leq p\leq 6$. For $p=2$, it has been shown in Lemma 2.1.

For this purpose, we shall use the rescaling and grouping estimation method. Let $U(t,x)=u(t,x)$ and $V(t,x)=v(t,x)/G$. Then the reversible Gray-Scott system (3)–(4) is equivalently written as

$$\frac{\partial U}{\partial t}=d_1\Delta U-(F+k)U+GU^2V-GU^3,\quad t>0,\ x\in\Omega, \tag{21}$$

$$\frac{\partial V}{\partial t}=d_2\Delta V+\frac{F}{G}-FV-U^2V+U^3,\quad t>0,\ x\in\Omega. \tag{22}$$

The same Dirichlet boundary condition as (5) is satisfied by $U(t,x)$ and $V(t,x)$.

Lemma 2.2. *There exists a constant $K_1 > 0$ such that the solution semiflow $\{S(t)\}_{t\geq 0}$ of the evolutionary equation* (9) *satisfies the following dissipative inequality in the space $L^6(\Omega) \times L^6(\Omega)$,*

$$\limsup_{t\to\infty} \|S(t)\,(u_0, v_0)\|^6_{L^6(\Omega)\times L^6(\Omega)} < K_1, \quad \text{for any } w_0 = (u_0, v_0) \in H. \tag{23}$$

Therefore, the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$ has the $L^p \times L^p$ absorbing property for $p = 4$ and $p = 6$.

Proof. According to the solution property (10) of the reversible Gray–Scott evolutionary equation (9), in which $T_{\max} = \infty$ for all solutions, we see that for any $w_0 = (u_0, v_0) \in H$ there exists a time $t_0 \in (0,1)$ such that $S(t_0)\,w_0 \in E = H^1(\Omega) \times H^1(\Omega) \subset L^6(\Omega) \times L^6(\Omega)$. Then by the regularity of solutions of parabolic evolutionary equations shown in [39, Theorem 47.6], it holds that

$$S(\cdot)w_0 \in C\left([t_0, \infty), E\right) \subset C\left([t_0, \infty), L^6(\Omega) \times L^6(\Omega)\right).$$

for space dimension $n \leq 3$. Based on this observation, without loss of generality, we assume that $w_0 = (u_0, v_0) \in L^6(\Omega) \times L^6(\Omega)$. Then by the same reason of parabolic regularity we see that $S(t)\,(u_0, v_0) \in Z \subset L^8(\Omega) \times L^8(\Omega)$, for $t > 0$.

Taking the inner-product $\langle (21), U^5(t,\cdot)\rangle_{L^2(\Omega)}$, we obtain

$$\begin{aligned}\frac{1}{6}\frac{d}{dt}\int_\Omega U^6(t,x)\,dx + 5d_1 \left\|U^2(t,\cdot)\nabla U(t,\cdot)\right\|^2 &= -(F+k)\int_\Omega U^6(t,x)\,dx \\ &+ G\int_\Omega U^7(t,x)V(t,x)\,dx - G\int_\Omega U^8(t,x)\,dx, \quad t \geq 0.\end{aligned} \tag{24}$$

Then taking the inner-product $\langle (22), GV^5(t,\cdot)\rangle_{L^2(\Omega)}$, we see that for any $t \geq 0$,

$$\begin{aligned}&\frac{G}{6}\frac{d}{dt}\int_\Omega V^6(t,x)\,dx + 5Gd_2\left\|V^2(t,\cdot)\nabla V(t,\cdot)\right\|^2 = \int_\Omega F\,V^5(t,x)\,dx \\ &- FG\int_\Omega V^6(t,x)\,dx - G\int_\Omega U^2(t,x)V^6(t,x)\,dx + G\int_\Omega U^3(t,x)V^5(t,x)\,dx.\end{aligned} \tag{25}$$

Add up the above two equalities (24) and (25) to obtain that for $t \geq 0$,

$$\begin{aligned}&\frac{1}{6}\frac{d}{dt}\left(\|U(t,\cdot)\|^6_{L^6(\Omega)} + G\|V(t,\cdot)\|^6_{L^6(\Omega)}\right) \\ &\quad + 5d_0\left(\|U^2(t,\cdot)\nabla U(t,\cdot)\|^2 + G\left\|V^2(t,\cdot)\nabla V(t,\cdot)\right\|^2\right) \\ &= -(F+k)\int_\Omega U^6(t,x)\,dx + \int_\Omega FV^5(t,x)\,dx - FG\int_\Omega V^6(t,x)\,dx \\ &\quad - G\int_\Omega \left(U^8(t,x) - U^7(t,x)V(t,x) - U^3(t,x)V^5(t,x) + U^2(t,x)V^6(t,x)\right)dx \\ &= -(F+k)\int_\Omega U^6(t,x)\,dx + \int_\Omega FV^5(t,x)\,dx - FG\int_\Omega V^6(t,x)\,dx \\ &\quad + G\int_\Omega U^2(t,x)\left[-U^6(t,x) + U^5(t,x)V(t,x) + U(t,x)V^5(t,x) - V^6(t,x)\right]dx.\end{aligned} \tag{26}$$

By Young's inequality, we have

$$-U^6 + U^5V + UV^5 - V^6 \leq -U^6 + \left(\frac{5}{6}U^6 + \frac{1}{6}V^6\right) + \left(\frac{1}{6}U^6 + \frac{5}{6}V^6\right) - V^6 = 0,$$

and

$$\int_\Omega FV^5(t,x)\,dx - FG\int_\Omega V^6(t,x)\,dx \le -\frac{1}{6}\,FG\|V(t,\cdot)\|^6_{L^6(\Omega)} + \frac{F|\Omega|}{6G^5}.$$

Substituting the above two inequalities into (26), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|U(t,\cdot)\|^6_{L^6(\Omega)} + G\|V(t,\cdot)\|^6_{L^6(\Omega)}\right)\\ &\le -F\left(\|U(t,\cdot)\|^6_{L^6(\Omega)} + G\|V(t,\cdot)\|^6_{L^6(\Omega)}\right) + \frac{F|\Omega|}{G^5}, \quad t\ge 0.\end{aligned} \tag{27}$$

This Gronwall inequality yields the following estimate,

$$\|U(t,\cdot)\|^6_{L^6(\Omega)} + G\|V(t,\cdot)\|^6_{L^6(\Omega)} \le e^{-Ft}\left(\|U_0\|^6_{L^6(\Omega)} + G\,\|V_0\|^6_{L^6(\Omega)}\right) + \frac{|\Omega|}{G^5}, \quad t\ge 0.$$

Since $U(t,x)=u(t,x)$ and $V(t,x)=v(t,x)/G$, this inequality can be written as

$$\|u(t,\cdot)\|^6_{L^6(\Omega)} + \frac{1}{G^5}\|v(t,\cdot)\|^6_{L^6(\Omega)} \le e^{-Ft}\left(\|u_0\|^6_{L^6(\Omega)} + \frac{1}{G^5}\,\|v_0\|^6_{L^6(\Omega)}\right) + \frac{|\Omega|}{G^5}, \quad t\ge 0, \tag{28}$$

for any $(u_0,v_0)\in E$. From (28) it follows that, for any $w_0=(u_0,v_0)\in H$,

$$\begin{aligned}\limsup_{t\to\infty}\|S(t)\,(u_0,v_0)\|^6_{L^6(\Omega)\times L^6(\Omega)} &= \limsup_{t\to\infty}\left(\|u(t,\cdot)\|^6_{L^6(\Omega)} + \|v(t,\cdot)\|^6_{L^6(\Omega)}\right)\\ &\le \frac{|\Omega|}{\min\{1,G^{-5}\}G^5} + 1 = \frac{|\Omega|}{\min\{1,G^5\}} + 1.\end{aligned}$$

Let K_1 be the constant

$$K_1 = \frac{|\Omega|}{\min\{1,G^5\}} + 1. \tag{29}$$

Therefore, (23) is proved with K_1 given by (29). □

Note that the absorbing property (23) shown in Lemma 2.2 is a little weaker than the existence of an absorbing set in $L^6(\Omega)\times L^6(\Omega)$ for the semiflow $\{S(t)\}_{t\ge 0}$ on H since we did not claim a uniform absorbing time for any bounded set in H. However, by the same approach we can prove even more as stated in the following proposition.

Lemma 2.3. *For any positive integer $p\ge 1$, there exists a universal constant $\mathcal{K}_p>0$ such that the solution semiflow $\{S(t)\}_{t\ge 0}$ of the evolutionary equation (9) satisfies the following inequality,*

$$\limsup_{t\to\infty}\|S(t)\,(u_0,v_0)\|_{L^{2p}(\Omega)\times L^{2p}(\Omega)} < \mathcal{K}_p, \quad \textit{for any } w_0=(u_0,v_0)\in H.$$

Thus the reversible Gray–Scott semiflow $\{S(t)\}_{t\ge 0}$ has the $L^{2p}\times L^{2p}$ absorbing property for any $p\ge 1$.

3. **Decomposition for asymptotic compactness.** According to the basic existence theory of global attractor stated in Proposition 1.6, we need to show that the reversible Gray-Scott semiflow $\{S(t)\}_{t\ge 0}$ is asymptotically compact in H. However, due to the two pairs of oppositely opposite-signed cubic terms in (3)–(4), it is challenging for this attempt. Here we shall adaptively use a generic result shown in [51, 52, 54] through a new decomposition approach to proving that the semiflow generated by a system of two coupled reaction-diffusion equations is asymptotically compact.

Kuratowski measure of noncompactness for bounded sets in a Banach space $\mathcal{X}$ is defined by

$$\kappa(B) \stackrel{\text{def}}{=} \inf\{\delta : B \text{ has a finite cover by open sets in } \mathcal{X} \text{ of diameters} < \delta\}.$$

If B is an unbounded set, then define $\kappa(B) = \infty$.

Definition 3.1. A semiflow $\{S(t)\}_{t\geq 0}$ on a Banach space $\mathcal{X}$ is called *κ-contracting* if for every bounded subset B in $\mathcal{X}$, one has

$$\lim_{t\to\infty} \kappa(S(t)B) = 0.$$

The basic properties of the Kuratowski measure can be found in [39, Lemma 22.2]. The following lemma provides the connection of the κ-contracting property to the asymptotic compactness and the existence of a global attractor for the semiflow, cf. [39, Lemma 23.8].

Lemma 3.2. *Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a real Banach space $\mathcal{X}$. If the following conditions are satisfied:*

(i) *$\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $\mathcal{X}$, and*
(ii) *$\{S(t)\}_{t\geq 0}$ is κ-contracting,*

then $\{S(t)\}_{t\geq 0}$ is asymptotically compact and there exists a global attractor $\mathscr{A}$ in $\mathcal{X}$ for this semiflow.

A generically good idea in dealing with the issue of asymptotic compactness or κ-contracting property is through a decomposition. There have been different decomposition methods used in different settings [39, 41–44, 49, 50].

Here we present a new decomposition method, which is stated in the next lemma and will be used to check the κ-contracting property of the reversible Gray–Scott semiflow.

We shall use the notation

$$\Omega^u_M = \Omega(|u(t)| \geq M) = \{x \in \Omega : |u(t,x)| \geq M\},$$
$$\Omega_{u,M} = \Omega(|u(t)| < M) = \{x \in \Omega : |u(t,x)| < M\}.$$

Lebesgue measure of a subset Ω_s of Ω is denoted by $m(\Omega_s)$ or $|\Omega_s|$.

Lemma 3.3. *For the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$, there exists a global attractor $\mathscr{A}$ in H if and only if the following two conditions are satisfied:*

(i) *There exists a bounded absorbing set $\mathcal{B}_0$ in H for this semiflow.*
(ii) *For any $\varepsilon > 0$, there are positive constants $M = M(\varepsilon)$, $T = T(\varepsilon)$, and a uniform constant $C > 0$ such that*

$$\int_{\Omega(|u(t)|\geq M)} |S(t)w_0|^2\, dx < C\varepsilon, \quad \text{for any } t \geq T,\ w_0 \in \mathcal{B}_0, \tag{30}$$

and

$$\kappa\left((S(t)\mathcal{B}_0)_{\Omega(|u(t)|<M)}\right) \longrightarrow 0, \quad \text{as } t \to \infty, \tag{31}$$

where

$$(S(t)\mathcal{B}_0)_{\Omega(|u(t)|<M)} \stackrel{\text{def}}{=} \{(S(t)w_0)(\cdot)\theta_M(\cdot\,; t, w_0) : w_0 \in \mathcal{B}_0\}, \tag{32}$$

in which $\theta_M(x; t, w_0)$, $x \in \Omega$, is the characteristic function of the subset $\Omega(|u(t)| < M)$, and $u(t) = u(t, x; w_0)$ is the u-component of the strong solution of the reversible Gray–Scott evolutionary equations (9).

The proof of Lemma 3.3 is similar to the corresponding result in [52, Theorem 2] as well as in [51, 54], with the only difference that the truncation in terms of v-component used in [51, 52, 54] is replaced by the truncation in terms of the u-component. Therefore the proof is omitted.

Here the decomposition in terms of the truncation of the u-component instead of the v-component makes a nontrivial difference in view of the new feature of appearance of the $\pm Gu^3$ terms in the reversible Gray–Scott equations, which are not involved in the nonreversible systems in [51, 52, 54].

For reversible cubic autocatalytic reaction-diffusion systems, the process of checking the first condition (30) and the second condition (31) in part (ii) of Lemma 3.3 is more challenging than and quite different from what was done for the corresponding nonreversible systems. Here we shall use the *rescaling and grouping estimation method* to show that the conditions (30) and (31) in Lemma 3.3 are satisfied.

Recall that $U(t,x) = u(t,x)$ and $V(t,x) = v(t,x)/G$ satisfy the rescaled systems (21)–(22).

Lemma 3.4. *For any $\varepsilon > 0$, there exists positive constants $M_1 = M_1(\varepsilon)$ and $T_1 = T_1(\varepsilon)$ such that the u-component $u(t) = u(t,x;w_0)$ of the solution of the RGSE* (3)–(4) *with the boundary condition* (5) *satisfies*

$$\int_{\Omega^u_{M_1}} |u(t)|^2 dx < C_1 \varepsilon, \qquad \text{for} \ \ t > T_1, \, w_0 = (u_0, v_0) \in \mathcal{B}_0, \tag{33}$$

where $\mathcal{B}_0$ is the absorbing set shown in Lemma 2.1, *and $C_1 = C_1(K_0)$ is a positive constant.*

Proof. Since $\mathcal{B}_0$ attracts itself, there is $T_0 = T_0(\mathcal{B}_0) \geq 0$ such that

$$\{S(t)w_0 : t \geq T_0, w_0 \in \mathcal{B}_0\} \subset \mathcal{B}_0.$$

Thus $\|u(t)\|^2 \leq \|S(t)w_0\|^2 \leq K_0$ for $t \geq T_0$ and $w_0 \in \mathcal{B}_0$.

For any given $\varepsilon > 0$, there is a sufficiently large $M = M(\varepsilon) > 0$, such that

$$m(\Omega(|u(t)| \geq M)) \leq \frac{K_0}{M^2} < \varepsilon, \quad \text{for } t \geq T_0, \, w_0 \in \mathcal{B}_0. \tag{34}$$

Let M be the constant given by (34). Define

$$(\varphi - M)_+ = \begin{cases} \varphi - M, & \text{if } \varphi \geq M, \\ 0, & \text{if } \varphi < M. \end{cases}$$

Taking the inner-product $\langle (21), (u(t)-M)_+ \rangle_{\Omega^u_M} = \langle (21), (U(t)-M)_+ \rangle_{\Omega^u_M}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(U-M)_+\|^2_{L^2(\Omega^u_M)} + d_1\|\nabla(U-M)_+\|^2_{L^2(\Omega^u_M)} \leq -(F+k)\|(U-M)_+\|^2_{L^2(\Omega^u_M)} \\ &\quad + \int_{\Omega(|u(t)|\geq M)} GU^2(V-M)_+(U-M)_+ dx + MG\int_{\Omega(|u(t)|\geq M)} U^2(U-M)_+ dx \\ &\quad - G\int_{\Omega(|u(t)|\geq M)} U^2(U-M)^2_+ dx - MG\int_{\Omega(|u(t)|\geq M)} U^2(U-M)_+ dx. \end{aligned} \tag{35}$$

Taking the inner-product $\langle (22), G(V(t)-M)_+\rangle_{\Omega^u_M}$, where M is the same as above, we get

$$\begin{aligned}
&\frac{G}{2}\frac{d}{dt}\|(V-M)_+\|^2_{L^2(\Omega^u_M)} + d_2 G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)} \\
&\le \int_{\Omega(|u(t)|\ge M)} F(V-M)_+ dx - FG\int_{\Omega(|u(t)|\ge M)} V(V-M)_+ dx \\
&\quad - \int_{\Omega(|u(t)|\ge M)} GU^2(V-M)_+^2 dx - MG\int_{\Omega(|u(t)|\ge M)} U^2(V-M)_+ dx \\
&\quad + G\int_{\Omega(|u(t)|\ge M)} U^2(U-M)_+(V-M)_+ dx + MG\int_{\Omega(|u(t)|\ge M)} U^2(V-M)_+ dx.
\end{aligned} \tag{36}$$

Recall that $d_0 = \min\{d_1, d_2\}$. Add up the above two inequalities (35) and (36) (in each of them the two integrals with the coefficient MG are cancelled out) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|(V-M)_+\|^2_{L^2(\Omega^u_M)}\right) \\
&\quad + d_0\left(\|\nabla(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)}\right) \\
&\le \int_{\Omega(|u(t)|\ge M)} F(V-M)_+ dx - FG\int_{\Omega(|u(t)|\ge M)} V(V-M)_+ dx \\
&\quad - G\int_{\Omega(|u(t)|\ge M)} \left[U(U-M)_+ - U(V-M)_+\right]^2 dx \\
&\le \int_{\Omega(|u(t)|\ge M)} F(V-M)_+ dx - FG\int_{\Omega(|u(t)|\ge M)} V(V-M)_+ dx
\end{aligned} \tag{37}$$

By Young's inequality, we have

$$\begin{aligned}
\int_{\Omega(|u(t)|\ge M)} F(V-M)_+ dx &\le \frac{F^2}{d_0 G\eta}|\Omega^u_M| + \frac{1}{4}d_0 G\eta\|(V-M)_+\|^2_{L^2(\Omega^u_M)} \\
&\le \frac{F^2}{d_0 G\eta}|\Omega^u_M| + \frac{1}{4}d_0 G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)}.
\end{aligned}$$

Using (16) and noting that $V(t,x) = v(t,x)/G$, we can get

$$\begin{aligned}
-FG\int_{\Omega(|u(t)|\ge M)} V(V-M)_+ dx &\le \frac{F^2 G}{d_0\eta}\|V\|^2_{L^2(\Omega^u_M)} + \frac{1}{4}d_0 G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)} \\
&\le \frac{2F^2}{d_0 G\eta}|\Omega^u_M| + \frac{1}{4}d_0 G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)},
\end{aligned}$$

provided that $t > \tau_0$, where $\tau_0 > 0$ is such a constant that

$$e^{-F\tau_0}K_0 \le |\Omega^u_M|.$$

Substitute these two estimates into (37) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|(V-M)_+\|^2_{L^2(\Omega^u_M)}\right) \\
&\quad + d_0\left(\|\nabla(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)}\right) \\
&\le \frac{F^2}{d_0 G\eta}|\Omega^u_M| + \frac{F^2 G}{d_0\eta}\|V\|^2_{L^2(\Omega^u_M)} + \frac{1}{2}d_0 G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)} \\
&\le \frac{F^2}{d_0 G\eta}|\Omega^u_M| + \frac{2F^2}{d_0 G\eta}|\Omega^u_M| + \frac{1}{2}d_0 G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)} \\
&= \frac{3F^2}{d_0 G\eta}|\Omega^u_M| + \frac{1}{2}d_0 G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)}, \quad \text{for } t > \max\{T_0, \tau_0\}.
\end{aligned} \tag{38}$$

By (34), $|\Omega^u_M| < \varepsilon$. It follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\|(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|(V-M)_+\|^2_{L^2(\Omega^u_M)}\right) \\
&\quad + d_0\left(\|\nabla(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|\nabla(V-M)_+\|^2_{L^2(\Omega^u_M)}\right) \le C_2\,\varepsilon,
\end{aligned} \tag{39}$$

where

$$C_2 = \frac{6F^2}{d_0 G\eta}$$

is a positive constant independent of M. By Poincaré inequality (11) and Gronwall inequality, from (39) we get

$$\begin{aligned}
&\|(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|(V-M)_+\|^2_{L^2(\Omega^u_M)} \\
&\le e^{-d_0\eta t}\left(\|(U_0-M)_+\|^2_{L^2(\Omega^u_M)} + G\|(V_0-M)_+\|^2_{L^2(\Omega^u_M)}\right) + \frac{C_2\,\varepsilon}{d_0\eta} \\
&\le (1+G)K_0 e^{-d_0\eta t} + \frac{C_2\,\varepsilon}{d_0\eta}.
\end{aligned}$$

Hence, there exists a time $T_+(\varepsilon)$ ($\ge \max\{T_0, \tau_0\}$) such that for any $t > T_+$ and any $w_0 \in B_0$, one has

$$\|(U-M)_+\|^2_{L^2(\Omega^u_M)} \le \|(U-M)_+\|^2_{L^2(\Omega^u_M)} + G\|(V-M)_+\|^2_{L^2(\Omega^u_M)} < \frac{2C_2\,\varepsilon}{d_0\eta}. \tag{40}$$

By the symmetric argument we can show that there exists a time $T_-(\varepsilon)$ such that for any $t > T_-$ and any $w_0 \in B_0$, it holds that

$$\|(U+M)_-\|^2_{L^2(\Omega^u_M)} \le \|(U+M)_-\|^2_{L^2(\Omega^u_M)} + G\|(V+M)_-\|^2_{L^2(\Omega^u_M)} < \frac{2C_2\,\varepsilon}{d_0\eta}, \tag{41}$$

where

$$(\varphi+M)_- = \begin{cases} \varphi(x)+M, & \text{if } \varphi(x) \le -M, \\ 0, & \text{if } \varphi(x) > -M. \end{cases}$$

By (40) and (41), for any $t > T_1(\varepsilon) = \max\{T_+, T_-\}$ and any $w_0 \in B_0$, it holds that

$$\int_{\Omega(|u(t)|\ge M)} (|U(t)| - M)^2 dx < \frac{4C_2\,\varepsilon}{d_0\eta}. \tag{42}$$

Then

$$\int_{\Omega(u(t)|\geq \ell M)} |U(t)|^2 dx \leq 2\int_{\Omega(|u(t)|\geq M)} (|U(t)|-M)^2 dx + 2M^2 m(\Omega(|u(t)|\geq \ell M))$$
$$< \frac{8C_2\,\varepsilon}{d_0\eta} + \frac{2M^2K_0}{\ell^2M^2} < \frac{8C_2\,\varepsilon}{d_0\eta} + \varepsilon \tag{43}$$

for a sufficiently large $\ell = \ell(\varepsilon)$. Finally, since the rescaling preserves $U(t) = u(t)$, (33) is proved with $C_1 = 1 + 8C_2/(d_0\eta)$ and $M_1(\varepsilon) = \ell(\varepsilon)M(\varepsilon)$. □

Although $\|(V-M)_+\|^2_{L^2(\Omega^u_M)}$ and $\|(V+M)_-\|^2_{L^2(\Omega^u_M)}$ are involved in the grouped estimates (40) and (41) in the proof of Lemma 3.4, we cannot draw the same conclusion on $V(t,x) = v(t,x)/G$ as in (43) and (33), because unlike $(U-M)_+ = U - M$ on Ω^u_M, $(V-M)_+$ and $V-M$ (which can be nonnegative or negative) may not be identical on the subset $\Omega^u_M = \Omega(|u(t)| \geq M)$ so that

$$\int_{\Omega(|u(t)|\geq M)} (V-M)^2_+ dx \quad \text{and} \quad \int_{\Omega(|u(t)|\geq M)} (V-M)^2 dx$$

may not be equal, and same for the integrals of $(V+M)^2_-$ and $(V+M)^2$ on Ω^u_M. Thus we need a different treatment for the v-component in order to prove the following lemma.

Lemma 3.5. *For any $\varepsilon > 0$, there exists positive constants $M_2 = M_2(\varepsilon)$ and $T_2 = T_2(\varepsilon)$ such that the v-component $v(t) = v(t,x;w_0)$ of the solution of the RGSE* (3)–(4) *with the boundary condition* (5) *satisfies*

$$\int_{\Omega^u_{M_2}} |v(t)|^2 dx < C_3\,\varepsilon, \quad \text{for } t > T_2,\ w_0 = (u_0,v_0) \in \mathcal{B}_0, \tag{44}$$

where $\mathcal{B}_0$ is the absorbing set shown in Lemma 2.1, *and $C_3 = C_3(K_0)$ is a positive constant.*

Proof. We work on the sum $y(t) = u(t,x;w_0) + v(t,x;w_0)$, where (u,v) is the solution of (3)–(4) with the boundary condition (5) and the initial data $w_0 = (u_0,v_0) \in B_0$. Indeed, $y(\cdot)$ satisfies the equation

$$\frac{\partial y}{\partial t} = d_2\Delta y + (d_1 - d_2)\,\Delta u - Fy + F - ku. \tag{45}$$

Taking the inner-product $\langle (45), y(t)\rangle_{\Omega^u_M}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega^u_M)} + d_2\|\nabla y(t)\|^2_{L^2(\Omega^u_M)} + F\|y(t)\|^2_{L^2(\Omega^u_M)} \\
&= -(d_1-d_2)\int_{\Omega(|u(t)|\geq M)} \nabla u(t)\nabla y(t)\,dx + F\int_{\Omega(|u(t)|\geq M)} y(t)\,dx \\
&\quad - k\int_{\Omega(|u(t)|\geq M)} u(t)y(t)\,dx \\
&\leq \frac{F}{2}\,|\Omega^u_M| + \frac{F}{2}\|y(t)\|^2_{L^2(\Omega^u_M)} + \frac{k^2}{2F}\|u(t)\|^2_{L^2(\Omega^u_M)} + \frac{F}{2}\|y(t)\|^2_{L^2(\Omega^u_M)} \\
&\quad + \frac{d_2}{2}\|\nabla y(t)\|^2_{L^2(\Omega^u_M)} + \frac{|d_1-d_2|^2}{2d_2}\|\nabla u(t)\|^2_{L^2(\Omega^u_M)}.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|y(t)\|^2_{L^2(\Omega^u_M)} + d_2\|\nabla y(t)\|^2_{L^2(\Omega^u_M)} \le \frac{|d_1-d_2|^2}{d_2}\|\nabla u(t)\|^2_{L^2(\Omega^u_M)} + F\,|\Omega^u_M| + \frac{k^2}{F}\|u(t)\|^2_{L^2(\Omega^u_M)}. \tag{46}$$

An exponential multiplication on (46) yields

$$\frac{d}{dt}\left(e^{d_2\eta t}\|y(t)\|^2_{L^2(\Omega^u_M)}\right) \le \frac{|d_1-d_2|^2}{d_2}e^{d_2\eta t}\|\nabla u(t)\|^2_{L^2(\Omega^u_M)} + e^{d_2\eta t}\left(\frac{k^2}{F}\|u(t)\|^2_{L^2(\Omega^u_M)} + F\,|\Omega^u_M|\right). \tag{47}$$

Similar to the inequality (15), we have

$$\frac{d}{dt}\left(G\|u(t)\|^2_{L^2(\Omega^u_M)} + \|v(t)\|^2_{L^2(\Omega^u_M)}\right) + d_0\left(G\|\nabla u(t)\|^2_{L^2(\Omega^u_M)} + \|\nabla v(t)\|^2_{L^2(\Omega^u_M)}\right) \le F\,|\Omega^u_M|,$$

which implies that

$$\|u(t)\|^2_{L^2(\Omega^u_M)} \le e^{-d_0\eta t}\left(\|u_0\|^2_{L^2(\Omega^u_M)} + G^{-1}\|v_0\|^2_{L^2(\Omega^u_M)}\right) + \frac{F}{d_0\eta}\,|\Omega^u_M|. \tag{48}$$

From (47) and (48) It follows that

$$\begin{aligned}
&\|y(t)\|^2_{L^2(\Omega^u_M)} \\
&\le e^{-d_2\eta t}\|u_0+v_0\|^2_{L^2(\Omega^u_M)} + \frac{|d_1-d_2|^2}{d_2}e^{-d_2\eta t}\int_0^t e^{d_2\eta s}\|\nabla u(s)\|^2_{L^2(\Omega^u_M)}\,ds \\
&\quad + e^{-d_2\eta t}\alpha(t)\frac{k^2}{F}\left(\|u_0\|^2_{L^2(\Omega^u_M)} + G^{-1}\|v_0\|^2_{L^2(\Omega^u_M)}\right) + \frac{1}{d_2\eta}\left(\frac{k^2}{d_0\eta}+F\right)|\Omega^u_M|,
\end{aligned} \tag{49}$$

where

$$\alpha(t) = \int_0^t e^{(d_2-d_0)\eta s}\,ds \le \begin{cases} t, & \text{if } d_2 = d_0; \\ \dfrac{e^{(d_2-d_0\eta)t}}{(d_2-d_0)\,\eta}, & \text{if } d_2 > d_0; \\ \dfrac{1}{(d_0-d_2)\,\eta}, & \text{if } d_2 < d_0. \end{cases}$$

For the integral term in (49), similar to (20), we have

$$\begin{aligned}
\int_0^t e^{d_2\eta s}\|\nabla u(s)\|^2_{L^2(\Omega^u_M)}\,ds \le{}& \frac{1}{d_1 G}\left(G\|u_0\|^2_{L^2(\Omega^u_M)} + \|v_0\|^2_{L^2(\Omega^u_M)}\right) + \frac{e^{d_2\eta t}F}{d_1 d_2\eta G}\,|\Omega^u_M| \\
&+ \frac{d_2\eta}{d_1 G}\int_0^t e^{d_2\eta s}\left[e^{-d_0\eta s}\left(G\|u_0\|^2_{L^2(\Omega^u_M)} + \|v_0\|^2_{L^2(\Omega^u_M)}\right) + \frac{F}{d_0\eta}\,|\Omega^u_M|\right]ds.
\end{aligned} \tag{50}$$

Substitute (50) into (49) to get

$$
\begin{aligned}
&\|y(t)\|^2_{L^2(\Omega^u_M)} \\
\le\, & e^{-d_2\eta t}\|u_0+v_0\|^2 + e^{-d_2\eta t}\alpha(t)\frac{k^2}{F}\left(\|u_0\|^2 + G^{-1}\|v_0\|^2\right) + \frac{1}{d_2\eta}\left(\frac{k^2}{d_0\eta}+F\right)|\Omega^u_M| \\
&+ \frac{|d_1-d_2|^2}{d_2}e^{-d_2\eta t}\left[\frac{1}{d_1 G}\left(G\|u_0\|^2+\|v_0\|^2\right) + \frac{e^{d_2\eta t}F}{d_1 d_2 \eta G}|\Omega^u_M|\right] \\
&+ \frac{|d_1-d_2|^2}{d_2}e^{-d_2\eta t}\frac{d_2\eta}{d_1 G}\int_0^t e^{d_2\eta s}\left[e^{-d_0\eta s}\left(G\|u_0\|^2+\|v_0\|^2\right)+\frac{F}{d_0\eta}|\Omega|\right] ds \\
=\, & e^{-d_2\eta t}\|u_0+v_0\|^2 + e^{-d_2\eta t}\alpha(t)\frac{k^2}{F}\left(\|u_0\|^2 + G^{-1}\|v_0\|^2\right) \\
&+ \frac{|d_1-d_2|^2}{d_2}e^{-d_2\eta t}\frac{1}{d_1 G}\left(G\|u_0\|^2+\|v_0\|^2\right) \\
&+ \frac{|d_1-d_2|^2\eta}{d_1 G}e^{-d_2\eta t}\alpha(t)\left(G\|u_0\|^2+\|v_0\|^2\right) \\
&+ \left[\frac{1}{d_2\eta}\left(\frac{k^2}{d_0\eta}+F\right) + \frac{F|d_1-d_2|^2}{d_1 d_2\eta G}\left(\frac{1}{d_2}+\frac{1}{d_0}\right)\right]|\Omega^u_M|.
\end{aligned} \tag{51}
$$

Note that $e^{-d_2\eta t}\alpha(t)\to 0$ as $t\to\infty$. Therefore, there exists two positive constant $C_4 = C_4(K_0)$ and C_5, where K_0 is determined by the absorbing set $\mathcal{B}_0$ in (12), such that

$$
\|y(t)\|^2_{L^2(\Omega^u_M)} \le C_4\, e^{-d_2\eta t}(1+\alpha(t)) + C_5\,|\Omega^u_M|, \tag{52}
$$

which implies that there exist sufficiently large $T_2(\varepsilon)$ and $M_2(\varepsilon) = M$ given by (34), such that

$$
\int_{\Omega^u_{M_2}} |y(t)|^2 dx < (C_4+C_5)\,\varepsilon, \quad \text{for } t > T_2,\ w_0=(u_0,v_0)\in B_0. \tag{53}
$$

Finally, since $v(t) = y(t) - u(t)$, combining (33) and (53), the inequality (44) is valid with $C_3 = 2(C_1 + C_4 + C_5)$. Thus the lemma is proved. □

By Lemmas 3.4 and 3.5, we have proved that the condition (30) in Lemma 3.3 is satisfied with

$$
C = C_1 + C_3, \quad M = \max\{M_1(\varepsilon), M_2(\varepsilon)\}, \quad \text{and} \quad T = \max\{T_1(\varepsilon), T_2(\varepsilon)\},
$$

by the solution semiflow $\{S(t)\}_{t\ge 0}$ of the reversible Gray–Scott evolutionary equation (9).

4. **The existence of global attractor.** In this section we shall check that the κ-contraction condition (31) in Lemma 3.3 is also satisfied by the solution semiflow of the reversible Gray–Scott evolutionary equation (9). Then by Lemma 3.3, the existence of a global attractor is proved for the reversible Gray–Scott semiflow $\{S(t)\}_{t\ge 0}$.

Lemma 4.1. *For any given $M>0$, the solution semiflow of the reversible Gray-Scott evolutionary equation* (9) *satisfies the κ-contracting property* (31),

$$
\kappa\left((S(t)\mathcal{B}_0)_{\Omega(|u(t)|<M)}\right) \longrightarrow 0 \ \text{ as } t\to\infty,
$$

where $\mathcal{B}_0$ *is the absorbing set in* (12) *and* $(S(t)\mathcal{B}_0)_{\Omega(|u(t)|<M)}$ *is defined in* (32).

Proof. Taking the inner-product $\langle (3), -\Delta u\rangle_{\Omega_{u,M}}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2_{\Omega_{u,M}} + d_1\|\Delta u(t)\|^2_{\Omega_{u,M}} \\
&= -(F+k)\|\nabla u(t)\|^2_{\Omega_{u,M}} - \int_{\Omega_{u,M}} u^2 v\Delta u\, dx + G\int_{\Omega_{u,M}} u^3\Delta u\, dx \\
&\le GM^3\int_{\Omega_{u,M}} |\Delta u(t)|\, dx - (F+k)\|\nabla u(t)\|^2_{\Omega_{u,M}} + M^2\int_{\Omega_{u,M}} |v(t)||\Delta u(t)|\, dx.
\end{aligned}$$

By the Cauchy–Schwarz inequality and a subsequent cancellation of the terms $d_1\|\Delta u(t)\|^2_{\Omega_{u,M}}$ on both sides, it follows that

$$\begin{aligned}
\frac{d}{dt}\|\nabla u(t)\|^2_{\Omega_{u,M}} + 2(F+k)\|\nabla u(t)\|^2_{\Omega_{u,M}} &\le \frac{G^2M^6}{d_1}|\Omega_{u,M}| + \frac{M^4}{d_1}\|v(t)\|^2_{\Omega_{u,M}} \\
&\le \frac{1}{d_1}\left(G^2M^6|\Omega| + M^4K_0\right),
\end{aligned} \tag{54}$$

for $w_0 \in \mathcal{B}_0$ and $t > T_0$, where $T_0 = T_0(B_0)$ is given in the beginning of the proof of Lemma 3.4.

From (15) we can get that, for $t \ge 0$ and any $w_0 = (u_0, v_0) \in H$,

$$\int_t^{t+1} \|\nabla u(t)\|^2_{\Omega_{u,M}}\, ds \le \int_t^{t+1} \|\nabla u(t)\|^2\, ds \le \frac{1}{d_1 G}\left[\left(G\|u(t)\|^2 + \|v(t)\|^2\right) + F|\Omega|\right].$$

Here for any initial data $(u_0, v_0) \in \mathcal{B}_0$, it holds that

$$\int_t^{t+1} \|\nabla u(t)\|^2_{\Omega_{u,M}}\, ds \le \frac{1}{d_1 G}\left((G+1)K_0 + F|\Omega|\right), \quad \text{for } t > T_0. \tag{55}$$

By the uniform Gronwall inequality [39,43], from (54) and (55) we can deduce that

$$\|\nabla u(t)\|^2_{\Omega_{u,M}} \le e^{2(F+k)}\left(C_6 + C_7\right), \quad \text{for } t > T_0 + 1,\ w_0 \in \mathcal{B}_0, \tag{56}$$

where

$$C_6 = \frac{1}{d_1 G}\left((G+1)K_0 + F|\Omega|\right)$$

and

$$C_7 = \frac{1}{d_1}\left(G^2M^6|\Omega| + M^4K_0\right)$$

are positive constants, and C_7 depends on the given constant M.

Next we can take inner-product $\langle (4), -\Delta v(t)\rangle_{\Omega_{u,M}}$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2_{\Omega_{u,M}} + d_2\|\Delta v(t)\|^2_{\Omega_{u,M}} \\
&= -\int_{\Omega_{u,M}} F\Delta v\, dx - F\|\nabla v(t)\|^2_{\Omega_{u,M}} + \int_{\Omega_{u,M}} u^2 v\Delta v\, dx - G\int_{\Omega_{u,M}} u^3\Delta v\, dx \\
&\le \left(GM^3 + F\right)\int_{\Omega_{u,M}} |\Delta v(t)|\, dx - F\|\nabla v(t)\|^2_{\Omega_{u,M}} + M^2\int_{\Omega_{u,M}} |v(t)||\Delta v(t)|\, dx.
\end{aligned}$$

Similarly by using the Cauchy–Schwarz inequality and the uniform Gronwall inequality we can prove that there exists a uniform constant $C_8 = C_8(K_0, M) > 0$ such that

$$\|\nabla v(t)\|^2_{\Omega_{u,M}} \le C_8, \quad \text{for } t > T_0 + 1,\ w_0 \in \mathcal{B}_0 \tag{57}$$

Due to the compact Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ for space dimension $n \le 3$, (56) and (57) together show that for any fixed $t > T_0 + 1$,

$$(S(t)\mathcal{B}_0)_{\Omega(|u(t)|<M)} \text{ is a precompact set in } L^2(\Omega).$$

Therefore, by the property of Kuratowski measure, in the product phase space H we have

$$\kappa\left((S(t)\mathcal{B}_0)_{\Omega(|u(t)|<M)}\right) = 0, \quad \text{for } t \ge T_0 + 1.$$

Thus the lemma is proved. □

Now we can prove the first main result stated in the following theorem.

Theorem 4.2. *For any positive parameters d_1, d_2, F, k and G, there exists a global attractor $\mathscr{A}$ in H for the solution semiflow $\{S(t)\}_{t\ge 0}$ generated by the reversible Gray–Scott equations* (3)–(4) *with the Dirichlet boundary condition* (5). *Moreover, the global attractor $\mathscr{A}$ has a finite Hausdorff dimension $d_H(\mathscr{A})$ and a finite fractal dimension $d_{\mathcal{F}}(\mathscr{A})$.*

Proof. By Lemma 2.1, the solution semiflow $\{S(t)\}_{t\ge 0}$ of the reversible Gray–Scott equations (3)–(5) has a bounded absorbing set $\mathcal{B}_0$ in H and the condition (i) in Lemma 3.3 is satisfied. Then by Lemma 3.4, Lemma 3.5, and Lemma 4.1 this solution semiflow $\{S(t)\}_{t\ge 0}$ satisfies the conditions (30) and (31), so that the condition (ii) in Lemma 3.3 is also satisfied. Thus by Lemma 3.3, there exists a global attractor $\mathscr{A}$ in H for the reversible Gray–Scott semiflow $\{S(t)\}_{t\ge 0}$.

The proof of the finite dimensionality $d_H(\mathscr{A}) < \infty$ and $d_{\mathcal{F}}(\mathscr{A}) < \infty$ is parallel to the proof of the corresponding results in [52, Theorem 3] and [54, Theorem 2], based on the $L^4(\Omega) \times L^4(\Omega)$ boundedness of the global attractor $\mathscr{A}$ (which is the consequence of Lemma 2.2 and the invariance of $\mathscr{A}$), and shown by the Kaplan–Yorke formula and the estimation of Lyapunov exponents by the global trace formula. Here the detail of that proof is omitted. □

Furthermore, based on Lemma 2.3 and the invariance property of the global attractor, we can prove the following result on the regularity of the global attractor.

Corollary 4.3. *The global attractor $\mathscr{A}$ of the reversible Gray–Scott semiflow $\{S(t)\}_{t\ge 0}$ is a bounded set in the space $H_\infty = L^\infty(\Omega) \times L^\infty(\Omega)$.*

The proof of this corollary is similar to the proof of Lemma 19 in [53] combined with the invariance of the global attractor $\mathscr{A}$. The detail is also omitted.

5. The (H, E) global attractor.

The following definition of (X, Y) global attractor for a semiflow on a Banach space X, where Y is a compactly imbedded subspace of X, was initially introduced by A. V. Babin and M. I. Vishik [2,3]. This concept actually plays a role of bridge linking a global attractor and an exponential attractor in X, as will be shown in this paper.

Definition 5.1. Let $\{S(t)\}_{t\ge 0}$ be a semiflow on a Banach space X. Let Y be a compactly imbedded subspace of X, denoted by $Y \hookrightarrow X$. A subset $\mathscr{A}$ of Y is called an (X, Y) global attractor for this semiflow, if $\mathscr{A}$ has the following properties:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in Y.

(ii) $\mathscr{A}$ attracts any bounded set $B \subset X$ with respect to the Y-norm, namely, there is a finite time $\tau = \tau(B)$ such that $S(t)B \subset Y$ for $t > \tau$ and

$$\operatorname{dist}_Y(S(t)B, \mathscr{A}) = \sup_{x\in B} \inf_{y\in\mathscr{A}} \|S(t)x - y\|_Y \to 0, \quad \text{as } t \to \infty.$$

The following two lemmas address the ultimate boundedness of trajectories in E and the (H,E) absorbing property of the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$.

Lemma 5.2. *The reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$ has the following property: for any $R>0$ there exists a constant $M(R)>0$ such that if the initial data $w_0=(u_0,v_0)\in E$ and $\|w_0\|_E^2\leq R$, then $S(t)w_0=(u(t,\cdot),v(t,\cdot))\in E$ for all $t\geq 0$, and*

$$\|S(t)w_0\|_E^2 \leq M(R), \quad \text{for } t\geq 0. \tag{58}$$

Proof. Taking the $L^2(\Omega)$ inner-product $\langle(3),-\Delta u\rangle$ and $\langle(4),-\Delta v\rangle$, by the homogeneous Dirichlet boundary condition, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 + (F+k)\|\nabla u\|^2 &= -\int_\Omega u^2 v\Delta u\,dx + \int_\Omega G u^3\Delta u\,dx \\
&\leq \left(\frac{d_1}{2}+\frac{d_1}{2}\right)\|\Delta u\|^2 + \frac{1}{2d_1}\int_\Omega \left(u^4v^2+G^2u^6\right)dx \\
&\leq d_1\|\Delta u\|^2 + \frac{1}{2d_1}\left(\frac{2}{3}\int_\Omega u^6\,dx + \frac{1}{3}\int_\Omega v^6\,dx + \int_\Omega G^2u^6\,dx\right) \\
&= d_1\|\Delta u\|^2 + \frac{1}{6d_1}\left(\left(2+3G^2\right)\int_\Omega u^6\,dx + \int_\Omega v^6\,dx\right), \quad t>0,
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 + F\|\nabla v\|^2 &= -\int_\Omega F\Delta v\,dx + \int_\Omega u^2v\Delta v\,dx - \int_\Omega Gu^3\Delta v\,dx \\
&\leq \left(\frac{d_2}{2}+\frac{d_2}{2}\right)\|\Delta v\|^2 + \frac{1}{2d_2}\int_\Omega\left(u^4v^2+G^2u^6\right)dx \\
&\leq d_2\|\Delta v\|^2 + \frac{1}{6d_2}\left(\left(2+3G^2\right)\int_\Omega u^6\,dx + \int_\Omega v^6\,dx\right), \quad t>0.
\end{aligned} \tag{60}$$

By Lemma 2.2 and (28), we have

$$\int_\Omega u^6(t,x)\,dx + \int_\Omega v^6(t,x)\,dx \leq K_2 e^{-Ft}\left(\int_\Omega u_0^6\,dx + \int_\Omega v_0^6\,dx\right) + K_1, \quad t\geq 0, \tag{61}$$

for any $w_0=(u_0,v_0)\in E$, where K_1 is the positive constant given in Lemma 2.2 and (29), and

$$K_2 = \max\left\{1, G^{-5}\right\}/\min\left\{1,G^{-5}\right\}.$$

Substituting (61) into (59) and (60), we get

$$\begin{aligned}
\frac{d}{dt}\|\nabla u\|^2 + 2(F+k)\|\nabla u\|^2 &\leq \frac{2+3G^2}{3d_1}\left(K_2e^{-Ft}\left(\int_\Omega u_0^6\,dx + \int_\Omega v_0^6\,dx\right)+K_1\right), \\
\frac{d}{dt}\|\nabla v\|^2 + 2F\|\nabla v\|^2 &\leq \frac{2+3G^2}{3d_2}\left(K_2e^{-Ft}\left(\int_\Omega u_0^6\,dx + \int_\Omega v_0^6\,dx\right)+K_1\right),
\end{aligned} \tag{62}$$

for $t\geq 0$ and $w_0=(u_0,v_0)\in E$. By Gronwall inequality and obviously $F+k>F$, (62) implies that

$$\|(\nabla u(t,\cdot),\nabla v(t,\cdot))\|^2 \leq \frac{2K_2\left(2+3G^2\right)e^{-Ft}}{3d_0F}\left(\int_\Omega u_0^6\,dx + \int_\Omega v_0^6\,dx\right) + K_3(R), \tag{63}$$

where

$$K_3(R) = R + \frac{K_1\left(2+3G^2\right)}{3d_0F}.$$

Finally, combining (63) and (16) we conclude that, for $t \geq 0$ and any $(u_0, v_0) \in E$,

$$\begin{aligned}\|(u(t,\cdot),v(t,\cdot)\|_E^2 \leq &e^{-Ft}\left[\frac{\max\{1,G\}}{\min\{1,G\}}\|(u_0,v_0)\|^2 + \frac{2K_2\left(2+3G^2\right)\gamma^6}{3d_0F}\|(u_0,v_0)\|_E^6\right] \\ &+ K_0 + K_3(R),\end{aligned} \tag{64}$$

where $\gamma > 0$ is the universal constant of the continuous imbedding inequality

$$\|w\|_{L^6(\Omega)\times L^6(\Omega)} \leq \gamma\|w\|_E. \tag{65}$$

Therefore, if $\|w_0\|_E^2 = \|w_0\|^2 + \|\nabla w_0\|^2 \leq R$, then the strong solution $S(t)w_0 = (u(t,\cdot), v(t,\cdot)) \in E$ for all $t \geq 0$, and

$$\|S(t)w_0\|_E^2 \leq M(R), \quad \text{for } t \geq 0,$$

where $M(R)$ is the constant given by

$$M(R) = \frac{\max\{1,G\}}{\min\{1,G\}}R + \frac{2K_2\left(2+3G^2\right)\gamma^6}{3d_0F}R^3 + K_0 + K_3(R). \tag{66}$$

Thus (58) is proved. □

Lemma 5.3. *For the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$, there exists a universal constant $K^* > 0$ with the property that for any $R > 0$ there is a $T(R) > 0$ such that if $w_0 = (u_0, v_0) \in H$ with $\|w_0\|^2 \leq R$, then the solution of the reversible Gray–Scott evolutionary equation (9) satisfies $S(t)w_0 \in E$ for $t \geq T(R)$ and*

$$\|S(t)w_0\|_E^2 = \|(u(t,\cdot),v(t,\cdot)\|_E^2 \leq K^*, \quad \text{for } t \geq T(R). \tag{67}$$

Proof. From the inequality (15), for any $t_0 > 0$ we have

$$\begin{aligned}&2d_0\int_{t_0}^{t_0+1}\left(\|\nabla u(s,\cdot)\|^2 + \|\nabla v(s,\cdot)\|^2\right)ds \\ \leq &\frac{\max\{1,G\}}{\min\{1,G\}}\|(u(t_0,\cdot),v(t_0,\cdot))\|^2 + \frac{F|\Omega|}{\min\{1,G\}}.\end{aligned} \tag{68}$$

By the absorbing property shown in Lemma 2.1, there exists a time $t_0(R) > 0$ such that for any $w_0 = (u_0, v_0) \in H$ with $\|w_0\|^2 \leq R$ we have

$$\|(u(t,\cdot),v(t,\cdot))\|^2 \leq K_0, \quad \text{for } t \geq t_0(R). \tag{69}$$

According to the Mean Value Theorem, since the solution regularity of the parabolic evolutionary equation (9) implies that $(u, v) \in C([t_0(R), \infty), E)$, by (68) there exists a time $t_1(R) \in [t_0(R), t_0(R) + 1]$ such that

$$\|\nabla(u(t_1(R),\cdot),v(t_1(R),\cdot))\|^2 \leq \frac{1}{2d_0}\left(\frac{\max\{1,G\}}{\min\{1,G\}}K_0 + \frac{F|\Omega|}{\min\{1,G\}}\right). \tag{70}$$

Combining (69) and (70) we get

$$\|(u(t_1(R),\cdot),v(t_1(R),\cdot))\|_E^2 \leq K_0 + \frac{1}{2d_0}\left(\frac{\max\{1,G\}}{\min\{1,G\}}K_0 + \frac{F|\Omega|}{\min\{1,G\}}\right). \tag{71}$$

Finally, we utilize Lemma 5.2 to conclude that for $T(R) = t_0(R) + 1$,

$$\|(u(t,\cdot), v(t,\cdot)\|_E^2 \le M\left(K_0 + \frac{1}{2d_0}\left[\frac{\max\{1,G\}}{\min\{1,G\}}K_0 + \frac{F|\Omega|}{\min\{1,G\}}\right]\right), \quad t \ge T(R),$$

where $M(\cdot)$ is the function given in (58) or specifically by (66). Thus (67) is proved with K^* given by

$$K^* = M\left(K_0 + \frac{1}{2d_0}\left[\frac{\max\{1,G\}}{\min\{1,G\}}K_0 + \frac{F|\Omega|}{\min\{1,G\}}\right]\right), \tag{72}$$

which is a universal positive constant. The proof is completed. □

Lemma 5.4. *Let $\{w_m\}_{m=1}^\infty$ be a sequence in E. Assume that $\{w_m\}$ converges to $w_0 \in E$ weakly in E and $\{w_m\}$ converges to w_0 strongly in H, as $m \to \infty$. Then the sequence of solutions $\{S(t)w_m\}_{m=1}^\infty$ of the Gray-Scott evolutionary equation* (9) *with the initial data $S(0)w_m = w_m$ has the convergence property that for each $t > 0$,*

$$\lim_{m\to\infty} S(t)w_m = S(t)w_0 \ \text{ strongly in } E \tag{73}$$

and the convergence is uniform on any compact interval $[t_0, t_1] \subset (0, \infty)$.

Proof. Let $[t_0, t_1]$ be any fixed compact interval in $(0,\infty)$. Since the weakly convergent sequence $\{w_m\}$ is a bounded set in E, by Lemma 5.2 there exists a constant $K > 0$ such that

$$\|S(t)w_m\|_E \le K \quad \text{and} \quad \|S(t)w_0\|_E \le K. \tag{74}$$

Let N_K be the bounded ball in the space E of radius K centered at the origin. Let $L > 0$ be a Lipschitz constant of the Nemytskii mapping $f : E \to H$ defined by (8) relative to this bounded ball N_K. Since any strong solution of (9) is a mild solution, we have

$$S(t)w_m = e^{At}w_m + \int_0^t e^{A(t-\sigma)} f\left(S(\sigma)w_m\right) d\sigma, \quad t \in [0, t_1], \ \ m = 0, 1, 2, \cdots. \tag{75}$$

By the smoothing property of the analytic C_0-semigroup $\left\{e^{At}\right\}_{t\ge 0}$, cf. [39, Theorem 37.5], there is a constant $M_1 > 0$ such that

$$\left\|e^{At}\right\|_{\mathcal{L}(H,E)} \le M_1 t^{-1/2}, \quad \text{for } t > 0,$$

since the generator $A : D(A)(= Z) \to H$ is a nonpositive, self-adjoint operator and $E = D\left((-A)^{1/2}\right)$. Thus, from (75) it follows that

$$\begin{aligned}\|S(t)w_m - S(t)w_0\|_E &\le M_1 t^{-1/2}\|w_m - w_0\|_H \\ &\quad + M_1 L \int_0^t (t-\sigma)^{-1/2}\|S(\sigma)w_m - S(\sigma)w_0\|_E \, d\sigma, \ \ t \in [0, t_1].\end{aligned}$$

By the Gronwall–Henry inequality, cf. [19, Lemma 7.1.1] and [39, Lemma D.4], the above inequality implies that there exists a positive constant $C(M_1, L, t_1)$ such that

$$\begin{aligned}
&\|S(t)w_m - S(t)w_0\|_E \le M_1 t^{-1/2} \|w_m - w_0\|_H \\
&+ C(M_1, L, t_1) \int_0^t \left(1 + (t-\sigma)^{-1/2}\right) \left(M_1 \sigma^{-1/2} \|w_m - w_0\|_H\right) d\sigma \\
=&M_1 \left(t^{-1/2} + C(M_1, L, t_1) \int_0^t \left(1 + (t-\sigma)^{-1/2}\right) \sigma^{-1/2} d\sigma\right) \|w_m - w_0\|_H \\
=&M_1 \left(t^{-1/2} + C(M_1, L, t_1) \left[2t^{1/2} + \int_0^t (t-\sigma)^{-1/2} \sigma^{-1/2} d\sigma\right]\right) \|w_m - w_0\|_H \\
=&M_1 \left(t^{-1/2} + C(M_1, L, t_1) \left[2t^{1/2} + \Gamma(1/2)^2\right]\right) \|w_m - w_0\|_H \\
\le&C^*(M_1, L, t_1) \left(1 + t^{-1/2}\right) \|w_m - w_0\|_H,
\end{aligned} \tag{76}$$

where $\Gamma(\cdot)$ is the Gamma function (here we used $B(1/2, 1/2) = |\Gamma(1/2)|^2 = \pi$), and $C^*(M_1, L, t_1)$ is a positive constant given by

$$C^*(M_1, L, t_1) = M_1 \max\left\{1, C(M_1, L, t_1)\left(2t_1^{1/2} + \pi\right)\right\}.$$

By the condition that

$$\|w_m - w_0\|_H \to 0, \quad \text{as } m \to \infty,$$

from (76) it follows that, on any compact interval $[t_0, t_1] \subset (0, \infty)$,

$$\sup_{t \in [t_0, t_1]} \|S(t)w_m - S(t)w_0\|_E \le C^*(M_1, L, t_1)\left(1 + t_0^{-1/2}\right) \|w_m - w_0\|_H \to 0, \tag{77}$$

as $m \to \infty$. Thus the proof is completed. □

Based on the above lemmas, we can show the asymptotic compactness of the solution semiflow $\{S(t)\}_{t \ge 0}$ of the reversible Gray-Scott evolutionary equation (9) not only in the space H as already proved in Section 3 and Section 4 but also in the space E.

Lemma 5.5. *The reversible Gray-Scott semiflow $\{S(t)\}_{t \ge 0}$ is asymptotically compact in E with respect to the strong topology of E.*

Proof. Let $T > 0$ be arbitrarily given. For any time sequence $\{t_n\}_{n=1}^\infty$, $t_n \to \infty$, and any bounded sequence $\{w_n\}_{n=1}^\infty \subset E$, there is an integer $n_0 \ge 1$ such that $t_n > T$ for all $n \ge n_0$.

Since $\{w_n\}_{n=1}^\infty$ is bounded in E, by Lemma 5.2,

$$\{S(t_n - T) w_n\}_{n \ge n_0} \text{ is also a bounded set in } E. \tag{78}$$

Since E is a Hilbert space, (78) implies that there is an increasing sequence of integers $\{n_k\}_{k=1}^\infty$, with $n_k \ge n_0$, such that

$$(\text{weak}) \lim_{k \to \infty} S(t_{n_k} - T) w_{n_k} = w^* \in E. \tag{79}$$

By the fact that $E \hookrightarrow H$ is a compact imbedding, we can choose a further subsequence of $\{n_k\}_{k=1}^\infty$, but relabel it as the same as $\{n_k\}_{k=1}^\infty$, such that

$$(\text{strong}) \lim_{k \to \infty} S(t_{n_k} - T) w_{n_k} = w^* \in H. \tag{80}$$

Then by Lemma 5.4, we obtain the convergence

$$\text{(strong)} \lim_{k\to\infty} S(t_{n_k})\, w_{n_k} = \text{(strong)} \lim_{k\to\infty} S(T)\left[S(t_{n_k} - T)\, w_{n_k}\right] = S(T)w^* \in E,$$

with respect to the strong topology of E. According to Definition 1.5, the lemma is proved. □

Now we can assemble these lemmas to prove the existence of an (H, E) global attractor and to identify it in the following theorem. This is the second main result of this paper.

Theorem 5.6. *For the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$, the global attractor $\mathscr{A}$ in H is an (H, E) global attractor.*

Proof. By Lemma 5.2 and Lemma 5.3, there exists a bounded absorbing set $\mathcal{B}_1 \subset E$ for the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$ and the absorbing is in the E-norm. By Lemma 5.5, the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact in E. Therefore, by the basic theory shown in Proposition 1.6, there exists a global attractor $\mathscr{A}_E$ for the reversible Gray–Scott semiflow in the space E.

According to Definition 5.1, Lemma 5.3 also shows that this global attractor $\mathscr{A}_E$ is an (H, E) global attractor. Since $\mathscr{A}_E$ attracts any bounded set in H with respect to the E-norm and, on the other hand, the global attractor $\mathscr{A}$ is a bounded set in H, we have

$$\mathscr{A}_E \text{ attracts } \mathscr{A} \text{ in } E. \tag{81}$$

Since $\mathscr{A}$ is an invariant set, (81) implies that $\mathscr{A} \subset \mathscr{A}_E$. Moreover, by definition we also have

$$\mathscr{A} \text{ attracts } \mathscr{A}_E \text{ in } H. \tag{82}$$

Since $\mathscr{A}_E$ is also an invariant set, (82) implies that $\mathscr{A}_E \subset \mathscr{A}$. Therefore, it holds that $\mathscr{A}_E = \mathscr{A}$. The proof is completed. □

Corollary 5.7. *For the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$, there exists a compact, positively invariant, absorbing set $\mathcal{B}_E$ in H.*

Proof. As indicated in the proof of Theorem 5.6, by the definition of the (H, E) attractor $\mathscr{A}_E$, there exists a bounded absorbing set $\mathcal{B}_1 \subset E$, which not only absorbs any bounded set in E but also absorbs any bounded set in H as well. Thus there is a finite time $T_E = T_E(\mathcal{B}_1) > 0$ such that

$$S(t)\mathcal{B}_1 \subset \mathcal{B}_1, \quad \text{for } t \geq T_E.$$

Let

$$\mathcal{B}_E = \bigcup_{t\in[0,T_E]} S(t)\mathcal{B}_1. \tag{83}$$

We can verify that this set $\mathcal{B}_E$ is a compact, positively invariant, absorbing set in H. Indeed, the compactness of $\mathcal{B}_E$ in H follows from the fact that $\mathcal{B}_E = \Pi([0, T_E] \times \mathcal{B}_1)$, where $\Pi(t, w) = S(t)w$ is a continuous mapping on $\mathbb{R} \times H$ and $[0, T_E] \times \mathcal{B}_1$ is a compact set in $\mathbb{R} \times H$. The positive invariance of $\mathcal{B}_E$ can be easily verified. That B_E absorbs any bounded set in H can be checked by using Lemma 5.3 and the absorbing property of $\mathcal{B}_1$. Thus the corollary is proved. □

6. The existence of an exponential attractor. In this final section, we shall prove the existence of an exponential attractor for the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$.

Definition 6.1. Let X be a real Banach space and $\{S(t)\}_{t\geq 0}$ be a semiflow on X. A set $\mathscr{E} \subset X$ is an exponential attractor for the semiflow $\{S(t)\}_{t\geq 0}$ in X, if the following conditions are satisfied:

(i) $\mathscr{E}$ is a nonempty, compact, positively invariant set in X,
(ii) $\mathscr{E}$ has a finite fractal dimension, and
(iii) $\mathscr{E}$ attracts every bounded set $B \subset X$ exponentially: there exist positive constants μ and $C(B)$ which depends on B, such that

$$\operatorname{dist}_X(S(t)B, \mathscr{E}) \leq C(B)e^{-\mu t}, \quad \text{for } t \geq 0.$$

The basic theory and construction of exponential attractors were established in [12] for discrete and continuous semiflows on Hilbert spaces. The existence theory was generalized to semiflows on Banach spaces in [10, 11] and extended to some nonlinear reaction-diffusion equations on unbounded domains in [4, 13, 14] and others. The existence of exponential attractors has also been shown for chemotaxis equations [30] and for some quasilinear parabolic equstions [25]. More references and results in this area can be found in [12, 27, 34].

Global attractors, exponential attractors, and inertial manifolds are the three major research topics in the area of infinite dimensional dynamical systems. For a continuous semiflow on a Hilbert space, if all the three objects (a global attractor $\mathscr{A}$, an exponential attractor $\mathscr{E}$, and an inertial manifold $\mathscr{M}$ of the same exponential attraction rate) exist, then the following inclusion relationship holds,

$$\mathscr{A} \subset \mathscr{E} \subset \mathscr{M}.$$

Since the structure of an exponential attractor is to some extend more constructive and informative than a global attractor and, on the other hand, the existence of an exponential attractor is less restrictive than the existence of an inertial manifolds (due to the hurdle of the spectral gap condition for the latter), in recent years more attentions have been focused on the topics of exponential attractors and its approximations.

Here we can take either the Hilbert space approach or the Banach space approach as aforementioned to show the existence of an exponential attractor for the reversible Gray–Scott semiflow $\{S(t)\}_{t\geq 0}$. To fix the idea, we shall tackle it by the argument of squeezing property [12, 27].

Definition 6.2. For a spectral (orthogonal) projection P_N relative to a nonnegative, self-adjoint, linear operator $\Lambda : D(\Lambda) \to \mathcal{H}$ with compact resolvent, which maps the Hilbert space $\mathcal{H}$ onto the N-dimensional subspace $\mathcal{H}_N$ spanned by a set of the first N eigenvectors of the operator Λ, we defined a cone

$$\mathscr{C}_{P_N} = \{y \in X : \|(I - P_N)(y)\|_{\mathcal{H}} \leq \|P_N(y)\|_{\mathcal{H}}\}.$$

A continuous mapping S_* satisfies the *discrete squeezing property* relative to a set $B \subset \mathcal{H}$ if there exist a constant $\delta \in (0, 1/2)$ and a spectral projection P_N on $\mathcal{H}$ such that for any pair of points $y_0, z_0 \in B$, if

$$S_*(y_0) - S_*(z_0) \notin \mathscr{C}_{P_N},$$

then

$$\|S_*(y_0) - S_*(z_0)\|_{\mathcal{H}} \leq \delta \|y_0 - z_0\|_{\mathcal{H}}.$$

We first present the following lemma, which is a modified version of the basic result [27, Theorem 4.5] on the sufficient conditions for the existence of an exponential attractor of a semiflow on a Hilbert space. In some sense, this lemma provides a more accessible way to check these sufficient conditions if we are sure there exists an (W, V) global attractor, such as the (H, E) global attractor for the reversible Gray–Scott semiflow in this paper.

Lemma 6.3. *Let W be a real Banach space and V be a compactly embedded subspace of W. Consider a semilinear evolutionary equation*

$$\frac{dw}{dt} + \Lambda w = g(w), \quad t > 0, \tag{84}$$

where $\Lambda : D(\Lambda) \to W$ is a nonnegative, self-adjoint, linear operator with compact resolvent, and $g : V(= D(\Lambda^{1/2})) \to W$ is a locally Lipschitz continuous mapping. Suppose that the weak solution of (84) for each initial point $w(0) = w_0 \in W$ uniquely exists for all $t \geq 0$ and it turns out to be a strong solution for $t > 0$, which altogether form a semiflow denoted by $\{\Phi(t)\}_{t\geq 0}$. Assume that the following four conditions are satisfied:

(i) *There exist a compact, positively invariant, absorbing set $\mathcal{B}_c$ in W.*
(ii) *There is a positive integer N such that the norm quotient $Q(t)$ defined by*

$$Q(t) = \frac{\left\|\Lambda^{1/2}\left(w_1(t) - w_2(t)\right)\right\|_W^2}{\|w_1(t) - w_2(t)\|_W^2} \tag{85}$$

for any two trajectories $w_1(\cdot)$ and $w_2(\cdot)$ starting from the set $\mathcal{B}_c \backslash \mathscr{C}_{P_N}$ satisfies a differential inequality

$$\frac{dQ}{dt} \leq \rho\left(\mathcal{B}_c\right) Q(t), \quad t > 0,$$

where $\rho\left(\mathcal{B}_c\right)$ is a positive constant only depending on $\mathcal{B}_c$.
(iii) *For any given finite $T > 0$ and any given $w \in \mathcal{B}_c$, $\Phi(\cdot)w : [0, T] \to \mathcal{B}_c$ is Hölder continuous with exponent $\theta = 1/2$ and the coefficient of Hölder continuity, $K(w) : \mathcal{B}_c \to (0, \infty)$, is a bounded function.*
(iv) *For any $t \in [0, T]$ where $T > 0$ is arbitrarily given, $\Phi(t)(\cdot) : \mathcal{B}_c \to \mathcal{B}_c$ is Lipschitz continuous and the Lipschitz constant $L(t) : [0, T] \to (0, \infty)$ is a bounded function.*

Then there exists an exponential attractor $\mathscr{E}$ in W for this semiflow $\{\Phi(t)\}_{t\geq 0}$.

Proof. The proof of this lemma can be made parallel to [27, Theorem 4.5] except the following modifications. All the other details are omitted here.

First modification is that here the condition (ii) on the norm quotient $Q(t)$ implies that the semiflow $\{\Phi(t)\}_{t\geq 0}$ satisfies the discrete squeezing property (instead of directly assuming this property) relative to the nonempty, compact, positively invariant set $\mathcal{B}_c$ in W. The proof of this implication can be seen in the proof of [27, Theorem 4.7] or in the proof of [12, Proposition 3.1].

Second modification is that the condition (iii) requires the uniform Hölder continuity with exponent $\theta = 1/2$ for the mapping $\Phi(\cdot)w : [0, T] \to \mathcal{B}_c$ instead of the uniform Lipschitz continuity. This modification works through because the condition (iii) is used only in proving that the fractal dimension of the set $\Phi\left([0, t_*] \times \mathscr{E}_*\right)$ is finite, cf. [27, Theorem 4.5], by using the fractal dimension $(d_{\mathcal{F}})$ property

$$d_{\mathcal{F}}\left(\Phi\left([0, t_*] \times \mathscr{E}_*\right)\right) \leq d_{\mathcal{F}}\left(\Phi\left([0, t_*] \,|w \in \mathscr{E}_*\right)\right) + d_{\mathcal{F}}\left(\Phi\left(\mathscr{E}_* | t \in [0, T]\right)\right),$$

where $\mathscr{E}_*$ is the exponential attractor for the discrete semiflow generated by the mapping $S(t_*)$. Originally the uniform Lipschitz continuity implies that

$$d_{\mathcal{F}}\left(\Phi\left([0,t_*]\left|w\in\mathscr{E}_*\right.\right)\right)\le d_{\mathcal{F}}\left([0,t_*]\right).$$

Here, by the following auxiliary lemma, the uniform Hölder continuity with exponent 1/2 can also guarantee $d_{\mathcal{F}}\left(\Phi\left([0,t_*]\left|w\in\mathscr{E}_*\right.\right)\right)$ to be finite. □

Lemma 6.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces and let $\mathcal{K}$ be any given compact set in $\mathcal{X}$. If a continuous mapping $\Psi:\mathcal{X}\to\mathcal{Y}$ is uniformly Hölder continuous with exponent $\theta=1/2$ on the set $\mathcal{K}$, then the fractal dimension of the image set $\Psi(\mathcal{K})$ in $\mathcal{Y}$ is finite.*

Proof. Let $0<\delta<1$. One can cover the set $\mathcal{K}$ with exactly the smallest number $N_\delta(\mathcal{K})$ of balls of diameter δ. By the described uniform Hölder continuity, the image set $\Psi(\mathcal{K})$ can be covered by $N_\delta^2(\mathcal{K})$ balls with diameter $\sqrt{\delta}$. Hence, the smallest number $N_{\sqrt{\delta}}(\Psi(\mathcal{K}))$ of balls of diameter at most $\sqrt{\delta}$ that can cover $\Psi(\mathcal{K})$ satisfies

$$N_{\sqrt{\delta}}(\Psi(\mathcal{K}))\le N_\delta^2(\mathcal{K}).$$

Therefore, by definition of fractal dimension, we have

$$\begin{aligned} d_{\mathcal{F}}(\Psi(\mathcal{K})) &= \limsup_{\sqrt{\delta}\to 0}\frac{\log N_{\sqrt{\delta}}(\Psi(\mathcal{K}))}{-\log\sqrt{\delta}} \le \limsup_{\delta\to 0}\frac{\log\delta}{\log\sqrt{\delta}}\cdot\frac{\log N_\delta^2(\mathcal{K})}{-\log\delta} \\ &= 2\limsup_{\delta\to 0}\frac{\log N_\delta^2(\mathcal{K})}{-\log\delta} = 4\,d_{\mathcal{F}}(\mathcal{K})<\infty, \end{aligned}$$

which proves the conclusion. □

The next theorem is the third main result in this paper.

Theorem 6.5. *For any positive parameters d_1, d_2, F, k and G, there exists an exponential attractor $\mathscr{E}$ in H for the solution semiflow $\{S(t)\}_{t\ge 0}$ of the evolutionary equation* (9) *generated by the reversible Gray–Scott equations* (3)–(4) *on a bounded domain of space dimension $n\le 3$ with the Dirichlet boundary conditions* (5).

Proof. By Theorem 5.6, there exists an (H,E) global attractor $\mathscr{A}$, which is exactly the global attractor of the reversible Gray-Scott semiflow $\{S(t)\}_{t\ge 0}$ in H. By Corollary 5.7, there exists a compact, positively invariant, absorbing set $\mathcal{B}_E$ in H for this semiflow.

Next we prove that the second condition in Lemma 6.3 is satisfied for this reversible Gray-Scott semiflow. Consider any two points $w_1(0),w_2(0)\in\mathcal{B}_E$ and let $w_i(t)=(u_i(t),v_i(t))$, $i=1,2$, be the corresponding solutions, respectively. Let $y(t)=w_1(t)-w_2(t)$, $t\ge 0$. The associated norm quotient of (w_1,w_2), where $w_1(0)\ne w_2(0)$, is given by

$$Q(t)=\frac{\left\|(-A)^{1/2}y(t)\right\|^2}{\|y(t)\|^2},\quad t\ge 0.$$

Directly we can calculate

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}Q(t) &= \frac{1}{\|y(t)\|^4}\left[\langle(-A)^{1/2}y(t),(-A)^{1/2}y_t\rangle\|y(t)\|^2 - \left\|(-A)^{1/2}y(t)\right\|^2\langle y(t),y_t\rangle\right] \\
&= \frac{1}{\|y(t)\|^2}\left[\langle(-A)y(t),y_t\rangle - Q(t)\langle y(t),y_t\rangle\right] \\
&= \frac{1}{\|y(t)\|^2}\langle(-A)y(t)-Q(t)y(t), Ay(t)+f(w_1(t))-f(w_2(t))\rangle \\
&= \frac{1}{\|y(t)\|^2}\langle(-A)y(t)-Q(t)y(t), Ay(t)+Q(t)y(t)+f(w_1(t))-f(w_2(t))\rangle \\
&= \frac{1}{\|y(t)\|^2}\left[-\|Ay(t)+Q(t)y(t)\|^2 - \langle Ay(t)+Q(t)y(t), f(w_1(t))-f(w_2(t))\rangle\right] \\
&\le \frac{1}{\|y(t)\|^2}\left(-\frac{1}{2}\|Ay(t)+Q(t)y(t)\|^2 + \frac{1}{2}\|f(w_1(t))-f(w_2(t))\|\right)
\end{aligned}
\tag{86}
$$

where we used the identity

$$-\langle Ay(t)+Q(t)y(t), Q(t)y(t)\rangle = 0.$$

Note that $\mathcal{B}_E$ is a bounded set in E and that $E \hookrightarrow L^6(\Omega)\times L^6(\Omega)$ is a continuous imbedding so that there is a universal constant $R>0$ only depending on $\mathcal{B}_E$ such that

$$\|(u,v)\|^2_{L^6(\Omega)} \le R, \quad \text{for any } (u,v)\in\mathcal{B}_E. \tag{87}$$

We have

$$
\begin{aligned}
\|f(w_1(t))-f(w_2(t))\| \le{}& \left\|-(F+k)(u_1-u_2)+\left(u_1^2v_1-u_2^2v_2\right)-G\left(u_1^3-u_2^3\right)\right\| \\
&+\left\|-F(v_1-v_2)-\left(u_1^2v_1-u_2^2v_2\right)+G\left(u_1^3-u_2^3\right)\right\|.
\end{aligned}
\tag{88}
$$

Using the Hölder inequality, the imbedding inequality (65), and Poincaré inequality orderly, we have

$$
\begin{aligned}
\|u_1-u_2\|^2 &\le |\Omega|^{2/3}\|u_1-u_2\|^2_{L^6(\Omega)} \le |\Omega|^{2/3}\gamma^2(1+\eta)\|\nabla(u_1-u_2)\|^2 \\
&= c_1\left\|(-A)^{1/2}(w_1-w_2)\right\|^2,
\end{aligned}
$$

where $c_1 = |\Omega|^{2/3}\gamma^2(1+\eta)\,d_1$. Similarly, we have

$$\|v_1-v_2\|^2 \le c_2\left\|(-A)^{1/2}(w_1-w_2)\right\|^2,$$

where $c_2 = |\Omega|^{2/3}\gamma^2(1+\eta)\,d_2$. Moreover,

$$
\begin{aligned}
\left\|u_1^3-u_2^3\right\|^2 &\le \left(\int_\Omega |u_1-u_2|^6\,dx\right)^{1/3}\left(\int_\Omega \left|u_1^2+u_1u_2+u_2^2\right|^3 dx\right)^{2/3} \\
&= \|u_1-u_2\|^2_{L^6(\Omega)}\left\|u_1^2+u_1u_2+u_2^2\right\|^2_{L^3(\Omega)} \\
&\le \gamma^2\|u_1-u_2\|^2_{H^1(\Omega)}\cdot 8\left(\|u_1\|^4_{L^6(\Omega)}+\|u_2\|^4_{L^6(\Omega)}\right) \\
&\le \gamma^2(1+\eta)\|\nabla(u_1-u_2)\|^2\cdot 16R^2 = c_3(R)\left\|(-A)^{1/2}(w_1-w_2)\right\|^2,
\end{aligned}
$$

where $c_3(R) = 16(1+\eta)\gamma^2 d_1 R^2$. By the generalized Hölder inequality and (87), we have

$$\begin{aligned}\left\|u_1^2 v_1 - u_2^2 v_2\right\|^2 \le & 2\left\|u_1 - u_2\right\|_{L^6(\Omega)}^2 \left\|u_1 + u_2\right\|_{L^6(\Omega)}^2 \left\|v_2\right\|_{L^6(\Omega)}^2 \\ & + 2\left\|v_1 - v_2\right\|_{L^6(\Omega)}^2 \left\|u_1\right\|_{L^6(\Omega)}^4 \\ \le & 8R^2 \left\|u_1 - u_2\right\|_{L^6(\Omega)}^2 + 2R^2 \left\|v_1 - v_2\right\|_{L^6(\Omega)}^2 \\ \le & c_4(R)\left\|(-A)^{1/2}(w_1 - w_2)\right\|^2,\end{aligned}$$

where $c_4(R) = 2\gamma^2(1+\eta)\,(4d_1 + d_2)\,R^2$.

Substituting these four inequalities into (88), we can get

$$\begin{aligned}&\left\|f(w_1(t)) - f(w_2(t))\right\| \\ &\le \left(\sqrt{c_1}(F+k) + \sqrt{c_2}\,F + 2\sqrt{c_3(R)}\,G + 2\sqrt{c_4(R)}\right)\left\|(-A)^{1/2} y(t)\right\|.\end{aligned} \tag{89}$$

Then substitution of (89) into (86) yields

$$\frac{d}{dt} Q(t) \le \frac{1}{\|y(t)\|^2}\left\|f(w_1(t)) - f(w_2(t))\right\| \le \rho(\mathcal{B}_E) Q(t), \quad t > 0, \tag{90}$$

where

$$\rho(\mathcal{B}_E) = \sqrt{c_1}(F+k) + \sqrt{c_2}F + 2\sqrt{c_3(R)}G + 2\sqrt{c_4(R)}$$

is a positive constant only depending on R which depends on $\mathcal{B}_E$. Thus the second condition in Lemma 6.3 is satisfied.

Now check the Hölder continuity of $S(\cdot)w : [0,T] \to \mathcal{B}_E$ for any given $w \in \mathcal{B}_E$ and any given compact interval $[0,T]$. Since $w_1(0), w_2(0) \in \mathcal{B}_E$, $w_1(t)$ and $w_2(t)$ are strong solutions, which must be mild solutions. By the mild solution formula, for any $0 \le t_1 < t_2 \le T$ we get

$$\begin{aligned}\left\|S(t_2)w - S(t_1)w\right\| \le & \left\|\left(e^{A(t_2-t_1)} - I\right)e^{At_1}w\right\| + \int_{t_1}^{t_2}\left\|e^{A(t_2-\sigma)} f(S(\sigma)w)\right\| d\sigma \\ & + \int_0^{t_1}\left\|\left(e^{A(t_2-t_1)} - I\right)e^{A(t_1-\sigma)} f(S(\sigma)w)\right\| d\sigma.\end{aligned} \tag{91}$$

By the proof of Corollary 5.7 and (83), $\mathcal{B}_E$ is a bounded set in E. Since $\mathcal{B}_E$ is positively invariant with respect to the reversible Gray-Scott semiflow $\{S(t)\}_{t\ge 0}$, there exists a constant $K_{\mathcal{B}_E} > 0$ such that for any $w \in \mathcal{B}_E$, we have

$$\|S(t)w\|_E \le K_{\mathcal{B}_E}, \quad t \ge 0.$$

Since $f : E \to H$ is locally Lipschitz continuous, there is a Lipschitz constant $L_{\mathcal{B}_E} > 0$ of f relative to this positively invariant set $\mathcal{B}_E$. Moreover, by [39, Theorem 37.5], for the analytic, contracting, linear semigroup $\{e^{At}\}_{t\ge 0}$, there exist positive constants M_0 and M_1 such that

$$\left\|e^{At}w - w\right\|_H \le M_0\, t^{1/2}\|w\|_E, \quad \text{for } t \ge 0, \;\; w \in E,$$

and (as used in the proof of Lemma 5.4)

$$\|e^{At}\|_{\mathcal{L}(H,E)} \le M_1 t^{-1/2}, \quad \text{for } t > 0.$$

It follows that

$$\left\|\left(e^{A(t_2-t_1)} - I\right)e^{At_1}w\right\| \le M_0\,(t_2 - t_1)^{1/2} K_{\mathcal{B}_E}$$

and

$$\int_{t_1}^{t_2} \left\| e^{A(t_2-\sigma)} f(S(\sigma)w) \right\| d\sigma \le \int_{t_1}^{t_2} \frac{M_1 L_{\mathcal{B}_E} K_{\mathcal{B}_E}}{\sqrt{t_2-\sigma}} \, d\sigma = 2K_{\mathcal{B}_E} L_{\mathcal{B}_E} M_1 \left(t_2-t_1\right)^{1/2}.$$

Moreover,

$$\begin{aligned}\int_0^{t_1} \left\| \left(e^{A(t_2-t_1)} - I\right) e^{A(t_1-\sigma)} f(S(\sigma)w) \right\| d\sigma &\le M_0 \left(t_2-t_1\right)^{1/2} \int_0^{t_1} \frac{M_1 L_{\mathcal{B}_E} K_{\mathcal{B}_E}}{\sqrt{t_1-\sigma}} \, d\sigma \\ &= 2K_{\mathcal{B}_E} L_{\mathcal{B}_E} M_0 M_1 \sqrt{T} \left(t_2-t_1\right)^{1/2}.\end{aligned}$$

Substituting the above three inequalities into (91), we obtain

$$\left\| S\left(t_2\right) w - S\left(t_1\right) w \right\| \le K_{\mathcal{B}_E} \left(M_0 + 2L_{\mathcal{B}_E} M_1 (1 + M_0 \sqrt{T}) \right) \left(t_2 - t_1\right)^{1/2}, \qquad (92)$$

for $0 \le t_1 < t_2 \le T$. Thus the third condition in Lemma 6.3 is satisfied. Namely, for any given $T > 0$, the mapping $S(\cdot)w : [0,T] \to \mathcal{B}_E$ is Hölder continuous with the exponent $1/2$ and with a uniformly bounded coefficient independent of $w \in \mathcal{B}_E$.

We can use Theorem 47.8 (specifically (47.20) therein) in [39] to confirm the Lipschitz continuity of the mapping $S(t)(\cdot) : \mathcal{B}_E \to \mathcal{B}_E$ for any $t \in [0,T]$ where $T > 0$ is arbitrarily given. Thus the fourth condition in Lemma 6.3 is also satisfied. Finally, we apply Lemma 6.3 to reach the conclusion of this theorem. □

As a remark, the results on the existence of global attractor, (H, E) global attractor, and exponential attractor shown respectively in Theorem 4.2, Theorem 5.6, and Theorem 6.5 are also valid for reversible Brusselator equations, reversible Selkov equations, and reversible Schnackenberg equations.

REFERENCES

[1] J. F. G. Auchmuty and G. Nicolis, *Bifurcation analysis of nonlinear reaction-diffusion equations. I. Evolution equations and the steady state solutions*, Bull. Math. Biology, **37** (1975), 323–365.

[2] A. V. Babin and M. I. Vishik, *Regular attractors of semigroups and evolution equations*, J. Math. Pures Appl., **62** (1983), 441–491.

[3] A. V. Babin and M. I. Vishik, "Attractors of Evolutionary Equations," Nauka, Moskow, 1989; English translation, North-Holland, Amsterdam, 1992.

[4] A. V. Babin and B. Nicolaenko, *Exponential attractors of reaction-diffusion systems in an unbounded domain*, J. Dyn. Diff. Eqns., **7** (1995), 567–590.

[5] D. L. Benson, J. A. Sherratt and P. K. Maini, *Diffusion driven instability in an inhomogeneous domain*, Bull. Math. Biology, **55** (1993), 365–384.

[6] K. J. Brown and F. A. Davidson, *Global bifurcation in the Brusselator system*, Nonlinear Analysis, **24** (1995), 1713–1725.

[7] V. V. Chepyzhov and M.I. Vishik, "Attractors for Equations of Mathematical Physics," AMS Colloquium Publications, Vol. 49, AMS, Providence, RI, 2002.

[8] E. J. Crampin and P. K. Maini, *Reaction-diffusion models for biological pattern formation*, Methods Appl. Anal., **8** (2001), 415–428.

[9] A. Doelman, T. J. Kaper and P. A. Zegeling, *Pattern formation in the one-dimensional Gray-Scott model*, Nonlinearity, **10** (1997), 523–563.

[10] L. Dung, *Exponential attractors for a chemotaxis growth system on domains of arbitrary dimension*, in "Proceedings of the 4th International Conference on Dynamical Systems and Differential Equations," Wilmington, NC, USA, (2002), 536–543.

[11] L. Dung and B. Nicolaenko, *Exponential attractors in Banach spaces*, J. Dynamics and Diff. Eqns., **13** (2001), 791–806.

[12] A. Eden, C. Foias, B. Nicolaenko and R. Temam, "Exponential Attractors for Dissipative Evolution Equations," John Wiley & Sons, Chichester; Masson, Paris, 1994.

[13] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a nonlinear reaction diffusion systems in* $\mathbb{R}^3$, C.R. Acad. Sci., **330** (2000), 713–718.

[14] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a singularly perturbed Cahn-Hilliard system*, Math. Nachr., **272** (2004), 11–31.

[15] I. R. Epstein, *Complex dynamical behavior in simple chemical systems*, J. Phys. Chemistry, **88** (1984), 187–198.

[16] P. Gray and S. K. Scott, *Autocatalytic reactions in the isothermal continuous stirred tank reactor: Isolas and other forms of multistability*, Chem. Eng. Sci., **38** (1983), 29–43.

[17] P. Gray and S. K. Scott, *Autocatalytic reactions in the isothermal, continuous stirred tank reactor: Oscillations and instabilities in the system* $a + 2b \to 3b, b \to c$, Chem. Eng. Sci., **39** (1984), 1087–1097.

[18] P. Gray and S. K. Scott, "Chemical Oscillations and Instabilities," Clarendon, Oxford, 1994.

[19] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Springer-Verlag, Berlin, 1981.

[20] T. Kolokolnikov, T. Erneux, and J. Wei, *Mesa-type patterns in one-dimensional Brusselator and their stability*, Physica D, **214** (2006), 63–77.

[21] T. Kolokolnikov and J. Wei, *On ring-like solutions for the Gray-Scott model: Existence, instability and self-replicating rings*, Euro. J. Appl. Math., **16** (2005), 201–237.

[22] K. J. Lee, W. D. McCormick, Q. Ouyang and H. Swinney, *Pattern formation by interacting chemical fronts*, Science, **261** (1993), 192–194.

[23] K. J. Lee, W. D. McCormick, J. E. Pearson and H. L. Swinney, *Experimental observation of self-replicating spots in areaction-diffusion system*, Nature, **369** (1994), 215–218.

[24] K. J. Lee and H. L. Swinney, *Replicating spots in reaction-diffusion systems*, Int. J. Bifurcation and Chaos, **7** (1997), 1149–1158.

[25] K. Matsuura and M. Ôtani, *Exponential attarctors for a quasilinear parabolic equation*, Disc. Cont. Dyn. Sys. Suppl., (2007), 713-720.

[26] J. S. McGough and K. Riley, *Pattern formation in the Gray-Scott model*, Nonlinear Analysis: Real World Applications, **5** (2004), 105–121.

[27] A. J. Milani and N. J. Koksch, "An Introduction to Semiflows," Chapman & Hall/CRC, 2005.

[28] D. Morgan and T. Kaper, *Axisymmetric ring solutions of the 2D Gray-Scott model and their destabilization into spots*, Physica D, **192** (2004), 33–62.

[29] C. B. Muratov and V. V. Osipov, *Static spike autosolitons in the Gray-Scott model*, J. Phys. A, **33** (2000), 8893–8916.

[30] K. Osaki, T. Tsujikawa, A. Yagi and M. Minura, *Exponential attractor for a chemotaxis-growth system of equations*, Nonlinear Analysis, **51** (2002), 119–144.

[31] J. E. Pearson, *Complex patterns in a simple system*, Science, **261** (1993), 189–192.

[32] I. Prigogine and R. Lefever, *Symmetry-breaking instabilities in dissipative systems*, J. Chem. Physics, **48** (1968), 1695–1700.

[33] W. Reynolds, J. E. Pearson and S. Ponce-Dawson, *Dynamics of self-replicating patterns in reaction-diffusion systems*, Phys. Rev. E, **56** (1997), 185–198.

[34] J. C. Robinson, "Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors," Cambridge University Press, Cambridge, UK, 2001.

[35] F. Rothe, "Global Solutions of Reaction-Diffusion Systems," Lecture Notes in Math, 1072, Springer-Verlag, Berlin, 1984.

[36] J. Schnackenberg, *Simple chemical reaction systems with limit cycle behavior*, J. Theor. Biology, **81** (1979), 389–400.

[37] S. K. Scott and K. Showalter, *Simple and complex reaction-diffusion fronts*, in "Chemical Waves and Patterns" (R. Kapral and K. Showalter, eds.), Springer, **10** (1995), 485–516.

[38] E. E. Selkov, *Self-oscillations in glycolysis: a simple kinetic model*, European J. Biochem., **4** (1968), 79–86.

[39] George R. Sell and Yuncheng You, "Dynamics of Evolutionary Equations," Applied Mathematical Sciences, vol. **143**, Springer-Verlag, New York, 2002.

[40] L. J. Shaw and J. D. Murray, *Analysis of a model for complex skin patterns*, SIAM J. Appl. Math., **50** (1990), 628–648.

[41] M. Stanislavova, A. Stefanov and B. Wang, *Asymptotic smoothing and attractors for the generalized Benjamin-Bona-Mahonay equations on* $\Re^3$, J. Dff. Eqns., **219** (2005), 451–483.

[42] C. Sun and C. Zhong, *Attractors for the semilinear reaction-diffusion equation with distributed derivatives in unbounded domains*, Nonlinear Analysis, **63** (2005), 49–65.

[43] Roger Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Applied Mathematical Sciences, vol. **68**, Springer-Verlag, New York, 1988.

[44] B. Wang, *Attractors for reaction-diffusion equation in unbounded domains*, Physica D, **128** (1999), 41–52.

[45] M. J. Ward and J. Wei, *The existence and stability of asymmetric spike patterns for the Schnackenberg model*, Stud. Appl. Math., **109** (2002), 229–264.

[46] J. Wei and M. Winter, *Asymmetric spotty patterns for the Gray-Scott model in* $\Re^2$, Stud. Appl. Math., **110** (2003), 63–102.

[47] J. Wei and M. Winter, *Existence and stability of multiple-spot solutions for the Gray-Scott model in* $\Re^2$, Physica D, **176** (2003), 147–180.

[48] L. Yang, A. M. Zhabotinsky and I. R. Epstein, *Stable square and other oscillatory Turing patterns in a reaction-diffusion model*, Phys. Rev. Lett., **92** (2004), 198303–1–4.

[49] Y. You, *Global dynamics of nonlinear wave equations with cubic non-monotone damping*, Dynamics of PDE, **1** (2004), 65–86.

[50] Y. You, *Finite dimensional reduction of global dynamics and lattice dynamics of a damped nonlinear wave equation*, in "Control Theory and mathematical Finance" (S. Tang and J. Yong, eds.), World Scientific, (2007), 367–386.

[51] Y. You, *Global dynamics of the Brusselator equations*, Dynamics of PDE, **4** (2007), 167–196.

[52] Y. You, *Global attractor of the Gray-Scott equations*, Comm. Pure Appl. Anal., **7** (2008), 947–970.

[53] Y. You, *Inertial manifolds for nonautonomous skew product semiflows*, Far East J. Appl. Math., **32** (2008), 141–188.

[54] Y. You, *Asymptotic dynamics of Selkov equations*, Disc. Cont. Dyn. Systems, Series S, **2** (2009), 193–219.

[55] Y. You, *Asymptotic dynamics of the modified Schnackenberg equations*, Disc. Cont. Dyn. Systems, Supplement 2009, 857–868.

Received May 2009; revised March 2010.

E-mail address: you@math.usf.edu