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Classical Potential Theory

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To

Deborah and Lindsey

Preface

This book is about the potential theory of Laplace's equation,

$$\frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} + \cdots + \frac{\partial^2 h}{\partial x_N^2} = 0,$$

in Euclidean space $\mathbb{R}^N$, where $N \geq 2$; in brief, classical potential theory. It involves the whole circle of ideas concerning harmonic and subharmonic functions, maximum principles and analyticity, Green functions, potentials and capacity, the Dirichlet problem and boundary integral representations. From its origins in Newtonian physics, the subject has developed into a major field of research in its own right, intimately connected with several other areas of real and complex analysis. Over the past half-century, new lines of investigation have emerged and come to maturity, largely inspired by classical potential theory: examples are non-linear potential theory, probabilistic potential theory, axiomatic potential theory and pluripotential theory. For a proper appreciation of these subjects an understanding of the classical theory is essential. There is also a close relationship between potential theory in the plane and complex analysis: concepts from potential theory are important and natural tools for the study of holomorphic functions. Further, this connection suggests potential theoretic analogues of theorems concerning functions of one complex variable, ranging from elementary results such as the maximum modulus theorem and Laurent's theorem, to the approximation theorems of Runge and Mergelyan and the theory of prime ends.

We treat our subject at a level intended to be accessible to graduate students. Prerequisite knowledge does not go beyond what is commonly taught in undergraduate or first-year graduate courses. The reader will need a good grasp of the limiting processes of analysis, some facility with calculus in higher dimensions, and some measure theory. A few well-known theorems from functional analysis are required, and only very basic topology and linear algebra. Some of the less elementary results that are employed are stated in the Appendix, where convenient references to proofs are supplied. As we sometimes indicate connections with the theory of holomorphic functions, familiarity with the rudiments of one-variable complex analysis would enrich the reader's appreciation of this aspect of the subject.

We have set out to present rigorously and economically many of the results and techniques that are central to potential theory and are the everyday tools of researchers in the field. Occasionally we have taken the opportunity to present some lesser known results that we have found useful and interesting. The collection of theorems in Chapter 3 connecting convexity and subharmonicity, some of which are not widely known but have elegant proofs, falls into this category; another example from the same chapter is the characterization of open sets in which the maximum principle holds (and, surprisingly, these include some unbounded domains). In our own research we have sometimes needed a standard result in a form not easily found in the literature. This is no doubt a common experience, so we have given strong and general versions of theorems when it has been feasible to do so without excessively prolonging proofs. For example, the Dirichlet problem is discussed for the most general open sets possible (which, when $N \geq 3$, include all open sets), and the main removable singularity result (Theorem 5.2.1) does not require that the exceptional polar set is closed. Obviously, we have had to decide to omit certain topics, and among these are the notion of energy, and families of capacities associated with various function spaces.

The first six chapters are of quite a concrete character, dealing with harmonic and subharmonic functions and potentials, and their particular properties. Here the underlying topology is always the standard Euclidean one. Each of these chapters concludes with a set of exercises, some fairly routine and others leading step-by-step to results from the research literature. The material in these chapters is especially appropriate to readers seeking a background knowledge of the subject for wider application. In the final three chapters the level of abstraction deepens as we introduce topological concepts specially created for potential theory, such as the fine topology, the Martin boundary and minimal thinness. Our aim here is to give the reader a firm grounding in these more advanced topics on which to base future reading and research. At the back of the book we have provided brief historical notes for each chapter indicating, to the best of our knowledge, the original sources of results and ideas, and pointing to further developments which lie beyond the scope of this book.

In preparing this book we have, of course, benefitted from the work of earlier authors. In particular, we acknowledge our indebtedness to Brelot [12, 1965], Helms [1, 1969], Hayman and Kennedy [1, 1976], Doob [6, 1984] and Axler, Bourdon and Ramey [1, 1992]. Other related texts include Brelot [13, 1971], Landkof [1, 1972], Hayman [2, 1989], Ransford [1, 1995], and the older works of Kellogg [1, 1929], Radó [1, 1937] and Tsuji [1, 1959]. We are also grateful to Professors Hiroaki Aikawa, Ivan Netuka and Jiří Veselý for reading various parts of the manuscript in draft form and making helpful suggestions. Any defects that remain are, of course, the responsibility of the authors. Finally, we express our appreciation to Michael Elliott, Sheila O'Brien, Siobhán Purcell, Gerhard Schick and Thomas Unger for their careful typesetting of the book, and to the staff of Springer-Verlag (UK) for their courteous efficiency and helpfulness.

Contents

Notation and Terminology

Some general notation and conventions are summarized below. However, most notation will be explained as it is introduced in the course of the book, and can be traced using the notation index.

Sets

We work mostly in the context of Euclidean space $\mathbb{R}^N$, where $N \geq 2$, and denote a typical point by x or $(x_1, \ldots, x_N)$. We write

$$||x|| = (x_1^2 + \cdots + x_N^2)^{1/2} \text{ and } \langle x, y \rangle = x_1 y_1 + \cdots + x_N y_N,$$

where $y = (y_1, \ldots, y_N)$, for the usual norm and inner product. The symbol Ω always denotes a non-empty open subset of $\mathbb{R}^N$. The connected components of Ω will be referred to simply as *components*, and we sometimes use the term *domain* as an abbreviation for "non-empty connected open set". All topological concepts will be relative to the Euclidean topology on $\mathbb{R}^N$ (that is, the topology associated with the above norm) unless otherwise indicated. By a G_δ set we mean one which can be expressed as a countable intersection of open sets, and by an F_σ set we mean one which can be expressed as a countable union of closed sets. (In general, a collection of objects will be called *countable* if it is either finite or countably infinite.)

If $E \subseteq \mathbb{R}^N$, then the closure, interior and boundary of E are denoted respectively by $\overline{E}$, E° and ∂E. We write the one-point compactification of $\mathbb{R}^N$ as $\mathbb{R}^N \cup \{\infty\}$, and use $\partial^\infty \Omega$ to denote the boundary of Ω in $\mathbb{R}^N \cup \{\infty\}$. Thus $\infty \in \partial^\infty \Omega$ if and only if Ω is unbounded.

The open ball of centre x and radius r in $\mathbb{R}^N$ is denoted by $B(x,r)$, and $S(x,r)$ denotes the sphere $\partial B(x,r)$. We abbreviate $B(0,1)$ to B and $S(0,1)$ to S. By a *hyperplane* we mean a set of the form $\{x \in \mathbb{R}^N : \langle x, y \rangle = a\}$, where $y \in S$ and $a \in \mathbb{R}$. A linear mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ that satisfies $T(S) = S$ will be called an *orthogonal transformation*. (Thus T is orthogonal if and only if the columns of the matrix associated with T form an orthonormal basis of $\mathbb{R}^N$.) An isometry (a distance-preserving map) of $\mathbb{R}^N$ can be expressed as a composition of a translation and an orthogonal transformation (see Fleming [1, 1965], p.98).

The extended real numbers will be denoted by $[-\infty,+\infty]$. The natural ordering and topology apply, and arithmetic involving $\pm\infty$ and $x \in \mathbb{R}$ will follow the conventions:

$$(\pm\infty)+(\pm\infty) = x+(\pm\infty) = (\pm\infty)+x = \pm\infty,$$

$$(\pm\infty).(\pm\infty) = +\infty, \quad (\pm\infty).(\mp\infty) = -\infty,$$

$$\frac{\pm\infty}{x} = \begin{cases} \pm\infty & \text{if } x>0 \\ \mp\infty & \text{if } x<0, \end{cases} \qquad x.(\pm\infty) = \begin{cases} \pm\infty & \text{if } x>0 \\ 0 & \text{if } x=0 \\ \mp\infty & \text{if } x<0. \end{cases}$$

Expressions such as $(\pm\infty)+(\mp\infty)$ are left undefined, and quotients with 0 or $\pm\infty$ in the denominator will be interpreted as they arise. We assign the value $+\infty$ to $\inf\emptyset$ and the value $-\infty$ to $\sup\emptyset$.

Functions and function spaces

If $E \subseteq \mathbb{R}^N \cup \{\infty\}$, then we write $C(E)$ for the space of all (real-valued) continuous functions on E. If $n \in \mathbb{N} = \{1,2,\ldots\}$, then $C^n(\Omega)$ stands for the space of n-times continuously differentiable functions on Ω, while $C^\infty(\Omega)$ represents the space of infinitely differentiable functions on Ω. We write ∇f for the gradient $(\partial f/\partial x_1,\ldots,\partial f/\partial x_N)$ of a function f whenever it exists. A function which is continuous as a mapping from E into $[-\infty,+\infty]$ is said to be *continuous in the extended sense.* The *characteristic function* of E is the function valued 1 on E and 0 elsewhere, and is usually denoted by χ_E.

Given a function f defined at least on a set A, and a limit point y of A, we define

$$\liminf_{x\to y, x\in A} f(x) = \sup_{U\in\mathcal{N}_y}\left(\inf_{x\in(U\cap A)\setminus\{y\}} f(x)\right)$$

and

$$\limsup_{x\to y, x\in A} f(x) = \inf_{U\in\mathcal{N}_y}\left(\sup_{x\in(U\cap A)\setminus\{y\}} f(x)\right),$$

where $\mathcal{N}_y$ denotes the collection of all neighbourhoods of y. (The function f, and the suprema and infima here, may take infinite values.) Thus

$$\lim_{x\to y, x\in A} f(x) \text{ exists}$$

if and only if

$$\liminf_{x\to y, x\in A} f(x) \text{ and } \limsup_{x\to y, x\in A} f(x) \text{ have a common value } l \in [-\infty,+\infty],$$

and then

$$\lim_{x\to y, x\in A} f(x) = l.$$

If A is the domain of definition of f, or if $A \in \mathcal{N}_y$, then we may drop the qualification "$x \in A$".

Given a function f on Ω and a point $y \in \partial^\infty\Omega$, we say that f *vanishes continuously at* y if

$$\lim_{x\to y, x\in\Omega} f(x) = 0.$$

If $E \subseteq \partial^\infty\Omega$ and f vanishes continuously at each point of E, then we say that f *vanishes continuously on* E.

Let $x_0 \in \mathbb{R}^N \cup \{\infty\}$ and let f and g be real-valued functions on $U \setminus \{x_0\}$, where $g > 0$ and U is a neighbourhood of x_0. We say that $f(x) = O(g(x))$ as $x \to x_0$ if there exist a neighbourhood $V \subseteq U$ of x_0 and a positive number M such that $|f(x)| \le Mg(x)$ whenever $x \in V \setminus \{x_0\}$. Also, we say that $f(x) = o(g(x))$ as $x \to x_0$ if $f(x)/g(x) \to 0$ as $x \to x_0$. (In the case of real sequences (a_j) and (b_j), where $b_j > 0$ for all j, we write $a_j = O(b_j)$ if there exists $M > 0$ such that $|a_j| \le Mb_j$ for all j.)

Given a number or function f, we define $f^+ = \max\{f,0\}$ and $f^- = \max\{-f,0\}$. Thus $f = f^+ - f^-$ and $|f| = f^+ + f^-$. We write $f|_A$ for the restriction of a function f to a subset A of its domain of definition.

A function $f : \Omega \to \mathbb{R}$ is said to be *locally bounded* on Ω if f is bounded on every compact subset of Ω. The phrases *locally bounded above* and *locally bounded below* should be analogously interpreted. A family $\mathcal{F}$ of functions is said to be *uniformly bounded* on a set A if there exists $M > 0$ such that $|f(x)| \le M$ for all $f \in \mathcal{F}$ and for all $x \in A$. Further, we say that $\mathcal{F}$ is *locally uniformly bounded* on Ω if $\mathcal{F}$ is uniformly bounded on each compact subset of Ω. Obvious definitions apply to phrases such as *uniformly bounded below, locally uniformly bounded below,* etc. A sequence (f_n) is said to *converge locally uniformly* on Ω if it converges uniformly on each compact subset of Ω. The phrase "(f_n) is locally uniformly Cauchy on Ω" has an analogous meaning. Finally, the *support* of f is defined by

$$\text{supp } f = \Omega \setminus \{x \in \Omega : f = 0 \text{ on a neighbourhood of } x\}.$$

Clearly supp f is closed relative to Ω.

Measures and mean values

Let X be a locally compact Hausdorff space and suppose that there is a countable base for the topology. The class $\mathcal{B}$ of Borel sets is the smallest σ-algebra of subsets of X which contains the open sets. We say that a function $f : X \to [-\infty,+\infty]$ is *Borel measurable* if the set $\{x \in X : f(x) > a\}$ belongs to $\mathcal{B}$ for every $a \in \mathbb{R}$. By a *measure* on X we mean a countably additive set function μ, defined on $\mathcal{B}$ (or a larger σ-algebra) and taking values in $[0,+\infty]$, such that $\mu(\emptyset) = 0$ and $\mu(K) < +\infty$ for every compact subset K of X. Such a measure μ is regular in the sense that, if $E \in \mathcal{B}$, then

$$\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ and } U \text{ is open}\}$$
$$= \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}.$$

(See Chapter 2 of Rudin [1, 1974] for further details.)

The *support* of a measure μ is defined by

$$\operatorname{supp} \mu = \{x \in X : \mu(U) > 0 \text{ for every open neighbourhood } U \text{ of } x\}.$$

It is easy to see that $\operatorname{supp}\mu$ is the smallest closed subset F of X such that $\mu(X \setminus F) = 0$. If $A \in \mathcal{B}$, then the *restriction* of μ to A is defined by $\mu|_A(E) = \mu(E \cap A)$ for all $E \in \mathcal{B}$.

By a *signed measure* on X we mean a countably additive set function $\mu : \mathcal{B} \to \mathbb{R}$ such that $\mu(\emptyset) = 0$. (Thus we do not allow the values $\pm\infty$.) In view of the Hahn–Jordan decomposition theorem there are disjoint sets $P, N \in \mathcal{B}$ such that $P \cup N = X$, and finite measures μ^+, μ^- on X such that $\mu^+(N) = 0 = \mu^-(P)$ and $\mu = \mu^+ - \mu^-$. The *total variation* of μ is defined by $\|\mu\| = \mu^+(X) + \mu^-(X)$.

We use λ to denote Lebesgue measure on $\mathbb{R}^N$, and σ to denote surface area measure on a given smooth surface, usually a sphere. We define $\lambda_N = \lambda(B)$ and $\sigma_N = \sigma(S)$. Thus

$$\sigma_N = \begin{cases} \dfrac{\pi^{N/2} N}{(N/2)!} & (N \text{ is even}) \\[2ex] \dfrac{2^{(N+1)/2}\pi^{(N-1)/2}}{1.3.5\ldots(N-2)} & (N \text{ is odd}, N \geq 3) \end{cases}$$

and $\lambda_N = \sigma_N / N$. We write the surface mean value of a σ-integrable function f on $S(x,r)$ as

$$\mathcal{M}(f; x, r) = \frac{1}{\sigma_N r^{N-1}} \int_{S(x,r)} f \, d\sigma,$$

and the volume mean value of a λ-integrable function f on $B(x,r)$ as

$$\mathcal{A}(f; x, r) = \frac{1}{\lambda_N r^N} \int_{B(x,r)} f \, d\lambda.$$

A function $f : \Omega \to [-\infty, +\infty]$ is said to be *locally integrable* on Ω if it is integrable with respect to λ on each compact subset of Ω.

Chapter 1. Harmonic Functions

1.1. Laplace's equation

Our starting point is Laplace's equation $\Delta h = 0$ on an open subset Ω of $\mathbb{R}^N$, where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_N^2$.

Definition 1.1.1. A function $h : \Omega \to \mathbb{R}$ is called *harmonic* on Ω if $h \in C^2(\Omega)$ and $\Delta h \equiv 0$. The set of all harmonic functions on Ω is denoted by $\mathcal{H}(\Omega)$.

Laplace's equation is simple to state but profound in its implications. In this opening chapter we begin to explore the properties of its solutions h on an open set Ω. For example, although we require only that h is in $C^2(\Omega)$, we will deduce that h is in $C^\infty(\Omega)$ and even real-analytic. The first main step will be to show that, for any closed ball $\overline{B(x,r)}$ in Ω, the mean value $\mathcal{M}(h; x, r)$ of a harmonic function h over the sphere $S(x,r)$ equals $h(x)$ and, conversely, that any function h in $C(\Omega)$ which has this mean value property must be harmonic. The mean value property leads to the maximum principle, which is analogous to the maximum modulus theorem for holomorphic functions. We will focus for a while on the cases where Ω is a ball or a half-space, solving the Dirichlet problem and establishing integral representation theorems. Harnack's inequalities will demonstrate the rigidity of positive harmonic functions and help us to establish convergence theorems for sequences of harmonic functions. We will also look at the preservation of harmonicity under certain transformations and discuss the close relationship between harmonic and holomorphic functions when $N = 2$.

It is clear that $\mathcal{H}(\Omega)$ is a vector subspace of $C(\Omega)$ and contains all the constant functions. Further, the chain rule shows that if $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is an isometry or dilation ($\phi(x) = \alpha x$, where $\alpha > 0$) and $h \in \mathcal{H}(\phi(\Omega))$, then $h \circ \phi \in \mathcal{H}(\Omega)$. Also, if ω is an open subset of Ω, then $\mathcal{H}(\Omega) \subseteq \mathcal{H}(\omega)$ in the sense that $h|_\omega \in \mathcal{H}(\omega)$ whenever $h \in \mathcal{H}(\Omega)$. The rotation-invariant harmonic function U_y introduced by the following result is known as the *fundamental harmonic function with pole* y and plays an important role throughout the book. We denote by $A(y; r_1, r_2)$ the annular region

$$\{x \in \mathbb{R}^N : r_1 < \|x - y\| < r_2\} \qquad (y \in \mathbb{R}^N; 0 \leq r_1 < r_2 \leq +\infty).$$

Theorem 1.1.2. *If $y \in \mathbb{R}^N$, then the function U_y defined on $\mathbb{R}^N \backslash \{y\}$ by*

$$U_y(x) = \begin{cases} -\log \|x-y\| & (N=2) \\ \|x-y\|^{2-N} & (N \geq 3) \end{cases} \tag{1.1.1}$$

is harmonic on $\mathbb{R}^N \backslash \{y\}$. Moreover, if h is harmonic on some annular region $A(y; r_1, r_2)$ and $h(x)$ depends only on $\|x-y\|$, then $h = \alpha U_y + \beta$ for some constants α, β.

Proof. Suppose that $f \in C^2(A(y; r_1, r_2))$ and that $f(x)$ depends only on $\|x-y\|$. We write $\rho = \|x-y\|$ and $f(x) = F(\rho)$. Elementary calculations yield $\Delta f(x) = F''(\rho) + (N-1)\rho^{-1}F'(\rho)$. Since the general solution of the differential equation $F''(\rho) + (N-1)\rho^{-1}F'(\rho) = 0$ on the interval (r_1, r_2) is $F(\rho) = -\alpha \log \rho + \beta$ if $N = 2$ and $F(\rho) = \alpha\rho^{2-N} + \beta$ if $N \geq 3$, both parts of the theorem follow immediately. □

The formula (1.1.1) forewarns us that potential theory in the plane differs significantly from that in higher dimensions. In the case of the plane (identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way) we have extra tools available because of the relationship between harmonic and holomorphic functions described below. Part (i) of the following theorem may be used to write down many examples of harmonic functions; for example, with $z = x_1 + ix_2$,

$$x_1^4 - 6x_1^2x_2^2 + x_4^2 = \operatorname{Re} z^4, \qquad e^{x_1}\cos x_2 = \operatorname{Re}(e^z).$$

Theorem 1.1.3. *(i) If $f = u + iv$ is holomorphic on a plane open set Ω, then u and v are harmonic on Ω.*
(ii) If u is harmonic on a simply connected plane domain Ω, then u is the real part of a holomorphic function on Ω.
(iii) If f is holomorphic on a plane open set Ω and $f \not\equiv 0$, then $\log|f|$ is harmonic on $\{z \in \Omega : f(z) \neq 0\}$.

Proof. (i) We know that $u, v \in C^\infty(\Omega)$, and the Cauchy–Riemann equations give

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial v}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial v}{\partial x_1}\right) \equiv 0.$$

A similar equation holds for v.

(ii) Let $g = \partial u/\partial x - i\partial u/\partial y$. Since $u \in C^2(\Omega)$ and $\Delta u = 0$, the real and imaginary parts of g belong to $C^1(\Omega)$ and satisfy the Cauchy–Riemann equations. Thus g is holomorphic on Ω. Since Ω is simply connected, there is a holomorphic function $F = U + iV$ on Ω such that $F' = g$. Hence $\partial U/\partial x = \partial u/\partial x$ and $\partial U/\partial y = -\partial V/\partial x = \partial u/\partial y$. Thus $U - u$ has a constant value c on Ω, and so $u = \operatorname{Re}(F - c)$.

(iii) Let ω be an open disc contained in $\{z \in \Omega : f(z) \neq 0\}$. Then there is a holomorphic function g on ω such that $f = e^g$ there and so $\log|f| = \operatorname{Re} g \in \mathcal{H}(\omega)$, by (i). □

1.2. The mean value property

Let f be a C^2 function on an open set which contains $\overline{B(x,r)}$. We can use first Green's formula and then differentiation under the integral sign (see Appendix, Theorems A.15 and A.1) to obtain

$$\int_{B(x,t)} \Delta f \, d\lambda = \int_{S(x,t)} \frac{\partial f}{\partial n_e} d\sigma \qquad (0 < t \leq r),$$

where $\partial/\partial n_e$ denotes the exterior normal derivative, and then

$$\int_{B(x,t)} \Delta f \, d\lambda = t^{N-1} \int_S \frac{\partial}{\partial t} f(x+ty)\, d\sigma(y) = t^{N-1} \frac{d}{dt} \int_S f(x+ty)\, d\sigma(y).$$

This can be rewritten as

$$\lambda_N t^N \mathcal{A}(\Delta f; x, t) = \sigma_N t^{N-1} \frac{d}{dt} \mathcal{M}(f; x, t),$$

where $\mathcal{A}(f; x, t)$ denotes the mean value of f over $B(x,t)$ and $\mathcal{M}(f; x, t)$ denotes the mean value of f over $S(x,t)$. Since $\sigma_N = N\lambda_N$, we obtain

$$N \frac{d}{dt} \mathcal{M}(f; x, t) = t\mathcal{A}(\Delta f; x, t) \qquad (0 < t \leq r). \tag{1.2.1}$$

In particular, if $\Delta f \equiv 0$, then $\mathcal{M}(f; x, \cdot)$ is constant on $(0, r]$ and this constant value must be $f(x)$ by continuity.

The above discussion shows that if $h \in \mathcal{H}(\Omega)$, then h satisfies

$$h(x) = \mathcal{M}(h; x, r) \quad \text{whenever } \overline{B(x,r)} \subset \Omega. \tag{1.2.2}$$

We refer to this as the *(spherical) mean value property* of harmonic functions. Conversely, and rather surprisingly, we will see that any continuous function on Ω which has this mean value property is harmonic.

Lemma 1.2.1. *If $h \in C(\Omega)$ and (1.2.2) holds, then $h \in C^\infty(\Omega)$.*

Proof. We define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \begin{cases} C_N e^{-1/t} & (t > 0) \\ 0 & (t \leq 0), \end{cases}$$

where the constant C_N is chosen so that

$$\sigma_N \int_0^1 t^{N-1} \phi(1-t^2)\, dt = 1.$$

It is easy to verify that $\phi \in C^\infty(\mathbb{R})$. For each $n \in \mathbb{N}$ we define ϕ_n on $\mathbb{R}^N$ by

$$\phi_n(x) = n^N \phi(1 - n^2\|x\|^2).$$

Since a composition of smooth functions is smooth, $\phi_n \in C^\infty(\mathbb{R}^N)$. We write $\Omega_n = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > n^{-1}\}$ if $\Omega \neq \mathbb{R}^N$ and $\Omega_n = \mathbb{R}^N$ otherwise, and we define

$$H_n(x) = \int_\Omega \phi_n(x-y)h(y)\,d\lambda(y) \quad (x \in \Omega_n).$$

Since ϕ_n and all its partial derivatives are bounded on $\mathbb{R}^N$ and vanish outside $B(0, n^{-1})$, we see that $H_n \in C^\infty(\Omega_n)$. Also,

$$\begin{aligned} H_n(x) &= \int_0^{n^{-1}} n^N \phi(1 - n^2t^2) \int_{S(x,t)} h\, d\sigma dt \\ &= h(x)\sigma_N \int_0^{n^{-1}} n^N t^{N-1}\phi(1-n^2t^2)dt \\ &= h(x)\sigma_N \int_0^1 \tau^{N-1}\phi(1-\tau^2)d\tau = h(x), \end{aligned}$$

in view of (1.2.2). Hence $h \in C^\infty(\Omega)$, since $\Omega = \bigcup_n \Omega_n$. □

We established (1.2.1) for any C^2 function f on an open set containing $\overline{B(x,r)}$. Integrating this equation we obtain

$$N\{\mathcal{M}(f;x,t) - f(x)\} = \int_0^t \tau\mathcal{A}(\Delta f; x, \tau)\,d\tau$$

and, since $\mathcal{A}(\Delta f; x, \tau) \to \Delta f(x)$ as $\tau \to 0+$,

$$\Delta f(x) = 2N \lim_{t\to 0+} t^{-2}\{\mathcal{M}(f;x,t) - f(x)\}. \qquad (1.2.3)$$

We are now in a position to show that the mean value property (1.2.2) characterizes the harmonic functions among all continuous functions on Ω.

Theorem 1.2.2. *The following are equivalent:*
(a) $h \in \mathcal{H}(\Omega)$;
(b) $h \in C(\Omega)$ *and* $h(x) = \mathcal{M}(h;x,r)$ *whenever* $\overline{B(x,r)} \subset \Omega$;
(c) $h \in C(\Omega)$ *and* $h(x) = \mathcal{A}(h;x,r)$ *whenever* $\overline{B(x,r)} \subset \Omega$.

Proof. We observed at the beginning of this section that (a) implies (b). Conversely, if (b) holds, then $h \in C^\infty(\Omega)$ by Lemma 1.2.1, and (a) follows from (1.2.3). To prove the equivalence of (b) and (c), let $h \in C(\Omega)$. Then

$$r^N\mathcal{A}(h;x,r) = N\int_0^r t^{N-1}\mathcal{M}(h;x,t)\,dt \qquad (1.2.4)$$

whenever $\overline{B(x,r)} \subset \Omega$, so (b) implies (c). Conversely, the continuity of h implies that the integrand in (1.2.4) is continuous as a function of t. Thus, if (c) holds, we can differentiate (1.2.4) to obtain (b). □

Corollary 1.2.3. *If* $h \in \mathcal{H}(\Omega)$, *then* $h \in C^\infty(\Omega)$ *and all partial derivatives of* h *are in* $\mathcal{H}(\Omega)$.

Proof. Lemma 1.2.1 and Theorem 1.2.2 together show that $h \in C^\infty(\Omega)$. The second part follows by induction from the observation that $\Delta(\partial h/\partial x_k) = (\partial/\partial x_k)(\Delta h)$. □

Parts (i) and (ii) of the next result, with "maximum" in place of "extremum", and the second inequality of (iii), are all forms of the *maximum principle* for harmonic functions; the remainder of the result is referred to as the *minimum principle* for harmonic functions.

Theorem 1.2.4. *Let* $h \in \mathcal{H}(\Omega)$ *and* $x \in \Omega$.
(i) If h *attains a local extremum at* x, *then* h *is constant on some neighbourhood of* x.
(ii) If Ω *is connected and* h *attains an extremum at* x, *then* h *is constant.*
(iii) If $h \in C(\Omega \cup \partial^\infty\Omega)$, *then* $\inf_{\partial^\infty\Omega} h \le h \le \sup_{\partial^\infty\Omega} h$ *on* Ω.

Proof. (i) We may assume that h attains a local maximum at x. We choose r small enough so that $\overline{B(x,r)} \subset \Omega$ and $h \le h(x)$ on $B(x,r)$. Since $h(x) = \mathcal{A}(h;x,r)$, continuity implies that $h = h(x)$ on $B(x,r)$.

(ii) From (i) the set $\{y \in \Omega : h(y) = h(x)\}$ is open, and continuity implies that it is closed relative to Ω, so by connectedness it must be all of Ω.

(iii) Let ω be a component of Ω. Since h is continuous on the compact set $\omega \cup \partial^\infty\omega$, it attains finite extrema there. If either of these occurs at a point of ω, then by (ii) h is constant on ω and hence on $\omega \cup \partial^\infty\omega$. Otherwise both extrema occur on $\partial^\infty\omega$. Since $\partial^\infty\omega \subseteq \partial^\infty\Omega$, the result follows. □

Remark 1.2.5. (a) We note, for future reference, that the proof of Theorem 1.2.4 remains valid under the apparently weaker hypothesis that $h \in C(\Omega)$ and that, for each $x \in \Omega$, there is a positive number r_x such that $h(x) = \mathcal{A}(h;x,r)$ whenever $0 < r < r_x$.
(b) We will see later that harmonic functions are real-analytic. It will then follow that, in (i) above, h is constant on the component of Ω which contains x.

The following result is an analogue of the classical result of Liouville concerning holomorphic functions.

Theorem 1.2.6. *If* $h \in \mathcal{H}(\mathbb{R}^N)$ *and* h *is bounded below (or above), then* h *is constant.*

Proof. We may suppose that $h > 0$. If $x, y \in \mathbb{R}^N$ and $\rho = \|x-y\|$, then $B(x,r) \subseteq B(y, r+\rho)$. Hence

$$h(x) = \mathcal{A}(h;x,r) \le (\lambda_N r^N)^{-1} \int_{B(y,r+\rho)} h\, d\lambda$$
$$= \left(1+\frac{\rho}{r}\right)^N h(y) \to h(y) \quad (r \to +\infty).$$

Thus $h(x) \le h(y)$. Similarly we obtain $h(y) \le h(x)$, so h is constant. □

1.3. The Poisson integral for a ball

The Dirichlet problem is one of the classical problems of potential theory. In its simplest form it can be stated as follows: given a continuous function $f : \partial\Omega \to \mathbb{R}$, find a function $h \in \mathcal{H}(\Omega)$ such that $h(x) \to f(y)$ as $x \to y$ for each $y \in \partial\Omega$. A detailed account of this problem will be given in Chapter 6, but we can deal now with the special case where Ω is a ball. We will also give below important integral representation theorems for certain classes of harmonic functions on a ball.

Definition 1.3.1. The *Poisson kernel* of $B(x_0,r)$ is the function

$$K_{x_0,r}(x,y) = \frac{1}{\sigma_N r}\frac{r^2 - \|x-x_0\|^2}{\|x-y\|^N} \qquad (y \in S(x_0,r); x \in \mathbb{R}^N\backslash\{y\}). \quad (1.3.1)$$

It is clear from (1.3.1) that, if $y \in S(x_0,r)$, then $K_{x_0,r}(\cdot\,,y)$ is positive, zero and negative, respectively, on the sets $B(x_0,r)$, $S(x_0,r)\backslash\{y\}$ and $\mathbb{R}^N\backslash\overline{B(x_0,r)}$. Also, if we write $\sigma_N r K_{x_0,r}(x,y)$ as the product of $r^2 - \|x-x_0\|^2$ and $\|x-y\|^{-N}$ and use the identity $\Delta(uv) = u\Delta v + v\Delta u + 2\langle\nabla u, \nabla v\rangle$, we see after some calculation that $K_{x_0,r}(\cdot\,,y) \in \mathcal{H}(\mathbb{R}^N\backslash\{y\})$. In what follows we use the functions $\{K_{x_0,r}(\cdot\,,y) : y \in S(x_0,r)\}$ as "building blocks" to construct more general harmonic functions on $B(x_0,r)$.

Definition 1.3.2. If μ is a signed measure on $S(x_0,r)$, then the *Poisson integral* of μ is defined by

$$I_{\mu,x_0,r}(x) = \int_{S(x_0,r)} K_{x_0,r}(x,y)\, d\mu(y) \quad (x \in B(x_0,r)). \qquad (1.3.2)$$

In the special case where $d\mu = f d\sigma$ for some σ-integrable function f on $S(x_0,r)$ (that is, the case where μ is absolutely continuous with respect to σ), we write $I_{f,x_0,r}$ instead of $I_{\mu,x_0,r}$. When there is no risk of confusion we write K for $K_{x_0,r}$, and I_μ and I_f for the corresponding Poisson integrals.

Theorem 1.3.3. *(i) If μ is a signed measure on $S(x_0,r)$, then $I_\mu \in \mathcal{H}(B(x_0,r))$.*
(ii) If f is a σ-integrable function on $S(x_0,r)$, then

$$\limsup_{x\to y, x\in B(x_0,r)} I_f(x) \le \limsup_{z\to y, z\in S(x_0,r)} f(z) \qquad (y \in S(x_0,r)); \qquad (1.3.3)$$

further, if f is continuous in the extended sense at $y \in S(x_0,r)$, then

$$I_f(x) \to f(y) \quad (x \to y; x \in B(x_0,r)). \qquad (1.3.4)$$

Proof. (i) Since the function $(x,y) \mapsto K(x,y)$ and all its partial derivatives with respect to the coordinates of x are bounded on $B(x_0,\rho) \times S(x_0,r)$ when $0 < \rho < r$, we may pass the Laplace operator Δ under the integral sign in (1.3.2) to see that $I_\mu \in \mathcal{H}(B(x_0,r))$.

(ii) We first show that $I_c \equiv c$ for any finite constant function c. By (i), $I_c \in \mathcal{H}(B(x_0,r))$, and $I_c(x)$ clearly depends only on $\|x-x_0\|$. Since I_c is finite at x_0, it follows from the latter part of Theorem 1.1.2 that I_c is constant on $B(x_0,r)$. Hence $I_c \equiv I_c(x_0) = c$.

To prove (1.3.3), we suppose that

$$\limsup_{z\to y, z\in S(x_0,r)} f(z) < A < +\infty \qquad (1.3.5)$$

and will deduce that

$$\limsup_{x\to y, x\in B(x_0,r)} I_f(x) \le A. \qquad (1.3.6)$$

From (1.3.5) there exists $\delta > 0$ such that $f(z) < A$ whenever $z \in B(y,2\delta) \cap S(x_0,r)$. If $x \in B(y,\delta) \cap B(x_0,r)$, then by the result of the previous paragraph

$$\begin{aligned} I_f(x) - A &= I_{f-A}(x) \\ &\le \int_{S(x_0,r)\backslash B(y,2\delta)} K(x,z)|f(z)-A|\, d\sigma(z) \\ &\quad + \int_{S(x_0,r)\cap B(y,2\delta)} K(x,z)\{f(z)-A\}\, d\sigma(z). \end{aligned}$$

The second integral here is negative, and the first does not exceed

$$\frac{1}{\sigma_N r}\frac{r^2 - \|x-x_0\|^2}{\delta^N}\int_{S(x_0,r)} (|f(z)| + |A|)\, d\sigma(z),$$

which tends to 0 as $x \to y$. Hence (1.3.6) holds.

If f is continuous at y, then (1.3.4) follows by applying (1.3.3) to f and $-f$. □

Corollary 1.3.4. *If $h \in C(\overline{B(x_0,r)}) \cap \mathcal{H}(B(x_0,r))$, then $h = I_h$ on $B(x_0,r)$.*

Proof. By Theorem 1.3.3, $h - I_h$ is harmonic on $B(x_0,r)$ and tends to 0 at each point of $S(x_0,r)$. Hence, by the maximum principle, $h - I_h \equiv 0$. □

Theorem 1.3.3 shows that the Poisson integral solves the Dirichlet problem (in the form stated above) for a ball: if $f \in C(S(x_0,r))$, then I_f is harmonic on $B(x_0,r)$ and satisfies (1.3.4) for each $y \in S(x_0,r)$. Further, it follows from Corollary 1.3.4 that I_f is the unique solution.

We give below some applications of Theorem 1.3.3. The first adds two further equivalent conditions to Theorem 1.2.2.

Theorem 1.3.5. *Let $h \in C(\Omega)$. The following are equivalent:*
(a) $h \in \mathcal{H}(\Omega)$;
(b) for each $x \in \Omega$, there exists a positive number r_x such that $h(x) = \mathcal{M}(h;x,r)$ whenever $0 < r < r_x$;
(c) for each $x \in \Omega$, there exists a positive number r_x such that $h(x) = \mathcal{A}(h;x,r)$ whenever $0 < r < r_x$.

Proof. Theorem 1.2.2 shows that (a) implies (b), and (b) implies (c), by (1.2.4). We now suppose that (c) holds. It is enough to show that, if $\overline{B(y,\rho)} \subset \Omega$, then $h = I_{h,y,\rho}$ on $B(y,\rho)$. We define $H = h - I_{h,y,\rho}$ on $B(y,\rho)$ and $H = 0$ on $S(y,\rho)$. By Theorem 1.3.3, $H \in C(\overline{B(y,\rho)})$ and for each $x \in B(y,\rho)$ there exists $\rho_x > 0$ such that $H(x) = \mathcal{A}(H;x,r)$ whenever $0 < r \le \rho_x$. It now follows from Remark 1.2.5(a) and Theorem 1.2.4(iii) that $H \equiv 0$ on $B(y,\rho)$, as required. □

Theorem 1.3.5 is the key to the following result concerning harmonic continuation across a flat boundary. For each point $x = (x_1,\ldots,x_N)$ in $\mathbb{R}^N$, we denote the point $(x_1,\ldots,x_{N-1})$ in $\mathbb{R}^{N-1}$ by x', and we write $\overline{x} = (x', -x_N)$ for the image of x under reflection in the hyperplane $\{y \in \mathbb{R}^N : y_N = 0\}$.

Theorem 1.3.6. (The reflection principle) *Let Ω be such that $\overline{x} \in \Omega$ whenever $x \in \Omega$, and let $\Omega_+, \Omega_0, \Omega_-$ denote the sets of points x in Ω for which x_N is respectively positive, zero, negative. If $h \in \mathcal{H}(\Omega_+)$ and $h(x) \to 0$ as $x \to y$ for each $y \in \Omega_0$, then the function $\overline{h}$, defined by*

$$\overline{h}(x) = h(x)\ (x \in \Omega_+),\ \overline{h}(x) = 0\ (x \in \Omega_0),\ \overline{h}(x) = -h(\overline{x})\ (x \in \Omega_-),$$

is harmonic on Ω.

Proof. The continuity of $\overline{h}$ is clear, and it is easy to see that $\overline{h}(x) = \mathcal{M}(\overline{h};x,r)$ if $\overline{B(x,r)} \subset \Omega_+$ or $\overline{B(x,r)} \subset \Omega_-$ or if $x \in \Omega_0$ and $\overline{B(x,r)} \subset \Omega$. By Theorem 1.3.5, this is sufficient to show that $\overline{h} \in \mathcal{H}(\Omega)$. □

The next application of Theorem 1.3.3 is a removable singularity result for harmonic functions. It involves the fundamental harmonic function introduced in Section 1.1.

Theorem 1.3.7. *If h is harmonic on $B(x_0,r)\backslash\{x_0\}$ and $h(x)/U_{x_0}(x) \to 0$ as $x \to x_0$, then h has a harmonic continuation to $B(x_0,r)$.*

Proof. Fix $\rho \in (0,r)$ and let I_h denote the Poisson integral of h in $B(x_0,\rho)$. It is enough to show that $h = I_h$ on $B(x_0,\rho)\backslash\{x_0\}$. Let $U = U_{x_0} - c_N$, where the constant c_N is chosen such that $U = 0$ on $S(x_0,\rho)$. For each real number α, define $H_\alpha = h - I_h + \alpha U$. Then $H_\alpha \in \mathcal{H}(B(x_0,r)\backslash\{x_0\})$ and by Theorem 1.3.3, $H_\alpha(x) \to 0$ as $x \to y$ for each $y \in S(x_0,\rho)$. If $\alpha > 0$, then $H_\alpha(x) \to +\infty$ as $x \to x_0$ and so $H_\alpha > 0$ on $B(x_0,\rho)\backslash\{x_0\}$ by the minimum principle; similarly, $H_\alpha < 0$ on $B(x_0,\rho)\backslash\{x_0\}$ if $\alpha < 0$. Hence, letting $\alpha \to 0$, we obtain $h = I_h$ on $B(x_0,\rho)\backslash\{x_0\}$, as required. □

We know from Theorem 1.3.3 that, if μ is a signed measure on $S(x_0,r)$, then $I_\mu \in \mathcal{H}(B(x_0,r))$. Next we characterize those elements of $\mathcal{H}(B(x_0,r))$ which are equal to I_μ for some such μ; in particular, we show that all positive elements of $\mathcal{H}(B(x_0,r))$ can be represented in this way with $\mu \ge 0$. The set of all non-negative harmonic functions on Ω will be denoted by $\mathcal{H}_+(\Omega)$.

Theorem 1.3.8. (Riesz–Herglotz) *Let $h \in \mathcal{H}(B(x_0,r))$. The following are equivalent:*
(a) there exist $h_1, h_2 \in \mathcal{H}_+(B(x_0,r))$ such that $h = h_1 - h_2$;
(b) there exists $h_0 \in \mathcal{H}(B(x_0,r))$ such that $|h| \le h_0$ on $B(x_0,r)$;
(c) $\mathcal{M}(|h|;x_0,\cdot)$ is bounded on $(0,r)$;
(d) $h = I_\mu$ for some signed measure μ on $S(x_0,r)$.
Further, if any of the above holds, then μ is unique, and $\mu \ge 0$ if $h \ge 0$.

Proof. If (d) holds, then μ is the difference of two (positive) measures and h is the difference of their Poisson integrals, so (a) holds. If (a) holds, then $|h| \le h_1 + h_2$ on $B(x_0,r)$, so (b) holds. If (b) holds and $0 < t < r$, then $\mathcal{M}(|h|;x_0,t) \le \mathcal{M}(h_0;x_0,t) = h_0(x_0)$, so (c) holds. We next show that (c) implies (d). It is enough to deal with the case where $x_0 = 0$ and $r = 1$, for the general case then follows by means of a simple transformation.

Suppose that (c) holds and let $h_t(x) = h(tx)$ when $0 < t < 1$ and $\|x\| < t^{-1}$. Then h_t is harmonic on $B(0,t^{-1})$, so $h_t = I_{h_t}$ by Corollary 1.3.4. Let μ_t denote the signed measure on the unit sphere S defined by

$$d\mu_t(y) = h_t(y)d\sigma(y) = h(ty)d\sigma(y),$$

and let $\|\mu_t\|$ denote its total variation. By hypothesis there is a constant M such that

$$\|\mu_t\| = \sigma_N \mathcal{M}(|h|;0,t) \le M \quad (0 < t < 1).$$

Hence (see Appendix) there is a signed measure μ and a sequence (t_n) such that $t_n \uparrow 1$ and

$$\int \psi \, d\mu_{t_n} \to \int \psi \, d\mu \quad (\psi \in C(S)). \tag{1.3.7}$$

Thus, for any x in the unit ball B, we obtain as desired

$$h(x) = \lim_{n\to\infty} h_{t_n}(x) = \lim_{n\to\infty} \int_S K_{0,1}(x,y)\, d\mu_{t_n}(y) = I_\mu(x).$$

To prove uniqueness, suppose that ν is another signed measure such that $I_\nu = h$. If $x, y \in S$ and $0 < t < 1$, then $\|y - tx\| = \|ty - x\|$, so $K_{0,1}(tx, y) = K_{0,1}(ty, x)$. Thus, if $\psi \in C(S)$, then

$$\begin{aligned}\int_S \psi d\mu_t &= \int_S \psi(x) h(tx)\, d\sigma(x)\\ &= \int_S \psi(x) \int_S K_{0,1}(tx,y)\, d\nu(y)\, d\sigma(x)\\ &= \int_S \int_S K_{0,1}(ty,x)\psi(x)\, d\sigma(x)\, d\nu(y)\\ &= \int_S I_\psi(ty)\, d\nu(y)\\ &\to \int_S \psi(y)\, d\nu(y) \quad (t \to 1-),\end{aligned}$$

using first Fubini's theorem and then Theorem 1.3.3(ii) and dominated convergence. In view of the uniqueness of μ in (1.3.7), we have $\nu = \mu$.

Finally, if $h \geq 0$, then $\mu_t \geq 0$ for all t and so $\mu \geq 0$. □

We can similarly characterize those harmonic functions on a ball which are expressible as the Poisson integral of an integrable boundary function. Convex functions will be discussed in Section 3.4. For now we simply recall that a real-valued function ϕ on an interval J is called *convex* if

$$\phi(t) \leq \frac{t_2 - t}{t_2 - t_1}\phi(t_1) + \frac{t - t_1}{t_2 - t_1}\phi(t_2) \qquad (t_1, t_2 \in J; t_1 < t < t_2).$$

Theorem 1.3.9. *Let $h \in \mathcal{H}(B(x_0, r))$. The following are equivalent:*
(a) there is a convex increasing function $\phi : [0, +\infty) \to [0, +\infty)$ such that $t^{-1}\phi(t) \to +\infty$ as $t \to +\infty$ and $\mathcal{M}(\phi \circ |h|; x_0, \cdot)$ is bounded on $(0, r)$;
(b) $h = I_f$ for some integrable function $f : S(x_0, r) \to [-\infty, +\infty]$.
If $h \geq 0$, then a further equivalent condition is that h is the limit of an increasing sequence of bounded non-negative harmonic functions.

Proof. We again give the argument for $h \in \mathcal{H}(B)$. Suppose first that (a) holds. Then there exists $c > 0$ such that $t \leq \phi(t) + c$ for all $t \geq 0$, and so $\mathcal{M}(|h|; 0, \cdot)$ is bounded on $(0,1)$ by hypothesis. Thus, by the Riesz–Herglotz theorem, there is a signed measure μ on S such that $h = I_\mu$. We will show that μ is absolutely continuous with respect to σ (see Appendix, Definition A.2). Let $\varepsilon > 0$ and $a_\varepsilon = \sup\{t/\phi(t) : t \geq \varepsilon^{-1/2}\}$. Then $|t| \leq a_\varepsilon \phi(|t|) + \varepsilon^{-1/2}$ on $\mathbb{R}$. Let A_ε be a relatively open subset of S such that $\sigma(A_\varepsilon) < \varepsilon$ and let $\psi : S \to [0,1]$ be a continuous function such that $\psi = 0$ on $S \backslash A_\varepsilon$. Then

$$\begin{aligned}\left|\int_S h(\rho y)\psi(y)\, d\sigma(y)\right| &\leq \int_S |h(\rho y)|\psi(y)\, d\sigma(y)\\ &\leq a_\varepsilon \int_S \phi(|h(\rho y)|)\psi(y)\, d\sigma(y) + \varepsilon^{-1/2}\int_{A_\varepsilon} \psi(y)\, d\sigma(y).\end{aligned}$$

We let $\rho \to 1-$ and use (1.3.7) and the facts that $\psi \leq 1$ and $\sigma(A_\varepsilon) < \varepsilon$ to obtain

$$\left|\int_S \psi\, d\mu\right| \leq a_\varepsilon \sigma_N \sup_{(0,1)} \mathcal{M}(\phi \circ |h|; 0, \cdot) + \varepsilon^{1/2}.$$

Hence any measurable subset A of A_ε satisfies

$$|\mu(A)| \leq a_\varepsilon \sigma_N \sup_{(0,1)} \mathcal{M}(\phi \circ |h|; 0, \cdot) + \varepsilon^{1/2}.$$

Since $a_\varepsilon \to 0$ as $\varepsilon \to 0$, by the growth hypothesis on ϕ, it follows that μ is absolutely continuous with respect to σ, and (b) is proved in view of the Radon–Nikodým theorem (see Appendix).

Conversely, suppose that (b) holds and let

$$S_n = \{y \in S : n - 1 \leq |f(y)| < n\} \quad (n \in \mathbb{N}).$$

Then $\sum_n n\sigma(S_n) < +\infty$, and it follows that there is an increasing sequence (b_n) of positive numbers such that $b_n \to +\infty$ and $\sum_n nb_n\sigma(S_n) < +\infty$. Let $\phi : [0, +\infty) \to [0, +\infty)$ be the function whose graph consists of line segments joining the points $\{(0,0), (1, b_1), (2, b_1 + b_2), (3, b_1 + b_2 + b_3), \ldots\}$. Then ϕ is convex and increasing, $t^{-1}\phi(t) \to +\infty$ as $t \to +\infty$ and

$$\begin{aligned}\int \phi \circ |f|\, d\sigma &= \sum_{n=1}^{\infty} \int_{S_n} \phi \circ |f|\, d\sigma\\ &\leq \sum_{n=1}^{\infty} \phi(n)\sigma(S_n)\\ &\leq \sum_{n=1}^{\infty} nb_n\sigma(S_n) < +\infty. \qquad (1.3.8)\end{aligned}$$

Next we note that $\phi(\int g\, d\nu) \leq \int \phi \circ g\, d\nu$ for any unit measure ν on S and any integrable function $g : S \to [0, +\infty)$. (This is known as Jensen's inequality, and follows from the fact that by convexity $\phi(t) = \sup_{\alpha,\beta}(\alpha t + \beta)$, where the supremum is over all $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha t + \beta \leq \phi(t)$ when $t \geq 0$.) We can use this inequality with $g = |f|$ and $d\nu(y) = K(x,y)\, d\sigma(y)$, since $\nu(S) = I_1(x) = 1$. Thus

$$\phi(I_{|f|}(x)) \leq I_{\phi \circ |f|}(x) \qquad (x \in B).$$

Hence

$$\mathcal{M}(\phi\circ|h|;0,\cdot) \le \mathcal{M}(\phi\circ I_{|f|};0,\cdot) \le \mathcal{M}(I_{\phi\circ|f|};0,\cdot) = I_{\phi\circ|f|}(0) = \sigma_N^{-1}\int \phi\circ|f|\,d\sigma$$

on $(0,1)$, and so (a) holds in view of (1.3.8).

Finally, we consider the special case where $h \ge 0$. If (b) holds, then (suitably redefining f on a set of σ-measure 0) we may assume that $f \ge 0$, and $(I_{\min\{f,n\}})_{n\ge1}$ is an increasing sequence of bounded non-negative harmonic functions with limit h. Conversely, suppose that $h = \lim h_n$, where each h_n is a bounded non-negative harmonic function and (h_n) is increasing. For each n in $\mathbb{N}$ we use the fact that (a) implies (b) to see that $h_n = I_{f_n}$ for some integrable function f_n on $S(x_0,r)$. Further, since $h_{n+1} - h_n = I_{f_{n+1}-f_n}$, it follows from the final assertion of the Riesz–Herglotz theorem that (f_n) can be chosen to be increasing. Hence h is the Poisson integral of $\lim f_n$ and so (b) holds. □

Later, in Theorem 4.6.6, we will see that the function f in condition (b) of the above result is determined at σ-almost every point $z \in S(x_0,r)$ by the limit of $h(x)$ as x approaches z in a "non-tangential" manner.

Corollary 1.3.10. *If h is a bounded harmonic function on $B(x_0,r)$, then there is a σ-measurable function f on $S(x_0,r)$ such that $h = I_f$ and $\sup|f| = \sup|h|$.*

Proof. Let $M = \sup|h|$. Since $\mathcal{M}(h^2;x_0,\cdot) \le M^2$, it follows from Theorem 1.3.9 that there is an integrable function f on $S(x_0,r)$ such that $h = I_f$. Further, $I_{f+M} = h+M \ge 0$, so $f+M \ge 0$ almost everywhere (σ) on $S(x_0,r)$ by Theorem 1.3.8, and we can redefine f on a set of zero σ-measure so that $f \ge -M$. Similarly we can arrange that $f \le M$. Thus $\sup|f| \le \sup|h|$, and the reverse inequality is clear from the fact that $I_1 \equiv 1$. □

If $h \in \mathcal{H}(B(x,r))$ and h is integrable on $B(x,r)$, then it follows from Theorem 1.2.2 and dominated convergence that $h(x) = \mathcal{A}(h;x,r)$. Thus, if Ω is a ball of centre x,

$$h(x) = \frac{1}{\lambda(\Omega)}\int_\Omega h\,d\lambda \tag{1.3.9}$$

for every integrable harmonic function on Ω. We conclude this section by exploiting the properties of the Poisson kernel to prove a converse result.

Theorem 1.3.11. *If $x \in \Omega$ and $\lambda(\Omega) < +\infty$ and if (1.3.9) holds for every integrable harmonic function h on Ω, then Ω is a ball of centre x.*

Proof. We choose $r > 0$ such that $B(x,r) \subseteq \Omega$ and $S(x,r)\cap\partial\Omega \ne \emptyset$. Let $y \in S(x,r)\cap\partial\Omega$ and let $h = K_{x,r}(\,\cdot\,,y)$. Then $h \in \mathcal{H}(\mathbb{R}^N\backslash\{y\})$ and h is bounded on $\mathbb{R}^N\backslash B(y,1)$, and the function $z \mapsto \|z-y\|^{N-1}h(z)$ is bounded on $B(y,1)$. Hence h is integrable on Ω. Clearly $h > 0$ on $B(x,r)$ and $h < 0$ on $\mathbb{R}^N\backslash\overline{B(x,r)}$. If $\Omega\backslash\overline{B(x,r)} \ne \emptyset$, then

$$\frac{1}{\lambda(\Omega)}\int_\Omega h\,d\lambda < \frac{1}{\lambda(\Omega)}\int_{B(x,r)} h\,d\lambda < \frac{1}{\lambda(B(x,r))}\int_{B(x,r)} h\,d\lambda = h(x),$$

which contradicts (1.3.9). Hence $B(x,r) \subseteq \Omega \subseteq \overline{B(x,r)}$ and so $\Omega = B(x,r)$. □

1.4. Harnack's inequalities

We will now use the Poisson integral representation to make important observations about the "rigidity" of positive harmonic functions.

Theorem 1.4.1. (Harnack's inequalities) *If $h \in \mathcal{H}_+(B(x_0,r))$, then*

$$\frac{(r-\|x-x_0\|)r^{N-2}}{(r+\|x-x_0\|)^{N-1}}h(x_0) \le h(x) \le \frac{(r+\|x-x_0\|)r^{N-2}}{(r-\|x-x_0\|)^{N-1}}h(x_0) \tag{1.4.1}$$

for each $x \in B(x_0,r)$. In particular, if $0 < \alpha < 1$, then

$$\frac{1-\alpha}{(1+\alpha)^{N-1}}h(x_0) \le h(x) \le \frac{1+\alpha}{(1-\alpha)^{N-1}}h(x_0) \quad (x \in B(x_0,\alpha r)). \tag{1.4.2}$$

Proof. The Riesz–Herglotz theorem shows that $h = I_\mu$ for some measure μ on $S(x_0,r)$ and clearly

$$h(x_0) = \int_{S(x_0,r)} K(x_0,y)\,d\mu(y) = \frac{\mu(S(x_0,r))}{\sigma_N r^{N-1}}. \tag{1.4.3}$$

The Poisson kernel K satisfies

$$\frac{r-\|x-x_0\|}{\sigma_N r(r+\|x-x_0\|)^{N-1}} \le K(x,y) \le \frac{r+\|x-x_0\|}{\sigma_N r(r-\|x-x_0\|)^{N-1}}$$

when $x \in B(x_0,r)$ and $y \in S(x_0,r)$. Integration with respect to $d\mu(y)$ yields (1.4.1) in view of (1.4.3), and (1.4.2) follows easily. □

Corollary 1.4.2. *If $h \in \mathcal{H}_+(B(x_0,r))$, then $\|\nabla h(x_0)\| \le (N/r)h(x_0)$.*

Proof. From (1.4.1),

$$\left\{\frac{(r-\|x-x_0\|)r^{N-2}}{(r+\|x-x_0\|)^{N-1}} - 1\right\}h(x_0) \le h(x) - h(x_0)$$
$$\le \left\{\frac{(r+\|x-x_0\|)r^{N-2}}{(r-\|x-x_0\|)^{N-1}} - 1\right\}h(x_0),$$

so that

$$\left|h(x)-h(x_0)\right| \le \{(N/r)\|x-x_0\| + O(\|x-x_0\|^2)\}h(x_0) \qquad (x \to x_0),$$

and the conclusion follows. □

Corollary 1.4.3. *Let E be a compact subset of Ω and D be a linear partial differential operator with constant coefficients. Then there is a constant C, depending only on E, Ω and D with the following property: if $h \in \mathcal{H}(\Omega)$ and $|h| \le M$ on Ω, then $|Dh| \le CM$ on E.*

Proof. Let ω be a bounded open set such that $E \subset \omega$ and $\overline{\omega} \subset \Omega$, and let $r > 0$ be such that $B(x,r) \subset \Omega$ for each x in ω. If $h \in \mathcal{H}(\Omega)$ and $|h| \le M$ on Ω, then $(h+M) \in \mathcal{H}_+(\omega)$ and we can apply Corollary 1.4.2 to the function $h+M$ to obtain

$$\left|\frac{\partial h}{\partial x_j}(x)\right| = \left|\frac{\partial (h+M)}{\partial x_j}(x)\right| \le \frac{N}{r}(h(x)+M) \le \frac{2N}{r}M \qquad (x \in \omega).$$

This implies the result in the case where $D = \partial/\partial x_j$. The general case follows using Corollary 1.2.3 and induction. □

Corollary 1.4.4. *If Ω is connected and E is a compact subset of Ω, then there is a constant C such that*

$$C^{-1}h(x) \le h(y) \le Ch(x) \qquad (x, y \in E)$$

for every $h \in \mathcal{H}_+(\Omega)$.

Proof. We treat first the case where E is a 2-point set $\{w, z\}$. Let $x_0 = w$ and let $B(x_1, 2r_1), \ldots, B(x_n, 2r_n)$ be open balls in Ω such that $x_n = z$ and $x_{j-1} \in B(x_j, r_j)$ when $j \in \{1, \ldots, n\}$. If $h \in \mathcal{H}_+(\Omega)$, then we apply (1.4.2) with $\alpha = 1/2$ to obtain

$$h(w) = h(x_0) \le C_1 h(x_1) \le C_1^2 h(x_2) \le \cdots \le C_1^n h(x_n) = C_1^n h(z),$$

where $C_1 = 3.2^{N-2}$. Thus the result holds when E is a 2-point set and hence when E is finite. In the general case let $\{B(y_k, t_k) : k = 1, \ldots, m\}$ be a cover of E such that $B(y_k, 2t_k) \subset \Omega$ for each k. By the finite case, there is a positive constant C_2 such that $h(y_j) \le C_2 h(y_k)$ when $j, k \in \{1, \ldots, m\}$. If $x, y \in E$, then we choose j, k such that $x \in B(y_j, t_j)$ and $y \in B(y_k, t_k)$. Then, again using (1.4.2) with $\alpha = 1/2$, we find that

$$h(x) \le C_1 h(y_j) \le C_1 C_2 h(y_k) \le 3^N C_2 h(y).$$

□

Remark 1.4.5. The first part of the above proof shows that if, in the general case, for each pair of points $w, z \in E$ there exist n balls $B(x_1, 2r_1), \ldots, B(x_n, 2r_n)$ as described, n being independent of w and z, then we can take the constant C in Corollary 1.4.4 to be $(3.2^{N-2})^n$.

1.5. Families of harmonic functions: convergence properties

We take as our starting point the following simple result.

Theorem 1.5.1. *If (h_n) is a sequence in $\mathcal{H}(\Omega)$ which converges locally uniformly on Ω to a function h, then $h \in \mathcal{H}(\Omega)$. If, further, D is any linear partial differential operator with constant coefficients, then (Dh_n) converges locally uniformly on Ω to Dh.*

Proof. By local uniform convergence $h \in C(\Omega)$ and $h(x) = \mathcal{M}(h; x, r)$ whenever $\overline{B(x,r)} \subset \Omega$, so $h \in \mathcal{H}(\Omega)$. Now let E be a compact set and ω be a bounded open set such that $E \subset \omega$ and $\overline{\omega} \subset \Omega$. If $\varepsilon > 0$, then $|h_n - h| < \varepsilon$ on $\overline{\omega}$ for all sufficiently large n. Hence, by Corollary 1.4.3, $|Dh_n - Dh| < C\varepsilon$ on E for all such n, where C depends only on E, ω and D. This completes the proof. □

Definition 1.5.2. Let $\mathcal{F}$ be a family of functions from a set E into $[-\infty, +\infty]$. Then $\mathcal{F}$ is said to be *up-directed* if for each pair of functions f_1, f_2 in $\mathcal{F}$ there exists f in $\mathcal{F}$ such that $\max\{f_1, f_2\} \le f$ on E. Also, $\mathcal{F}$ is said to be *down-directed* if $\{-f : f \in \mathcal{F}\}$ is up-directed.

Theorem 1.5.3. *Let Ω be connected. If $\mathcal{F}$ is an up-directed family of harmonic functions on Ω, then either $\sup \mathcal{F} \equiv +\infty$ on Ω or $\sup \mathcal{F} \in \mathcal{H}(\Omega)$.*

Proof. Let $H = \sup \mathcal{F}$ and suppose that $H \not\equiv +\infty$. We choose x_0 in Ω such that $H(x_0) < +\infty$. Let E be any compact subset of Ω and for each n in $\mathbb{N}$ choose h_n in $\mathcal{F}$ such that $h_n(x_0) > H(x_0) - n^{-1}$. Let $h \in \mathcal{F}$. Since $\mathcal{F}$ is up-directed, for each n there exists $g_n \in \mathcal{F}$ such that $\max\{h_n, h\} \le g_n$ on Ω. Hence, by Corollary 1.4.4, there is a positive constant C, depending only on x_0, E and Ω such that

$$\begin{aligned} h(y) - h_n(y) &\le g_n(y) - h_n(y) \\ &\le C\{g_n(x_0) - h_n(x_0)\} \\ &\le C\{H(x_0) - h_n(x_0)\} \le Cn^{-1} \qquad (y \in E). \end{aligned}$$

Taking the supremum over all $h \in \mathcal{F}$, we obtain $H - h_n \le Cn^{-1}$ on E. It now follows from Theorem 1.5.1 that $H \in \mathcal{H}(\Omega)$. □

Corollary 1.5.4. *If Ω is connected and (h_n) is an increasing sequence in $\mathcal{H}(\Omega)$, then either $\lim h_n \equiv +\infty$ or $\lim h_n \in \mathcal{H}(\Omega)$.*

Proof. The family $\{h_n : n \in \mathbb{N}\}$ is up-directed. □

Definition 1.5.5. A family $\mathcal{F}$ of real-valued functions on a set E in $\mathbb{R}^N$ is said to be *equicontinuous* at $x \in E$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for each $f \in \mathcal{F}$ and each $y \in E \cap B(x, \delta)$. Further, $\mathcal{F}$ is called equicontinuous *on* E if it is equicontinuous at each point of E.

The family $\mathcal{F}$ is said to be *uniformly equicontinuous* on E if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $f \in \mathcal{F}$ and $x, y \in E$ and $\|x - y\| < \delta$.

Lemma 1.5.6. *Let Ω be connected and $\mathcal{F}$ be a family of harmonic functions locally uniformly bounded below on Ω. Then either* $\sup \mathcal{F} \equiv +\infty$ *on Ω or $\mathcal{F}$ is uniformly bounded and uniformly equicontinuous on each compact subset of Ω.*

Proof. Suppose that $\sup \mathcal{F} \not\equiv +\infty$ on Ω and choose $x_0 \in \Omega$ such that $(\sup \mathcal{F})(x_0) < +\infty$. Let E be a compact set and let ω be a bounded connected open set such that $E \cup \{x_0\} \subset \omega$ and $\overline{\omega} \subset \Omega$. Then $\mathcal{F}$ is uniformly bounded below on $\overline{\omega}$. By adding a suitable constant, we may assume that all members of $\mathcal{F}$ are positive on ω. By Corollary 1.4.4 there is a constant C such that $0 < h < C$ on E for all $h \in \mathcal{F}$ and so $\mathcal{F}$ is uniformly bounded on E. Let $r > 0$ be such that $B(x, r) \subseteq \omega$ for all $x \in E$. If $\varepsilon > 0$, then by (1.4.2) there is a positive constant α, depending only on ε and N, such that

$$(1 - \varepsilon)h(x) \le h(y) \le (1 + \varepsilon)h(x) \qquad (y \in B(x, \alpha r); x \in E)$$

for all $h \in \mathcal{F}$. Hence

$$|h(x) - h(y)| < \varepsilon h(x) < C\varepsilon \qquad (y \in B(x, \alpha r); x \in E)$$

for all $h \in \mathcal{F}$, and so $\mathcal{F}$ is uniformly equicontinuous on E. □

Lemma 1.5.7. *If (f_n) is a uniformly equicontinuous sequence of functions on a bounded set E in $\mathbb{R}^N$ and (f_n) converges pointwise to a function $f : E \to \mathbb{R}$, then f is uniformly continuous on E and $f_n \to f$ uniformly on E.*

Proof. We start with the uniform continuity of f. Let $\varepsilon > 0$ and let δ be as in the definition of uniform equicontinuity. If $x, y \in E$ and $\|x - y\| < \delta$, then for some n we have

$$|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 3\varepsilon. \quad (1.5.1)$$

To prove the uniform convergence of (f_n), let $x_1, \dots, x_m$ be points of E such that $E \subseteq \bigcup_j B(x_j, \delta)$. There exists n_0 such that $|f_n(x_j) - f(x_j)| < \varepsilon$ for all $n \ge n_0$ and all $j \in \{1, \dots, m\}$. If $y \in E$, then $y \in B(x_j, \delta)$ for some j and

$$\begin{aligned}|f_n(y) - f(y)| &\le |f_n(y) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(y)| \\ &< \varepsilon + \varepsilon + 3\varepsilon = 5\varepsilon \qquad (n \ge n_0),\end{aligned}$$

by (1.5.1). □

Theorem 1.5.8. *If (h_n) is a locally uniformly bounded sequence in $\mathcal{H}(\Omega)$ and (h_n) converges pointwise on Ω to a function h, then (h_n) converges locally uniformly on Ω and $h \in \mathcal{H}(\Omega)$.*

Proof. By Lemma 1.5.6, (h_n) is uniformly equicontinuous on every compact subset of Ω. Hence, by Lemma 1.5.7, the convergence of (h_n) is locally uniform on Ω, and by Theorem 1.5.1, $h \in \mathcal{H}(\Omega)$. □

Definition 1.5.9. Let $\mathcal{F} \subseteq C(\Omega)$. We say that $\mathcal{F}$ is *normal* if every sequence in $\mathcal{F}$ has a subsequence which converges locally uniformly on Ω.

Lemma 1.5.10. *Let $\mathcal{F}$ be a family of real-valued functions on Ω which is uniformly bounded and uniformly equicontinuous on each compact subset of Ω. Then $\mathcal{F}$ is normal.*

Proof. Let (f_n) be a sequence in $\mathcal{F}$, let $E \subset \Omega$ be compact and let $\{x_j : j \in \mathbb{N}\}$ be a dense subset of E. Since $(f_n(x_1))$ is a bounded sequence of numbers, it has a convergent subsequence $(f_{1,n}(x_1))$. Similarly $(f_{1,n}(x_2))$ has a convergent subsequence $(f_{2,n}(x_2))$, and in general there are sequences $(f_{m,n})_{n \ge 1}$ such that $(f_{m,n}(x_m))$ converges as $n \to \infty$ and $(f_{m+1,n})_{n \ge 1}$ is a subsequence of $(f_{m,n})_{n \ge 1}$. Let $g_n = f_{n,n}$ for each n. Then $(g_n(x_j))$ converges for each j.

Now suppose that $y \in E$ and $\varepsilon > 0$. Let δ be as in the definition of uniform equicontinuity. Since $\{x_j : j \in \mathbb{N}\}$ is dense in E, we have $\|x_j - y\| < \delta$ for some j. Since $(g_n(x_j))$ converges,

$$\begin{aligned}|g_k(y) - g_n(y)| &\le |g_k(y) - g_k(x_j)| + |g_k(x_j) - g_n(x_j)| + |g_n(x_j) - g_n(y)| \\ &< 3\varepsilon\end{aligned}$$

when n and k are sufficiently large. Hence $(g_n(y))$ is Cauchy and therefore convergent. It follows from Lemma 1.5.7, applied to arbitrary compact subsets E of Ω, that $\mathcal{F}$ is normal. □

Theorem 1.5.11. *Let Ω be connected and let $\mathcal{F}$ be a family in $\mathcal{H}(\Omega)$ which is locally uniformly bounded below on Ω. If (h_n) is a sequence in $\mathcal{F}$, then there is a subsequence (h_{n_j}) such that either (h_{n_j}) is locally uniformly convergent to a harmonic function on Ω or* $\lim h_{n_j} \equiv +\infty$ *on Ω.*

Proof. Let (h_n) be a sequence in $\mathcal{F}$. By Lemma 1.5.6, either $\sup h_n \equiv +\infty$ on Ω or (h_n) is uniformly bounded and uniformly equicontinuous on each compact subset of Ω. In the latter case, it follows from Lemma 1.5.10 that there is a subsequence (h_{n_j}) which converges locally uniformly on Ω to a function h, and $h \in \mathcal{H}(\Omega)$ by Theorem 1.5.1. In the case where $\sup h_n \equiv +\infty$

we fix $x_0 \in \Omega$ and choose (h_{n_j}) such that $h_{n_j}(x_0) \to +\infty$. Given $x \in \Omega$ we choose a bounded connected open set ω such that $x, x_0 \in \omega$ and $\overline{\omega} \subset \Omega$, and choose $M \in \mathbb{R}$ such that $h_n \geq M$ on ω for all n. By Corollary 1.4.4, with $E = \{x, x_0\}$,

$$h_{n_j}(x) - M \geq C^{-1}\{h_{n_j}(x_0) - M\} \to +\infty,$$

so $h_{n_j}(x) \to +\infty$ as required. □

We conclude this section with an application of Theorem 1.5.8. We denote a point of $\mathbb{R}^{N-1}$ by x' and write λ' for $(N-1)$-dimensional Lebesgue measure.

Theorem 1.5.12. *Let $\Omega = \mathbb{R}^{N-1} \times (a,b)$, where $-\infty \leq a < b \leq +\infty$. If $h \in \mathcal{H}(\Omega)$ and the function*

$$t \mapsto \int_{\mathbb{R}^{N-1}} |h(x',t)|\, d\lambda'(x') \qquad (a < t < b)$$

is locally bounded on (a,b), then the equation

$$\mathcal{L}(h;t) = \int_{\mathbb{R}^{N-1}} h(x',t)\, d\lambda'(x') \qquad (a < t < b)$$

defines a polynomial $\mathcal{L}(h;\cdot)$ of degree at most 1.

Proof. For each $m \in \mathbb{N}$ we define

$$h_m(x', x_N) = \int_{\{y' \in \mathbb{R}^{N-1}: \|y'\| < m\}} h(x'+y', x_N)\, d\lambda'(y') \qquad ((x', x_N) \in \Omega).$$

By dominated convergence, h_m is continuous, and using Fubini's theorem to justify a change of order of integration, we see that h_m has the mean value property. Hence $h_m \in \mathcal{H}(\Omega)$. Also

$$|h_m(x', x_N)| \leq \int_{\mathbb{R}^{N-1}} |h(y', x_N)|\, d\lambda'(y') \qquad (m \in \mathbb{N}; (x', x_N) \in \Omega),$$

so the sequence (h_m) is locally uniformly bounded on Ω. Since

$$h_m(x', x_N) \to \mathcal{L}(h; x_N) \quad \text{as} \quad m \to \infty,$$

it follows from Theorem 1.5.8 that the function

$$(x', x_N) \mapsto \mathcal{L}(h; x_N)$$

is harmonic on Ω and therefore $\mathcal{L}(h;\cdot)$ is a polynomial of degree at most 1. □

1.6. The Kelvin transform

We begin by observing that, in the plane, harmonic functions are preserved under composition with holomorphic and anti-holomorphic functions.

Theorem 1.6.1. *Let Ω_1, Ω_2 be plane domains, let $f : \Omega_1 \to \Omega_2$ and let $h \in \mathcal{H}(\Omega_2)$. Then $h \circ f \in \mathcal{H}(\Omega_1)$ if either f or its complex conjugate $\overline{f}$ is holomorphic on Ω_1.*

Proof. If f is holomorphic, then it follows from the Cauchy–Riemann equations and the harmonicity of $\operatorname{Re} f$ and $\operatorname{Im} f$, that

$$\Delta(h \circ f) = ((\Delta h) \circ f)|f'|^2 = 0$$

on Ω_1, and a similar argument applies if $\overline{f}$ is holomorphic. □

In all dimensions we know that, if $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is an isometry or dilation, and if $h \in \mathcal{H}(\phi(\Omega))$, then $h \circ \phi \in \mathcal{H}(\Omega)$. However, the inversion map $\psi(x) = \|x\|^{-2}x$ on $\mathbb{R}^N \backslash \{0\}$, which is potentially so useful for mapping between bounded and unbounded domains, fails to preserve harmonicity when $N \geq 3$. Indeed, even in the special case of a spherically symmetric harmonic function $h(x) = a\|x\|^{2-N} + b$, where $a \neq 0$, we see that $(h \circ \psi)(x) = a\|x\|^{N-2} + b$, which is not harmonic. To obtain a harmonic function in this case, we would instead have to consider the function $x \mapsto \|x\|^{2-N}(h \circ \psi)(x)$. We will see below that this formula is the appropriate one even when h is not spherically symmetric.

Definition 1.6.2. Let $S(y,a)$ be a fixed sphere in $\mathbb{R}^N$. If $x \in \mathbb{R}^N \backslash \{y\}$, then the *inverse* of x with respect to $S(y,a)$ is the point

$$x^* = \frac{a^2}{\|x-y\|^2}(x-y) + y.$$

Thus x^* lies on the ray emanating from y and passing through x, and is determined by the condition $\|x-y\|\|x^*-y\| = a^2$ (see Figure 1.1). The *inverse* of a set E in $\mathbb{R}^N$ with respect to $S(y,a)$ is the set $E^* = \{x^* : x \in E \backslash \{y\}\}$.

If f is a function defined at least on E, then we define f^* on E^* by

$$f^*(x) = \left(\frac{a}{\|x-y\|}\right)^{N-2} f(x^*).$$

The mapping $f \mapsto f^*$ is called the *Kelvin transform (with respect to $S(y,a)$).*

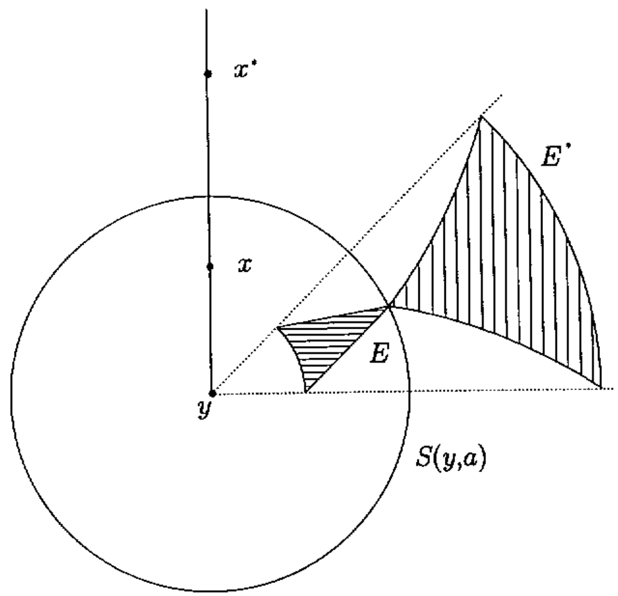

Figure 1.1.

If $x \in \mathbb{R}^N \setminus \{y\}$, then clearly $(x^*)^* = x$. Hence, if $y \notin E$, then $(E^*)^* = E$. The inverse of a sphere or hyperplane is a sphere or hyperplane (not necessarily respectively), possibly with one point deleted. This can be seen in the case where $y = 0$ by observing that a set E in $\mathbb{R}^N$ is a sphere or hyperplane if and only if

$$E = \{x \in \mathbb{R}^N : \alpha\|x\|^2 + \langle x, z\rangle + \beta = 0\},$$

where α and β are real numbers, $z \in \mathbb{R}^N$ and $\|z\|^2 > 4\alpha\beta$. Under inversion with respect to $S(0,a)$ the image of this set E is

$$E^* = \{x \in \mathbb{R}^N \setminus \{0\} : \beta\|x\|^2 + \langle x, a^2 z\rangle + a^4\alpha = 0\}.$$

This establishes the claim with $y = 0$, and to deal with the general case, we can apply a translation.

Theorem 1.6.3. *If $f \in C^2(\Omega)$ and f^* is the image of f under the Kelvin transform with respect to S, then*

$$\Delta f^*(x) = \|x\|^{-2-N}(\Delta f)(x^*) \qquad (x \in \Omega^*). \tag{1.6.1}$$

Proof. By definition $x^* = \|x\|^{-2}x$ and $f^*(x) = \|x\|^{2-N} f(x^*)$ when $x \in \Omega^*$. Writing f_j for $\partial f/\partial x_j$ and f_{kj} for $\partial^2 f/\partial x_j \partial x_k$, we have for $j \in \{1, \dots, N\}$:

$$\frac{\partial}{\partial x_j} f(x^*) = \|x\|^{-2} f_j(x^*) - 2x_j\|x\|^{-4}\sum_{k=1}^N x_k f_k(x^*), \tag{1.6.2}$$

$$\begin{aligned}\frac{\partial^2}{\partial x_j^2} f(x^*) = &-4x_j\|x\|^{-4} f_j(x^*) - 2\|x\|^{-4}\sum_{k=1}^N x_k f_k(x^*)\\ &+ 8x_j^2\|x\|^{-6}\sum_{k=1}^N x_k f_k(x^*)\\ &+ \|x\|^{-4} f_{jj}(x^*) - 4x_j\|x\|^{-6}\sum_{k=1}^N x_k f_{kj}(x^*)\\ &+ 4x_j^2\|x\|^{-8}\sum_{k=1}^N\sum_{n=1}^N x_k x_n f_{kn}(x^*).\end{aligned} \tag{1.6.3}$$

Summing (1.6.3) over j, we see that

$$\Delta\big(f(x^*)\big) = (4-2N)\|x\|^{-4}\sum_{j=1}^N x_j f_j(x^*) + \|x\|^{-4}\big(\Delta f\big)(x^*). \tag{1.6.4}$$

From (1.6.2) we obtain

$$\sum_{j=1}^N \left(\frac{\partial}{\partial x_j}\|x\|^{2-N}\right)\left(\frac{\partial}{\partial x_j} f(x^*)\right) = (N-2)\|x\|^{-N-2}\sum_{j=1}^N x_j f_j(x^*). \tag{1.6.5}$$

Using (1.6.4), (1.6.5) and the harmonicity of the function $x \mapsto \|x\|^{2-N}$ on $\mathbb{R}^N \setminus \{0\}$, and applying the identity $\Delta(uv) = u\Delta v + 2\langle\nabla u, \nabla v\rangle + v\Delta u$, we obtain

$$\begin{aligned}\Delta f^*(x) &= 2\sum_{j=1}^N\Big(\frac{\partial}{\partial x_j}\|x\|^{2-N}\Big)\Big(\frac{\partial}{\partial x_j} f(x^*)\Big) + \|x\|^{2-N}\Delta\big(f(x^*)\big)\\ &= \|x\|^{-2-N}(\Delta f)(x^*),\end{aligned}$$

that is, (1.6.1) holds. □

Corollary 1.6.4. *If $h \in \mathcal{H}(\Omega)$ and h^* is the image of h under the Kelvin transform with respect to $S(y,a)$, then $h^* \in \mathcal{H}(\Omega^*)$.*

Proof. By means of a dilation and an isometry we can reduce the proof to the special case $S(y,a) = S$, which is contained in Theorem 1.6.3. □

1.7. Harmonic functions on half-spaces

In this section we will study the Poisson integral for the half-space D given by $\{x = (x_1, \ldots, x_N) : x_N > 0\}$ and obtain analogues of some of the representation theorems in Section 1.3. In particular, we will see how the Kelvin transform can be used to obtain an analogue of the Riesz–Herglotz theorem for D.

Definition 1.7.1. The *Poisson kernel* $\mathcal{K}$ of D is defined by

$$\mathcal{K}(x,y) = \frac{2}{\sigma_N} \frac{x_N}{\|x-y\|^N} \qquad (y \in \partial D; x \in \mathbb{R}^N \backslash \{y\}).$$

If μ is a measure on ∂D, then the *Poisson integral* $\mathcal{I}_\mu$ of μ is defined by

$$\mathcal{I}_\mu(x) = \int_{\partial D} \mathcal{K}(x,y)\, d\mu(y) \qquad (x \in D).$$

We note that $\mathcal{K}(\cdot\,, y) \in \mathcal{H}_+(D)$ for each y in ∂D; the harmonicity here follows from the observation that $\mathcal{K}(x,y)$ is a multiple of $\partial U_y / \partial x_N$.

Theorem 1.7.2. *Let μ be a measure on ∂D. If*

$$\int_{\partial D} \frac{1}{1+\|y\|^N}\, d\mu(y) < +\infty, \tag{1.7.1}$$

then $\mathcal{I}_\mu \in \mathcal{H}_+(D)$; otherwise $\mathcal{I}_\mu = +\infty$ on D.

Proof. Let Ω be a bounded open set such that $\overline{\Omega} \subset D$. Then there is a positive constant C such that

$$C^{-1} \frac{1}{1+\|y\|^N} \le \mathcal{K}(x,y) \le C \frac{1}{1+\|y\|^N} \quad (x \in \Omega; y \in \partial D).$$

It follows that $\mathcal{I}_\mu$ is non-negative and finite-valued on Ω if (1.7.1) holds, and $\mathcal{I}_\mu = +\infty$ on Ω otherwise. Also, in the former case, by Lebesgue's dominated convergence theorem, $\mathcal{I}_\mu$ is continuous on Ω, and taking a ball $\overline{B(x_0,r)}$ in Ω, we can change the order of integration and use the harmonicity of $\mathcal{K}(\cdot\,, y)$ to see that $\mathcal{M}(\mathcal{I}_\mu; x_0, r) = \mathcal{I}_\mu(x_0)$. Thus $\mathcal{I}_\mu \in \mathcal{H}_+(\Omega)$. The conclusion follows in view of the arbitrary nature of Ω. □

Theorem 1.7.3. *If $h \in \mathcal{H}_+(D)$, then there exists a measure μ on ∂D and a number $c \ge 0$ such that*

$$h(x) = \mathcal{I}_\mu(x) + c x_N \quad (x \in D). \tag{1.7.2}$$

Proof. Let $z = (0, \ldots, 0, -1)$ and $w = (0, \ldots, 0, -1/2)$. We will deduce this result from the Riesz–Herglotz theorem using the Kelvin transform with respect to the sphere $S(z,1)$. Thus $x^* = z + \|x-z\|^{-2}(x-z)$ and $\|x^* - z\| = \|x-z\|^{-1}$ whenever $x \ne z$, and

$$\begin{aligned}\|x^* - y\|^2 &= \|x^* - z\|^2 + \|y-z\|^2 - 2\langle x^* - z, y - z\rangle \\ &= \frac{1}{\|x-z\|^2} + \frac{1}{\|y^*-z\|^2} - \frac{2\langle x-z, y^*-z\rangle}{\|x-z\|^2\|y^*-z\|^2} \\ &= \frac{\|x-z\|^2 + \|y^*-z\|^2 - 2\langle x-z, y^*-z\rangle}{\|x-z\|^2\|y^*-z\|^2} \\ &= \frac{\|x-y^*\|^2}{\|x-z\|^2\|y^*-z\|^2} \quad (x, y \in \mathbb{R}^N \backslash \{z\}).\end{aligned} \tag{1.7.3}$$

Noting that $w^* = -z$, we obtain from (1.7.3) that

$$\begin{aligned}\frac{1}{4} - \|x^* - w\|^2 &= \frac{1}{4} - \frac{\|x+z\|^2}{\|x-z\|^2 2^2} \\ &= \frac{-\langle z, x\rangle}{\|x-z\|^2} = \frac{x_N}{\|x-z\|^2} \qquad (x \in \mathbb{R}^N \backslash \{z\}).\end{aligned} \tag{1.7.4}$$

It follows from (1.7.4) that $x_N > 0$ if and only if $\|x^* - w\| < 1/2$ and so $D^* = B(w, 1/2)$. Similarly, the mapping $x \mapsto x^*$ takes $S(w,1/2)\backslash\{z\}$ homeomorphically onto ∂D. (See Figure 1.2.)

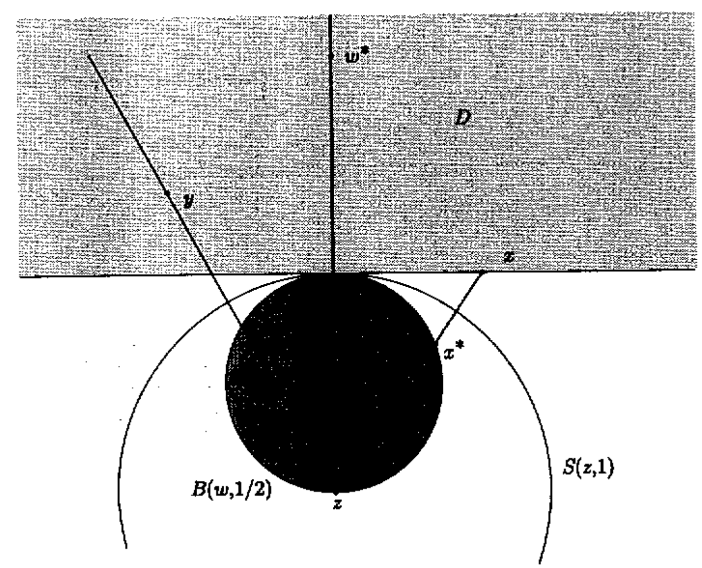

Figure 1.2.

Suppose now that $h \in \mathcal{H}_+(D)$, and let h^* be the image of h under the Kelvin transform. By Corollary 1.6.4, $h^* \in \mathcal{H}_+(B(w,1/2))$ and hence, by the Riesz–Herglotz theorem, there is a measure ν on $S(w,1/2)$ such that

$$h^*(x^*) = \int_{S(w,1/2)} \frac{1/4 - \|x^* - w\|^2}{\|x^* - y\|^N}\, d\nu(y) \qquad (x \in D). \tag{1.7.5}$$

Writing the integral in (1.7.5) as an integral over $S(w,1/2)\backslash\{z\}$ plus an integral over $\{z\}$ and using (1.7.3), (1.7.4), we obtain

$$h^*(x^*) = x_N\|x-z\|^{N-2}\int_{S(w,1/2)\backslash\{z\}} \frac{\|y^*-z\|^N}{\|x-y^*\|^N}\, d\nu(y) + \nu(\{z\})x_N\|x-z\|^{N-2}.$$

Since $h(x) = \|x-z\|^{2-N}h^*(x^*)$, this yields

$$h(x) = x_N \int_{S(w,1/2)\backslash\{z\}} \frac{\|y^*-z\|^N}{\|x-y^*\|^N} d\nu(y) + \nu(\{z\})x_N. \tag{1.7.6}$$

If we define the measure μ on ∂D by writing

$$\mu(E) = (\sigma_N/2)\int_{E^*} \|y^* - z\|^N\, d\nu(y)$$

for each Borel subset E of ∂D, and also put $c = \nu(\{z\})$, then we can rewrite (1.7.6) as (1.7.2). □

The remainder of this section is concerned with the boundary behaviour of Poisson integrals of functions. Let λ' denote $(N-1)$-dimensional Lebesgue measure on ∂D (which can be identified with $\mathbb{R}^{N-1}$). If f is a non-negative measurable function on ∂D then we write $\mathcal{I}_f$ in place of $\mathcal{I}_{f\lambda'}$. If f is measurable on ∂D and

$$\int_{\partial D} \frac{|f(y)|}{1+\|y\|^N} d\lambda'(y) < +\infty, \tag{1.7.7}$$

then $\mathcal{I}_{f^+}, \mathcal{I}_{f^-} \in \mathcal{H}_+(\Omega)$, by Theorem 1.7.2, and we can define $\mathcal{I}_f = \mathcal{I}_{f^+} - \mathcal{I}_{f^-}$. Thus $\mathcal{I}_f \in \mathcal{H}(\Omega)$.

Lemma 1.7.4. $\mathcal{I}_1 \equiv 1$.

Proof. By considering spheres in $\mathbb{R}^{N-1}$ of centre $(x_1, \dots, x_{N-1})$, we see that

$$\begin{aligned}\mathcal{I}_1(x) &= \frac{2}{\sigma_N}\int_{\partial D} \frac{x_N}{\|x-y\|^N}\, d\lambda'(y) \qquad (x \in D)\\ &= \frac{2\sigma_{N-1}}{\sigma_N}\int_0^{+\infty} \frac{x_N t^{N-2}}{(x_N^2+t^2)^{N/2}}\, dt\\ &= \frac{2\sigma_{N-1}}{\sigma_N}\int_0^{+\infty} \frac{s^{N-2}}{(1+s^2)^{N/2}}\, ds, \end{aligned} \tag{1.7.8}$$

where σ_{N-1} is interpreted as 2 when $N = 2$. Denote the integral in (1.7.8) by $J(N)$. Easy calculations yield $J(2) = \pi/2$ and $J(3) = 1$. Integration by parts shows that $J(N+2) = N^{-1}(N-1)J(N)$. These equations give

$$J(N) = \begin{cases} \dfrac{1.3.5\dots(N-3)}{2^{N/2}(N/2-1)!}\pi & (N \text{ even}; N \geq 4) \\[2ex] \dfrac{2^{(N-3)/2}((N-3)/2)!}{1.3.5\dots(N-2)} & (N \text{ odd}; N \geq 5). \end{cases}$$

Hence, using the explicit value of σ_N (see p. xvi), we see that $J(N) = \sigma_N/(2\sigma_{N-1})$, and so $\mathcal{I}_1 \equiv 1$. □

Theorem 1.7.5. *Let f be a measurable function on ∂D satisfying (1.7.7). Then*

$$\limsup_{x\to y, x\in D} \mathcal{I}_f(x) \leq \limsup_{z\to y, z\in\partial D} f(z) \quad (y \in \partial^\infty D). \tag{1.7.9}$$

Further, if f is continuous in the extended sense at $y \in \partial D$, then $\mathcal{I}_f(x) \to f(y)$ as $x \to y$; also $\mathcal{I}_f(x) \to \lim_{z\to\infty} f(z)$ as $x \to \infty$ if f has a limit (finite or infinite) at ∞.

Proof. Once (1.7.9) is established the rest of the theorem will follow by applying this inequality to f and $-f$. In proving (1.7.9), we suppose that $\limsup_{z\to y} f(z) < A < +\infty$ and show that $\limsup_{x\to y} \mathcal{I}_f(x) \leq A$. We treat the cases $y \in \partial D$ and $y = \infty$ separately.

Suppose first that $y \in \partial D$. There exists $\delta > 0$ such that $f(z) < A$ whenever $z \in B(y, 2\delta) \cap D$. If $x \in B(y,\delta) \cap D$, then by Lemma 1.7.4

$$\begin{aligned}\mathcal{I}_f(x) - A &= \mathcal{I}_{f-A}(x)\\ &\leq \int_{\partial D\backslash B(y,2\delta)} \mathcal{K}(x,z)\left|f(z)-A\right| d\lambda'(z)\\ &\quad + \int_{\partial D\cap B(y,2\delta)} \mathcal{K}(x,z)(f(z)-A)\, d\lambda'(z). \end{aligned} \tag{1.7.10}$$

The second integral here is negative and the first does not exceed

$$Cx_N\int_{\partial D} \frac{|f(z)|+|A|}{1+\|z\|^N}\, d\lambda'(z),$$

where C depends only on y, δ and N. Hence the right-hand side of (1.7.10) has non-positive upper limit as $x \to y$ and therefore $\limsup_{x\to y} \mathcal{I}_f(x) \leq A$, as required.

Now suppose that $y = \infty$. There exists $R > 0$ such that $f(z) < A$ when $z \in \partial D\backslash B(0,R)$. Hence, if $x \in D\backslash \overline{B(0,R)}$, then

$$\begin{aligned}\mathcal{I}_f(x) - A &\le \int_{\partial D \cap B(0,R)} \mathcal{K}(x,z)\big(f(z) - A\big)\, d\lambda'(z) \\ &\le \frac{2x_N}{\sigma_N\big(\|x\| - R\big)^N} \int_{\partial D \cap B(0,R)} \big|f(z) - A\big|\, d\lambda'(z) \\ &\to 0 \qquad (x \to \infty).\end{aligned}$$

Thus $\limsup_{x\to\infty} \mathcal{I}_f(x) \le A$ again. □

In formulating the Dirichlet problem for unbounded open sets Ω, it is usual to work with a continuous function on the boundary $\partial^\infty \Omega$ in the compactified space $\mathbb{R}^N \cup \{\infty\}$ rather than the Euclidean boundary $\partial\Omega$. If $\partial\Omega$ is used instead, then uniqueness of solutions will be lost if there are non-constant harmonic functions on Ω which vanish on $\partial\Omega$; when $\Omega = D$, an example of such a function is $x \mapsto x_N$.

Corollary 1.7.6. *Let $f : \partial^\infty D \to [-\infty, +\infty]$ be continuous and suppose that (1.7.7) holds. Then there exists $h \in \mathcal{H}(D)$ such that $h(x) \to f(y)$ as $x \to y$ for each $y \in \partial^\infty D$. Further, if f is finite-valued, then h is unique.*

Proof. By Theorem 1.7.5 the function $h = \mathcal{I}_f$ has the required properties. The uniqueness assertion follows from the maximum principle. □

Note that uniqueness fails if f is allowed to take infinite values. For example, if $f(y_0) = +\infty$ for some $y_0 \in \partial D$ and if $h \in \mathcal{H}(D)$ is a solution of the Dirichlet problem in the sense that $h(x) \to f(y)$ as $x \to y$ for each $y \in \partial^\infty D$, then $h + a\mathcal{K}(\,\cdot\,, y_0)$ will be another such solution for any $a > 0$.

1.8. Real-analyticity of harmonic functions

Real-analyticity of functions on open sets in $\mathbb{R}^N$ is defined in terms of multiple power series, so we will first discuss these. If $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index (an ordered N-tuple of non-negative integers), then we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_N, \quad \alpha! = \alpha_1! \ldots \alpha_N!,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \quad D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}.$$

By a multiple power series we mean a series of the form $\sum a_\alpha x^\alpha$, where the coefficients a_α are real numbers and the sum is over all N-tuples α. We need to consider only the case where such a series is absolutely convergent and so the ordering of terms need not be specified.

Lemma 1.8.1. *If the coordinates $y_1, \ldots, y_N$ of a point y are positive and $\sum |a_\alpha| y^\alpha$ converges, then $\sum |a_\alpha x^\alpha|$ converges uniformly on the set $E = \{x : |x_j| \le y_j$ for each $j\}$. Further, for any multi-index β, the series $\sum |a_\alpha D^\beta x^\alpha|$ converges locally uniformly on E° and*

$$D^\beta \sum a_\alpha x^\alpha = \sum a_\alpha D^\beta x^\alpha \qquad (x \in E^\circ). \tag{1.8.1}$$

Proof. The uniform convergence of $\sum |a_\alpha x^\alpha|$ on E follows immediately from the Weierstrass M-test. For the convergence of $\sum |a_\alpha D^\beta x^\alpha|$, it is enough to work with the case where $|\beta| = 1$, since the general result will then follow by induction. Without loss of generality we work with $\beta = (1, 0, \ldots, 0)$. Let $E_c = \{x : cx \in E\}$, where $c > 1$. If $\alpha_1 \ge 1$ and $|\alpha|$ is large enough, then

$$|a_\alpha D^\beta x^\alpha| = |a_\alpha \alpha_1 x^{\alpha - \beta}| \le |a_\alpha| \alpha_1 c^{1-|\alpha|} y_1^{-1} y^\alpha \le |a_\alpha| y^\alpha \quad (x \in E_c);$$

if $\alpha_1 = 0$, then $D^\beta x^\alpha = 0$. It follows from the M-test that $\sum |a_\alpha D^\beta x^\alpha|$ converges uniformly on E_c, and hence locally uniformly on E° in view of the arbitrary nature of c. Equation (1.8.1) now follows by repeated application of a well-known result on term-by-term differentiation of series of functions of one real variable. □

Definition 1.8.2. A function $f : \Omega \to \mathbb{R}$ is called *real-analytic* on Ω if for each $y \in \Omega$ there exists $r > 0$ such that f has a representation of the form

$$f(y + x) = \sum a_\alpha x^\alpha \tag{1.8.2}$$

when $\|x\| < r$ and the series is absolutely convergent for such x.

In view of Lemma 1.8.1 we can differentiate the series in (1.8.2) term by term, and then take $x = 0$, to see that $f \in C^\infty(\Omega)$ and $D^\beta f(y) = \beta! a_\beta$ for each multi-index β. The series in (1.8.2) is called the *Taylor series* for f at y.

Lemma 1.8.3. *If f is real-analytic on a connected open set Ω and $f = 0$ on a non-empty open subset of Ω, then $f = 0$ on Ω.*

Proof. Let Ω_0 be the set of points x in Ω such that $f = 0$ on some neighbourhood of x. Then Ω_0 is open and non-empty. If $y \in \overline{\Omega_0} \cap \Omega$, then f and all its partial derivatives vanish at y by continuity. Thus the Taylor series for f at y vanishes identically and so $y \in \Omega_0$. It follows that Ω_0 is relatively closed in Ω and hence $\Omega_0 = \Omega$ by connectedness. □

In fact, the space of real-analytic functions on Ω is strictly contained in $C^\infty(\Omega)$. For example, the function $f : \mathbb{R}^N \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & (x_1 \le 0) \\ \exp(-x_1^{-1}) & (x_1 > 0) \end{cases}$$

belongs to $C^\infty(\mathbb{R}^N)$, but is not real-analytic on $\mathbb{R}^N$ in view of Lemma 1.8.3.

In order to prove that harmonic functions are real-analytic, it is enough to show that ball Poisson integrals are real-analytic (see Corollary 1.3.4). With this in mind we first study Poisson kernels.

Lemma 1.8.4. *The Poisson kernel K of $B(x_0,r)$ has an expansion of the form*

$$K(x_0+x,y)=\sum a_\alpha(y)x^\alpha \qquad (\|x\|<(\sqrt{2}-1)r;\, y\in S(x_0,r)), \qquad (1.8.3)$$

where $a_\alpha(y)$ is a polynomial in the coordinates of y and $\sum|a_\alpha(y)x^\alpha|$ is uniformly convergent on $\{(x,y): x\in B(0,cr), y\in S(x_0,r)\}$ if $0<c<\sqrt{2}-1$.

Proof. Let $y\in S(x_0,r)$ and $z=y-x_0$. Then

$$\sigma_N r K(x_0+x,y)=(r^2-\|x\|^2)(\|x\|^2+r^2-2\langle x,z\rangle)^{-N/2}. \qquad (1.8.4)$$

If $\|x\|<(\sqrt{2}-1)r$, then

$$\left|\|x\|^2-2\langle x,z\rangle\right|\le \|x\|^2+2\|x\|\,\|z\|<r^2$$

and hence

$$\left(\|x\|^2+r^2-2\langle x,z\rangle\right)^{-N/2}=r^{-N}\sum_{n=0}^{\infty}d_n\left(\frac{\|x\|^2-2\langle x,z\rangle}{r^2}\right)^n, \qquad (1.8.5)$$

where the d_n are binomial coefficients. Further, if $\|x\|<cr$, where $c<\sqrt{2}-1$, then

$$\sum_{j=1}^{N}x_j^2+2\sum|x_jz_j|<c^2r^2+2cr^2,$$

so that $\left(\|x\|^2-2\langle x,z\rangle\right)/r^2$ is the sum of $2N$ monomial terms the sum of whose moduli is less than $c^2+2c<1$. Since $\sum d_n(c^2+2c)^n<+\infty$, we see that (1.8.5) yields an expansion of the form

$$\left(\|x\|^2+r^2-2\langle x,z\rangle\right)^{-N/2}=\sum b_\alpha(y)x^\alpha,$$

where $b_\alpha(y)$ is a polynomial in the coordinates of y and $\sum|b_\alpha(y)x^\alpha|$ converges uniformly on $B(0,cr)\times S(x_0,r)$. The corresponding result for K is now obtained from (1.8.4). □

Theorem 1.8.5. *If $h\in\mathcal{H}(\Omega)$, then h is real-analytic on Ω.*

Proof. If $\overline{B(x_0,r)}\subset\Omega$, then h is equal on $B(x_0,r)$ to its Poisson integral:

$$h(x_0+x)=\int_{S(x_0,r)}K(x_0+x,y)\,h(y)\,d\sigma(y) \qquad (\|x\|<r).$$

If $\|x\|<(\sqrt{2}-1)r$, then (1.8.3) holds with the stated convergence properties. Integrating term by term we obtain $h(x_0+x)=\sum d_\alpha x^\alpha$, where

$$d_\alpha=\int_{S(x_0,r)}a_\alpha(y)\,h(y)\,d\sigma(y);$$

moreover $\sum|d_\alpha x^\alpha|$ converges uniformly on $B(0,cr)$ when $c<\sqrt{2}-1$. □

It follows from Theorem 1.8.5 and Lemma 1.8.3 that, if h is harmonic on a connected open set Ω and $h=0$ on some ball in Ω, then $h\equiv 0$. Thus the improvement of Theorem 1.2.4(i) promised in Remark 1.2.5(b) is verified. Another consequence of Theorem 1.8.5 is the following result which will be improved in Chapter 2 (see Theorem 2.4.4). We recall that a polynomial P in $x_1,\dots,x_N$ is *homogeneous* of degree j if it is a finite linear combination of monomials x^α where $|\alpha|=j$.

Corollary 1.8.6. *If $h\in\mathcal{H}(\Omega)$ and $x_0\in\Omega$, then there is a unique sequence (H_j) of harmonic polynomials such that H_j is homogeneous of degree j, and such that for some positive number r,*

$$h(x_0+x)=\sum_{j=0}^{\infty}H_j(x) \qquad (\|x\|<r) \qquad (1.8.6)$$

and $\sum|H_j|$ converges uniformly on $B(0,r)$.

Proof. By Theorem 1.8.5 and Lemma 1.8.1, there exists $r>0$ such that $h(x_0+x)=\sum a_\alpha x^\alpha$ when $\|x\|<r$ and $\sum|a_\alpha x^\alpha|$ converges uniformly on $B(0,r)$. Let $H_j(x)=\sum_{|\alpha|=j}a_\alpha x^\alpha$. Then (1.8.6) holds and $\sum|H_j|$ converges uniformly on $B(0,r)$. To show that each H_j is harmonic, we differentiate under the summation sign in (1.8.6), using Lemma 1.8.1 for justification, and obtain

$$0=(\Delta h)(x_0+x)=\sum_{j=0}^{\infty}(\Delta H_j)(x) \qquad (\|x\|<r).$$

If $0<\|x\|<r$ and $-1<t<1$, then by homogeneity

$$0=\sum_{j=0}^{\infty}(\Delta H_j)(tx)=\sum_{j=2}^{\infty}t^{j-2}(\Delta H_j)(x).$$

Hence, by the uniqueness property of single-variable power series, $\Delta H_j=0$ on $B(0,r)$ for each j. Thus each H_j is a harmonic polynomial.

For the uniqueness assertion it is enough to show that if Q_j is a homogeneous polynomial of degree j and $\sum Q_j=0$ on some ball $B(0,\rho)$, then $Q_j\equiv 0$ for each j. Since $\sum t^jQ_j(x)=\sum Q_j(tx)=0$ when $\|x\|<\rho$ and $-1<t<1$, the result again follows from the uniqueness property of single-variable power series. □

1.9. Exercises

Exercise 1.1. Show that, if $h \in \mathcal{H}_+(B)$, then

$$\liminf_{r \to 1-} h(ry) < +\infty$$

for σ-almost every $y \in S$. (Hint: consider $\mathcal{M}(h;0,r_n)$ for a sequence (r_n) with $\lim r_n = 1$.)

Exercise 1.2. Let (y_n) be a dense sequence of points in S, and let h be defined on B by

$$h = \sum_{n=1}^{\infty} 2^{-n} K_{0,1}(\cdot, y_n).$$

Show that $h \in \mathcal{H}_+(B)$ and $\limsup_{x \to y} h(x) = +\infty$ for each $y \in S$. Deduce, using the result of Exercise 1.1, that $\lim_{x \to y} h(x)$ fails to exist for σ-almost every $y \in S$.

Exercise 1.3. Let $D = \{x \in \mathbb{R}^N : x_N > 0\}$, and let h be defined on D by

$$h(x) = \frac{x_1 x_N}{(x_1^2 + x_N^2)^2}.$$

Show that:
 (i) $h \in \mathcal{H}(D)$,
 (ii) $\lim_{t \to 0+} h(x', t) = 0$ for each $x' \in \mathbb{R}^{N-1}$,
 (iii) h has no harmonic continuation to $\mathbb{R}^N$.
(This shows that in the reflection principle, Theorem 1.3.6, limits along normals to ∂D cannot replace limits.)

Exercise 1.4. Suppose that $h \in \mathcal{H}(D)$ and $h(x) \to 0$ as $x \to y$ for each $y \in \partial D$. By using the reflection principle and considering the harmonic function $\partial \overline{h}/\partial x_N$, show that if $t^{-1}h(x_1, \ldots, x_{N-1}, t) \to 0$ as $t \to 0+$ whenever $x_1, \ldots, x_{N-1}$ are rational, then $h \equiv 0$.

Exercise 1.5. Let h be harmonic on the open unit disc and suppose that $h = 0$ on $R_1 \cup R_2$, where R_1, R_2 are radii of the disc meeting at an angle $\alpha\pi$ $(0 < \alpha \le 1)$. Use the reflection principle to show that if α is irrational, then $h \equiv 0$. Show also that if α is rational, then h need not be identically zero.

Exercise 1.6. Let $P = \{x \in \mathbb{R}^N : |x_N| = 1\}$.

(i) Give an example of a function $h \in \mathcal{H}(\mathbb{R}^N), h \not\equiv 0$, such that $h = 0$ on P.

(ii) Show that if H is a harmonic polynomial and $H = 0$ on P, then $H \equiv 0$.

Exercise 1.7. Suppose that $h \in \mathcal{H}(B(x_0, r))$ and $|h| \le M$. Show that for each multi-index α there is a constant C, depending only on $|\alpha|$ and N, such that

$$|D^\alpha h(x_0)| \le CMr^{-|\alpha|}.$$

(See Section 1.8 for notation.)

Exercise 1.8. Use the result of Exercise 1.7 to show that, if $h \in \mathcal{H}(\mathbb{R}^N)$ and $|h(x)| \le A(1 + \|x\|)^m$ for all $x \in \mathbb{R}^N$ and some constants $A > 0$ and $m \ge 0$, then h is a polynomial of degree at most m.

Exercise 1.9. The Harnack metric. Let Ω be connected, and let d be defined on $\Omega \times \Omega$ by

$$d(x,y) = \log(\inf\{C > 0 : C^{-1} \le h(x)/h(y) \le C \text{ for all } h \in \mathcal{H}_+(\Omega)\backslash\{0\}\}).$$

Show that d is a semi-metric (that is,

$$d(x,x) = 0 \le d(x,y) = d(y,x) < +\infty, \quad d(x,z) \le d(x,y) + d(y,z)$$

for all $x, y, z \in \Omega$). Show further that, if Ω is bounded, then d is a metric.

Exercise 1.10. Transfer of smallness. Let Ω be connected and let K, ω be non-empty subsets of Ω with K compact and ω open. Show that for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $h \in \mathcal{H}(\Omega), |h| \le 1$ on Ω and $|h| < \delta$ on ω, then $|h| < \varepsilon$ on K. (Hint: suppose not and let (h_n) be a sequence in $\mathcal{H}(\Omega)$ such that $|h_n| \le 1$ on $\Omega, |h_n| < n^{-1}$ on ω and $|h_n(x_n)| \ge \varepsilon$ for some $x_n \in K$.)

Exercise 1.11. Let Ω be unbounded and connected. Using the result of Exercise 1.10, show that there exists a continuous function $\eta : [0, +\infty) \to (0, 1]$ with the following property: if $h \in \mathcal{H}(\Omega)$ and $|h(x)| < \eta(\|x\|)$ for each $x \in \Omega$, then $h \equiv 0$.

Exercise 1.12. Spherical reflection. Let ω be an open set such that $\omega \cap S \neq \emptyset$. Suppose that $h \in \mathcal{H}(\omega \cap B)$ and $h(x) \to 0$ as $x \to y$ for each $y \in \omega \cap S$. Use the Kelvin transform and the reflection principle to show that h has a harmonic continuation to some open set containing $\omega \cap \overline{B}$.

Exercise 1.13. Suppose that $h \in \mathcal{H}(\mathbb{R}^N \backslash B)$ and $h(x) \to 0$ as $x \to \infty$. Show that the image h^* of h under the Kelvin transform with respect to S has a harmonic continuation to B, and deduce that $\|x\|^{N-2}h(x)$ has a finite limit as $x \to \infty$.

Exercise 1.14. Suppose that $h \in \mathcal{H}_+(\mathbb{R}^N \backslash \{0\})$ and $h(x) \to 0$ as $x \to \infty$. Show that $h(x) = c\|x\|^{2-N}$ for some constant $c \ge 0$.

Exercise 1.15. Show that if h is the Poisson integral on $D = \mathbb{R}^{N-1} \times (0, +\infty)$ of a finite measure on ∂D, then the function

$$t \mapsto \int_{\mathbb{R}^{N-1}} h(y', t) d\lambda'(y')$$

is constant on $(0, +\infty)$.

Exercise 1.16. Let $N = 2$ and let f be an integrable function on ∂D such that $f(\xi, 0) \to 0$ as $\xi \to +\infty$ and $f(\xi, 0) \to \pi$ as $\xi \to -\infty$. Show that the Poisson integral of f on D satisfies

$$\lim_{r \to +\infty} \mathcal{I}_f(r\cos\theta, r\sin\theta) = \theta \qquad (0 < \theta < \pi).$$

Exercise 1.17. Let $D = \mathbb{R}^{N-1} \times (0, +\infty)$ and suppose that $h \in \mathcal{H}_+(D)$ and $h(x) \to 0$ as $x \to y$ for each $y \in \partial D$. Write

$$h(x) = \mathcal{I}_\mu(x) + cx_N \qquad (x \in D)$$

as in Theorem 1.7.3. Let K be a compact subset of ∂D. Show that $\mathcal{I}_{\mu|_K}$ has a harmonic continuation H to $\mathbb{R}^N$ and that $H(x) \to 0$ as $x \to \infty$. Deduce that $h(x) = cx_N$.

Exercise 1.18. (i) Let $h \in \mathcal{H}_+(B)$. By considering functions of the form $x \mapsto h(rx)$, show that h is the limit on B of some sequence of bounded elements of $\mathcal{H}_+(B)$.

(ii) We say that $h \in \mathcal{H}_+(B)$ is *quasi-bounded* if h is the limit of some increasing sequence of bounded elements of $\mathcal{H}_+(B)$. Let $K(\cdot, y)$ denote the Poisson kernel of B with some fixed pole $y \in S$. Show (without using Theorem 1.3.9) that $K(\cdot, y)$ is not quasi-bounded. (Hint: show that if h is a bounded element of $\mathcal{H}_+(B)$ and $h \le K(\cdot, y)$, then $h \equiv 0$.)

(iii) Let $h \in \mathcal{H}(B)$. Using Theorem 1.3.9 or otherwise, show that h is the Poisson integral of an integrable function on S if and only if there exists a quasi-bounded function $h_0 \in \mathcal{H}_+(B)$ such that $|h| \le h_0$ on B.

Exercise 1.19. Let $h \in C(\Omega)$ and suppose that for each $x \in \Omega$ there exists a positive sequence (r_n) with $r_n \to 0$ such that $h(x) = \mathcal{A}(h; x, r_n)$ for each n. Show that h satisfies the maximum and minimum principles (that is, h is constant in some neighbourhood of any local extremum). Deduce that $h \in \mathcal{H}(\Omega)$.

Chapter 2. Harmonic Polynomials

2.1. Spaces of homogeneous polynomials

We start with an algebraic study of harmonic polynomials as elements of vector spaces equipped with inner products. This leads quickly to information about the structure of the spaces and the behaviour of individual elements. The special role of axially symmetric polynomials is emphasized. We next extend the study of polynomial expansions of harmonic functions which concluded Chapter 1 and give an expansion for harmonic functions on annular domains, analogous to the Laurent expansion for holomorphic functions. This Laurent-type expansion is then used to obtain basic results on harmonic approximation. These results will be applied firstly to establish the existence of harmonic functions with prescribed singular parts at a sequence of isolated singularities, and secondly to construct harmonic functions on $\mathbb{R}^N$ with unexpected properties.

Let $\mathcal{P}_m$ denote the vector space of all real-valued homogeneous polynomials of degree m on $\mathbb{R}^N$, where $m \ge 0$ and $N \ge 2$; in the multi-index notation of Section 1.8 the elements of $\mathcal{P}_m$ are those functions $P : \mathbb{R}^N \to \mathbb{R}$ of the form

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha. \tag{2.1.1}$$

A useful inner product is defined on $\mathcal{P}_m$ as follows. First, to an element P of $\mathcal{P}_m$ given by (2.1.1) we associate the differential operator D_P given by

$$D_P = \sum_{|\alpha|=m} a_\alpha D^\alpha.$$

Clearly if $Q(x) = \sum_{|\alpha|=m} b_\alpha x^\alpha$, then $D_P Q = \sum_{|\alpha|=m} \alpha! a_\alpha b_\alpha$ and it follows easily that the equation

$$[P, Q]_m = D_P Q$$

defines an inner product on $\mathcal{P}_m$.

Let $\mathcal{H}_m$ be the set of all harmonic elements of $\mathcal{P}_m$, and let $\mathcal{Q}_m$, when $m \ge 2$, be the set of all polynomials of the form $Q(x) = \|x\|^2 P(x)$ where $P \in \mathcal{P}_{m-2}$; we also put $\mathcal{Q}_0 = \mathcal{Q}_1 = \{0\}$. Clearly $\mathcal{H}_m$ and $\mathcal{Q}_m$ are subspaces of $\mathcal{P}_m$.

First we aim to demonstrate an important relationship between the spaces $\mathcal{P}_m, \mathcal{H}_m$ and $\mathcal{Q}_m$. This needs some notions from elementary vector space theory. We recall that a space V is said to be the *direct sum* of subspaces V_1, V_2 if every element $v \in V$ has a unique representation of the form $v = v_1 + v_2$, where $v_1 \in V_1$ and $v_2 \in V_2$; we then write $V = V_1 \oplus V_2$. In particular, if V is equipped with an inner product and U is a subspace of V, then $V = U \oplus U^{\perp}$, where $U^{\perp}$ is the subspace of V comprising those vectors which are orthogonal to (every vector in) U.

Theorem 2.1.1. *With the inner product $[\cdot,\cdot]_m$ on $\mathcal{P}_m$ we have $\mathcal{H}_m = \mathcal{Q}_m^{\perp}$ and hence $\mathcal{P}_m = \mathcal{H}_m \oplus \mathcal{Q}_m$.*

Proof. The result is trivial when $m = 0, 1$, so we suppose that $m \geq 2$. If $P \in \mathcal{P}_m$ and $Q_0 \in \mathcal{P}_{m-2}$, then writing $Q(x) = ||x||^2 Q_0(x)$, we have

$$[Q, P]_m = D_{Q_0} \Delta P = [Q_0, \Delta P]_{m-2}. \tag{2.1.2}$$

Hence, if $P \in \mathcal{Q}_m^{\perp}$, then taking $Q_0 = \Delta P$ in (2.1.2), we see that $[\Delta P, \Delta P]_{m-2} = 0$, so that $\Delta P \equiv 0$ and $P \in \mathcal{H}_m$. Conversely, (2.1.2) also implies that if $P \in \mathcal{H}_m$ then $P \in \mathcal{Q}_m^{\perp}$. Hence $\mathcal{H}_m = \mathcal{Q}_m^{\perp}$, as required. □

We denote the greatest integer not exceeding a real number a by $[a]$.

Corollary 2.1.2. *If $P \in \mathcal{P}_m$, then there exist harmonic polynomials $H_{m-2j} \in \mathcal{H}_{m-2j}$ $(j = 0, 1, \ldots, [m/2])$ such that*

$$P(x) = \sum_{j=0}^{[m/2]} ||x||^{2j} H_{m-2j}(x). \tag{2.1.3}$$

Proof. By Theorem 2.1.1, we can write $P(x) = H_m(x) + ||x||^2 Q(x)$, where $H_m \in \mathcal{H}_m$ and $Q \in \mathcal{P}_{m-2}$. This observation forms the basis of a simple induction argument. □

The next corollary shows that a Poisson integral on the unit ball is a polynomial if the boundary function is a polynomial.

Corollary 2.1.3. *If P is a polynomial of degree m on $\mathbb{R}^N$, then there exists a harmonic polynomial H of degree at most m such that $H = P$ on S.*

Proof. Since P is the sum of its homogeneous parts (that is, $P = \sum_{k=0}^m P_k$, where $P_k \in \mathcal{P}_k$), it is enough to treat the case where P is homogeneous, and we suppose without loss of generality that $P \in \mathcal{P}_m$. The function H that we seek is then obtained by writing P in the form (2.1.3) and defining $H = \sum H_{m-2j}$. □

We define

$$d_{m,N} = \dim \mathcal{H}_m.$$

Corollary 2.1.4. $d_{0,N} = 1$; $d_{1,N} = N$;

$$d_{m,N} = \binom{m+N-1}{m} - \binom{m+N-3}{m-2} \quad (m \geq 2). \tag{2.1.4}$$

In particular,

$$d_{m,2} = 2, \quad d_{m,3} = 2m+1, \quad d_{m,4} = (m+1)^2 \quad (m \geq 1). \tag{2.1.5}$$

Proof. With $m = 0, 1$ the results are trivial. Suppose then that $m \geq 2$. By Theorem 2.1.1,

$$\dim \mathcal{P}_m = \dim \mathcal{H}_m + \dim \mathcal{Q}_m = \dim \mathcal{H}_m + \dim \mathcal{P}_{m-2}. \tag{2.1.6}$$

Now $\dim \mathcal{P}_m$ is the number of linearly independent monomial functions of degree m in N variables, and this is equal to the number of arrangements in which m indistinguishable objects together with $N-1$ indistinguishable division-markers are put in a row of $m+N-1$ places. An arrangement of this type is determined by selecting m places for the objects from the $m+N-1$ available. This selection can be made in $\binom{m+N-1}{m}$ ways. Hence $\dim \mathcal{P}_m = \binom{m+N-1}{m}$, and (2.1.4) now follows from (2.1.6). The values in (2.1.5) are easily calculated from (2.1.4). □

Remark 2.1.5. The space $\mathcal{H}_m$ has a very simple form when $N = 2$. In this case, with $m \geq 1$, the functions

$$h_1(x_1, x_2) = \operatorname{Re}(x_1 + ix_2)^m \quad \text{and} \quad h_2(x_1, x_2) = \operatorname{Im}(x_1 + ix_2)^m$$

are linearly independent elements of $\mathcal{H}_m$; indeed $[h_1, h_2]_m = 0$. Since $d_{m,2} = 2$, they form a basis for $\mathcal{H}_m$. Note that, writing $x_1 + ix_2 = re^{i\theta}$, we have

$$h_1(r\cos\theta, r\sin\theta) = r^m \cos m\theta \quad \text{and} \quad h_2(r\cos\theta, r\sin\theta) = r^m \sin m\theta.$$

2.2. Another inner product on a space of polynomials

The inner product $[\cdot,\cdot]_m$ has served its main purpose by enabling us to give a short proof that $\mathcal{P}_m = \mathcal{H}_m \oplus \mathcal{Q}_m$, and another inner product $\langle \cdot,\cdot \rangle_2$, defined by an integral, will be more useful from now on. It is convenient to work here with surface measure $\hat{\sigma}$ on the unit sphere S, normalized so that $\hat{\sigma}(S) = 1$. It is easy to see that the equation

$$\langle P, Q \rangle_2 = \int_S PQ \, d\hat{\sigma} \qquad (P, Q \in \mathcal{P}_m)$$

defines an inner product on $\mathcal{P}_m$; the associated norm is given by $||P||_2 = \sqrt{\langle P, P \rangle_2}$.

Lemma 2.2.1. *If $P \in \mathcal{P}_k$ and $H \in \mathcal{H}_m$, where $0 \le k < m$, then*

$$\int_S HP\, d\hat{\sigma} = 0. \tag{2.2.1}$$

In particular,

$$\int_S H_m H_n d\hat{\sigma} = 0 \qquad (H_m \in \mathcal{H}_m; H_n \in \mathcal{H}_n) \tag{2.2.2}$$

whenever $m \neq n$.

Proof. We start with (2.2.2). Using the equation

$$\sum_{j=1}^N x_j \frac{\partial Q}{\partial x_j}(x) = qQ(x) \qquad (Q \in \mathcal{P}_q),$$

we find that

$$\begin{aligned}(m-n)\int_S H_m H_n\, d\hat{\sigma} &= \int_S \Big\{ H_n(x) \sum_{j=1}^N x_j \frac{\partial H_m}{\partial x_j}(x) \\ &\qquad - H_m(x) \sum_{j=1}^N x_j \frac{\partial H_n}{\partial x_j}(x) \Big\}\, d\hat{\sigma}(x) \\ &= \int_S \Big\{ H_n \frac{\partial H_m}{\partial n_e} - H_m \frac{\partial H_n}{\partial n_e} \Big\} d\hat{\sigma},\end{aligned}$$

where $\frac{\partial}{\partial n_e}$ denotes differentiation in the direction of the exterior normal to S. Green's formula (see Appendix) shows that the last-written integral is 0, and the proof of (2.2.2) is complete.

To prove (2.2.1), we note that by Corollary 2.1.3 there exists a harmonic polynomial J of degree at most k such that $P = J$ on S. Applying (2.2.2) to each of the homogeneous parts of J (which are also harmonic), we see that

$$\int_S HP\, d\hat{\sigma} = \int_S HJ\, d\hat{\sigma} = 0.$$ □

Theorem 2.2.2. *For each fixed m, the spaces $\mathcal{H}_m$ and $\mathcal{Q}_m$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_2$.*

Proof. The cases $m = 0, 1$ are trivial, so we suppose that $H \in \mathcal{H}_m$ and $Q \in \mathcal{Q}_m$ where $m \ge 2$, and let $Q(x) = ||x||^2 P(x)$, where $P \in \mathcal{P}_{m-2}$. By Lemma 2.2.1,

$$\langle H, Q \rangle_2 = \int_S HP\, d\hat{\sigma} = 0.$$ □

A further application of Lemma 2.2.1 is as follows.

Theorem 2.2.3. *Every non-constant factor of a non-zero harmonic polynomial takes both positive and negative values.*

Proof. This theorem is vacuous for polynomials of degree 0 and trivial for polynomials of degree 1. To deal with higher degrees, suppose initially that $H \in \mathcal{H}_m \backslash \{0\}$, where $m \ge 2$. Since $\mathcal{M}(H; 0, 1) = H(0) = 0$ and H is non-constant on S, it follows that H itself takes both positive and negative values. Now suppose that $H = PQ$, where P, Q are non-constant polynomials. Since deg $P < m$, Lemma 2.2.1 gives

$$0 = \int_S HP\, d\hat{\sigma} = \int_S P^2 Q\, d\hat{\sigma},$$

so that Q takes both positive and negative values on S. This completes the proof for homogeneous harmonic polynomials.

Now let H be an arbitrary harmonic polynomial of degree $m \ge 2$. If P, Q are polynomials such that $H = PQ$, then $\tilde{H} = \tilde{P}\tilde{Q}$, where $\tilde{P}$ denotes the homogeneous part of P of highest degree. If Q is non-constant, then since $\tilde{H} \in \mathcal{H}_m \backslash \{0\}$, the result of the previous paragraph implies that $\tilde{Q}$ takes a positive value at some point y of S. Denoting the degree of Q by k, we then have $Q(ry) = r^k \tilde{Q}(y) + O(r^{k-1})$ as $r \to +\infty$, so $Q(ry) > 0$ for sufficiently large values of r. Similarly, Q takes negative values. □

2.3. Axially symmetric harmonic polynomials

Definition 2.3.1. Let y be a point of S. A real-valued function f, defined either on $\mathbb{R}^N$ or on some ball $B(0, R)$ is called *y-axial* if $f = f \circ A$ for every orthogonal transformation A of $\mathbb{R}^N$ such that $A(y) = y$.

Theorem 2.3.2. *If $y \in S$, then there exists a unique element $J_{y,m}$ of $\mathcal{H}_m$ such that*

$$\langle H, J_{y,m} \rangle_2 = H(y) \qquad (H \in \mathcal{H}_m). \tag{2.3.1}$$

Further, $J_{y,m}$ is y-axial.

Proof. The first statement will be proved using elementary vector space theory. Let $\mathcal{H}_m^*$ denote the dual of $\mathcal{H}_m$, that is, the vector space of all real-valued linear functions on $\mathcal{H}_m$, and let $\Phi : \mathcal{H}_m \to \mathcal{H}_m^*$ be the function that associates with each element G of $\mathcal{H}_m$ the linear function $H \mapsto \langle H, G \rangle_2$. It is easy to see that Φ is injective, and so it is an isomorphism since $\dim \mathcal{H}_m = \dim \mathcal{H}_m^*$. Hence, if $y \in S$ and ϕ_y is the element of $\mathcal{H}_m^*$ given by $\phi_y(H) = H(y)$, then there is a unique element $J_{y,m}$ of $\mathcal{H}_m$ such that $\Phi(J_{y,m}) = \phi_y$, that is to say, (2.3.1) holds.

To show that $J_{y,m}$ is y-axial, let A be an orthogonal transformation of $\mathbb{R}^N$ with $A(y) = y$. Then by the rotation-invariance of $\hat{\sigma}$ and (2.3.1),

$$\langle H, J_{y,m} \circ A\rangle_2 = \langle H \circ A^{-1}, J_{y,m}\rangle_2 = (H \circ A^{-1})(y) = H(y)$$

for each $H \in \mathcal{H}_m$. From the uniqueness of $J_{y,m}$ it now follows that $J_{y,m} \circ A = J_{y,m}$, as required. □

Definition 2.3.3. We refer to the polynomials $J_{y,m}$ of Theorem 2.3.2 as *axial harmonics*. Specifically $J_{y,m}$ is called the *y-axial harmonic of degree m (on $\mathbb{R}^N$)*.

Axial harmonics are intimately related to certain well-studied classical polynomials. We make no use of this relationship here but will give details in Section 2.7. Below we investigate the properties of axial harmonics and then give some examples. The next result adds to the uniqueness statement in Theorem 2.3.2.

Theorem 2.3.4. *Every y-axial element of $\mathcal{H}_m$ is proportional to $J_{y,m}$.*

Proof. The result will follow easily once we have established that the zero function is the only y-axial element of $\mathcal{H}_m$ that vanishes at y. We therefore suppose that H is a y-axial element of $\mathcal{H}_m$ and $H(y) = 0$. Choosing coordinate axes with respect to which $y = (1, 0, \ldots, 0)$ we define

$$g(\theta) = H(\cos\theta, \sin\theta, 0, \ldots, 0).$$

By the axial symmetry of H, the function g is even and $g \equiv 0$ if and only if $H \equiv 0$. Suppose that $H \not\equiv 0$. Then g has only finitely many zeros in $[-\pi, \pi]$ and $g(0) = H(y) = 0$. Since g is even, it follows that g attains a strict local extremum at 0. The axial symmetry and homogeneity of H now imply that H attains a local extremum at y, contrary to the maximum principle. Hence $H \equiv 0$, as claimed.

If G is any y-axial element of $\mathcal{H}_m$, then the function $J_{y,m}(y)G - G(y)J_{y,m}$ is a y-axial element of $\mathcal{H}_m$ which vanishes at y. By the previous paragraph, this function is identically 0. Since $J_{y,m}(y) \neq 0$ by (2.3.1), we have $G = (G(y)/J_{y,m}(y))J_{y,m}$. □

Lemma 2.3.5. *If $y \in S$ and A is an orthogonal transformation of $\mathbb{R}^N$, then*

$$J_{A(y),m} = J_{y,m} \circ A^{-1}.$$

Proof. By (2.3.1) and the rotation invariance of $\hat{\sigma}$, we have for each $H \in \mathcal{H}_m$,

$$\langle H, J_{A(y),m}\rangle_2 = (H \circ A)(y) = \langle H \circ A, J_{y,m}\rangle_2 = \langle H, J_{y,m} \circ A^{-1}\rangle_2.$$

The result now follows from the uniqueness assertion of Theorem 2.3.2 applied to $J_{A(y),m}$. □

Theorem 2.3.6. *Let $\{H_j : j = 1, \ldots, d_{m,N}\}$ be an orthonormal basis for $\mathcal{H}_m$. If $x, y \in S$, then*

$$\sum_{j=1}^{d_{m,N}} H_j(x)H_j(y) = J_{y,m}(x) = J_{x,m}(y). \tag{2.3.2}$$

Proof. Fix a point $y \in S$. Using first the hypothesis that $\{H_j\}$ is an orthonormal basis for $\mathcal{H}_m$ and then (2.3.1), we obtain

$$J_{y,m}(x) = \sum_{j=1}^{d_{m,N}} \langle H_j, J_{y,m}\rangle_2 H_j(x) = \sum_{j=1}^{d_{m,N}} H_j(y)H_j(x) \quad (x \in \mathbb{R}^N).$$

Clearly the roles of x and y can be interchanged to yield the second equation in (2.3.2). □

Corollary 2.3.7. *If $\{H_j : j = 1, \ldots, d_{m,N}\}$ is an orthonormal basis for $\mathcal{H}_m$, then for each $y \in S$*

$$\sum_{j=1}^{d_{m,N}} (H_j(y))^2 = J_{y,m}(y) = ||J_{y,m}||_2^2 = d_{m,N}. \tag{2.3.3}$$

Proof. Taking $y = x$ in (2.3.2) yields

$$J_{y,m}(y) = \sum_{j=1}^{d_{m,N}} (H_j(y))^2.$$

Since, by Lemma 2.3.5, $J_{y,m}(y)$ is independent of y, we may integrate the right-hand side of the above equation with respect to $d\hat{\sigma}(y)$ to obtain $J_{y,m}(y) = d_{m,N}$. The middle equation in (2.3.3) is immediate from (2.3.1). □

Corollary 2.3.8. *If $H \in \mathcal{H}_m$, then*

$$|H(x)| \leq d_{m,N}^{1/2}||H||_2||x||^m \qquad (x \in \mathbb{R}^N). \tag{2.3.4}$$

Equality occurs when $H = J_{y,m}$ for some $y \in S$ and $x = ky$ for some $k \in \mathbb{R}$.

Proof. By homogeneity, it is enough to prove (2.3.4) in the case where $||x|| = 1$, and we may also suppose that $||H||_2 = 1$. Let $\{H_j\}$ be an orthonormal basis for $\mathcal{H}_m$ with $H_1 = H$. By Corollary 2.3.7,

$$(H(x))^2 \le \sum (H_j(x))^2 = d_{m,N},$$

whence (2.3.4) follows. To verify the stated case of equality, we note that if $y \in S$ and $x = ky$ for some $k \in \mathbb{R}$, then by the homogeneity of $J_{y,m}$ and (2.3.3),

$$|J_{y,m}(x)| = \|x\|^m |J_{y,m}(y)| = d_{m,N}\|x\|^m = d_{m,N}^{1/2}\|J_{y,m}\|_2\|x\|^m. \qquad \square$$

We conclude this section with some examples of axial harmonics. To verify the examples it is enough to check that the functions are indeed y-axial elements of $\mathcal{H}_m$ and take the value $d_{m,N}$ at y (see Theorem 2.3.4 and Corollary 2.3.7). The verifications are routine; compare the first example with Remark 2.1.5.

Example 2.3.9. Let $N = 2$ and write $y = (\cos\phi, \sin\phi), x = (r\cos\theta, r\sin\theta)$. The axial harmonic $J_{y,m}$ is given by $J_{y,m}(x) = 2r^m \cos m(\theta - \phi)$.

Example 2.3.10. Let $y = (1, 0, \ldots, 0) \in \mathbb{R}^N$. Then y-axial harmonics are given by

$$J_{y,0}(x) = 1,\ J_{y,1}(x) = Nx_1,\ J_{y,2}(x) = \frac{1}{2}(N+2)(Nx_1^2 - \|x\|^2),$$

$$J_{y,3}(x) = \frac{1}{6}N(N+4)\{(N+2)x_1^3 - 3x_1\|x\|^2\},$$

$$J_{y,4}(x) = \frac{1}{24}N(N+6)\{(N+4)(N+2)x_1^4 - 6(N+2)x_1^2\|x\|^2 + 3\|x\|^4\}.$$

2.4. Polynomial expansions of harmonic functions

We saw in Corollary 1.8.6 that if h is harmonic on some ball $B(x_0, r)$, then at least in some smaller ball h has a unique representation of the form

$$h(x) = \sum_{j=0}^{\infty} H_j(x - x_0), \tag{2.4.1}$$

where $H_j \in \mathcal{H}_j$. Our aim here is to show that this series converges absolutely and locally uniformly to h on the whole of $B(x_0, r)$. It is enough to work with $B(x_0, r) = B$, for the general result will then follow by translation and dilation. With a view to exploiting the Riesz–Herglotz theorem, we first seek an explicit expansion of the form (2.4.1) in the case $h = K(\cdot, y)$, where K is the Poisson kernel of B. For this we use the following uniqueness result.

Lemma 2.4.1. *Suppose that $y \in S$ and $h \in \mathcal{H}(B)$. If h is y-axial and $h(ty) = 0$ when $-1 < t < 1$, then $h \equiv 0$.*

Proof. By Corollary 1.8.6, there exist $H_j \in \mathcal{H}_j$ such that $h = \sum_{j=0}^{\infty} H_j$ in some ball $B(0, r_0)$. Let A be an orthogonal transformation of $\mathbb{R}^N$ with $A(y) = y$. Since h is y-axial, on $B(0, r_0)$ we have

$$\sum_{j=0}^{\infty} H_j = h = h \circ A = \sum_{j=0}^{\infty} H_j \circ A,$$

and from the uniqueness of the expansion for h it now follows that $H_j = H_j \circ A$ for each j. Hence each H_j is y-axial and therefore, by Theorem 2.3.4, proportional to $J_{y,j}$. Thus $h = \sum_{j=0}^{\infty} a_j J_{y,j}$ on $B(0, r_0)$ where (a_j) is a sequence of real numbers. In particular,

$$0 = h(ty) = \sum_{j=0}^{\infty} a_j J_{y,j}(ty) = \sum_{j=0}^{\infty} a_j J_{y,j}(y) t^j \qquad (-r_0 < t < r_0).$$

Since $J_{y,j}(y) \neq 0$, it follows from the uniqueness of power series expansions that $a_j = 0$ for each j, so that $h = 0$ on $B(0, r_0)$ and hence on B. $\square$

Definition 2.4.2. Let (f_n) be a sequence of real-valued functions defined on a non-empty subset E of $\mathbb{R}^N$. We shall say that the series $\sum f_n$ is *Weierstrass convergent* on E if $\sum \sup_E |f_n|$ converges; also $\sum f_n$ is *locally Weierstrass convergent* on E if it is Weierstrass convergent on every non-empty compact subset of E. (This terminology is suggested by the Weierstrass M-test.)

Recall that the Poisson kernel K of B is given by

$$K(x, y) = \frac{1}{\sigma_N} \frac{1 - \|x\|^2}{\|x - y\|^N} \qquad (x \in B; y \in S).$$

Theorem 2.4.3.

$$K(x, y) = \frac{1}{\sigma_N} \sum_{j=0}^{\infty} J_{y,j}(x) \quad (x \in B; y \in S). \tag{2.4.2}$$

Further, there exists a constant C, depending only on N, such that

$$|J_{y,j}(x)| \le C(j+1)^{N-2}\|x\|^j \quad (x \in B; y \in S; j = 0, 1, 2, \ldots) \tag{2.4.3}$$

so that the series in (2.4.2) is locally Weierstrass convergent on B for each $y \in S$.

Proof. We start with (2.4.3). In view of Corollaries 2.3.7 and 2.3.8, we have $|J_{y,j}(x)| \le d_{j,N}\|x\|^j$. From the explicit formula (2.1.4) for $d_{j,N}$ it follows easily that $d_{j,N} = O(j^{N-2})$ as $j \to \infty$.

We prove (2.4.2) first in the case where $x = ty$ with $-1 < t < 1$. We obtain

$$\begin{aligned}\sigma_N K(ty,y) &= (1-t^2)(1-t)^{-N}\\ &= (1-t^2)\sum_{j=0}^{\infty}\binom{j+N-1}{j}t^j\\ &= 1+Nt+\sum_{j=2}^{\infty}\left\{\binom{j+N-1}{j}-\binom{j+N-3}{j-2}\right\}t^j \quad (2.4.4)\\ &= \sum_{j=0}^{\infty}J_{y,j}(y)t^j = \sum_{j=0}^{\infty}J_{y,j}(ty)\end{aligned}$$

using the fact that the coefficient of t^j in (2.4.4) is equal to $d_{j,N}$ which in turn is equal to $J_{y,j}(y)$ (see Corollaries 2.1.4 and 2.3.7).

Since the series in (2.4.2) is locally uniformly convergent on B by (2.4.3), its sum, h say, is harmonic there, and since the terms of the series are y-axial, so also is h. Now $\sigma_N K(\cdot, y)$ is also y-axial and harmonic on B and agrees with h on the line segment $\{ty : -1 < t < 1\}$. Hence $h - \sigma_N K(\cdot, y) = 0$ on B, by Lemma 2.4.1, as required. □

A formal statement of the homogeneous polynomial expansion announced in the first paragraph of this section is as follows.

Theorem 2.4.4. *If $h \in \mathcal{H}(B(x_0, r))$, then there exist unique polynomials $H_j \in \mathcal{H}_j$ such that*

$$h(x) = \sum_{j=0}^{\infty} H_j(x - x_0) \qquad (x \in B(x_0, r)), \tag{2.4.5}$$

and the series is locally Weierstrass convergent on $B(x_0, r)$. In particular, if $h \in \mathcal{H}(\mathbb{R}^N)$, then the equation in (2.4.5) holds for all x in $\mathbb{R}^N$ and the series is locally Weierstrass convergent on $\mathbb{R}^N$.

Proof. It is enough to prove the results in the first sentence, for they clearly imply those in the second. Also, it is enough to work with B in place of $B(x_0, r)$. Suppose first that h is harmonic on some neighbourhood of $\overline{B}$. Then h is given on B by its Poisson integral:

$$h(x) = \int_S K(x,y)h(y)d\sigma(y).$$

Hence, by Theorem 2.4.3, $h = \sum_{j=0}^{\infty} H_j$ on B, where

$$H_j(x) = \frac{1}{\sigma_N}\int_S J_{y,j}(x)h(y)d\sigma(y). \tag{2.4.6}$$

By (2.4.3),

$$|H_j(x)| \le C(j+1)^{N-2}\|x\|^j \mathcal{M}(|h|; 0, 1),$$

so that $\sum_{j=0}^{\infty} H_j$ is locally Weierstrass convergent on B. By Lemma 2.3.5, the integrand in (2.4.6) and all its partial derivatives with respect to the coordinates of x are bounded when $\|x\| < r < 1$ and $y \in S$, so we may pass any linear partial differential operator under the integral sign. Using Laplace's operator Δ, we find that $H_j \in \mathcal{H}(B)$. Similarly, all the partial derivatives of H_j of order greater than j vanish on B and all such derivatives of order less than j vanish at 0, so that $H_j \in \mathcal{H}_j$. The uniqueness of the polynomials H_j follows from Corollary 1.8.6.

Now suppose only that $h \in \mathcal{H}(B)$. For $\rho \in (0,1)$ define h_ρ on $B(0, 1/\rho)$ by $h_\rho(x) = h(\rho x)$. Then $h_\rho \in \mathcal{H}(B(0, 1/\rho))$ and by the result of the previous paragraph, h_ρ has a unique polynomial expansion on B of the type described above which is locally Weierstrass convergent on B. This implies that h has the form (2.4.5) on $B(0,\rho)$ with local Weierstrass convergence there. Since the expansion is unique, it is independent of ρ and therefore locally Weierstrass convergent to h on B. □

Theorem 2.4.4 can be applied to obtain a further result concerning the Dirichlet problem for the half-space D. We know from Theorem 1.7.5 that, if $f \in C(\partial D)$ satisfies the integral condition (1.7.7), then there is a harmonic function h on D such that $h(x) \to f(y)$ as $x \to y$ for all $y \in \partial D$. We can now dispense with the integral condition.

Theorem 2.4.5. *If $f \in C(\partial D)$, then there exists $h \in \mathcal{H}(D)$ such that*

$$h(x) \to f(y) \qquad (x \to y; y \in \partial D). \tag{2.4.7}$$

Proof. We define continuous functions g_k ($k \in \mathbb{N}$) on ∂D by

$$g_1(y) = \max\{1 - \|y\|, 0\} \qquad (y \in \partial D),$$

$$g_k(y) = \left(\min\{k - \|y\|, \|y\| - k + 2\}\right)^+ \qquad (y \in \partial D; k \ge 2).$$

Thus $g_k(y) = 0$ when $\|y\| \ge k$ and when $\|y\| \le k-2$. By Theorem 1.7.5 the Poisson integral $h_k = \mathcal{I}_{fg_k}$ has limit $f(y)g_k(y)$ at $y \in \partial D$. In particular, if $k \ge 3$, then h_k has limit 0 at points of $B(0, k-2) \cap \partial D$ and, by the reflection principle, has a harmonic extension to $D \cup B(0, k-2)$. By Theorem 2.4.4 there exist $H_{j,k} \in \mathcal{H}_j$ such that

$$h_k(x) = \sum_{j=0}^{\infty} H_{j,k}(x) \qquad (x \in B(0, k-2))$$

and we can choose j_k such that the harmonic polynomial

$$H_k = \sum_{j=0}^{j_k} H_{j,k}$$

satisfies

$$|H_k(x) - h_k(x)| < 2^{-k} \qquad (||x|| \le k-3). \tag{2.4.8}$$

Since $D^\alpha h_k = 0$ on $B(0, k-2) \cap \partial D$ for any multi-index α of the form $(\alpha_1, \ldots, \alpha_{N-1}, 0)$, we see from Theorem 1.8.5 and Lemma 1.8.1 that x_N is a factor of each $H_{j,k}(x)$ and so $H_k = 0$ on ∂D. We now define

$$h = h_1 + h_2 + \sum_{k=3}^{\infty} (h_k - H_k)$$

on $\overline{D}$, where h_k is interpreted as fg_k on ∂D. By (2.4.8) the above series converges locally uniformly on $\overline{D}$. Since $h_k \in C(\overline{D}) \cap \mathcal{H}(D)$ for each k, the same is true of h. Finally, since $\sum g_k \equiv 1$ and $H_k = 0$ on ∂D for each k, we have $h = f$ on ∂D and so (2.4.7) holds. □

2.5. Laurent expansions of harmonic functions

Here we work on the annular domain

$$A(x_0; r_1, r_2) = \{x \in \mathbb{R}^N : r_1 < ||x - x_0|| < r_2\} \ (x_0 \in \mathbb{R}^N; 0 \le r_1 < r_2 \le +\infty).$$

Our aim is to obtain a series expansion for harmonic functions on such a domain which is analogous to the Laurent expansion for holomorphic functions on an annulus. We start by reinterpreting Theorem 2.4.4 for functions harmonic on $A(x_0; r_1, +\infty)$.

Lemma 2.5.1. *If $h \in \mathcal{H}(A(x_0; r_1, +\infty))$ and*

$$h(x) = \begin{cases} o(1) & (N \ge 3) \\ o(\log ||x||) & (N = 2) \end{cases} \qquad (x \to \infty), \tag{2.5.1}$$

then there exist unique harmonic polynomials $K_j \in \mathcal{H}_j$ such that

$$h(x) = \sum_{j=0}^{\infty} ||x - x_0||^{2-N-2j} K_j(x - x_0) \tag{2.5.2}$$

and the series is Weierstrass convergent on $A(x_0; \rho, +\infty)$ for each $\rho > r_1$.

Proof. Again we may suppose without loss of generality that $r_1 = 1$ and $x_0 = 0$. We use the Kelvin transform to reduce the proof to a problem about harmonic functions on B and then appeal to Theorem 2.4.4. Define h^* on $B \backslash \{0\}$ by $h^*(x) = ||x||^{2-N} h(x^*)$, where $x^* = ||x||^{-2}x$. By Corollary 1.6.4, h^* is harmonic on $B \backslash \{0\}$. Further, by (2.5.1), h^* satisfies the conditions of Theorem 1.3.7 on $B \backslash \{0\}$ and therefore has a harmonic continuation to B. Hence, by Theorem 2.4.4, there exist $K_j \in \mathcal{H}_j$ such that $h^* = \sum_{j=0}^{\infty} K_j$ on B. Hence if $x \in A(0; 1, +\infty)$, then

$$h(x) = ||x^*||^{N-2} h^*(x^*) = ||x||^{2-N} h^*(x^*) = ||x||^{2-N} \sum_{j=0}^{\infty} K_j(x^*)$$

$$= \sum_{j=0}^{\infty} ||x||^{2-N-2j} K_j(x). \tag{2.5.3}$$

The local Weierstrass convergence of $\sum_{j=0}^{\infty} K_j$ on B implies that the series in (2.5.3) is Weierstrass convergent on $A(0; \rho, +\infty)$ when $\rho > 1$.

The above argument is reversible: if h is given by (2.5.3), then $h^* = \sum_{j=0}^{\infty} K_j$ on B, so the uniqueness assertion in this lemma follows from that in Theorem 2.4.4. □

Lemma 2.5.2. *If $h \in \mathcal{H}(A(x_0; r_1, r_2))$, then there exist unique functions $h_1 \in \mathcal{H}(A(x_0; r_1, +\infty))$ and $h_2 \in \mathcal{H}(B(x_0, r_2))$ such that*

$$h_1(x) = \begin{cases} O(||x||^{2-N}) & (N \ge 3) \\ O(||x||^{-1}) & (N = 2) \end{cases} \qquad (x \to \infty) \tag{2.5.4}$$

and for each $x \in A(x_0; r_1, r_2)$

$$h(x) = \begin{cases} h_1(x) + h_2(x) & (N \ge 3) \\ h_1(x) + h_2(x) + \alpha \log ||x - x_0|| & (N = 2), \end{cases} \tag{2.5.5}$$

where $\alpha \in \mathbb{R}$.

Proof. We may suppose that $x_0 = 0$. For $\rho \in (r_1, r_2)$ and $x \in \mathbb{R}^N \backslash S(0, \rho)$, we write

$$g(\rho, x) = \frac{1}{a_N} \int_{S(0,\rho)} \left\{ U_x(y) \frac{\partial h}{\partial n_e}(y) - h(y) \frac{\partial U_x}{\partial n_e}(y) \right\} d\sigma(y), \tag{2.5.6}$$

where $a_N = \sigma_N \max\{1, N-2\}$ and $\partial / \partial n_e$ denotes differentiation with respect to the y-variable in the direction of the exterior normal to $B(0, \rho)$. If ω is a bounded open set such that $\overline{\omega} \cap S(0, \rho) = \emptyset$, then the integrand in (2.5.6) is bounded and continuous as a function of (x, y) on $\omega \times S(0, \rho)$, and for each fixed y in $S(0, \rho)$ it is harmonic as a function of x on ω. Hence $g(\rho, \cdot)$ is harmonic on $\mathbb{R}^N \backslash S(0, \rho)$ since it is continuous and, by Fubini's theorem, has the mean value property there.

Now suppose that $r_1 < \rho_1 < \rho_2 < r_2$. If $x \in A(0; \rho_1, \rho_2)$, then by applying Green's formula on $A(0; \rho_1, \rho_2) \backslash \overline{B(x, \delta)}$ and letting $\delta \to 0+$, we obtain $h(x) = g(\rho_2, x) - g(\rho_1, x)$. Since $g(\rho_2, x)$ is independent of ρ_1, it follows from this equation that $g(\rho_1, x)$ is independent of ρ_1. Similarly, $g(\rho_2, x)$ is independent of ρ_2. Since $g(\rho_1, \cdot)$ and $g(\rho_2, \cdot)$ are harmonic on $A(0; \rho_1, +\infty)$ and $B(0, \rho_2)$ respectively, it now follows that the functions defined by

$$g_1(x) = \lim_{\rho_1 \to r_1+} g(\rho_1, x) \qquad (\|x\| > r_1),$$
$$g_2(x) = \lim_{\rho_2 \to r_2-} g(\rho_2, x) \qquad (\|x\| < r_2)$$

are harmonic on their domains of definition and satisfy $h = g_2 - g_1$ on $A(0; r_1, r_2)$.

In the case where $N \geq 3$, easy estimates of $U_x(y)$ and its derivatives with respect to the coordinates of y show that $g(\rho_1, x) = O(\|x\|^{2-N})$ as $x \to \infty$. Since $g(\rho_1, x)$ is independent of ρ_1 when $\|x\| > \rho_1$, it follows that g_1 satisfies (2.5.4). Also, with $h_1 = -g_1$ and $h_2 = g_2$, (2.5.5) holds.

With $N = 2$ we have

$$\begin{aligned} -g(\rho_1, x) &= \frac{1}{2\pi} \int_{S(0,\rho_1)} \log \|x - y\| \frac{\partial h}{\partial n_e}(y) d\sigma(y) + O(\|x\|^{-1}) \\ &= \frac{1}{2\pi} \log \|x\| \int_{S(0,\rho_1)} \frac{\partial h}{\partial n_e}(y) d\sigma(y) \\ &\quad + \frac{1}{2\pi} \int_{S(0,\rho_1)} \log\Big(\frac{\|x-y\|}{\|x\|}\Big) \frac{\partial h}{\partial n_e}(y) d\sigma(y) + O(\|x\|^{-1}) \\ &= \alpha \log \|x\| + O(\|x\|^{-1}), \end{aligned}$$

say. Again, since $g(\rho_1, \cdot)$ is independent of ρ_1 when $\|x\| > \rho_1$, the function g_1 has the same behaviour. Hence, if we take $h_1(x) = -(g_1(x) + \alpha \log \|x\|)$ and $h_2 = g_2$, then (2.5.4) and (2.5.5) are satisfied.

To prove uniqueness, suppose that h_1', h_2' have all the properties of h_1, h_2. Then $h_1' + h_2' = h = h_1 + h_2$ on $A(0; r_1, r_2)$, so that $h_1 - h_1' = h_2' - h_2$ there. Thus $h_1 - h_1'$ has a harmonic continuation to $\mathbb{R}^N$. Hence, by (2.5.4) and the maximum principle, $h_1 = h_1'$. Therefore $h_2 = h_2'$ also. □

Theorem 2.5.3. (Laurent expansion) *If $h \in \mathcal{H}(A(x_0; r_1, r_2))$, then there exist unique harmonic polynomials $H_j, K_j \in \mathcal{H}_j$ such that*

$$h(x) = \sum_{j=0}^{\infty} H_j(x - x_0) + \sum_{j=1}^{\infty} \|x - x_0\|^{2-N-2j} K_j(x - x_0) + \alpha U_{x_0}(x), \quad (2.5.7)$$

where $\alpha \in \mathbb{R}$. If $0 < \delta < r_2$, then the first series in (2.5.7) is Weierstrass convergent on $B(x_0, r_2 - \delta)$ and the second on $A(x_0; r_1 + \delta, +\infty)$.

Proof. We decompose h as in Lemma 2.5.2. By Theorem 2.4.4, h_2 has the form (2.4.5) and the series in (2.4.5) is Weierstrass convergent on $B(x_0, r_2 - \delta)$. By Lemma 2.5.1 and (2.5.4), the function h_1 has the form (2.5.2) and the series in (2.5.2) is Weierstrass convergent on $A(x_0; r_1 + \delta, +\infty)$. In the case where $N \geq 3$, the term with $j = 0$ in the series (2.5.2) has the form $\alpha U_{x_0}(x)$;

when $N = 2$ the corresponding term is constant and is in fact zero, since $h_1(x) \to 0$ as $x \to \infty$. These remarks establish (2.5.7). The uniqueness of the polynomials in (2.5.7) follows from the uniqueness results in Theorem 2.4.4 and Lemmas 2.5.1 and 2.5.2. □

2.6. Harmonic approximation

In this section we give an approximation theorem for harmonic functions inspired by Runge's theorem on holomorphic approximation; it includes as a special case the result that if K is compact and $\mathbb{R}^N \backslash K$ is connected, then a harmonic function on a neighbourhood of K can be uniformly approximated on K by harmonic polynomials. We start with some lemmas. The second and third of these are given in a stronger form than is needed to prove the main approximation results. Their full strength will be used at the end of the section to show the existence of harmonic functions on $\mathbb{R}^N$ exhibiting surprising behaviour.

Lemma 2.6.1. *Let K be a compact subset of $\mathbb{R}^N$. If h is harmonic on a neighbourhood of K and $\varepsilon > 0$, then there exist points $y_1, \dots, y_m$ in $\mathbb{R}^N \backslash K$ and real numbers $\alpha_1, \dots, \alpha_m$ such that*

$$|h - \sum_{k=1}^{m} \alpha_k U_{y_k}| < \varepsilon$$

on K.

Proof. We can choose a compact set L on which Green's formula is applicable (see Appendix) such that $K \subset L^\circ$ and h is harmonic on a neighbourhood of L. By applying Green's formula on $L \backslash B(x, \delta)$ and letting $\delta \to 0+$, we obtain

$$h(x) = \frac{1}{a_N} \int_{\partial L} \Big\{ U_x(y) \frac{\partial h}{\partial n_e}(y) - h(y) \frac{\partial U_x}{\partial n_e}(y) \Big\} d\sigma(y) \qquad (x \in K),$$

where $\partial / \partial n_e$ denotes differentiation in the direction of the exterior normal to ∂L at y and $a_N = \sigma_N \max\{1, N-2\}$. Since the integrand is uniformly continuous as a function of (x, y) on $K \times \partial L$, the integral can be approximated uniformly for $x \in K$ by a Riemann sum

$$\sum_{j=1}^{n} b_j \Big\{ U_x(y_j) \frac{\partial h}{\partial n_e}(y_j) - h(y_j) \frac{\partial U_x}{\partial n_e}(y_j) \Big\}.$$

Also, by definition of a derivative, $(\partial / \partial n_e) U_x(y_j)$ can be approximated by a linear combination of $U_x(y_j')$ and $U_x(y_j)$ for some y_j' in $\mathbb{R}^N \backslash K$, and this approximation too will be uniform for $x \in K$. The lemma now follows after a relabelling of the points y_j, y_j' $(j = 1, \dots, n)$ as $y_1, \dots, y_m$. □

Lemma 2.6.2. *Let $y \in \mathbb{R}^N$ and suppose that ε, p, r_1, r_2 are numbers such that $r_2 > r_1 > 0$, $\varepsilon > 0$, $p \geq 0$. If $h \in \mathcal{H}(A(y; r_1, +\infty))$, then there exists $h_0 \in \mathcal{H}(\mathbb{R}^N \setminus \{y\})$ such that*

$$|(h - h_0)(x)| < \varepsilon(1 + ||x||)^{-p} \quad (x \in A(y; r_2, +\infty)).$$

Proof. By Theorem 2.5.3, the function h has a Laurent expansion

$$h(x) = h_1(x) + \alpha U_y(x) + \sum_{j=1}^{\infty} ||x - y||^{2-N-2j} K_j(x - y) \quad (x \in A(y; r_1, +\infty)),$$

where α is a constant, $h_1 \in \mathcal{H}(\mathbb{R}^N)$ and $K_j \in \mathcal{H}_j$. We write $\rho = (r_1 + r_2)/2$ and note that the above series is Weierstrass convergent on $S(y, \rho)$. Let C be a constant such that

$$C||x - y|| > \rho(1 + ||x||) \quad (x \in A(y; r_2, +\infty)). \tag{2.6.1}$$

There exists m_0 such that

$$\sum_{j=m_0+1}^{\infty} ||x - y||^{2-N-2j} |K_j(x - y)| < C^{-p}\varepsilon \quad (x \in S(y, \rho)). \tag{2.6.2}$$

Defining

$$h_0(x) = h_1(x) + \alpha U_y(x) + \sum_{j=1}^{m} ||x - y||^{2-N-2j} K_j(x - y),$$

we see that $h_0 \in \mathcal{H}(\mathbb{R}^N \setminus \{y\})$ and that if $m \geq \max\{m_0, p + 1 - N\}$ and $x \in A(y; r_2, +\infty)$, then

$$\begin{aligned}
|(h - h_0)(x)| &= \Big| \sum_{j=m+1}^{\infty} ||x - y||^{2-N-2j} \Big(\frac{||x - y||}{\rho} \Big)^j K_j\Big(\frac{\rho(x - y)}{||x - y||} \Big) \Big| \\
&\leq \sum_{j=m+1}^{\infty} \Big(\frac{\rho}{||x - y||} \Big)^{j+N-2} \rho^{2-N-2j} \Big| K_j\Big(\frac{\rho(x - y)}{||x - y||} \Big) \Big| \\
&\leq (\rho/||x - y||)^p C^{-p} \varepsilon \\
&< \varepsilon(1 + ||x||)^{-p},
\end{aligned}$$

using (2.6.2) and then (2.6.1). □

If $y \in \mathbb{R}^N$ and $f : [0, +\infty) \to \mathbb{R}^N$ is a continuous function such that $f(0) = y$ and $f(t) \to \infty$ as $t \to +\infty$, then we call $f([0, +\infty))$ a *path* from y to ∞, and any open set containing such a path we call a *tract* from y to ∞.

Lemma 2.6.3. *Let T be a tract from some point y to ∞. If $\varepsilon > 0, p \geq 0$ and $h \in \mathcal{H}(\mathbb{R}^N \setminus \{y\})$, then there exists $g \in \mathcal{H}(\mathbb{R}^N)$ such that*

$$|(h - g)(x)| < \varepsilon(1 + ||x||)^{-p} \quad (x \in \mathbb{R}^N \setminus T).$$

Proof. Let $(B_j) = (B(y_j, r_j))$ be a sequence of balls such that

$$y_0 = y, \ y_j \in B_{j+1}, \ B_j \subset T, \ r_j < 1 \quad (j \in \{0, 1, 2, ...\})$$

and $y_j \to \infty$. By Lemma 2.6.2, we can proceed recursively to find a sequence (g_j) of functions such that $g_0 = h, g_j \in \mathcal{H}(\mathbb{R}^N \setminus \{y_j\})$ and

$$|(g_{j+1} - g_j)(x)| < 2^{-j-1}\varepsilon(1 + ||x||)^{-p} \quad (x \in \mathbb{R}^N \setminus B_{j+1}; j \in \{0, 1, 2, ...\}).$$

Thus (g_j) is locally uniformly Cauchy on $\mathbb{R}^N$ and therefore converges to a limit function $g \in \mathcal{H}(\mathbb{R}^N)$. Also,

$$|(h - g)(x)| \leq \sum_{j=0}^{\infty} |(g_{j+1} - g_j)(x)| < \varepsilon(1 + ||x||)^{-p} \quad (x \in \mathbb{R}^N \setminus T).$$

□

Theorem 2.6.4. *Let K be a compact subset of an open set Ω in $\mathbb{R}^N$ such that every bounded component of $\mathbb{R}^N \backslash K$ contains a point of $\mathbb{R}^N \backslash \Omega$. If h is harmonic on an open set containing K and if $\varepsilon > 0$, then there exists $H \in \mathcal{H}(\Omega)$ such that $|h - H| < \varepsilon$ on K.*

Proof. In view of Lemma 2.6.1, it is enough to show that if $y_0 \in \mathbb{R}^N \backslash K$, then U_{y_0} can be uniformly approximated on K by functions in $\mathcal{H}(\Omega)$. Suppose first that y_0 belongs to a bounded component, ω say, of $\mathbb{R}^N \backslash K$. By hypothesis, there exists a point z in $\omega \backslash \Omega$. There are balls $B(y_1, r_1), \ldots, B(y_m, r_m)$, where $y_m = z$, with closures contained in ω, such that $y_{k-1} \in B(y_k, r_k)$ for each $k = 1, \ldots, m$. Define $H_0 = U_{y_0}$. By repeated applications of Lemma 2.6.2, we find that there exist functions $H_k \in \mathcal{H}(\mathbb{R}^N \backslash \{y_k\})$ for $k = 1, \ldots, m$ such that $|H_{k-1} - H_k| < \varepsilon/m$ on $A(y_k; r_k, +\infty)$, which contains K. Hence, on K,

$$|U_{y_0} - H_m| \leq \sum_{k=1}^{m} |H_{k-1} - H_k| < \varepsilon.$$

Finally, we note that H_m is harmonic on $\mathbb{R}^N \backslash \{z\}$, which contains Ω.

If y_0 lies in the unbounded component of $\mathbb{R}^N \backslash K$ then, by Lemma 2.6.3, U_{y_0} can be uniformly approximated on K by functions in $\mathcal{H}(\mathbb{R}^N)$. □

Corollary 2.6.5. *Let K be a compact subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \backslash K$ is connected. If h is harmonic on a neighbourhood of K and $\varepsilon > 0$, then there exists a harmonic polynomial H such that $|h - H| < \varepsilon$ on K.*

Proof. The topological hypothesis of Theorem 2.6.4 is vacuously satisfied with $\Omega = \mathbb{R}^N$. Hence there exists $H' \in \mathcal{H}(\mathbb{R}^N)$ such that $|h - H'| < \varepsilon/2$ on K. By

Theorem 2.4.4, there exists a harmonic polynomial H such that $|H' - H| < \varepsilon/2$ on K. Hence $|h - H| < \varepsilon$ on K. □

For the next result in this section, which is inspired by the Mittag–Leffler theorem for holomorphic functions, we require some terminology. If h is harmonic on some deleted neighbourhood of a point x_0 (that is, on $U\backslash\{x_0\}$ for some neighbourhood U of x_0), then it has a Laurent expansion of the form (2.5.7). We refer to the terms

$$\sum_{j=1}^{\infty} ||x - x_0||^{2-N-2j} K_j(x - x_0) + \alpha U_{x_0}(x)$$

in (2.5.7) as the ***singular part*** of h at x_0; note that by Theorem 2.5.3 this singular part is uniquely determined. If this singular part is identically zero, then h has a harmonic continuation to a neighbourhood of x_0, and x_0 is then called a ***removable singularity*** of h.

Theorem 2.6.6. *Let (y_m) be a sequence of distinct points in Ω with no limit point in Ω, and for each m let h_m be harmonic on a deleted neighbourhood of y_m. Then there is a harmonic function h on $\Omega\backslash\{y_m : m \in \mathbb{N}\}$ such that the singular parts of h and h_m are equal (and therefore y_m is a removable singularity of $h - h_m$) for each m.*

Proof. For each $n \in \mathbb{N}$ define

$$K_n = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{n} \text{ and } ||x|| \leq n\}.$$

(If $\Omega = \mathbb{R}^N$, then $K_n = \overline{B(0,n)}$.) It is easy to verify the following properties: each K_n is compact; $K_n \subset K_{n+1}^\circ$ for each n; $\bigcup_{n=1}^{\infty} K_n = \Omega$; each bounded component of $\mathbb{R}^N\backslash K_n$ contains a point of $\mathbb{R}^N\backslash\Omega$. Also, define

$$I_1 = \{m : y_m \in K_1\}, \qquad I_n = \{m : y_m \in K_n\backslash K_{n-1}\} \qquad (n \geq 2).$$

Since (y_m) has no limit point in Ω the sets I_n are finite. Let s_m denote the singular part of h_m at y_m. We note that $s_m \in \mathcal{H}(\mathbb{R}^N\backslash\{y_m\})$ and define

$$g_n = \sum_{m \in I_n} s_m \qquad (n \in \mathbb{N}).$$

(If $I_n = \emptyset$, then $g_n = 0$.) For $n \geq 2$, the function g_n is harmonic on a neighbourhood of K_{n-1}, and by Theorem 2.6.4 there exists $G_n \in \mathcal{H}(\Omega)$ such that $|g_n - G_n| < 2^{-n}$ on K_{n-1}. Define h on $\Omega\backslash\{y_m : m \in \mathbb{N}\}$ by

$$h = g_1 + \sum_{n=2}^{\infty}(g_n - G_n).$$

It follows that the series is uniformly convergent on $K_n\backslash\{y_m : m \in \mathbb{N}\}$ for all n, so that h is harmonic. Finally, if $k \in \mathbb{N}$, then there exists $\delta > 0$ such that $||y_k - y_m|| > \delta$ for $m \neq k$, and on $A(y_k; 0, \delta)$ the function h has the form $g + s_k$, where $g \in \mathcal{H}(B(y_k, \delta))$. Thus the singular part of h at y_k is s_k. □

So far we have used only the case $p = 0$ of Lemma 2.6.3. However, the full power of the lemma is used in the following examples. If $y \in S$ and $-\infty \leq a < b \leq +\infty$, then we write

$$W(y; a, b) = \{x \in \mathbb{R}^N : a < \langle x, y\rangle < b\}.$$

By a ***strip*** we mean a set of the form $W(y; a, b)$, where a, b are finite. If $h \in \mathcal{H}(\mathbb{R}^N)$ and $h(rx) \to 0$ as $r \to +\infty$ uniformly for $x \in S$, then $h \equiv 0$ by the maximum principle. The following example shows that uniformity cannot be dispensed with.

Example 2.6.7. If $p \geq 0$ then there exists a non-constant function $h \in \mathcal{H}(\mathbb{R}^N)$ such that

$$\lim_{x\to\infty,\, x\in W} ||x||^p h(x) = 0 \tag{2.6.3}$$

for every strip W. In particular, $||x||^p h(x)$ decays to 0 on every ray:

$$\lim_{t\to+\infty} t^p h(x_0 + ty_0) = 0 \qquad (x_0 \in \mathbb{R}^N; y_0 \in S).$$

To see this we start by defining

$$\psi(t) = (t, t^2, \ldots, t^N) \qquad (t \geq 0)$$

and

$$T = \bigcup_{t\geq 0} B(\psi(t), 1) \tag{2.6.4}$$

so that T is a tract in $\mathbb{R}^N$ from 0 to ∞. If $y = (y_1, \ldots, y_N) \in S$, then the function

$$g(t) = \sum_{j=1}^{N} y_j t^j \qquad (t \geq 0) \tag{2.6.5}$$

satisfies either $g(t) \to +\infty$ or $g(t) \to -\infty$ as $t \to +\infty$ and it follows that $T \cap W(y; -a, a)$ is bounded for any $a \in (0, +\infty)$.

We now choose a positive integer m such that $m + N - 2 > p$ and a harmonic polynomial $H \in \mathcal{H}_m \setminus \{0\}$, and we define

$$h_0(x) = ||x||^{2-N-2m} H(x).$$

Then $h_0 \in \mathcal{H}(\mathbb{R}^N \setminus \{0\})$ since h_0 is the image of H under the Kelvin transform with respect to S, and by Lemma 2.6.3 there exists, for any $\varepsilon > 0$, a function $h \in \mathcal{H}(\mathbb{R}^N)$ such that

$$|(h-h_0)(x)| < \varepsilon(1+\|x\|)^{2-N-m} \qquad (x \in \mathbb{R}^N \setminus T).$$

By choosing ε to be sufficiently small, we ensure that $h \not\equiv 0$. We have

$$|h(x)| \le |(h-h_0)(x)| + |h_0(x)| < C\|x\|^{2-N-m} \quad (x \in \mathbb{R}^N \setminus T),$$

where C is a constant. The observation at the end of the preceding paragraph shows that $T \cap W$ is bounded for every strip W, so h satisfies (2.6.3).

Let $\mathcal{P}^N$ denote the set of all $(N-1)$-dimensional hyperplanes, that is, sets of the form $\{x \in \mathbb{R}^N : \langle x, y\rangle = a\}$, where $y \in S$ and $a \in \mathbb{R}$. If $f \in C(\mathbb{R}^N)$ and f is integrable with respect to $(N-1)$-dimensional measure λ' on every hyperplane P, then the *Radon transform* $\mathcal{R}_f$ is defined on $\mathcal{P}^N$ by

$$\mathcal{R}_f(P) = \int_P f\,d\lambda' \qquad (P \in \mathcal{P}^N).$$

The question naturally arises whether such a function f is uniquely determined by $\mathcal{R}_f$. The following example, based on the previous one, shows that the answer is negative.

Example 2.6.8. There exists $h \in \mathcal{H}(\mathbb{R}^N)$ such that h is λ'-integrable on every hyperplane P and $\mathcal{R}_h \equiv 0$ on $\mathcal{P}^N$ but $h \not\equiv 0$.

To see this we again take T to be the tract defined by (2.6.4). The construction of Example 2.6.7 shows, in particular, that there exists a non-constant harmonic function h on $\mathbb{R}^N$ such that

$$|h(x)| \le (1+\|x\|)^{-N-1} \qquad (x \in \mathbb{R}^N \setminus T). \tag{2.6.6}$$

We fix an arbitrary point $y \in S$ and for each real number t define $P(y,t)$ to be the hyperplane $\{x \in \mathbb{R}^N : \langle x, y\rangle = t\}$. Since $T \cap W(y; -a, a)$ is bounded for every positive number a, it follows from (2.6.6) that the function

$$t \mapsto \int_{P(y,t)} |h|\,d\lambda' \qquad (t \in \mathbb{R})$$

is locally bounded on $\mathbb{R}$. Hence by Theorem 1.5.12, with a suitable rotation, $\mathcal{R}_h(P(y,\cdot))$ is a polynomial of degree at most 1. We have already remarked that the function g given by (2.6.5) satisfies either $g(t) \to +\infty$ or $g(t) \to -\infty$ as $t \to +\infty$. This implies that there exists a number t_y such that either $T \cap W(y; -\infty, t_y) = \emptyset$ or $T \cap W(y; t_y, +\infty) = \emptyset$. When $T \cap P(y,t) = \emptyset$ it follows from (2.6.6) that

$$\begin{aligned}|\mathcal{R}_h(P(y,t))| &\le \int_{\mathbb{R}^{N-1}} \frac{d\lambda'(x')}{\left(1+(\|x'\|^2+t^2)^{1/2}\right)^{N+1}}\\ &< \frac{1}{1+|t|}\int_{\mathbb{R}^{N-1}} \frac{d\lambda'(x')}{(1+\|x'\|)^N}.\end{aligned}$$

This integral is finite, and therefore either $\mathcal{R}_h(P(y,t)) \to 0$ as $t \to -\infty$ or $\mathcal{R}_h(P(y,t)) \to 0$ as $t \to +\infty$. Since $\mathcal{R}_h(P(y,\cdot))$ is a polynomial of degree at most 1, we must have $\mathcal{R}_h(P(y,\cdot)) \equiv 0$, and since y is an arbitrary point of S, it follows that $\mathcal{R}_h(P) = 0$ for all $P \in \mathcal{P}^N$.

2.7. Harmonic polynomials and classical polynomials

Most classical treatments of harmonic polynomials rely upon a close relationship between the axial harmonics $J_{y,m}$, introduced in Section 2.3, and certain special polynomials in one variable. We make no use of this relationship but for the sake of completeness we give here some brief indications for the interested reader.

The classical functions with which we are mainly concerned are the ultraspherical polynomials. For these we refer to the standard text of Szegö [1, 1967]. The results that we require are the explicit formula for the ultraspherical polynomial $P_m^{(\lambda)}$, where $\lambda > 0$ and $m = 0, 1, 2, \ldots$:

$$P_m^{(\lambda)}(t) = 2^m \sum_{j=0}^{[m/2]} (-1)^j \frac{\Gamma(m-j+\lambda)}{\Gamma(\lambda) j! 4^j (m-2j)!} t^{m-2j} \tag{2.7.1}$$

(Szegö, Formula (4.7.31)) and the equation (Szegö, Formula (4.7.3))

$$P_m^{(\lambda)}(1) = \binom{m+2\lambda-1}{m}. \tag{2.7.2}$$

Although they are not required for the present discussion, we mention some other characterizations of ultraspherical polynomials. They are orthogonal with respect to the weight function $(1-t^2)^{\lambda-\frac{1}{2}}$ on $(-1,1)$; that is,

$$\int_{-1}^{1} P_m^{(\lambda)}(t) P_n^{(\lambda)}(t)(1-t^2)^{\lambda-\frac{1}{2}}\,dt = 0 \qquad (0 \le m < n);$$

this together with (2.7.2) determines the ultraspherical polynomials uniquely. Also, $P_m^{(\lambda)}$ is the unique polynomial of degree m satisfying (2.7.2) and the differential equation

$$(1-t^2)f''(t) - (2\lambda+1)tf'(t) + m(m+2\lambda)f(t) = 0.$$

Finally, the ultraspherical polynomials are characterized by the equation

$$\sum_{n=0}^{\infty} P_n^{(\lambda)}(t)\xi^n = (1-2t\xi+\xi^2)^{-\lambda} \qquad (t, \xi \in (-1,1)).$$

(See Szegö, Formula (4.7.23).)

The relationship between axial harmonics and ultraspherical polynomials in dimension $N \geq 3$ is as follows; we defer consideration of the simpler case where $N = 2$.

Theorem 2.7.1. *Suppose that $N \geq 3$ and $y \in S$. Then*

$$J_{y,m}(x) = \frac{2m+N-2}{N-2}\|x\|^m P_m^{((N-2)/2)}\left(\frac{\langle x,y\rangle}{\|x\|}\right) \qquad (x \neq 0).$$

Proof. Let

$$h(x) = \|x\|^m P_m^{((N-2)/2)}\left(\frac{\langle x,y\rangle}{\|x\|}\right) \qquad (x \neq 0)$$

and $h(0) = 0$. By (2.7.1),

$$h(x) = \frac{2^m}{\Gamma\left(\frac{N-2}{2}\right)} \sum_{j=0}^{[m/2]} (-1)^j \frac{\Gamma(m-j+\frac{1}{2}(N-2))}{j!4^j(m-2j)!}\|x\|^{2j}\langle x,y\rangle^{m-2j}.$$

Clearly h is a y-axial homogeneous polynomial of degree m. Direct calculation shows that for $j = 0, 1, \ldots, [m/2]$,

$$\begin{aligned}\Delta(\|x\|^{2j}\langle x,y\rangle^{m-2j}) &= 2j(2m-2j+N-2)\|x\|^{2j-2}\langle x,y\rangle^{m-2j} \\ &\quad + (m-2j)(m-2j-1)\|x\|^{2j}\langle x,y\rangle^{m-2j-2}.\end{aligned}$$

A calculation using this equation shows that $\Delta h = 0$. Hence h is a y-axial element of $\mathcal{H}_m$ and therefore $J_{y,m} = kh$ for some real k, by Theorem 2.3.4. We know that $J_{y,m}(y) = d_{m,N}$ by Corollary 2.3.7 and $h(y) = P_m^{((N-2)/2)}(1)$ by the definition of h. These values are given explicitly by Corollary 2.1.4 and (2.7.2), and using them it is easy to show that $k = (2m+N-2)/(N-2)$. □

Remark 2.7.2. We saw in Example 2.3.9 that in the case $N = 2$ if $y = (\cos\phi, \sin\phi)$ and $x = (r\cos\theta, r\sin\theta)$, then

$$J_{y,m}(x) = 2r^m \cos mt,$$

where

$$\cos t = \cos\theta \cdot \cos\phi + \sin\theta \cdot \sin\phi = \frac{\langle x,y\rangle}{\|x\|}.$$

Thus

$$J_{y,m}(x) = 2\|x\|^m T_m\left(\frac{\langle x,y\rangle}{\|x\|}\right) \qquad (x \neq 0),$$

where T_m is the Chebyshev polynomial given by

$$T_m(t) = \cos(m\cos^{-1} t) \qquad (-1 \leq t \leq 1)$$

(Szegö, Section 2.4). Chebyshev polynomials and ultraspherical polynomials are related by the equation

$$T_m(t) = \frac{m}{2}\lim_{\lambda\to 0+} \lambda^{-1} P_m^{(\lambda)}(t)$$

(Szegö, Formula (4.7.8)).

Remark 2.7.3. Except for a normalizing factor, the ultraspherical polynomial $P_m^{(\lambda)}$, where $\lambda > 0$, is identical to the Jacobi polynomial $P_m^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}$ (Szegö, Formula (4.7.1)); also T_m is proportional to $P_m^{(-\frac{1}{2},-\frac{1}{2})}$ (Szegö, Section 4.1(3)). Hence, for all dimensions $N \geq 2$, there is a constant k such that

$$J_{y,m}(x) = k\|x\|^m P_m^{(\alpha,\alpha)}\left(\frac{\langle x,y\rangle}{\|x\|}\right), \text{ where } \alpha = \frac{1}{2}(N-3).$$

Using explicit values for $J_{y,m}(y)$ and $P_m^{(\alpha,\alpha)}(1)$ (Szegö, Formula (4.1.1)), we find that

$$k = d_{m,N}\binom{m+\frac{1}{2}(N-3)}{m}^{-1}.$$

2.8. Exercises

Exercise 2.1. Let P be a non-constant polynomial on $\mathbb{R}^N$ such that $P(x) > 0$ whenever $\|x\|$ is sufficiently large. Show that P cannot be a factor of any non-zero harmonic polynomial. (Hint: suppose that P is a factor of $H_0 + H_1 + \ldots + H_m$, where $H_j \in \mathcal{H}_j$, and consider the factors of H_m.)

Exercise 2.2. Let $R(x) = c_1^2x_1^2 + \ldots + c_N^2x_N^2$, where $c_1, \ldots, c_N$ are non-zero real numbers, and let $\mathcal{R}_m$, where $m \geq 2$, denote the vector space $\{RP : P \in \mathcal{P}_{m-2}\}$. Show that $\mathcal{P}_m = \mathcal{H}_m \oplus \mathcal{R}_m$. Deduce that if E is the ellipsoid $\{x : R(x) = 1\}$ and T is a polynomial of degree m on $\mathbb{R}^N$, then there is a harmonic polynomial H of degree at most m such that $H = T$ on E.

Exercise 2.3. Let $\mathcal{P}_m' = \{P \in \mathcal{P}_m : \partial P/\partial x_N \equiv 0\}$. For each $P \in \mathcal{P}_m'$, define

$$\begin{aligned}H_P(x) &= \sum_{j=0}^{[m/2]} \frac{(-1)^j}{(2j)!}x_N^{2j}\Delta^j P(x), \\ K_P(x) &= \sum_{j=0}^{[m/2]} \frac{(-1)^j}{(2j+1)!}x_N^{2j+1}\Delta^j P(x).\end{aligned}$$

Show that $H_P \in \mathcal{H}_m$ and $K_P \in \mathcal{H}_{m+1}$. Show also that the mappings $P \mapsto H_P$ and $P \mapsto K_P$ are bijections from $\mathcal{P}_m'$ onto the spaces $\{H \in \mathcal{H}_m : \partial H/\partial x_N = 0$ on $\mathbb{R}^{N-1} \times \{0\}\}$ and $\{H \in \mathcal{H}_{m+1} : H = 0$ on $\mathbb{R}^{N-1} \times \{0\}\}$ respectively.

Exercise 2.4. Let $P \in \mathcal{P}_m$, where $m \geq 2$, and define

$$H_P(x) = \sum_{j=0}^{[m/2]} (-1)^j \{4^j j!(m-j+\frac{N}{2}-1)(m-j+\frac{N}{2})\cdots$$
$$(m+\frac{N}{2}-2)\}^{-1} \|x\|^{2j} \Delta^j P(x).$$

Show that $H_P \in \mathcal{H}_m$ and $P - H_P \in \mathcal{Q}_m$. (Note: Theorem 2.1.1 tells us that there exist unique polynomials $H \in \mathcal{H}_m$ and $Q \in \mathcal{Q}_m$ such that $P = H + Q$, and we are now saying that $H = H_P$ and $Q = P - H_P$.)

Exercise 2.5. Use Corollary 2.1.4 to show that

$$d_{m,N+1} = d_{0,N} + d_{1,N} + \ldots + d_{m,N}$$

and

$$\frac{d_{m,N}}{m^{N-2}} \to \frac{2}{(N-2)!} \qquad (m \to \infty).$$

Exercise 2.6. Let $h \in \mathcal{H}_+(B)$ and suppose that $h = \sum_{j=0}^{\infty} H_j$, where $H_j \in \mathcal{H}_j$. By considering

$$\int_{S(0,r)} (\sup_S H_m - H_m) h \, d\sigma \qquad (0 < r < 1),$$

show that $\|H_m\|^2 \leq h(0) \sup_S H_m$ for each m. Verify that equality holds for each m if h is the Poisson kernel $K(\cdot, y)$ of B for some fixed $y \in S$.

Exercise 2.7. Let H be the Poisson integral on B of a polynomial $P \in \mathcal{P}_m$ (so that H is a polynomial of degree at most m; see Corollary 2.1.3). Show that

$$\int_B H^2 \, d\lambda \leq N^{-1}(N+2m) \int_B P^2 \, d\lambda.$$

Exercise 2.8. Give an example of a real-analytic function on $\mathbb{R}^N$ and a series $\sum P_j$, where $P_j \in \mathcal{P}_j$ such that the series converges to f on B but on no larger open ball of centre 0.

Exercise 2.9. Show that if $y = (1, 0, \ldots, 0)$ and $m \in \mathbb{N} \cup \{0\}$, then

$$J_{y,m} = \frac{d_{m,N}}{(m+1)d_{m+1,N}} \frac{\partial}{\partial x_1} J_{y,m+1}.$$

Exercise 2.10. Show that if $N \geq 3$ and $y \in S$, then

$$U_y(x) = \sum_{m=0}^{\infty} \binom{m+N-3}{m} d_{m,N}^{-1} J_{y,m}(x) \qquad (x \in B).$$

What is the corresponding equation with $N = 2$?

Exercise 2.11. Bôcher's theorem. Let $x_0 \in \Omega$. Show that if $h \in \mathcal{H}_+(\Omega \backslash \{x_0\})$, then there exist $h_0 \in \mathcal{H}(\Omega)$ and a non-negative constant c such that $h = h_0 + cU_{x_0}$ on Ω. (Hint: use Lemma 2.5.2 and the result of Exercise 1.14.)

Exercise 2.12. Show that if $h : \overline{B} \to \mathbb{R}$ is continuous on $\overline{B}$ and harmonic on B, then there is a series $\sum H_m$ of harmonic polynomials which converges uniformly to h on $\overline{B}$. (Hint: use the Stone–Weierstrass theorem to show that there is a series $\sum P_m$ of polynomials which converges uniformly to h on S.)

Exercise 2.13. Let $K = \{x \in \mathbb{R}^N : 1 \leq \|x\| \leq 2\}$. Show that U_0 cannot be uniformly approximated arbitrarily closely on K by elements of $\mathcal{H}(\mathbb{R}^N)$. (This shows that the topological hypotheses in Theorem 2.6.4 and Corollary 2.6.5 are indispensable.)

Exercise 2.14. We define the *hull* $\widetilde{K}$ of a compact set K in $\mathbb{R}^N$ to be the union of K with all the bounded components of $\mathbb{R}^N \backslash K$. We say that disjoint sets E, F are *separated* by a family $\mathcal{F}$ of real-valued functions if there exists $\phi \in \mathcal{F}$ such that $\phi > 0$ on E and $\phi < 0$ on F. Show that if K_1, K_2 are disjoint compact subsets of $\overline{B}$, then the following are equivalent:
(a) K_1, K_2 are separated by $\mathcal{H}(\mathbb{R}^N)$;
(b) K_1, K_2 are separated by $C(\overline{B}) \cap \mathcal{H}(B)$;
(c) $\widetilde{K_1} \cap \widetilde{K_2} = \emptyset$.

Exercise 2.15. Let K_1, K_2 be compact subsets of $\overline{B}$, and let $f \in C(\overline{B})$ and $h^* \in C(\overline{B}) \cap \mathcal{H}(B)$ be such that $f - h^* > 0$ on K_1 and $f - h^* < 0$ on K_2. Show that:
(i) if K_1, K_2 are separated by $C(\overline{B}) \cap \mathcal{H}(B)$ and

$$|(f-h^*)(x)| < \sup_{K_1 \cup K_2} |f - h^*| \qquad (x \in \overline{B} \backslash (K_1 \cup K_2)),$$

then there exists $h \in C(\overline{B}) \cap \mathcal{H}(B)$ such that

$$\sup_{\overline{B}} |f - h| < \sup_{\overline{B}} |f - h^*|;$$

(ii) if K_1, K_2 are not separated by $C(\overline{B}) \cap \mathcal{H}(B)$, then

$$\sup_{K_1 \cup K_2} |f - h| \geq \inf_{K_1 \cup K_2} |f - h^*|$$

for all $h \in C(\overline{B}) \cap \mathcal{H}(B)$.

Exercise 2.16. Let $f \in C(\overline{B})$, where $f \notin \mathcal{H}(B)$, and $h^* \in C(\overline{B}) \cap \mathcal{H}(B)$. We call h^* a *best harmonic approximant* to f if

$$\sup_{\overline{B}} |f - h^*| \le \sup_{\overline{B}} |f - h| \qquad (h \in C(\overline{B}) \cap \mathcal{H}(B)).$$

Use Exercises 2.14 and 2.15 to show that h^* is a best harmonic approximant to f if and only if $\widetilde{K_+} \cap \widetilde{K_-} \ne \emptyset$, where

$$K_\pm = \{x \in \overline{B} : (f - h^*)(x) = \pm \sup_{\overline{B}} |f - h^*|\}.$$

Exercise 2.17. Let $\mathcal{B}_k$ be a basis for $\mathcal{H}_k$, and let $\{h_m : m \in \mathbb{N}\}$ be the set of all finite linear combinations of elements of $\bigcup_{k=0}^\infty \mathcal{B}_k$ with rational coefficients. Show that $\{h_m\}$ is dense in $\mathcal{H}(\mathbb{R}^N)$ in the topology of local uniform convergence; that is, for every $h \in \mathcal{H}(\mathbb{R}^N)$, every compact set K, and every $\varepsilon > 0$, there exists m such that $|h - h_m| < \varepsilon$ on K.

Exercise 2.18. A universal harmonic function. For each $m \in \mathbb{N}$, let $y_m = (2^{m+3}, 0, \ldots, 0) \in \mathbb{R}^N, B_m = \overline{B(0, 3.2^{m+1})}, C_m = \overline{B(y_m, 2^m)}$. (Note that $B_m \cap C_m = \emptyset$ and $C_m \subset B_{m+1}$.) Let U_m and V_m be disjoint open neighbourhoods of B_m and C_m respectively, and let $\{h_m\}$ be the set of harmonic polynomials defined in Exercise 2.14. Define $g_1(x) = h_1(x - y_1)$ on V_1 and $g_1 = 0$ on U_1. Use Corollary 2.6.5 to show that there exists $f_1 \in \mathcal{H}(\mathbb{R}^N)$ such that $|f_1 - g_1| < 2^{-2}$ on $B_1 \cup C_1$. Suppose now that functions $f_1, \ldots, f_{m-1} \in \mathcal{H}(\mathbb{R}^N)$ are given and define

$$g_m(x) = \begin{cases} h_m(x - y_m) - f_1(x) - \ldots - f_{m-1}(x) & (x \in V_m) \\ 0 & (x \in U_m). \end{cases}$$

Show that there exists $f_m \in \mathcal{H}(\mathbb{R}^N)$ such that $|f_m - g_m| < 2^{-m-1}$ on $B_m \cup C_m$. Let $H = \sum_{m=1}^\infty f_m$. Check that H is defined and harmonic on $\mathbb{R}^N$. Show also that

$$|H(x) - h_m(x - y_m)| < 2^{-m} \qquad (x \in C_m, m \in \mathbb{N}).$$

Deduce that if $h \in \mathcal{H}(\mathbb{R}^N), K$ is a compact set in $\mathbb{R}^N$, and $\varepsilon > 0$, then there exists m such that

$$|h(x) - H(x + y_m)| < \varepsilon \qquad (x \in K).$$

(Such a harmonic function H, whose translates uniformly approximate all elements of $\mathcal{H}(\mathbb{R}^N)$ on compact sets is called *universal*.)

Chapter 3. Subharmonic Functions

3.1. Elementary properties

We have seen that harmonic functions on an open set Ω can be characterized as those finite-valued, continuous functions h on Ω which satisfy the mean value property: $h(x) = \mathcal{M}(h; x, r)$ whenever $\overline{B(x,r)} \subset \Omega$. Subharmonic functions correspond to one half of this definition – they are upper-finite, upper semicontinuous functions s which satisfy the mean value inequality $s(x) \le \mathcal{M}(s; x, r)$ whenever $\overline{B(x,r)} \subset \Omega$. They are allowed to take the value $-\infty$ so that we can include such fundamental examples as $\log ||x||$ ($N = 2$) and $-||x||^{2-N}$ ($N \ge 3$). Also, semicontinuity (rather than continuity) is the appropriate condition for certain key results (for example, Theorems 3.1.4 and 3.3.1) to hold. The reason for the name "subharmonic" will become apparent in Section 3.2.

Some of the properties of subharmonic functions given in this chapter, such as the maximum principle and convergence theorems, are closely related to properties established for harmonic functions in Chapter 1. On the other hand, in sharp contrast to the analyticity of harmonic functions, it will be seen that a subharmonic function may be suitably modified on a subset of its domain of definition and still be subharmonic. It is this flexibility of subharmonic functions that makes them so useful even when we are proving results concerning harmonic functions, as is the case with the Dirichlet problem in Chapter 6.

Laplace's equation on $\mathbb{R}$ is simply $d^2h/dt^2 \equiv 0$ with general solution $h(t) = at + b$. Below we shall see that the subharmonic functions on $\mathbb{R}$ are simply the convex functions and, in particular, are continuous. It is, therefore, not surprising that notions of convexity will appear at several points of this chapter. Also, just as convexity among the smooth functions s on $\mathbb{R}$ is characterized by the condition $\Delta s = d^2 s/dt^2 \ge 0$, so subharmonicity among the smooth functions s on $\mathbb{R}^N$ will be characterized by the condition $\Delta s \ge 0$, and it will be shown that any subharmonic function can be expressed as the limit of a decreasing sequence of smooth subharmonic functions. This sometimes allows us to reduce a problem about subharmonic functions to one about smooth subharmonic functions.

Definition 3.1.1. Let $E \subseteq \mathbb{R}^N \cup \{\infty\}$. A function $f: E \to [-\infty, +\infty]$ is called *upper semicontinuous* (on E) if $\{x \in E: f(x) < a\}$ is a relatively open subset of E for each $a \in \mathbb{R}$. Also, $f: E \to [-\infty, +\infty]$ is called *lower semicontinuous* if $-f$ is upper semicontinuous.

Thus a function $f: E \to [-\infty, +\infty]$ is continuous if and only if it is both upper and lower semicontinuous. It is easy to check that f is upper semicontinuous if and only if

$$\limsup_{x \to y} f(x) \le f(y) \tag{3.1.1}$$

for each limit point y of E. If K is a compact set and $f: K \to [-\infty, +\infty)$ is upper semicontinuous, then a simple covering argument shows that f is bounded above, and it follows from (3.1.1) that f attains its supremum on K. Finally, if $f: E \to [-\infty, +\infty]$ is upper semicontinuous and $F \subseteq E$, then $f|_F$ is clearly upper semicontinuous on F.

Definition 3.1.2. A function $s: \Omega \to [-\infty, +\infty)$ is called *subharmonic* on Ω if:
(i) s is upper semicontinuous on Ω,
(ii) $s(x) \le \mathcal{M}(s; x, r)$ whenever $\overline{B(x,r)} \subset \Omega$, and
(iii) $s \not\equiv -\infty$ on each component of Ω.
Also, a function $u: \Omega \to (-\infty, +\infty]$ is called *superharmonic* on Ω if $-u$ is subharmonic on Ω. We refer to (ii) above as the *subharmonic mean value property;* with the inequality reversed we call it the *superharmonic mean value property.*

The set of all subharmonic (respectively superharmonic) functions on Ω will be denoted by $\mathcal{S}(\Omega)$ (respectively $\mathcal{U}(\Omega)$). It is easy to see that $\mathcal{H}(\Omega) = \mathcal{S}(\Omega) \cap \mathcal{U}(\Omega)$, and that $\mathcal{S}(\Omega)$ and $\mathcal{U}(\Omega)$ are cones: that is, $as + bu \in \mathcal{S}(\Omega)$ (respectively $\mathcal{U}(\Omega)$) whenever $a, b \in [0, +\infty)$ and $s, u \in \mathcal{S}(\Omega)$ (respectively $\mathcal{U}(\Omega)$). Also, it follows easily from the definition that, if $s, u \in \mathcal{S}(\Omega)$, then $\max\{s, u\} \in \mathcal{S}(\Omega)$. In particular, $s^+ = \max\{s, 0\} \in \mathcal{S}(\Omega)$, and if $h \in \mathcal{H}(\Omega)$, then $|h| = \max\{h, -h\} \in \mathcal{S}(\Omega)$.

Theorem 3.1.3. *If $s \in \mathcal{S}(\Omega)$, then:*
(i) $\limsup_{x \to y} s(x) = s(y)$ for each $y \in \Omega$;
(ii) $s(x) \le \mathcal{A}(s; x, r)$ whenever $\overline{B(x,r)} \subset \Omega$;
(iii) s is locally integrable (and hence finite almost everywhere) on Ω.

Proof. (i) The upper semicontinuity of s implies that $\limsup s(x) \le s(y)$ as $x \to y \in \Omega$. If this inequality were strict, then we would have $s < s(y)$ on $B(y, r) \backslash \{y\}$ for some r, which would contradict the subharmonic mean value property of s.

(ii) This follows from the subharmonic mean value property and the relation

$$r^N \mathcal{A}(s; x, r) = N \int_{(0,r]} t^{N-1} \mathcal{M}(s; x, t) dt; \tag{3.1.2}$$

the above mean values are defined since s is bounded above on the compact set $\overline{B(x,r)}$.

(iii) It is enough to consider the case where Ω is connected. Let

$$\Omega_0 = \{y \in \Omega: s \text{ is integrable on some neighbourhood of } y\}.$$

Suppose that $y \in \Omega \backslash \Omega_0$ and choose ρ such that $\overline{B(y, 2\rho)} \subseteq \Omega$. If $z \in B(y, \rho)$, then $B(z, \rho)$ is a neighbourhood of y and $\overline{B(z, \rho)} \subset \Omega$. Hence s is bounded above and non-integrable on $B(z, \rho)$, and so $s(z) \le \mathcal{A}(s; z, \rho) = -\infty$. Thus $s = -\infty$ on $B(y, \rho)$, whence $B(y, \rho) \subseteq \Omega \backslash \Omega_0$. It follows that $\Omega \backslash \Omega_0$ is open and clearly Ω_0 is open. Since $s \not\equiv -\infty$, we see that $\Omega_0 \neq \emptyset$. Hence $\Omega_0 = \Omega$ by the connectedness of Ω. □

Theorem 3.1.4. *If Ω is connected and (s_n) is a decreasing sequence in $\mathcal{S}(\Omega)$, then either* $\lim s_n \equiv -\infty$ *or* $\lim s_n \in \mathcal{S}(\Omega)$.

Proof. The upper semicontinuity of $\lim s_n$ is clear, and the subharmonic mean value property of $\lim s_n$ follows by monotone convergence from the corresponding property of each s_n. □

The following is a generalization of the maximum principle for harmonic functions (Theorem 1.2.4).

Theorem 3.1.5. (Maximum principle) *Let $s \in \mathcal{S}(\Omega)$ and $x \in \Omega$.*
(i) If s attains a local maximum at x, then s is constant on some neighbourhood of x.
(ii) If Ω is connected and s attains a maximum at x, then s is constant.
(iii) If $u \in \mathcal{U}(\Omega)$ and

$$\limsup_{x \to y} (s - u)(x) \le 0 \tag{3.1.3}$$

for each $y \in \partial^\infty \Omega$, then $s \le u$ on Ω.

Proof. (i) Let r be small enough so that $\overline{B(x,r)} \subset \Omega$ and $s \le s(x)$ on $B(x, r)$. Since $s(x) \le \mathcal{A}(s; x, r)$, upper semicontinuity implies that $s = s(x)$ on $B(x, r)$.

(ii) From (i) the set $\{y \in \Omega: s(y) = s(x)\}$ is open, and upper semicontinuity implies that it is closed relative to Ω, so by connectedness it is all of Ω.

(iii) Since $s - u \in \mathcal{S}(\Omega)$, we may assume that $u \equiv 0$. Also, since $\partial^\infty \omega \subseteq \partial^\infty \Omega$ for each component ω of Ω, we may suppose that Ω is connected. If we define $\overline{s} = s$ on Ω and $\overline{s}(y) = \limsup_{x \to y} s(x)$ when $y \in \partial^\infty \Omega$, then $\overline{s}$ is upper semicontinuous on the compact set $\Omega \cup \partial^\infty \Omega$ and therefore attains its supremum. If this supremum is positive, then it is attained at a point of Ω, and s has a positive constant value on Ω, by (ii), contrary to (3.1.3). □

The obvious analogue of Theorem 3.1.5 for superharmonic functions is called the *minimum principle*. Part (iii) of the above result can be generalized as follows.

Theorem 3.1.6. *If $s \in \mathcal{S}(\Omega), u \in \mathcal{U}(\Omega), h \in \mathcal{H}(\Omega)$ and $h > 0$ on Ω, and if*

$$\limsup_{x \to y} \frac{(s-u)(x)}{h(x)} \leq 0 \qquad (y \in \partial^\infty \Omega),$$

then $s \leq u$ on Ω.

Proof. Again we may assume that $u \equiv 0$. Suppose that $s(x_0) > 0$ for some $x_0 \in \Omega$, and let $\varepsilon = s(x_0)/(2h(x_0))$. Then $(s - \varepsilon h)(x_0) > 0$, and yet $\limsup_{x \to y}(s - \varepsilon h) \leq 0$ on $\partial^\infty \Omega$. This contradicts Theorem 3.1.5(iii). □

By careful choice of the function h in the above result we can often relax condition (3.1.3) at some points of $\partial^\infty \Omega$, as the following examples illustrate. Results of this type are known as Phragmén–Lindelöf theorems.

Example 3.1.7. (i) Let $\Omega = B \backslash \{0\}$, let $s \in \mathcal{S}(\Omega)$ and $u \in \mathcal{U}(\Omega)$. If (3.1.3) holds for each $y \in S$ and if

$$\limsup_{x \to 0} \frac{(s-u)(x)}{U_0(x)} \leq 0,$$

then $s \leq u$ on Ω. To see this, apply Theorem 3.1.6 with $h = 1 + U_0$ $(N = 2)$ or $h = U_0$ $(N \geq 3)$.

(ii) Let $D = \{x = (x_1, \ldots, x_N): x_N > 0\}$, let $s \in \mathcal{S}(D)$ and $u \in \mathcal{U}(D)$. If (3.1.3) holds for each $y \in \partial D$ and if

$$\limsup_{x \to \infty} \frac{(s-u)(x)}{1 + x_N} \leq 0,$$

then $s \leq u$ on D. To see this, apply Theorem 3.1.6 with $h(x) = 1 + x_N$.

Let $s \in \mathcal{S}(\Omega)$ and suppose that $\limsup_{x \to y} s(x) \leq 0$ for each $y \in \partial\Omega$. If Ω is bounded, then it follows from Theorem 3.1.5(iii) that $s \leq 0$ on Ω. This implication breaks down for some unbounded sets Ω (for example, consider $s(x) = x_N$ and $\Omega = \mathbb{R}^{N-1} \times (0, +\infty)$), but not all such Ω, as will be seen below.

Definition 3.1.8. Let $\Omega \subseteq \Omega_0$, where Ω_0 is an open set. We say that $\partial^\infty \Omega_0$ is *accessible* from Ω if there is a continuous function $p: [0, +\infty) \to \Omega$ with the following property: for every compact $K \subseteq \Omega_0$ there exists t_K such that $p(t) \in \Omega \backslash K$ whenever $t \geq t_K$.

Example 3.1.9. An unbounded connected open subset of $\mathbb{R}^2$ from which $\{\infty\}$ is not accessible is defined by

$$\Omega = \{(x_1, x_2): 0 < x_1 < 1, -1 < x_2 < x_1^{-1} \sin^2(x_1^{-1})\}.$$

A similar example in higher dimensions can be obtained by rotating Ω about the x_1-axis.

Theorem 3.1.10. *Let $\Omega \subseteq \Omega_0$, where Ω_0 is open, and suppose that $\partial^\infty \Omega_0$ is not accessible from Ω. If $s \in \mathcal{S}(\Omega)$ and $\limsup_{x \to y} s(x) \leq 0$ for each $y \in \Omega_0 \cap \partial\Omega$, then $s \leq 0$ on Ω.*

Proof. Let (K_m) be a sequence of compact sets such that $K_m \subset K_{m+1}^\circ$ for each m and $\bigcup_m K_m = \Omega_0$. Now fix $\varepsilon > 0$ and $m \in \mathbb{N}$. It will be enough to show that $s \leq \varepsilon$ on $\Omega \cap K_m$. Let $W_0 = \Omega \backslash K_m$, let $U_{0,1}, U_{0,2}, \ldots$ denote those components of W_0 for which either $\Omega \cap \partial U_{0,k} = \emptyset$ or $s \leq \varepsilon$ on $\Omega \cap \partial U_{0,k}$, and let $V_{0,1}, V_{0,2}, \ldots$ denote the remaining components of W_0. Thus, for each $V_{0,k}$, there is a point y_k in $\Omega \cap \partial V_{0,k}$ such that $s(y_k) > \varepsilon$. Clearly $y_k \in \partial K_m$ for each k. There can only be finitely many, k_0 say, of the components $V_{0,k}$, for otherwise there is a subsequence of (y_k) which converges to some point of $\partial\Omega$, and this contradicts the hypothesis on s.

If $k_0 \geq 1$, then we define

$$W_1 = \left(\bigcup_{k=1}^{k_0} V_{0,k} \right) \backslash K_{m+1}$$

and divide the components of W_1 into two classes $\{U_{1,1}, U_{1,2}, \ldots\}$ and $\{V_{1,1}, V_{1,2}, \ldots, V_{1,k_1}\}$ as before. Similarly, if $k_1 \geq 1$, then we define

$$W_2 = \left(\bigcup_{k=1}^{k_1} V_{1,k} \right) \backslash K_{m+2},$$

and so on. If $j \geq 1$, then each $V_{j,k}$ is a component of W_j and so must be contained in some $V_{j-1,k'}$. Thus, if $j > j' \geq 0$, each $V_{j,k}$ is contained in some $V_{j',k'}$; in this case we say that $V_{j,k}$ is a *descendant* of $V_{j',k'}$.

Now suppose that, for each j, the collection $\{V_{j,1}, V_{j,2}, \ldots, V_{j,k_j}\}$ is non-empty. Then, for some choice of k, the set $V_{0,k}$ has infinitely many descendants: we call this set V_0. There must be a descendant $V_{1,k}$ of V_0 which also has infinitely many descendants: we call this set V_1. Proceeding in this manner, we obtain a sequence $(V_j)_{j \geq 0}$ of connected open subsets of Ω such that $V_0 \supset V_1 \supset \ldots$ and $V_j \cap K_{m+j} = \emptyset$. However, we can then construct a continuous function $p: [0, +\infty) \to \Omega$ such that $p(t) \in \Omega \backslash K_m$ whenever $t \geq m$, and this contradicts the hypothesis on Ω.

Thus there exists j' for which there are no sets $V_{j',k}$ as above, in which case $W_{j'} = \bigcup_k U_{j',k}$ and we do not construct $W_{j'+1}$. If we define

$$L_m = K_{m+j'} \backslash \left(\bigcup_{j=0}^{j'-1} \left[\bigcup_k U_{j,k} \right] \right),$$

then L_m is compact, $K_m \subseteq L_m$ and

$$\Omega \backslash L_m = \bigcup_{j=0}^{j'} \left[\bigcup_k U_{j,k} \right].$$

From the definition of the sets $U_{j,k}$, we see that $s \leq \varepsilon$ on $\Omega \cap \partial L_m$. Thus, by the maximum principle, $s \leq \varepsilon$ on $\Omega \cap L_m^\circ$ and hence on $\Omega \cap K_m$, as required. □

Our final result in this section shows the connection between Theorem 3.1.10 and the results of Section 2.6.

Corollary 3.1.11. *Let $\emptyset \neq E \subseteq \Omega$. The following are equivalent:*
(a) $\sup_E h = \sup_\Omega h$ for all $h \in \mathcal{H}(\Omega)$;
(b) $\partial^\infty \Omega$ is not accessible from $\Omega \backslash \overline{E}$.

Proof. First suppose that (b) holds, let $h \in \mathcal{H}(\Omega)$ and $M = \sup_E h$. To avoid triviality we may assume that $M < +\infty$. By continuity $h \leq M$ on $\overline{E}$. We can now apply Theorem 3.1.10 (with $\Omega \backslash \overline{E}$ in place of Ω) to see that $h \leq M$ on $\Omega \backslash \overline{E}$ and hence on Ω, as required.

Conversely, suppose that (b) fails to hold. Thus there is a continuous function $p : [0, +\infty) \to \Omega \backslash \overline{E}$ with the following property: for each compact $K \subset \Omega$ there exists t_K such that $p(t) \in \Omega \backslash K$ whenever $t \geq t_K$. We choose sequences $(t_n)_{n \geq 0}$ and $(r_n)_{n \geq 1}$ of positive numbers such that $t_n \to +\infty$, $r_n \to 0$ and

$$p(t_{n-1}) \in B(p(t_n), r_n) \subset \Omega \backslash \overline{E} \qquad (n \geq 1).$$

The function $u = U_{p(t_1)}$ satisfies $u(p(t_0)) > \sup_E u$. Let $\varepsilon > 0$. By repeated application of Lemma 2.6.2 (cf. the proof of Lemma 2.6.3) with $p = 0$ we obtain $h \in \mathcal{H}(\Omega)$ such that $|u - h| < \varepsilon$ on $E \cup \{p(t_0)\}$. By choosing ε sufficiently small, we can arrange that $h(p(t_0)) > \sup_E h$ and so (a) also fails. □

3.2. Criteria for subharmonicity

In this section we will establish several alternative criteria for subharmonicity and see some important examples of subharmonic functions. First we give a preliminary lemma concerning upper semicontinuous functions.

Lemma 3.2.1. *If E is a non-empty subset of $\mathbb{R}^N$ and $f: E \to [-\infty, +\infty)$ is upper semicontinuous and bounded above, then there is a decreasing sequence (f_n) in $C(\mathbb{R}^N)$ such that $f_n \to f$ pointwise on E.*

Proof. First we extend f to $\mathbb{R}^N$ by defining

$$\overline{f}(x) = \begin{cases} f(x) & (x \in E) \\ \limsup_{y \to x} f(y) & (x \in \overline{E} \backslash E) \\ -\infty & (x \in \mathbb{R}^N \backslash \overline{E}). \end{cases}$$

It is easy to see that $\overline{f}$ is upper semicontinuous and bounded above on $\mathbb{R}^N$. We will show that $\overline{f}$ is the pointwise limit of a decreasing sequence in $C(\mathbb{R}^N)$.

If $\overline{f} \equiv -\infty$, then we define $f_n \equiv -n$. Otherwise, we put

$$f_n(x) = \sup\{\overline{f}(y) - n\|x - y\| : y \in \mathbb{R}^N\} \qquad (x \in \mathbb{R}^N).$$

Clearly (f_n) is decreasing and $f_n \geq \overline{f}$ for all n. Also,

$$\overline{f}(y) - n\|y - x\| \leq f_n(x_0) + n\|x - x_0\| \qquad (x, y, x_0 \in \mathbb{R}^N)$$

and, taking suprema over all y and interchanging x and x_0, we see that

$$|f_n(x) - f_n(x_0)| \leq n\|x - x_0\| \qquad (x, x_0 \in \mathbb{R}^N),$$

so $f_n \in C(\mathbb{R}^N)$ for each n. Finally, if $\overline{f}(x) < a$, then there exists $\delta > 0$ such that $\overline{f} < a$ on $B(x, \delta)$, by upper semicontinuity, and so

$$\overline{f}(x) \leq f_n(x) \leq \max\{a, \sup \overline{f} - n\delta\} \to a \qquad (n \to \infty).$$

Hence $f_n(x) \to \overline{f}(x)$ for all x. □

Theorem 3.2.2. *Let $s: \Omega \to [-\infty, +\infty)$ be upper semicontinuous and suppose that $s \not\equiv -\infty$ on each component of Ω. The following are equivalent:*
(a) $s \in \mathcal{S}(\Omega)$;
(b) $s \leq I_{s,x,r}$ on $B(x, r)$ whenever $\overline{B(x,r)} \subset \Omega$;
(c) for each $x \in \Omega$ such that $s(x) > -\infty$, we have

$$\limsup_{t \to 0+} \frac{\mathcal{M}(s; x, t) - s(x)}{t^2} \geq 0;$$

(d) for each $x \in \Omega$ there exists $r_x > 0$ such that $s(x) \leq \mathcal{M}(s; x, r)$ whenever $0 < r < r_x$;
(e) for each $x \in \Omega$ there exists $r_x > 0$ such that $s(x) \leq \mathcal{A}(s; x, r)$ whenever $0 < r < r_x$;
(f) if ω is a bounded open set such that $\overline{\omega} \subset \Omega$ and if $h \in C(\overline{\omega}) \cap \mathcal{H}(\omega)$ is such that $s \leq h$ on $\partial\omega$, then $s \leq h$ on ω.

Proof. The implications (a) ⇒ (d) ⇒ (c) are obvious, and Theorem 3.1.3(ii) shows that (a) ⇒ (e). It remains to establish that (e) ⇒ (c) ⇒ (f) ⇒ (b) ⇒ (a).

"(e) ⇒ (c)". If (e) holds, then there are arbitrarily small values of t for which $s(x) \leq \mathcal{M}(s; x, t)$, in view of (3.1.2), and so (c) holds.

"(c) ⇒ (f)". Let ω be a bounded open set such that $\overline{\omega} \subset \Omega$, let $w(y) = \|y\|^2$ and let $a = \sup_\omega w$. Further, let $h \in C(\overline{\omega}) \cap \mathcal{H}(\omega)$ where $s \leq h$ on $\partial\omega$, and let $\varepsilon > 0$. If we define $u = h - s - \varepsilon(w - a)$ on $\overline{\omega}$, then u is lower semicontinuous on $\overline{\omega}$ and $u \geq 0$ on $\partial\omega$. Let $\alpha = \inf_{\overline{\omega}} u$. By (1.2.3)

$$\lim_{t\to 0+} \frac{\mathcal{M}(w;y,t)-w(y)}{t^2} = (2N)^{-1}\Delta w(y) = 1 \qquad (y \in \mathbb{R}^N).$$

Using this, the harmonicity of h and hypothesis (c), we find that for each $y \in \omega$ there are arbitrarily small values of t for which $\mathcal{M}(u;y,t) < u(y)$, and therefore $u > \alpha$ on ω. Hence u attains the value α at some point of $\partial\omega$, and so $u \geq 0$. Letting $\varepsilon \to 0$, we obtain $s \leq h$ on ω.

"(f) $\Rightarrow$ (b)". Suppose that $\overline{B(x,r)} \subset \Omega$. By Lemma 3.2.1 there exists a decreasing sequence (f_n) in $C(S(x,r))$ such that $f_n \to s$ on $S(x,r)$. The function h_n defined to be f_n on $S(x,r)$ and $I_{f_n,x,r}$ on $B(x,r)$ belongs to $C(\overline{B(x,r)}) \cap \mathcal{H}(B(x,r))$. Our hypothesis implies $s \leq h_n$ on $B(x,r)$ for each n. By monotone convergence, $h_n \to I_{s,x,r}$ on $B(x,r)$, so (b) holds.

"(b) $\Rightarrow$ (a)". If $\overline{B(x,r)} \subset \Omega$ and (b) holds, then $s(x) \leq I_{s,x,r}(x) = \mathcal{M}(s;x,r)$. □

Remark 3.2.3. (i) Criterion (f) above explains the name *sub*harmonic.
(ii) Criterion (b) implies that s is σ-integrable on $S(x,r)$ whenever $\overline{B(x,r)} \subset \Omega$.

Corollary 3.2.4. *Let ω be an open subset of Ω, let $s \in \mathcal{S}(\Omega)$ and $u \in \mathcal{S}(\omega)$, and suppose that*

$$\limsup_{x\to y,\, x\in\omega} u(x) \leq s(y) \qquad (y \in \partial\omega \cap \Omega). \tag{3.2.1}$$

Then the function

$$v(x) = \begin{cases} \max\{s(x),u(x)\} & (x \in \omega) \\ s(x) & (x \in \Omega\backslash\omega) \end{cases}$$

belongs to $\mathcal{S}(\Omega)$.

Proof. Clearly $v \in \mathcal{S}(\omega)$ and $v \in \mathcal{S}(\Omega\backslash\overline{\omega})$. Also, (3.2.1) ensures that v satisfies (3.1.1) at each $y \in \partial\omega\cap\Omega$. By criterion (d) of Theorem 3.2.2, it is now enough to check the subharmonic mean value property for v at points $x \in \partial\omega\cap\Omega$: if $\overline{B(x,r)} \subset \Omega$, then

$$v(x) = s(x) \leq \mathcal{M}(s;x,r) \leq \mathcal{M}(v;x,r).$$ □

Corollary 3.2.5. *If $s \in \mathcal{S}(\Omega)$ and $\overline{B(x,r)} \subset \Omega$, then the function $\overline{s}$, defined to be $I_{s,x,r}$ on $B(x,r)$ and s elsewhere on Ω, belongs to $\mathcal{S}(\Omega)$ and satisfies $\overline{s} \geq s$ on Ω.*

Proof. By Theorem 1.3.3, the hypotheses of Corollary 3.2.4 are satisfied with $\omega = B(x,r)$ and $u = I_{s,x,r}$. Since $s \leq I_{s,x,r}$ on $B(x,r)$ by Theorem 3.2.2, $\overline{s} \in \mathcal{S}(\Omega)$ and $s \leq \overline{s}$. □

Corollary 3.2.6. *If $s \in \mathcal{S}(\Omega)$ and $\overline{B(x,r_0)} \subset \Omega$, then the functions $\mathcal{M}(s;x,\cdot)$ and $\mathcal{A}(s;x,\cdot)$ are finite-valued and increasing on $(0,r_0]$. Also, $\mathcal{A}(s;x,\cdot) \leq \mathcal{M}(s;x,\cdot)$ and*

$$\lim_{r\to 0+} \mathcal{M}(s;x,r) = \lim_{r\to 0+} \mathcal{A}(s;x,r) = s(x). \tag{3.2.2}$$

Proof. The finiteness of the means was established in Theorem 3.1.3 and Remark 3.2.3. Now suppose that $0 < t < r \leq r_0$, and let $\overline{s}$ be the function introduced in Corollary 3.2.5. Then

$$\mathcal{M}(s;x,r) = \overline{s}(x) = \mathcal{M}(\overline{s};x,t) \geq \mathcal{M}(s;x,t),$$

so that $\mathcal{M}(s;x,\cdot)$ is increasing on $(0,r_0]$. It follows easily (see (3.1.2)) that $\mathcal{A}(s;x,\cdot) \leq \mathcal{M}(s;x,\cdot)$ and that $\mathcal{A}(s;x,\cdot)$ is also increasing on $(0,r_0]$. By upper semicontinuity the limits in (3.2.2) do not exceed $s(x)$, and by the subharmonic mean value property they are not less than $s(x)$. □

Corollary 3.2.7. *If $s,u \in \mathcal{S}(\Omega)$ and $s = u$ almost everywhere (λ), then $s = u$ on Ω.*

Proof. If $s = u$ almost everywhere, then $\mathcal{A}(s;x,r) = \mathcal{A}(u;x,r)$ whenever $\overline{B(x,r)} \subset \Omega$, so that $s = u$ by (3.2.2). □

Corollary 3.2.8. *Suppose that $s \in C^2(\Omega)$. Then $s \in \mathcal{S}(\Omega)$ if and only if $\Delta s \geq 0$ on Ω.*

Proof. This follows from criterion (c) of Theorem 3.2.2, since

$$\lim_{r\to 0+} \frac{\mathcal{M}(s;x,r)-s(x)}{r^2} = (2N)^{-1}\Delta s(x) \qquad (x \in \Omega)$$

by (1.2.3). □

Corollary 3.2.9. *Suppose that f is holomorphic on a plane domain Ω and that $f \not\equiv 0$. Then $\log|f|$ is harmonic on $\{z \in \Omega : f(z) \neq 0\}$ and subharmonic on Ω, provided we interpret $\log 0$ as $-\infty$.*

Proof. Let Z denote the set of zeros of f. We saw in Theorem 1.1.3 that $\log|f| \in \mathcal{H}(\Omega\backslash Z)$. Since $\log|f(z)| = -\infty$ when $z \in Z$, we have $\log|f(z)| \leq \mathcal{M}(\log|f|;z,r)$ when $z \in \Omega$ and $0 < r < r_z$, for some r_z. The result now follows from criterion (d) in Theorem 3.2.2. □

We saw in Theorem 1.1.2 that the function U_y is harmonic on $\mathbb{R}^N\backslash\{y\}$. We now observe that it is superharmonic on all of $\mathbb{R}^N$ provided we assign it the value $+\infty$ at y.

Corollary 3.2.10. *If $y \in \mathbb{R}^N$, then the function U_y defined on $\mathbb{R}^N$ by*

$$U_y(x) = \begin{cases} -\log\|x-y\| & (x \neq y; N=2) \\ \|x-y\|^{2-N} & (x \neq y; N \geq 3) \\ +\infty & (x=y) \end{cases}$$

is harmonic on $\mathbb{R}^N \backslash \{y\}$ and superharmonic on $\mathbb{R}^N$.

Proof. The harmonicity was proved in Theorem 1.1.2. Since $U_y(y) = +\infty$, we have $U_y(x) \geq \mathcal{M}(U_y; x, r)$ when $x \in \mathbb{R}^N$ and $0 < r < r_x$, where $r_x = \|x-y\|$ if $x \neq y$ and $r_y = +\infty$. Hence $U_y \in \mathcal{U}(\mathbb{R}^N)$ by criterion (d) of Theorem 3.2.2. □

Corollary 3.2.9 above provides the crucial link between subharmonic and holomorphic functions. In connection with Corollary 3.2.10 we mention that in Chapter 4 we will see how any superharmonic function can be locally represented in terms of the functions U_y.

3.3. Approximation of subharmonic functions by smooth ones

The definition of subharmonic functions does not require continuity and such functions can indeed be highly discontinuous (see Example 3.3.2 below). However, we will show below that, at least locally, a subharmonic function is the limit of a decreasing sequence of smooth subharmonic functions. This will allow us to take full advantage of the characterization of smooth subharmonic functions as functions with non-negative Laplacian. First we establish a general result which will prove useful also in later chapters.

Theorem 3.3.1. *Let Ω be connected, let μ be a σ-finite measure on a locally compact Hausdorff space Y, and let $f: \Omega \times Y \to (-\infty, +\infty]$ be measurable with respect to the σ-algebra generated by products of Borel sets. Further, suppose that there is a μ-integrable function $g: Y \to \mathbb{R}$ such that $f(x,y) \geq g(y)$ for all $(x,y) \in \Omega \times Y$. If $f(\cdot, y) \in \mathcal{U}(\Omega)$ for each $y \in Y$, then the function*

$$u(x) = \int_Y f(x,y)\, d\mu(y) \qquad (x \in \Omega)$$

is either identically $+\infty$ or is in $\mathcal{U}(\Omega)$.

Proof. By working with $f(x,y) - g(y)$ in place of $f(x,y)$, we may suppose that $f \geq 0$ on $\Omega \times Y$. If $x_0 \in \Omega$ and (x_n) is any sequence in $\Omega \backslash \{x_0\}$ converging to x_0, then by Fatou's lemma

$$\liminf_{n \to \infty} u(x_n) \geq \int_Y \left(\liminf_{n \to \infty} f(x_n, y) \right) d\mu(y) \geq u(x_0),$$

and so u is lower semicontinuous on Ω. If $\overline{B(x,r)} \subset \Omega$, then Fubini's theorem and the superharmonicity of $f(\cdot, y)$ yield

$$\mathcal{M}(u; x, r) = \int_Y \mathcal{M}(f(\cdot, y); x, r)\, d\mu(y) \leq u(x).$$

Since Ω is connected, the conclusion follows. □

Example 3.3.2. Let $Y = \{y_n : n \in \mathbb{N}\}$ be a dense subset of B and let

$$u(x) = \sum_{n=1}^{\infty} 2^{-n} U_{y_n}(x) \qquad (x \in \mathbb{R}^N).$$

Then we can apply Theorem 3.3.1 with $f(x,y) = U_y(x)$ to see that $u \in \mathcal{U}(\mathbb{R}^N)$. Hence u is finite almost everywhere, yet $u = +\infty$ on a dense subset of B. The function $-u$ is the promised example of a highly discontinuous subharmonic function.

In order to state the approximation theorem, we recall some notation used in the proof of Lemma 1.2.1. The functions $\phi_n \in C^\infty(\mathbb{R}^N)$ are defined by $\phi_n(x) = n^N \phi(1 - n^2\|x\|^2)$, where

$$\phi(t) = \begin{cases} C_N e^{-1/t} & (t > 0) \\ 0 & (t \leq 0) \end{cases}$$

and the constant C_N is chosen so that

$$\sigma_N \int_0^1 t^{N-1} \phi(1 - t^2) dt = 1.$$

Thus $\phi_n = 0$ outside $B(0, n^{-1})$.

Theorem 3.3.3. *Suppose that $s \in \mathcal{S}(\Omega)$ and ω is a bounded open set such that $\overline{\omega} \subset \Omega$. For all sufficiently large integers n the function*

$$s_n(x) = \int_\Omega \phi_n(x-y) s(y)\, d\lambda(y) \qquad (x \in \omega) \tag{3.3.1}$$

belongs to $\mathcal{S}(\omega) \cap C^\infty(\omega)$, and the sequence (s_n) is decreasing on ω with pointwise limit s.

Proof. Note first that s_n is well defined, provided that $n^{-1} < \operatorname{dist}(\omega, \partial\Omega)$ in the case where $\Omega \neq \mathbb{R}^N$. From now on we suppose that n satisfies this inequality. To prove the subharmonicity of s_n, we write s_n as

$$s_n(x) = \int_{B(0,n^{-1})} \phi_n(z)s(x-z)\,d\lambda(z) \qquad (x \in \omega).$$

The integrand here is upper semicontinuous and bounded above for $(x,z) \in \omega \times \Omega$, and for each fixed $z \in \Omega$ it is subharmonic as a function of $x \in \omega$. Hence $s_n \in \mathcal{S}(\omega)$ by Theorem 3.3.1. Further, since s is locally integrable on Ω and every partial derivative of $\phi_n(x)$ is bounded on $\mathbb{R}^N$, we can pass partial differential operators under the integral sign in (3.3.1) to show that $s_n \in C^\infty(\omega)$.

To prove the monotonicity of (s_n), we observe that

$$\begin{aligned} s_n(x) &= \sigma_N \int_0^{1/n} n^N \phi(1-n^2t^2)t^{N-1}\mathcal{M}(s;x,t)\,dt \qquad (3.3.2)\\ &= \sigma_N \int_0^{1/(n+1)} (n+1)^N\phi(1-(n+1)^2\tau^2)\tau^{N-1}\mathcal{M}(s;x,(n+1)\tau/n)\,d\tau \\ &\geq \sigma_N \int_0^{1/(n+1)} (n+1)^N\phi(1-(n+1)^2\tau^2)\tau^{N-1}\mathcal{M}(s;x,\tau)d\tau = s_{n+1}(x), \end{aligned}$$

since $\mathcal{M}(s;x,\cdot)$ is increasing. Finally, let $x \in \omega$ and let $a \in \mathbb{R}$ be such that $s(x) < a$. By Corollary 3.2.6 there exists $\delta > 0$ such that $s(x) \leq \mathcal{M}(s;x,t) < a$ when $0 < t < \delta$. Since

$$\sigma_N \int_0^{1/n} n^N\phi(1-n^2t^2)t^{N-1}dt = \sigma_N \int_0^1 \phi(1-t^2)t^{N-1}dt = 1,$$

it follows from (3.3.2) that $s(x) \leq s_n(x) < a$ when $n^{-1} < \delta$. Hence $s_n(x) \to s(x)$ as $n \to \infty$. □

Corollary 3.3.4. *Let Ω_1, Ω_2 be plane domains, let $f\colon \Omega_1 \to \Omega_2$ and let $s \in \mathcal{S}(\Omega_2)$. Then $s \circ f \in \mathcal{S}(\Omega_1)$, if either f or its complex conjugate $\overline{f}$ is a non-constant holomorphic function on Ω_1.*

Proof. Suppose first that f is holomorphic, let $z \in \Omega_1$, let B_0 be an open disc centred at $f(z)$ such that $\overline{B_0} \subset \Omega_2$, and let $\omega = f^{-1}(B_0)$. If $s \in \mathcal{S}(B_0) \cap C^2(B_0)$, then it follows from the Cauchy–Riemann equations that $\Delta(s \circ f) = ((\Delta s) \circ f)|f'|^2 \geq 0$ on ω, and so $s \circ f \in \mathcal{S}(\omega)$. In the general case we observe from Theorem 3.3.3 that there is a decreasing sequence (s_n) of functions in $\mathcal{S}(B_0) \cap C^2(B_0)$ such that $s_n \to s$. Hence $s \circ f = \lim(s_n \circ f)$. Since $s \not\equiv -\infty$ on $f(\omega)$, which is open by the open mapping theorem, $s \circ f \in \mathcal{S}(\omega)$ by Theorem 3.1.4. Hence $s \circ f \in \mathcal{S}(\Omega_1)$, in view of the arbitrary nature of B_0. A similar argument deals with the case where $\overline{f}$ is holomorphic. □

Corollary 3.3.5. *The Kelvin transform preserves subharmonicity. That is, with the notation of Section 1.6, if $s \in \mathcal{S}(\Omega)$, then $s^* \in \mathcal{S}(\Omega^*)$.*

Proof. In the case where $s \in \mathcal{S}(\Omega) \cap C^2(\Omega)$, the result follows immediately from Corollary 3.2.8 and the relation (1.6.1) between Δs and Δs^*. In the general case it suffices to work locally. Let ω be a bounded open set such that $\overline{\omega} \subset \Omega$. By Theorem 3.3.3, s is the limit on ω of a decreasing sequence (s_n) in $\mathcal{S}(\omega) \cap C^\infty(\omega)$. Since (s_n^*) is a decreasing sequence in $\mathcal{S}(\omega^*)$ with limit s^*, we see from Theorem 3.1.4 that $s^* \in \mathcal{S}(\omega^*)$, as required. □

If $v \in \mathcal{U}(\mathbb{R}^M \times \Omega)$, where $M \in \mathbb{N}$, and $v(x,y)$ depends only on y, then the function $y \mapsto v(0,y)$ is superharmonic on Ω: when $v \in C^2(\Omega)$, this is clear from Corollary 3.2.8, and the general case follows by Theorem 3.3.3. Conversely, similar reasoning shows that, if $u \in \mathcal{U}(\Omega)$, then the function $(x,y) \mapsto u(y)$ is superharmonic on $\mathbb{R}^M \times \Omega$. The next result deals with the more general situation where $(x,y) \mapsto u(x,y)$ is superharmonic as a function of each variable separately.

Theorem 3.3.6. *Let Ω_1 and Ω_2 be open sets in $\mathbb{R}^M$ and $\mathbb{R}^N$ respectively, and let $f : \Omega_1 \times \Omega_2 \to (-\infty, +\infty]$ be locally bounded below. If $f(x,\cdot) \in \mathcal{U}(\Omega_2)$ for each x in Ω_1 and $f(\cdot,y) \in \mathcal{U}(\Omega_1)$ for each y in Ω_2, then $f \in \mathcal{U}(\Omega_1 \times \Omega_2)$.*

Proof. We may assume, without loss of generality, that $f \geq 0$. In what follows, the function ϕ_n introduced above will be regarded as a function on $\mathbb{R}^M$ or $\mathbb{R}^N$ according to context. Let ω_1 and ω_2 be bounded open sets such that $\overline{\omega}_i \subset \Omega_i$ $(i = 1,2)$ and, for each $k \in \mathbb{N}$, let $f_k = \min\{f, k\}$. For all sufficiently large m in $\mathbb{N}$ the function

$$f_{k,m}(x,y) = \int_{\Omega_1} \phi_m(x-\tilde{x})f_k(\tilde{x},y)d\lambda(\tilde{x})$$

is defined on $\omega_1 \times \Omega_2$. By Fatou's lemma, $f_{k,m}(x,\cdot)$ is lower semicontinuous on Ω_2 for each $x \in \omega_1$, so for all sufficiently large n we can define

$$f_{k,m,n}(x,y) = \int_{\Omega_2} \phi_n(y-\tilde{y})f_{k,m}(x,\tilde{y})d\lambda(\tilde{y}) \qquad (x \in \omega_1; y \in \omega_2).$$

Since every partial derivative of $\phi_n(x)$ is bounded on $\mathbb{R}^N$, we can pass differential operators under the integral signs to see that $f_{k,m,n} \in C^\infty(\omega_1 \times \omega_2)$. It follows from Theorem 3.3.3 that $f_{k,m} \uparrow f_k$ as $m \to \infty$, so the function

$$(x,y) \mapsto \int_{\Omega_2} \phi_n(y-\tilde{y})f_k(x,\tilde{y})d\lambda(\tilde{y})$$

is lower semicontinuous on $\omega_1 \times \omega_2$. Letting $n \to \infty$ and then $k \to \infty$, we obtain the lower semicontinuity of f on $\omega_1 \times \omega_2$. Thus f is lower semicontinuous on $\Omega_1 \times \Omega_2$, in view of the arbitrary nature of ω_1 and ω_2.

By Theorems 3.3.1 and 3.3.3 the functions $f_{k,m}(\cdot,\cdot)$, and hence $f_{k,m,n}(\cdot,\cdot)$, are superharmonic in each variable separately. Thus, using Corollary 3.2.8,

$$\Delta f_{k,m,n}(x,y) = \sum_{i=1}^{M} \frac{\partial^2 f_{k,m,n}}{\partial x_i^2}(x,y) + \sum_{i=1}^{N} \frac{\partial^2 f_{k,m,n}}{\partial y_i^2}(x,y) \le 0,$$

so $f_{k,m,n} \in \mathcal{U}(\omega_1 \times \omega_2)$. Letting $m \to \infty$, $n \to \infty$ and then $k \to \infty$, it follows that $f \in \mathcal{U}(\omega_1 \times \omega_2)$, and hence $f \in \mathcal{U}(\Omega_1 \times \Omega_2)$. □

Corollary 3.3.7. *Let Ω_1 and Ω_2 be open sets in $\mathbb{R}^M$ and $\mathbb{R}^N$ respectively, and let $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be locally bounded. If $f(x,\cdot) \in \mathcal{H}(\Omega_2)$ for each $x \in \Omega_1$, and $f(\cdot,y) \in \mathcal{H}(\Omega_1)$ for each y in Ω_2, then $f \in \mathcal{H}(\Omega_1 \times \Omega_2)$.*

Proof. Apply Theorem 3.3.6 to f and $-f$. □

3.4. Convexity and subharmonicity

Throughout this section J denotes an interval in $\mathbb{R}$. We recall that a function $\phi: J \to \mathbb{R}$ is called *convex* on J if

$$\phi(t) \le \frac{t_2 - t}{t_2 - t_1}\phi(t_1) + \frac{t - t_1}{t_2 - t_1}\phi(t_2) \qquad (3.4.1)$$

whenever $t_1, t_2 \in J$ and $t_1 < t < t_2$. Also, a function $\psi: J \to \mathbb{R}$ is called *concave* on J if $-\psi$ is convex on J. We note that the right-hand side of (3.4.1) defines an affine function of t (that is, a function of the form $at + b$) whose values at t_1 and t_2 are $\phi(t_1)$ and $\phi(t_2)$ respectively. Thus a convex function is characterized by the property that its graph lies below each of its chords (not necessarily strictly). Since the harmonic functions on $\mathbb{R}$ are precisely the affine functions, (3.4.1) corresponds to criterion (f) of Theorem 3.2.2. Thus convex functions on open intervals, being continuous (see Lemma 3.4.1 below), correspond precisely to one-dimensional subharmonic functions. We will develop further connections between convexity and subharmonicity below.

Lemma 3.4.1. *If ϕ is convex on an open interval J, then the left and right derivatives ϕ'_-, ϕ'_+ exist, are increasing functions and satisfy $\phi'_- \le \phi'_+$ on J. In particular, ϕ is continuous on J. Further, if $t_0 \in J$, then there exists an affine function ψ such that $\psi(t_0) = \phi(t_0)$ and $\psi \le \phi$ on J.*

Proof. It follows from (3.4.1) that

$$\frac{\phi(t) - \phi(t_1)}{t - t_1} \le \frac{\phi(t_2) - \phi(t_1)}{t_2 - t_1} \le \frac{\phi(t_2) - \phi(t)}{t_2 - t} \qquad (3.4.2)$$

whenever $t_1, t_2 \in J$ and $t_1 < t < t_2$. Let $Q(t,\delta) = \{\phi(t+\delta) - \phi(t)\}/\delta$. Using (3.4.2), we find that $Q(t_1,\delta)$ is an increasing function of δ on the interval

$\{\delta > 0 : t_1 + \delta \in J\}$, and $Q(t_2,\delta)$ is increasing on $\{\delta < 0 : t_2 + \delta \in J\}$. Hence the one-sided derivatives ϕ'_-, ϕ'_+ exist on J and

$$Q(t,-\delta) \le \phi'_-(t), \qquad \phi'_+(t) \le Q(t,\delta) \qquad (3.4.3)$$

whenever $t \in J$, $\delta > 0$ and, respectively, $t - \delta \in J$ or $t + \delta \in J$. Letting t_1 and t_2 tend to t in (3.4.2), we obtain $\phi'_-(t) \le \phi'_+(t)$, which together with (3.4.3) shows that $\phi'_-(t), \phi'_+(t)$ are finite. If we let $t \to t_1$ in the left-hand inequality of (3.4.2) and $t \to t_2$ in the right-hand inequality, we obtain $\phi'_+(t_1) \le \phi'_-(t_2)$. This, together with the fact that $\phi'_- \le \phi'_+$, shows that ϕ'_- and ϕ'_+ are increasing. Finally, given $t_0 \in J$, we define

$$\psi(t) = \phi(t_0) + \phi'_+(t_0)(t - t_0).$$

It follows from (3.4.3) and the inequality $\phi'_-(t_0) \le \phi'_+(t_0)$ that $\psi \le \phi$ on J. □

Lemma 3.4.2. *If $\mathcal{F} \subseteq \mathcal{S}(\Omega)$ and $\sup \mathcal{F}$ is upper semicontinuous and less than $+\infty$ on Ω, then $\sup \mathcal{F} \in \mathcal{S}(\Omega)$.*

Proof. If $\overline{B(x,r)} \subset \Omega$, then $s(x) \le \mathcal{M}(s;x,r) \le \mathcal{M}(\sup \mathcal{F}; x, r)$ for all $s \in \mathcal{F}$, and the subharmonic mean value property for $\sup \mathcal{F}$ follows. □

We can now give some very general results involving convexity and subharmonicity.

Theorem 3.4.3. *(i) If $g, h \in \mathcal{H}(\Omega)$ and $h > 0$ on Ω, and if $\phi: \mathbb{R} \to \mathbb{R}$ is convex, then $h\phi(g/h) \in \mathcal{S}(\Omega)$.*
(ii) If $s \in \mathcal{S}(\Omega)$, $h \in \mathcal{H}(\Omega)$ and $h > 0$ on Ω, and if $\phi: [-\infty, +\infty) \to [-\infty, +\infty)$ is continuous on $[-\infty, +\infty)$ and increasing and convex on $\mathbb{R}$, then $h\phi(s/h) \in \mathcal{S}(\Omega)$.
(iii) If $s \in \mathcal{S}(\Omega)$, $u \in \mathcal{U}(\Omega)$ and $s \ge 0$, $u > 0$ on Ω, and if $\phi: [0, +\infty) \to [0, +\infty)$ is convex on $[0, +\infty)$ and $\phi(0) = 0$, then $u\phi(s/u) \in \mathcal{S}(\Omega)$. (Here $u\phi(s/u)(x)$ is interpreted as $s(x)\lim_{t\to 0}\phi(t)/t$ when $u(x) = +\infty$.)
(iv) If $u \in \mathcal{U}(\Omega)$, $h \in \mathcal{H}(\Omega)$ and $h > 0$ on Ω, and if $\psi: (-\infty, +\infty] \to (-\infty, +\infty]$ is continuous on $(-\infty, +\infty]$ and increasing and concave on $\mathbb{R}$, then $h\psi(u/h) \in \mathcal{U}(\Omega)$.
(v) If $u, v \in \mathcal{U}(\Omega)$ and $u > 0$, $v > 0$ on Ω, and if $\psi: [0, +\infty] \to [0, +\infty]$ is non-constant and continuous on $[0, +\infty]$ and concave on $[0, +\infty)$, then $v\psi(u/v) \in \mathcal{U}(\Omega)$. (Here $v\psi(u/v)(x)$ is interpreted as $u(x)\lim_{t\to 0}\psi(t)/t$ when $v(x) = +\infty$.)

Proof. (i) The final sentence in Lemma 3.4.1 implies that ϕ is the supremum of all its affine minorants; that is,

$$\phi(t) = \sup\{at + b : a, b \in \mathbb{R} \text{ and } a\tau + b \le \phi(\tau)\ \forall \tau \in \mathbb{R}\} \qquad (t \in \mathbb{R}).$$

Hence

$$h\phi(g/h) = h\sup\{a(g/h)+b\} = \sup\{ag+bh\},$$

where the suprema are taken over all real a, b such that $a\tau + b \le \phi(\tau)$ for all $\tau \in \mathbb{R}$. Since $h\phi(g/h)$ is continuous and $ag + bh \in \mathcal{H}(\Omega)$, it follows from Lemma 3.4.2 that $h\phi(g/h) \in \mathcal{S}(\Omega)$.

(ii) Since ϕ is increasing and convex on $\mathbb{R}$,

$$\phi(t) = \sup\{at+b: a \ge 0, b \in \mathbb{R} \text{ and } a\tau+b \le \phi(t)\ \forall\tau \in \mathbb{R}\} \qquad (t \in [-\infty, +\infty)).$$

Hence $h\phi(s/h) = \sup\{as + bh\}$, where the supremum is over certain real values of a, b, always with $a \ge 0$. For such a, b we have $as + bh \in \mathcal{S}(\Omega)$. Since ϕ is continuous and increasing, it is easy to verify that $h\phi(s/h)$ is upper semicontinuous. Hence, again by Lemma 3.4.2, $h\phi(s/h) \in \mathcal{S}(\Omega)$.

(iii) In this case

$$\phi(t) = \sup\{at + b: b \le 0 \le a \text{ and } a\tau + b \le \phi(\tau)\ \forall\tau \in \mathbb{R}\} \qquad (t \in [0, +\infty)),$$

so that $u\phi(s/u) = \sup\{as + bu\}$, where the supremum is over certain real values of a, b with $b \le 0 \le a$. For such a, b we have $as + bu \in \mathcal{S}(\Omega)$, so the conclusion will follow if we can show that $u\phi(s/u)$ is upper semicontinuous. To show this, we note first that $t^{-1}\phi(t)$ has a continuous increasing extension to $[0, +\infty)$. Also, it is easy to see that s/u is upper semicontinuous on Ω, provided we assign it the value 0 at points where $u = +\infty$. These observations imply that $(s/u)^{-1}\phi(s/u)$ is upper semicontinuous on Ω. Since $u\phi(s/u)$ is the product of the non-negative upper semicontinuous functions s and $(s/u)^{-1}\phi(s/u)$, we have

$$\begin{aligned}\limsup_{x\to y} u(x)\phi\left(\frac{s(x)}{u(x)}\right) &\le \left\{\limsup_{x\to y} s(x)\right\}\left\{\limsup_{x\to y}\left(\left(\frac{s}{u}\right)^{-1}\phi\left(\frac{s}{u}\right)\right)(x)\right\}\\ &\le s(y)\left(\left(\frac{s}{u}\right)^{-1}\phi\left(\frac{s}{u}\right)\right)(y) = (u\phi(s/u))(y)\ (y \in \Omega).\end{aligned}$$

Thus $u\phi(s/u)$ is upper semicontinuous on Ω, as required.

(iv) We apply (ii) with $\phi(t) = -\psi(-t)$ and $s = -u$.

(v) This can be proved by an argument similar to that given for (iii). □

Corollary 3.4.4. *(i) If $s \in \mathcal{S}(\Omega), u \in \mathcal{U}(\Omega)$ and $s \ge 0, u > 0$ on Ω, and if $1 \le p < +\infty$, then $s^p u^{1-p} \in \mathcal{S}(\Omega)$; in particular, $s^p, u^{1-p} \in \mathcal{S}(\Omega)$.*
(ii) If $u, v \in \mathcal{U}(\Omega)$ and $u > 0, v > 0$ on Ω, and if $0 < p < 1$, then $u^p v^{1-p} \in \mathcal{U}(\Omega)$; in particular, $u^p \in \mathcal{U}(\Omega)$.
(iii) If f is holomorphic on a plane open set Ω and $0 < p < +\infty$, then $|f|^p \in \mathcal{S}(\Omega)$.

Proof. Parts (i), (ii) follow from Theorem 3.4.3(iii), (v) with the obvious choices for ϕ and ψ. To prove (iii), we note that in any component of Ω on which $f \not\equiv 0$, the function $\log|f|$ is subharmonic by Corollary 3.2.9. The

subharmonicity of $|f|^p$ follows by taking $s = \log|f|, h \equiv 1$ and $\phi(t) = e^{pt}$ in Theorem 3.4.3(ii). □

Theorem 3.4.5. *Let $s, u \in \mathcal{S}(\Omega)$ and let $a, b \in [0, +\infty)$. Then:*
(i) $(as^p + bu^p)^{1/p} \in \mathcal{S}(\Omega)$, provided $s \ge 0, u \ge 0$ and $1 < p < +\infty$;
(ii) $\log(ae^s + be^u) \in \mathcal{S}(\Omega)$.

Proof. Let ω be a bounded open set such that $\overline{\omega} \subset \Omega$. To prove (i), let $h \in C(\overline{\omega}) \cap \mathcal{H}(\omega)$, where $(as^p + bu^p)^{1/p} \le h$ on $\partial\omega$. If $\varepsilon > 0$, then

$$a(h+\varepsilon)^{1-p}s^p + b(h+\varepsilon)^{1-p}u^p \le h + \varepsilon \tag{3.4.4}$$

on $\partial\omega$. By Corollary 3.4.4(i) and the maximum principle, (3.4.4) holds on ω and so $(as^p + bu^p)^{1/p} \le h + \varepsilon$ on ω. Since ε can be arbitrarily small, we conclude from criterion (f) of Theorem 3.2.2 that $(as^p + bu^p)^{1/p} \in \mathcal{S}(\Omega)$.

To prove (ii), let $h \in C(\overline{\omega}) \cap \mathcal{H}(\omega)$, where $\log(ae^s + be^u) \le h$ on $\partial\omega$. Then $ae^{s-h} + be^{u-h} \le 1$ on $\partial\omega$. It follows from the maximum principle that this inequality holds on ω, for by Theorem 3.4.3(ii) with $\phi(t) = e^t$, its left-hand side is subharmonic on ω. Hence $\log(ae^s + be^u) \le h$ on ω, and the conclusion again follows. □

3.5. Mean values and subharmonicity

In this section we will establish convexity properties for $\mathcal{M}(s; 0, \cdot)$ and related functions. For this we need a slight generalization of the concept of convexity.

Definition 3.5.1. If $\psi: J \to \mathbb{R}$ is a continuous, strictly monotone function, then $\phi: J \to \mathbb{R}$ is said to be a *convex function of* $\psi(t)$ if $\phi = \chi \circ \psi$, where χ is convex on the interval $\psi(J)$; that is, if

$$\phi(t) \le \frac{\psi(t_2) - \psi(t)}{\psi(t_2) - \psi(t_1)}\phi(t_1) + \frac{\psi(t) - \psi(t_1)}{\psi(t_2) - \psi(t_1)}\phi(t_2) \tag{3.5.1}$$

whenever $t_1, t_2 \in J$ and $t_1 < t < t_2$. The right-hand side of (3.5.1) is of the form $a\psi(t) + b$ where a, b are constants such that $\phi(t_j) = a\psi(t_j) + b$ $(j = 1, 2)$.

Let $V_2(r) = \log(1/r)$ and $V_N(r) = r^{2-N} (N \ge 3)$ when $r > 0$, and recall that

$$A(0; r_1, r_2) = \{x : r_1 < \|x\| < r_2\}.$$

Theorem 3.5.2. *If $s \in \mathcal{S}(A(0; r_1, r_2))$, where $0 \le r_1 < r_2 \le +\infty$, and $m(s; r) = \sup\{s(x): \|x\| = r\}$, then $m(s; r)$ is a convex function of $V_N(r)$ for $r \in (r_1, r_2)$.*

Proof. Let $r_1 < t_1 < t < t_2 < r_2$ and let a, b be such that

$$m(s;t_j) = aV_N(t_j) + b \qquad (j = 1,2).$$

Define $h(x) = aU_0(x) + b$. Then $h \in \mathcal{H}(\mathbb{R}^N\backslash\{0\})$ and $s \le h$ on $\partial A(0;t_1,t_2)$. By the maximum principle, $s \le h$ on $A(0;t_1,t_2)$ and in particular on $S(0,t)$, so that $m(s;t) \le aV_N(t) + b$, as required. □

Corollary 3.5.3. *Let $s: A(0;r_1,r_2) \to \mathbb{R}$, where $0 \le r_1 < r_2 \le +\infty$, be such that $s(x)$ depends only on $\|x\|$. Then $s \in \mathcal{S}(A(0;r_1,r_2))$ if and only if $s(0,\dots,0,r)$ is a convex function of $V_N(r)$ for $r \in (r_1,r_2)$.*

Proof. Since the function $U_0(x) = V_N(\|x\|)$ is harmonic on $\mathbb{R}^N\backslash\{0\}$, the sufficiency of the stated condition follows by taking $g = U_0$ and $h \equiv 1$ in Theorem 3.4.3(i). The necessity follows from Theorem 3.5.2. □

Corollary 3.5.4. *If $s \in \mathcal{S}(\mathbb{R}^2)$ and s is bounded above, then s is constant.*

Proof. We know that $m(s;r)$ is a convex function of $\log r$ for $r \in (0,+\infty)$ and is bounded above, so it has a constant value c, say. Hence, by the maximum principle, $s \le c$ on $\mathbb{R}^2$. Since s attains the value c on S, the result follows. □

Lemma 3.5.5. *(i) If (ϕ_n) is a decreasing sequence of convex functions on an interval J, then either $\lim \phi_n$ is convex on J or $\lim \phi_n \equiv -\infty$ on J°.*
(ii) Let J be an open interval and $\phi \in C^1(J)$. Then ϕ is convex if and only if ϕ' is increasing.

Proof. (i) Let $\phi = \lim \phi_n$. It is easy to see from (3.5.1) that, if $\phi(t) = -\infty$ for some $t \in J$, then $\phi = -\infty$ on J°. Suppose now that ϕ is finite-valued. If $t_1, t_2 \in J$ and $t_1 < t < t_2$, then the inequality (3.4.1) follows from the corresponding inequality for ϕ_n on taking limits as $n \to \infty$.

(ii) Now suppose that $\phi \in C^1(J)$. If ϕ is convex, then ϕ' is increasing, by Lemma 3.4.1. Conversely, if ϕ' is increasing and $t_1 < t < t_2$, then it follows from the mean value theorem of differential calculus that

$$\frac{\phi(t) - \phi(t_1)}{t - t_1} \le \frac{\phi(t_2) - \phi(t)}{t_2 - t},$$

which is equivalent to (3.4.1). □

Theorem 3.5.6. *Let $0 \le r_1 < r_2 \le +\infty$.*
(i) If $h \in \mathcal{H}(A(0;r_1,r_2))$, then $\mathcal{M}(h;0,r) = a + bV_N(r)$ for $r \in (r_1,r_2)$, where $a, b \in \mathbb{R}$.
(ii)If $s \in \mathcal{S}(A(0;r_1,r_2))$, then $\mathcal{M}(s;0,r)$ is a convex function of $V_N(r)$ for $r \in (r_1,r_2)$.

Proof. It is enough to prove (ii), since (i) then follows by applying (ii) to h and $-h$. Suppose first that s is C^2. By Green's formula applied to the functions s and 1, if $r_1 < t_1 < t_2 < r_2$, then

$$\int_{\partial A(0;t_1,t_2)} \frac{\partial s}{\partial n_e}\, d\sigma = \int_{A(0;t_1,t_2)} \Delta s \, d\lambda \ge 0,$$

where $\partial/\partial n_e$ denotes differentiation in the direction of the exterior normal to $A(0;t_1,t_2)$. Arguing as in the proof of (1.2.1) we obtain

$$t_2^{N-1}\left[\frac{d}{dt}\int_S s(ty)d\sigma(y)\right]_{t=t_2} - t_1^{N-1}\left[\frac{d}{dt}\int_S s(ty)d\sigma(y)\right]_{t=t_1} \ge 0$$

and so $r^{N-1}(d/dr)\mathcal{M}(s;0,r)$ is an increasing function of r on (r_1,r_2). Equivalently, $(d/dV_N(r))\mathcal{M}(s;0,r)$ increases as $V_N(r)$ increases. The required convexity now follows from Lemma 3.5.5(ii).

Now suppose only that $s \in \mathcal{S}(A(0;r_1,r_2))$. On $A(0;t_1,t_2)$ the function s is the limit of a decreasing sequence (s_n) of smooth subharmonic functions, by Theorem 3.3.3. By the result of the previous paragraph, each $\mathcal{M}(s_n;0,r)$ is a convex function of $V_N(r)$ for $r \in (t_1,t_2)$. Since $(\mathcal{M}(s_n;0,\cdot))$ decreases with limit $\mathcal{M}(s;0,\cdot)$, it follows from Lemma 3.5.5(i) that $\mathcal{M}(s;0,\cdot)$ has the same convexity property. Hence $\mathcal{M}(s;0,r)$ is a convex function of $V_N(r)$ for $r \in (r_1,r_2)$. □

Theorem 3.5.7. *If $s \in \mathcal{S}(A(0;r_1,r_2))$, where $0 \le r_1 < r_2 \le +\infty$, then the following are convex functions of $V_N(r)$ for $r \in (r_1,r_2)$:*
(i) $(\mathcal{M}(s^p;0,r))^{1/p}$, provided $s \ge 0$ and $1 \le p < +\infty$;
(ii) $\log \mathcal{M}(e^s;0,r)$.

Proof. (i) Suppose that $r_1 < t_1 < t < t_2 < r_2$ and let a, b be such that

$$(\mathcal{M}(s^p;0,t_j))^{1/p} = aV_N(t_j) + b \qquad (j = 1,2).$$

We define $h(x) = aU_0(x) + b$. If $\varepsilon > 0$, then the function $(h+\varepsilon)^{1-p}s^p$ is subharmonic on an open set containing $\overline{A(0;t_1,t_2)}$, by Corollary 3.4.4(i). Hence $\mathcal{M}((h+\varepsilon)^{1-p}s^p;0,r)$ is a convex function of $V_N(r)$ for $r \in (t_1,t_2)$, by Theorem 3.5.6. Now

$$\mathcal{M}((h+\varepsilon)^{1-p}s^p;0,t_j) \le aV_N(t_j) + b + \varepsilon \qquad (j = 1,2),$$

and so

$$\mathcal{M}((h+\varepsilon)^{1-p}s^p;0,t) \le aV_N(t) + b + \varepsilon,$$

from which it follows that $(\mathcal{M}(s^p;0,t))^{1/p} \le aV_N(t) + b + \varepsilon$. Since ε can be arbitrarily small, (i) follows.

(ii) Suppose that $r_1 < t_1 < t < t_2 < r_2$ and let a, b be such that

$$\log \mathcal{M}(e^s;0,t_j) = aV_N(t_j) + b \qquad (j = 1,2).$$

Define $h(x) = aU_0(x) + b$. Then $\mathcal{M}(e^{s-h};0,t_j) = 1$ $(j = 1,2)$. Since $e^{s-h} \in \mathcal{S}(A(0;r_1,r_2))$, we see that $\mathcal{M}(e^{s-h};0,r)$ is a convex function of $V_N(r)$, so $\mathcal{M}(e^{s-h};0,t) \le 1$ and hence $\log \mathcal{M}(e^s;0,t) \le aV_N(t) + b$, as required. □

Corollary 3.5.8. *If f is holomorphic on an annulus $A(0;r_1,r_2)$ and $f \not\equiv 0$, and if $0 < p < +\infty$, then $\log \mathcal{M}(|f|^p;0,r)$ and $\log m(|f|;0,r)$ are convex functions of $\log r$ for $r \in (r_1,r_2)$.*

Proof. This is obtained by taking $s = p\log|f|$ in Theorem 3.5.7(ii), and $s = \log|f|$ in Theorem 3.5.2. □

Integrals and suprema of subharmonic functions over hyperplanes also have convexity properties, and we give two such results below. (Compare Theorem 1.5.12.) We denote a point of $\mathbb{R}^{N-1}$ by x' or y', and $(N-1)$-dimensional Lebesgue measure by λ'.

Theorem 3.5.9. *Let s be subharmonic and bounded above on the strip $W = \mathbb{R}^{N-1} \times (0,1)$ and let $\mathcal{N}(s;t) = \sup\{s(x',t) : x' \in \mathbb{R}^{N-1}\}$ for $0 < t < 1$. Then $\mathcal{N}(s;\cdot)$ is convex on $(0,1)$.*

Proof. Let $0 < t_1 < t_2 < 1$ and

$$f(t) = \frac{t_2 - t}{t_2 - t_1}\mathcal{N}(s;t_1) + \frac{t - t_1}{t_2 - t_1}\mathcal{N}(s;t_2) \qquad (0 < t < 1),$$

and define $s_1(x',x_N) = s(x',x_N) - f(x_N)$ when $(x',x_N) \in W$ and

$$s_2(x',x_N) = \begin{cases} s_1^+(x',x_N) & (t_1 < x_N < t_2) \\ 0 & (\text{elsewhere in } \mathbb{R}^N). \end{cases}$$

Clearly $s_1 \in \mathcal{S}(W)$ and $s_1 \leq 0$ on $\mathbb{R}^{N-1} \times \{t_1,t_2\}$. It follows from Corollary 3.2.4 that $s_2 \in \mathcal{S}(\mathbb{R}^N)$. Since s_2 is bounded above, by a say, on W, we see that

$$s_2(x) \leq \mathcal{A}(s_2;x,r) \leq a\frac{\lambda(B(x,r)\cap W)}{\lambda(B(x,r))} \to 0 \qquad (r \to +\infty; x \in \mathbb{R}^N).$$

Hence $s_1 \leq 0$ on $\mathbb{R}^{N-1} \times (t_1,t_2)$ and so $\mathcal{N}(s;\cdot) \leq f$ on (t_1,t_2) as required. □

Theorem 3.5.10. *Let s be non-negative and subharmonic on the strip $W = \mathbb{R}^{N-1} \times (0,1)$ and let*

$$\mathcal{L}(s;t) = \int_{\mathbb{R}^{N-1}} s(y',t)\, d\lambda'(y') \qquad (0 < t < 1).$$

If $\mathcal{L}(s;\cdot)$ is locally bounded on $(0,1)$, then $\mathcal{L}(s;\cdot)$ is convex on $(0,1)$.

Proof. For each $m \in \mathbb{N}$ we define s_m on W by

$$s_m(x) = \int_{\{\|y'\|<m\}} s(x'+y',x_N)d\lambda'(y'),$$

where $x = (x',x_N)$. By Theorem 3.3.1, $s_m \in \mathcal{S}(W)$. Also, if $\varepsilon > 0$ and $2\varepsilon < x_N < 1 - 2\varepsilon$, then

$$\begin{aligned} s_m(x) \leq \mathcal{A}(s_m;x,\varepsilon) &= \int_{\{\|y'\|<m\}} \mathcal{A}(s;(x'+y',x_N),\varepsilon)d\lambda'(y') \\ &\leq \frac{\lambda_{N-1}m^{N-1}}{\lambda_N\varepsilon^N}\int_{A_x} s\, d\lambda, \end{aligned}$$

where

$$A_x = \{(y',y_N) \in \mathbb{R}^{N-1} \times (\varepsilon, 1-\varepsilon) : \|y'\| > \|x'\| - m - \varepsilon\}.$$

Since s is integrable on $\mathbb{R}^{N-1} \times (\varepsilon,1-\varepsilon)$, it follows that

$$s_m(x) \to 0 \qquad (x \to \infty; \delta < x_N < 1-\delta) \tag{3.5.2}$$

for each $\delta \in (0,1/2)$.

Now suppose that $0 < t_1 < t < t_2 < 1$ and let a,b be such that $\mathcal{L}(s;t_j) = at_j + b$ $(j = 1,2)$. Define $u_m(x) = s_m(x) - ax_N - b$. Then $u_m \in \mathcal{S}(W)$ and $u_m \leq 0$ on $\partial(\mathbb{R}^{N-1} \times (t_1,t_2))$. Further, by (3.5.2),

$$\limsup_{x\to\infty} u_m(x) \leq \limsup_{x\to\infty} s_m(x) = 0 \qquad (t_1 < x_N < t_2).$$

Hence, by the maximum principle, $u_m(x) \leq 0$ when $t_1 < x_N < t_2$; equivalently, $s_m(x) \leq ax_N + b$ for such x. Letting $m \to \infty$, we obtain $\mathcal{L}(s;t) \leq ax_N + b$. Thus $\mathcal{L}(s;\cdot)$ is convex on $(0,1)$. □

3.6. Harmonic majorants

If f,g are functions on a set E taking values in $[-\infty,+\infty]$ and $f \leq g$ on E, then f is called a *minorant* of g on E, and g is called a *majorant* of f on E. Let $s \in \mathcal{S}(\Omega)$. Below we will show that, if s has a superharmonic majorant on Ω, then it has a least superharmonic majorant which is, in fact, harmonic on Ω. We can thus subsequently refer to this function as the *least harmonic majorant* of s on Ω.

Definition 3.6.1. A non-empty family $\mathcal{F}$ of functions in $\mathcal{U}(\Omega)$ is called *saturated* if the following conditions are satisfied:
(i) if $u,v \in \mathcal{F}$, then $\min\{u,v\} \in \mathcal{F}$;
(ii) if $u \in \mathcal{F}$ and $\overline{B(x,r)} \subset \Omega$, then the function $\overline{u}$, equal to $I_{u,x,r}$ on $B(x,r)$ and equal to u elsewhere on Ω, belongs to $\mathcal{F}$.

Theorem 3.6.2. *If $\mathcal{F}$ is a saturated family in $\mathcal{U}(\Omega)$, then on each component of Ω, either $\inf\mathcal{F} \equiv -\infty$ or $\inf\mathcal{F} \in \mathcal{H}(\Omega)$.*

Proof. It is enough to consider the case where Ω is connected. Suppose that $\overline{B(x,r)} \subset \Omega$, and for each $u \in \mathcal{F}$, let $\overline{u}$ be as in Definition 3.6.1. By Theorem 3.2.2, we have $\overline{u} \le u$ on $B(x,r)$. Hence $\inf \mathcal{F} = \inf\{\overline{u}: u \in \mathcal{F}\}$ on $B(x,r)$. Since $\min\{u,v\} \in \mathcal{F}$ when $u,v \in \mathcal{F}$, the family $\mathcal{F}$ is down-directed (see Definition 1.5.2), and since $\overline{\min\{u,v\}} \le \min\{\overline{u},\overline{v}\}$ by the minimum principle, the family $\{\overline{u}: u \in \mathcal{F}\}$ is also down-directed. Since $\overline{u} \in \mathcal{H}(B(x,r))$ for each $u \in \mathcal{F}$, it follows from Theorem 1.5.3 that $\inf\{\overline{u}: u \in \mathcal{F}\}$ is either harmonic or identically $-\infty$ on $B(x,r)$. Hence the disjoint open sets

$$\Omega_1 = \{y \in \Omega: \inf \mathcal{F} \text{ is harmonic on a neighbourhood of } y\},$$
$$\Omega_2 = \{y \in \Omega: \inf \mathcal{F} \equiv -\infty \text{ on a neighbourhood of } y\}$$

satisfy $\Omega_1 \cup \Omega_2 = \Omega$. Since Ω is connected, either $\Omega_1 = \Omega$ or $\Omega_2 = \Omega$. In the former case, $\inf \mathcal{F} \in \mathcal{H}(\Omega)$. □

Theorem 3.6.3. *If $s \in \mathcal{S}(\Omega)$ and s has a superharmonic majorant on Ω, then s has a least superharmonic majorant v on Ω, and $v \in \mathcal{H}(\Omega)$.*

Proof. If u_1, u_2 are superharmonic majorants of s on Ω, then so also is $\min\{u_1,u_2\}$. Also, if u is a superharmonic majorant of s on Ω and $\overline{B(x,r)} \subset \Omega$, then with the notation of Definition 3.6.1,

$$\liminf_{y \to z}(\overline{u} - s)(y) = \liminf_{y \to z}(u - s)(y) \ge 0 \qquad (z \in \partial^\infty \Omega).$$

Since, by Corollary 3.2.5, $\overline{u} \in \mathcal{U}(\Omega)$, it follows from the minimum principle that $\overline{u}$ is a superharmonic majorant of s on Ω. Hence the superharmonic majorants of s on Ω form a saturated family and by Theorem 3.6.2 the infimum of this family is harmonic on Ω. □

Theorem 3.6.4. *Let $s_1, s_2 \in \mathcal{S}(\Omega)$ and suppose that each of these functions has a superharmonic majorant on Ω. Then the least harmonic majorant of s_1+s_2 on Ω is h_1+h_2, where h_j is the least harmonic majorant of s_j $(j = 1,2)$ on Ω.*

Proof. Clearly $h_1 + h_2$ is a harmonic majorant of $s_1 + s_2$. Let h be the least harmonic majorant of $s_1 + s_2$. Then $h \le h_1 + h_2$. Also, $h - s_1$ is a superharmonic majorant of s_2, so $h - s_1 \ge h_2$. Hence $h - h_2$ is a harmonic majorant of s_1, so $h - h_2 \ge h_1$, whence $h_1 + h_2 \le h$, as required. □

Theorem 3.6.5. *If $s \in \mathcal{S}(\Omega)$ and $\overline{B(x,r)} \subset \Omega$, then the least harmonic majorant of s on $B(x,r)$ is $I_{s,x,r}$.*

Proof. We know from Theorem 3.2.2 that $I_{s,x,r}$ is a harmonic majorant of s on $B(x,r)$. Let h be the least such majorant. If $0 < t < r$, then

$$h(x) = \mathcal{M}(h;x,t) \ge \mathcal{M}(s;x,t) \to \mathcal{M}(s;x,r) = I_{s,x,r}(x) \qquad (t \to r-).$$

Hence $h(x) = I_{s,x,r}(x)$ and so $h - I_{s,x,r} \equiv 0$ by the maximum principle. □

Theorem 3.6.6. *Let $s \in \mathcal{S}(B)$. Then s has a harmonic majorant on B if and only if $\mathcal{M}(s;0,\cdot)$ is bounded above on $(0,1)$. Further, if s has a harmonic majorant, then its least harmonic majorant is $\lim_{r\to 1-} I_{s,0,r}$.*

Proof. If s has a harmonic majorant h_0 on B, then $\mathcal{M}(s;0,\cdot) \le \mathcal{M}(h_0;0,\cdot) = h_0(0)$ on $(0,1)$. Conversely, if $\mathcal{M}(s;0,\cdot)$ is bounded above on $(0,1)$, let (r_n) be a positive strictly increasing sequence with limit 1, and let $h_n = I_{s,0,r_n}$ on $B(0,r_n)$. Theorem 3.6.5 implies that $s \le h_n \le h_{n+1}$ on $B(0,r_n)$, so $(h_n)_{n\ge m}$ is an increasing sequence on $B(0,r_m)$ for each m. Since

$$h_n(0) = \mathcal{M}(s;0,r_n) \le \sup_{(0,1)} \mathcal{M}(s;0,\cdot),$$

it follows that $\lim h_n \not\equiv +\infty$. Hence $\lim h_n$ is a harmonic majorant of s on B.

Further, if H is any harmonic majorant of s, then $h_n \le H$ on $B(0,r_n)$ by Theorem 3.6.5. Hence the least harmonic majorant of s is given by $\lim h_n$; that is, by $\lim_{r\to 1-} I_{s,0,r}$. □

Theorem 3.6.7. *Suppose that $s \in \mathcal{S}(B)$ has a harmonic majorant and let h be its least harmonic majorant. Then:*
(i) $h = I_{\mu,0,1}$ for some signed measure μ if and only if $\mathcal{M}(s^+;0,\cdot)$ is bounded on $(0,1)$;
(ii) $h = I_{f,0,1} - I_{\mu,0,1}$ for some non-negative integrable function f and some measure μ if and only if $\mathcal{M}(\phi \circ s^+;0,\cdot)$ is bounded on $(0,1)$ for some convex increasing function $\phi: [0,+\infty) \to \mathbb{R}$ such that $t^{-1}\phi(t) \to +\infty$ as $t \to +\infty$.

Proof. (i) If $h = I_{\mu,0,1}$ for some signed measure μ, then we can write h as $h_1 - h_2$, where $h_1, h_2 \in \mathcal{H}_+(B)$, and so $\mathcal{M}(s^+;0,\cdot) \le \mathcal{M}(h_1;0,\cdot) = h_1(0)$.

Conversely, if $\mathcal{M}(s^+;0,\cdot)$ is bounded, then s^+ has a harmonic majorant h_1 by Theorem 3.6.6, and clearly $h_1 \ge 0$. Also, $s - h_1 \le 0$, so $s - h_1$ has a least harmonic majorant $h_2 \le 0$. Thus $s \le h_1 + h_2$. Further, if $s \le H$ for some $H \in \mathcal{H}(B)$, then $s - h_1 \le H - h_1$, so $h_2 \le H - h_1$ and hence $h_1 + h_2 \le H$. Thus $h_1 + h_2$ is the least harmonic majorant of s, and it can be written as $I_{\mu,0,1}$ for some signed measure μ, in view of the Riesz–Herglotz theorem.

(ii) If $h = I_{f,0,1} - I_{\mu,0,1}$, where f and μ are as stated, then by Theorem 1.3.9 there is a convex increasing function $\phi: [0,+\infty) \to [0,+\infty)$ such that $t^{-1}\phi(t) \to +\infty$ as $t \to +\infty$ and such that $\mathcal{M}(\phi \circ I_{f,0,1};0,\cdot)$ is bounded on $(0,1)$. Since $s^+ \le I_{f,0,1}$, it follows that $\mathcal{M}(\phi \circ s^+;0,\cdot)$ is bounded.

Conversely, suppose that there is a function ϕ with the stated properties. Then there exists $c > 0$ such that $t \le \phi(t)+c$ for all $t \ge 0$, and so $\mathcal{M}(s^+;0,\cdot)$ is bounded on $(0,1)$. We may assume (by considering $\phi(t)+t$) that ϕ is strictly increasing. It follows from Theorem 3.6.6 that the subharmonic functions $\phi \circ s^+$ and s^+ have harmonic majorants. Let h_0 denote the least harmonic majorant of $\phi \circ s^+$ and h_1 denote the least harmonic majorant of s^+. Then

s^+ is majorized by the function $\phi^{-1} \circ h_0$, which is superharmonic since ϕ^{-1} is concave and increasing (see Theorem 3.4.3(iv)). Hence $h_1 \leq \phi^{-1} \circ h_0$, so $\mathcal{M}(\phi \circ h_1; 0, \cdot) \leq h_0(0)$. It follows from Theorems 1.3.8 and 1.3.9 that $h_1 = I_{f,0,1}$ for some non-negative integrable function f. If we now define h_2 to be the least harmonic majorant of $s - h_1$, then we see that $h_2 \leq 0$ and $h_1 + h_2$ is the least harmonic majorant of s. Since $h_2 = -I_{\mu,0,1}$ for some measure μ, by the Riesz–Herglotz theorem, the proof is complete. □

3.7. Families of subharmonic functions: convergence properties

Lemma 3.7.1. *Let $\{f_\alpha : \alpha \in I\}$ be a family of upper semicontinuous functions from Ω to $[-\infty, +\infty)$. Then there is a countable subset J of I such that* $\inf\{f_\alpha : \alpha \in I\} = \inf\{f_\alpha : \alpha \in J\}$.

Proof. Let $f = \inf\{f_\alpha : \alpha \in I\}$. Clearly f is upper semicontinuous. For each real number a we define open sets by

$$W_a(\alpha) = \{x \in \Omega : f_\alpha(x) < a\} \quad (\alpha \in I), \; W_a = \{x \in \Omega : f(x) < a\}.$$

Then $\bigcup_{\alpha \in I} W_a(\alpha) = W_a$. Since W_a can be expressed as a countable union of compact sets, there is a countable subset J_a of I such that $\bigcup_{\alpha \in J_a} W_a(\alpha) = W_a$. Let $J = \bigcup_{q \in \mathbb{Q}} J_q$ and $g = \inf\{f_\alpha : \alpha \in J\}$. Then J is countable and

$$\{x \in \Omega : f(x) < q\} = W_q = \bigcup_{\alpha \in J} W_q(\alpha) = \{x \in \Omega : g(x) < q\} \qquad (3.7.1)$$

for all $q \in \mathbb{Q}$. Clearly $f \leq g$ on Ω. If $f(y) < g(y)$ for some $y \in \Omega$ and we choose $q \in \mathbb{Q}$ such that $f(y) < q \leq g(y)$, then (3.7.1) is contradicted. Hence $f = g$ on Ω. □

We recall that a family $\mathcal{F}$ of functions is called down-directed if, for each pair of functions $f_1, f_2 \in \mathcal{F}$, there exists $f \in \mathcal{F}$ such that $f \leq \min\{f_1, f_2\}$.

Theorem 3.7.2. *Let $\mathcal{F}$ be a down-directed family in $\mathcal{S}(\Omega)$. Then, on each component of Ω, the function* $\inf \mathcal{F}$ *is either subharmonic or identically $-\infty$.*

Proof. By Lemma 3.7.1 there exists a sequence (s_n) such that $s_n \in \mathcal{F}$ for each n and $\inf\{s_n : n \in \mathbb{N}\} = \inf \mathcal{F}$ on Ω. Let $u_1 = s_1$. Given u_n, we can choose (since $\mathcal{F}$ is down-directed) $u_{n+1} \in \mathcal{F}$ such that $u_{n+1} \leq \min\{u_n, s_{n+1}\}$. Then (u_n) is a decreasing sequence in $\mathcal{S}(\Omega)$ with $\lim u_n = \inf \mathcal{F}$, and the result follows from Theorem 3.1.4. □

Definition 3.7.3. If $f : \Omega \to [-\infty, +\infty]$, then the *upper semicontinuous regularization* $\check{f}$ and *lower semicontinuous regularization* $\hat{f}$ of f are defined on Ω by

$$\check{f}(x) = \max\{f(x), \limsup_{y \to x} f(y)\} = \inf_{r>0} \sup\{f(y) : y \in \Omega \cap B(x,r)\}$$

and

$$\hat{f}(x) = \min\{f(x), \liminf_{y \to x} f(y)\} = \sup_{r>0} \inf\{f(y) : y \in \Omega \cap B(x,r)\}.$$

It is easy to see that $\check{f}$ is upper semicontinuous, $\hat{f}$ is lower semicontinuous and $\check{f} \geq f \geq \hat{f}$ on Ω. Further, if f is upper (respectively, lower) semicontinuous on Ω, then $\check{f} = f$ (respectively, $\hat{f} = f$).

Lemma 3.7.4. *Let $\{f_\alpha : \alpha \in I\}$ be a family of functions from Ω to $[-\infty, +\infty]$ and let $f = \sup\{f_\alpha : \alpha \in I\}$. Then there is a countable subset J of I such that $\check{g} = \check{f}$, where $g = \sup\{f_\alpha : \alpha \in J\}$.*

Proof. By considering the function $\tan^{-1} f_\alpha$ in place of f_α, we may suppose that $f_\alpha(\Omega) \subseteq [-\pi/2, \pi/2]$ for each α. Let (B_n) be a sequence of open balls in Ω such that $\{B_n : n \in \mathbb{N}\}$ forms a base for the Euclidean topology on Ω and such that each ball in this collection occurs infinitely often in the sequence. For each n we choose $x_n \in B_n$ such that

$$f(x_n) > \sup\{f(x) : x \in B_n\} - n^{-1}$$

and then choose $\alpha_n \in I$ such that $f_{\alpha_n}(x_n) \geq f(x_n) - n^{-1}$. Let $J = \{\alpha_n : n \in \mathbb{N}\}$ and $g = \sup\{f_\alpha : \alpha \in J\}$. Then

$$\sup\{g(x) : x \in B_n\} \geq \sup\{f(x) : x \in B_n\} - 2n^{-1} \qquad (n \in \mathbb{N}).$$

Hence, in view of the repetitious nature of (B_n),

$$\check{g}(x) = \inf_{\{n : x \in B_n\}} \sup\{g(y) : y \in B_n\} \geq \inf_{\{n : x \in B_n\}} \sup\{f(y) : y \in B_n\} = \check{f}(x),$$

so that $\check{g} \geq \check{f}$ on Ω. The reverse inequality is obvious. □

Theorem 3.7.5. *Let $\mathcal{F}$ be a family in $\mathcal{S}(\Omega)$ and let $s = \sup \mathcal{F}$. If s is locally bounded above, then:*
(i) $\check{s} \in \mathcal{S}(\Omega)$;
(ii) $\check{s} = s$ almost everywhere (λ);
(iii) $\check{s}(x) = \limsup_{y \to x} s(y) \quad (x \in \Omega)$.

Proof. The function $\check{s}$ takes values in $[-\infty, +\infty)$, is upper semicontinuous and is not identically $-\infty$ on any component of Ω. If $\overline{B(x,r)} \subset \Omega$, then

$$I_{\breve{s},x,r}(y) \geq I_{u,x,r}(y) \geq u(y) \qquad (y \in B(x,r); u \in \mathcal{F}). \tag{3.7.2}$$

Hence $I_{\breve{s},x,r} \geq s$, and it follows from the continuity of $I_{\breve{s},x,r}$ that $I_{\breve{s},x,r} \geq \breve{s}$ on $B(x,r)$. Thus $\breve{s} \in \mathcal{S}(\Omega)$, by Theorem 3.2.2.

By Lemma 3.7.4 there exists a sequence (u_n) in $\mathcal{F}$ such that $\breve{u} = \breve{s}$ on Ω, where $u = \sup u_n$. Define $v_n = \max\{u_1, \ldots, u_n\}$. Then (v_n) is an increasing sequence in $\mathcal{S}(\Omega)$ with limit u. Since $I_{\breve{s},x,r} \geq I_{v_n,x,r} \geq v_n$, it follows from monotone convergence that $I_{\breve{s},x,r} \geq I_{u,x,r} \geq u$. Thus $I_{\breve{s},x,r} \geq I_{u,x,r} \geq \breve{u} = \breve{s}$. By Theorem 3.6.5, the function $I_{\breve{s},x,r}$ is the least harmonic majorant of $\breve{s}$ on $B(x,r)$, and therefore $I_{\breve{s},x,r} = I_{u,x,r}$. Since $\breve{s} \geq u$, it follows that $\breve{s} = u$ almost everywhere (σ) on $S(x,r)$. In view of the arbitrary choice of $B(x,r)$, we conclude that $\breve{s} = u$ almost everywhere (λ) on Ω. Since $\breve{s} \geq s \geq u$, it follows that s is Lebesgue measurable and $\breve{s} = s$ almost everywhere (λ).

Finally, (i), (ii) and Corollary 3.2.6 give

$$\limsup_{y\to x} s(y) \leq \breve{s}(x) = \lim_{r\to 0+} \mathcal{A}(\breve{s};x,r) = \lim_{r\to 0+} \mathcal{A}(s;x,r) \leq \limsup_{y\to x} s(y) \; (x \in \Omega)$$

so (iii) holds. □

Assertion (ii) of the above result will be strengthened in Theorem 5.7.1.

3.8. Exercises

Exercise 3.1. Show that if $s \in C(\overline{B}) \cap \mathcal{S}(B)$, then $\mathbb{R}^N \backslash \{x \in \overline{B} : s(x) \leq 0\}$ is connected.

Exercise 3.2. Let $s \in \mathcal{S}(B)$ and suppose that $\limsup_{x\to y} s(x)$ is finite for every $y \in S$ and non-positive for almost every (σ) such y. Show that $s \leq 0$ on B.

Exercise 3.3. Phragmén–Lindelöf theorem for strips. Let $W = \mathbb{R}^{N-1} \times (0,1)$. Verify that if $0 < c < \pi$, then the function

$$\cosh\left(\frac{cx_1}{\sqrt{N-1}}\right) \cdots \cosh\left(\frac{cx_{N-1}}{\sqrt{N-1}}\right) \cos(c(x_N - \frac{1}{2}))$$

is harmonic on $\mathbb{R}^N$ and positive on $\overline{W}$. Hence show that if $s \in \mathcal{S}(W)$,

$$\limsup_{x\to y} s(x) \leq 0 \qquad (y \in \partial W)$$

and

$$s(x) \leq \exp\left(\frac{(1-\varepsilon)\pi(|x_1| + \ldots + |x_{N-1}|)}{\sqrt{N-1}}\right) \qquad (x \in W)$$

for some $\varepsilon \in (0,1)$, then $s \leq 0$ on W.

Exercise 3.4. Phragmén–Lindelöf theorem for cylinders. Let $\Omega = \mathbb{R} \times (-1,1)^{N-1}$. Show that if $s \in \mathcal{S}(\Omega)$,

$$\limsup_{x\to y} s(x) \leq 0 \qquad (y \in \partial\Omega)$$

and

$$s(x) \leq \exp((1-\varepsilon)|x_1|\pi\sqrt{N-1}/2) \qquad (x \in \Omega)$$

for some $\varepsilon \in (0,1)$, then $s \leq 0$ on Ω.

Exercise 3.5. Let Ω be unbounded and connected and suppose that

$$r^{-N}\lambda(\Omega \cap B(0,r)) \to 0 \text{ as } r \to +\infty.$$

Show that if $s \in \mathcal{S}(\Omega)$,

$$\limsup_{x\to y} s(x) \leq 0 \qquad (y \in \partial\Omega)$$

and s is bounded above on Ω, then $s \leq 0$ on Ω. (Hint: extend s^+ to be subharmonic on $\mathbb{R}^N$.)

Exercise 3.6. For $n \in \mathbb{N}$, define

$$s_n(x_1,x_2) = -\sin(2^{n+1}\pi x_1)\exp(2^{n+1}\pi x_2)$$

when $2^{-n-1} < x_1 < 2^{-n}$ and $s_n(x_1,x_2) = 0$ otherwise. Verify that $s_n \in \mathcal{S}(\mathbb{R}^2)$. Define s on $\mathbb{R}^4$ by

$$s(x_1,x_2,x_3,x_4) = \sum_{n=1}^{\infty} s_n(x_1,x_2)s_n(x_3,x_4).$$

Show that s is subharmonic as a function of (x_3,x_4) when (x_1,x_2) is fixed, and thus also as a function of (x_1,x_2) when (x_3,x_4) is fixed. Show also that

$$s(3.2^{-k}, 1, 3.2^{-k}, 1) \to +\infty \quad \text{as} \quad k \to \infty$$

and deduce that $s \notin \mathcal{S}(\mathbb{R}^4)$. (Compare this with Theorem 3.3.6.)

Exercise 3.7. Use two applications of Corollary 1.3.4 to give a direct proof of Corollary 3.3.7.

Exercise 3.8. Let u be positive and superharmonic on the annular region $A(0;r_1,r_2)$. Show that if $0 < p \leq 1$, then $(\mathcal{M}(u^p;0,r))^{1/p}$ is a concave function of $V_N(r)$ for $r \in (r_1,r_2)$. (Hint: let $r_1 < t_1 < t_2 < r_2$, choose a,b so that

$$(\mathcal{M}(u^p;0,t_j))^{1/p} = aV_N(t_j) + b \qquad (j = 1,2)$$

and consider $h^{1-p}u^p$, where $h = aU_0 + b$.) Show that the result also holds when $p < 0$.

Exercise 3.9. Let u be positive and superharmonic on $W = \mathbb{R}^{N-1} \times (0,1)$. Define v on W by

$$v(x_1, \ldots, x_N) = \int_{\mathbb{R}^{N-1}} u(x' + y', x_N)\, d\lambda'(y').$$

Use Theorem 3.3.1 to show that either $v \in \mathcal{U}(W)$ or $v \equiv +\infty$. Hence show that if the function

$$t \mapsto \int_{\mathbb{R}^{N-1}} u(y', t)\, d\lambda'(y')$$

is finite for some $t \in (0,1)$, then it is concave on $(0,1)$. (Note: the convexity result in Theorem 3.5.9 implies that if w superharmonic on W and $w(x)$ depends only on x_N, then the function $w(0,\ldots,0,t)$ is concave on $(0,1)$.)

Exercise 3.10. Let $s \in \mathcal{S}(B)$. Writing $\mathcal{M}(s;0,t) = (\phi \circ V_N)(t)$, where ϕ is convex (see Theorem 3.5.6), and recalling that

$$\mathcal{A}(s;0,t) = Nt^{-N} \int_0^t \tau^{N-1} \mathcal{M}(s;0,\tau)\, d\tau,$$

use Jensen's inequality (see Historical Notes on Section 1.3) to show that

$$\mathcal{M}(s;0,k_N t) \le \mathcal{A}(s;0,t) \qquad (0 < t < r),$$

where $k_2 = e^{-1/2}$ and $k_N = (2/N)^{1/(N-2)}$.

Exercise 3.11. Let s be a function on $\overline{B}$ such that $s \in \mathcal{S}(B)$ and

$$s(y) = \limsup_{x \to y} s(x) < +\infty \qquad (y \in S).$$

Let h be the least harmonic majorant of s on B. Show that $h \le I_{s,0,1}$ on B. Give an example in which this inequality is strict.

Exercise 3.12. Give an example of a decreasing sequence (s_n) of non-negative subharmonic functions on $D = \mathbb{R} \times (0, +\infty)$ such that each s_n has a least harmonic majorant h_n on D, $\lim_{n\to\infty} s_n = 0$ on D, but $\lim_{n\to\infty} h_n > 0$ on D.

Exercise 3.13. Give an example of an increasing sequence (s_n) of negative subharmonic functions on B such that $\lim s_n$ is not subharmonic on B. Give an example in which $\lim s_n$ is not subharmonic on any open subset of B.

Exercise 3.14. Let $\mathcal{F}$ be a family in $\mathcal{S}(\Omega)$ and let $s = \sup \mathcal{F}$. Show that if $\check{s}$ is locally integrable, then $\check{s} \in \mathcal{S}(\Omega)$. (Compare Theorem 3.7.5.)

Exercise 3.15. Let (s_n) be a sequence in $\mathcal{S}(\Omega)$, where Ω is connected, and let $s = \limsup_{n\to\infty} s_n$. Show that if (s_n) is locally uniformly bounded above, then $\check{s} = s$ almost everywhere (λ), and either $\check{s} \in \mathcal{S}(\Omega)$ or $\check{s} \equiv -\infty$.

Exercise 3.16. Let s be a non-negative function on Ω. Suppose that $p_0 > 0$ and $s^p \in \mathcal{S}(\Omega)$ for all $p > p_0$. Show that $s^{p_0} \in \mathcal{S}(\Omega)$.

Exercise 3.17. Suppose that $s \in C(\overline{B}) \cap \mathcal{S}(B)$. A function $h_0 \in C(\overline{B}) \cap \mathcal{H}(B)$ is called a *best harmonic approximant* (*b.h.a.*) to s if

$$\sup_{\overline{B}} |h_0 - s| \le \sup_{\overline{B}} |h - s| \qquad \text{for all } h \in C(\overline{B}) \cap \mathcal{H}(B).$$

Let $H = I_{s,0,1}$ on B, and $H = s$ on S. Use the maximum principle to show that the constant function $\frac{1}{2}\inf_{\overline{B}}(s - H)$ is the unique b.h.a. to $s - H$. Hence describe the b.h.a. to s.

Exercise 3.18. Let $f = u + iv$ be holomorphic on the unit disc. Let $s = (p-1)|f|^p - p|u|^p$, where $p \ge 2$. Show that $\Delta s \ge 0$ and deduce that if $f(0) = 0$, then

$$\int_0^{2\pi} |u(re^{i\theta})|^p\, d\theta \le \frac{p-1}{p} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \quad (0 < r < 1).$$

Exercise 3.19. Let $B_0 = B(0, 2^{-1/N})$, so that $\lambda(B_0) = \frac{1}{2}\lambda(B)$. Let u be superharmonic and integrable on B. Show that

$$\int_{B \setminus B_0} u\, d\lambda \le \int_{B_0} u\, d\lambda.$$

Show also that equality holds if and only if $u \in \mathcal{H}(B)$.

Exercise 3.20. Let Ω be a convex open proper subset of $\mathbb{R}^N$. Define u on Ω by $u(x) = \operatorname{dist}(x, \partial\Omega)$. Show that $u \in \mathcal{U}(\Omega)$. (Assume the result that for each y in $\partial\Omega$ there exists an $(N-1)$-dimensional hyperplane P_y such that $y \in P_y$ and Ω is contained in one component of $\mathbb{R}^N \setminus P_y$.)

Exercise 3.21. Let $E = \{x \in B : x_N = 0\}$. Suppose that $s \in C(B) \cap \mathcal{H}(B \setminus E)$ and at each point of E the left and right derivatives $(\partial s/\partial x_N)_-$ and $(\partial s/\partial x_N)_+$ exist and satisfy $(\partial s/\partial x_N)_- < (\partial s/\partial x_N)_+$. Show that $s \in \mathcal{S}(B)$. (Hint: it is enough to show that if $y \in E$ and $\overline{B(y,r)} \subset B$, then $s - I_{s,y,r} \le 0$ on $B(y,r)$. Suppose this inequality fails, deduce that $s - I_{s,y,r}$ attains a maximum at some point of E, and derive a contradiction.)

Exercise 3.22. Show that if we have "$\le$" in place of "$<$" in the hypothesis of Exercise 3.21, then the conclusion remains true. (Hint: consider $s_n(x) = s(x) + |x_N|/n$ for $n \in \mathbb{N}$.)

Exercise 3.23. Construct a superharmonic function u on B such that

$$\liminf_{x\to y} u(x) = -\infty, \quad \limsup_{x\to y} u(x) = +\infty$$

for each $y \in S$.

Exercise 3.24. Let E be a non-empty closed subset of $\mathbb{R}^N$ and let s be defined on $\mathbb{R}^N$ by $s(x) = \operatorname{dist}(x, E)$.
(i) Show that, if E is convex, then s is subharmonic on $\mathbb{R}^N$.
(ii) Let $x_0, y, z \in \mathbb{R}^N$ be distinct points such that $\|x_0 - y\| = \|x_0 - z\|$, and let $v(x) = \min\{\|x-y\|, \|x-z\|\}$. Show that $\{v(x_0)\}^2 > \mathcal{M}(v^2; x_0, r)$, and hence by Hölder's inequality that $v(x_0) > \mathcal{M}(v; x_0, r)$, for all sufficiently small values of r.
(iii) Motzkin's theorem tells us that E is convex if and only if to each point of $\mathbb{R}^N$ there is a unique nearest point of E. Use this result and (ii) to show that, if E is not convex, then s is not subharmonic on $\mathbb{R}^N$. (Hint: show that the subharmonic mean value property fails for small spheres centred at a point with no unique nearest point of E.)

Chapter 4. Potentials

4.1. Green functions

We recall that, if $y \in \mathbb{R}^N$, then the function defined by

$$U_y(x) = \begin{cases} -\log\|x-y\| & (x \neq y;\ N = 2) \\ \|x-y\|^{2-N} & (x \neq y;\ N \geq 3) \\ +\infty & (x = y) \end{cases}$$

is superharmonic on $\mathbb{R}^N$ and harmonic on $\mathbb{R}^N\backslash\{y\}$.

Let Ω be an open subset of $\mathbb{R}^N$. A function $G_\Omega : \Omega \times \Omega \to [0, +\infty]$, called the Green function of Ω, will be defined so that $G_\Omega(\cdot\,, y) = U_y - h_y$, where h_y is the greatest harmonic minorant of U_y on Ω. (To ensure the existence of h_y and hence G_Ω, a mild restriction on Ω will be required when $N = 2$.) It will follow that, if μ is a measure on Ω, then the equation

$$G_\Omega\mu(x) = \int_\Omega G_\Omega(x,y)\, d\mu(y) \quad (x \in \Omega)$$

defines a non-negative superharmonic function on Ω, provided only that $G_\Omega\mu$ is not identically $+\infty$ on any component of Ω. A superharmonic function of this form will be called a (Green) potential. The importance of potentials will become apparent in the Riesz decomposition theorem, which includes the result that every non-negative superharmonic function on Ω is the sum of a potential and a non-negative harmonic function. This reduces the local study of superharmonic functions to that of potentials and harmonic functions. Also, it shows that, in a ball or half-space, a non-negative superharmonic function is the sum of a potential and a Poisson integral; this is especially useful since in these domains the Green function, like the Poisson kernel, is known explicitly. Our knowledge of G_B will be used to show that potentials on the unit ball have radial limit 0 at almost every boundary point, and we will obtain related boundary limit theorems for harmonic and superharmonic functions on B. We shall also study continuity properties of potentials and examine how the nature of a measure μ affects the smoothness of $G_\Omega\mu$.

We begin by describing the type of open set on which we shall work.

Definition 4.1.1. An open set Ω in $\mathbb{R}^N$ is said to be *Greenian* if, for each y in Ω, the function U_y has a subharmonic minorant on Ω.

Theorem 4.1.2. *(i) All open sets in $\mathbb{R}^N$ $(N \geq 3)$ are Greenian.*
(ii) Any open subset of a Greenian open set is Greenian.
(iii) The set $\mathbb{R}^2$ is not Greenian.
(iv) If $\Omega \subseteq \mathbb{R}^2$ and there exists z in Ω such that U_z has a subharmonic minorant on Ω, then Ω is Greenian.
(v) If $\Omega \subseteq \mathbb{R}^2$ and $\mathbb{R}^2 \backslash \partial\Omega$ is not connected, then Ω is Greenian. In particular, all bounded open sets in $\mathbb{R}^2$ are Greenian.

Proof. Part (i) follows from the positivity of U_y when $N \geq 3$, and part (ii) is immediate from the definition.

To prove (iii), suppose that there is a subharmonic minorant s of U_0 on $\mathbb{R}^2$. Then $s \leq -\log r$ on $B(0,r)$ by the maximum principle and, since r can be arbitrarily large, we obtain the contradictory conclusion that $s \equiv -\infty$.

To prove (iv), let r be such that $\overline{B(z,r)} \subseteq \Omega$, let u be a subharmonic minorant of U_z on Ω and let $y \in \Omega$. We will show that U_y also has a subharmonic minorant on Ω. If $x \in \Omega \backslash B(z,r)$, then

$$\begin{aligned} U_y(x) - u(x) &\geq U_y(x) - U_z(x) \geq \log\left(\frac{\|z-x\|}{\|y-z\|+\|z-x\|}\right) \\ &\geq -\frac{\|y-z\|}{\|z-x\|} \geq -\frac{\|y-z\|}{r}. \end{aligned} \tag{4.1.1}$$

Since the lower semicontinuous function $U_y - u$ is also bounded below on $B(z,r)$, part (iv) follows.

It remains to prove (v). Suppose that $\Omega \subseteq \mathbb{R}^2$ and $\mathbb{R}^2\backslash\partial\Omega$ is not connected. Let $y \in \Omega$, let ω be the component of Ω which contains y, and let $\overline{B(z,r)} \subseteq \mathbb{R}^2 \backslash (\omega \cup \partial\Omega)$. It follows from (4.1.1) that the function

$$h(x) = \begin{cases} U_z(x) - \|y-z\|/r & (x \in \omega) \\ U_y(x) & (x \in \Omega\backslash\omega) \end{cases}$$

is a harmonic minorant of U_y on Ω. Hence Ω is Greenian, and so (v) holds. □

A complete characterization of the Greenian sets in $\mathbb{R}^2$ will be given in Theorem 5.3.8.

Definition 4.1.3. Let Ω be Greenian. Then, for each y in Ω, the function U_y has a greatest harmonic minorant h_y on Ω, by Theorem 3.6.3. The function $G_\Omega : \Omega \times \Omega \to [0, +\infty]$, defined by

$$G_\Omega(x,y) = U_y(x) - h_y(x), \tag{4.1.2}$$

is called the *Green function for Ω*. Clearly $G_\Omega(x,x) = +\infty$ for any x in Ω. In the case where $\Omega = \mathbb{R}^N$ $(N \geq 3)$ we simply write G for the Green function. The formula (4.1.2) has no meaning for open sets Ω which are not Greenian. We say that such sets Ω do not possess a Green function.

Before developing the general theory of Green functions, we give G_Ω explicitly for three specific domains Ω.

Theorem 4.1.4. *The Green function for $\mathbb{R}^N$ $(N \geq 3)$ is given by*

$$G(x,y) = U_y(x) \quad (x, y \in \mathbb{R}^N).$$

Proof. Let $y \in \mathbb{R}^N$. If h is any harmonic minorant of U_y, then $h \leq r^{2-N}$ on $B(y,r)$ by the maximum principle. Since r can be arbitrarily large, $h \leq 0$ on $\mathbb{R}^N$. Hence 0 is the greatest harmonic minorant of U_y on $\mathbb{R}^N$ and the Green function for $\mathbb{R}^N$ is as stated. □

Theorem 4.1.5. *Let $B_0 = B(x_0, r)$ and*

$$\phi(x,y) = \frac{(r^2 - \|x-x_0\|^2)(r^2 - \|y-x_0\|^2)}{r^2\|x-y\|^2} \qquad (x, y \in B_0;\ x \neq y),$$

and let y^* denote the inverse of a point $y \neq x_0$ with respect to ∂B_0; that is, y^* satisfies

$$y^* - x_0 = \left(\frac{r}{\|y-x_0\|}\right)^2 (y - x_0). \tag{4.1.3}$$

(i) If $N = 2$, then

$$G_{B_0}(x,y) = \begin{cases} \log\left(\dfrac{\|y-x_0\|}{r}\dfrac{\|x-y^*\|}{\|x-y\|}\right) & (x, y \in B_0; y \notin \{x, x_0\}) \\ \log\left(\dfrac{r}{\|x-y\|}\right) & (x \in B_0\backslash\{x_0\}; y = x_0) \\ +\infty & (x = y) \end{cases}$$

$$= \begin{cases} 2^{-1}\log\{1 + \phi(x,y)\} & (x, y \in B_0; x \neq y) \\ +\infty & (x = y). \end{cases}$$

(ii) If $N \geq 3$, then

$$G_{B_0}(x,y) = \begin{cases} \|x-y\|^{2-N} - \left(\dfrac{r}{\|y-x_0\|}\dfrac{1}{\|x-y^*\|}\right)^{N-2} & (x, y \in B_0; y \notin \{x, x_0\}) \\ \|x-y\|^{2-N} - r^{2-N} & (x \in B_0\backslash\{x_0\}; y = x_0) \\ +\infty & (x = y) \end{cases}$$

$$= \begin{cases} \{1 - (1 + \phi(x,y))^{1-N/2}\}\|x-y\|^{2-N} & (x, y \in B_0; x \neq y) \\ +\infty & (x = y). \end{cases}$$

Proof. Since

$$\|x-y^*\|^2 = \|x-x_0\|^2 + \|y^*-x_0\|^2 - 2\langle x-x_0, y^*-x_0\rangle \qquad (x\in\mathbb{R}^N; y\neq x_0),$$

we see from (4.1.3) that

$$\begin{aligned} r^{-2}\|y-x_0\|^2\|x-y^*\|^2 &= r^{-2}\|y-x_0\|^2\|x-x_0\|^2 + r^2 - 2\langle x-x_0, y-x_0\rangle \\ &= \|x-y\|^2\{1+\phi(x,y)\} \qquad (y\notin\{x,x_0\}), \quad (4.1.4)\end{aligned}$$

and, in particular,

$$r^{-1}\|y-x_0\|\|x-y^*\| = \|x-y\| \qquad (x\in\partial B_0; y\neq x_0). \tag{4.1.5}$$

If $y\in B_0\backslash\{x_0\}$, then the function

$$h_y(x) = \begin{cases} \log\left(\dfrac{r}{\|y-x_0\|}\dfrac{1}{\|x-y^*\|}\right) & (N=2) \\ \left(\dfrac{r}{\|y-x_0\|}\dfrac{1}{\|x-y^*\|}\right)^{N-2} & (N\geq 3)\end{cases}$$

is harmonic on $\mathbb{R}^N\backslash\{y^*\}$, which contains $\overline{B_0}$. Further, it can be seen from (4.1.5) that $h_y = U_y$ on ∂B_0. It follows from the maximum principle that $h_y \leq U_y$ on B_0 and that $h\leq h_y$ on B_0 for any harmonic minorant h of U_y on B_0. Hence h_y is the greatest harmonic minorant of U_y on B_0, and so $G_{B_0}(x,y) = U_y(x) - h_y(x)$ when $y\neq x_0$.

If $y = x_0$, then an argument similar to that of the previous paragraph applies with $h_y(x) = \log(1/r)$ if $N=2$, and $h_y(x) = r^{2-N}$ if $N\geq 3$. The first group of formulae in each part of the theorem are now established, and the formulae involving ϕ follow using (4.1.4). □

Theorem 4.1.6. *Let D denote the half-space $\{x=(x_1,\dots,x_N) : x_N>0\}$. Then*

$$G_D(x,y) = U_y(x) - U_{\overline{y}}(x) \quad (x,y\in D),$$

where $\overline{y} = (y,\dots,y_{N-1},-y_N)$.

Proof. Let $y\in D$ and $h_y(x) = U_{\overline{y}}(x)$. Then h_y is harmonic on $\mathbb{R}^N\backslash\{\overline{y}\}$, which contains $\overline{D}$. Further, $h_y = U_y$ on ∂D, $h_y\leq U_y$ on D, and $U_y(x)-h_y(x)$ has limit 0 as $x\to\infty$. It follows from the minimum principle that $h\leq h_y$ for any harmonic minorant h of U_y on D. Hence h_y is the greatest harmonic minorant of U_y on D, and so G_D is as stated above. □

Lemma 4.1.7. *Let Ω be Greenian, let $y\in\Omega$ and let ω be the component of Ω which contains y. Then:*
(i) $G_\Omega(\cdot,y)$ is positive and superharmonic on ω, and harmonic on $\Omega\backslash\{y\}$;
(ii)

$$G_\Omega(x,y) = \begin{cases} G_\omega(x,y) & (x\in\omega) \\ 0 & (x\in\Omega\backslash\omega);\end{cases}$$

(iii) the greatest harmonic minorant of $G_\Omega(\cdot,y)$ on Ω is the zero function;
(iv) $\inf_\omega G_\Omega(\cdot,y) = 0$.

Proof. It follows from the definition that $G_\Omega(\cdot,y)$ is non-negative and superharmonic on Ω, and harmonic on $\Omega\backslash\{y\}$. The positivity on ω is a consequence of the minimum principle and the fact that $G_\Omega(y,y)=+\infty$. Thus (i) holds.

Clearly U_y is harmonic on $\Omega\backslash\omega$. We thus obtain the greatest harmonic minorant of U_y on Ω by defining it to be equal on ω to the greatest harmonic minorant of U_y on ω, and equal on $\Omega\backslash\omega$ to U_y. This proves (ii).

Let v_y be a harmonic minorant of $G_\Omega(\cdot,y)$ on Ω. Then $U_y\geq h_y+v_y$ on Ω, where h_y is the greatest harmonic minorant of U_y on Ω, and so $v_y\leq 0$. This proves (iii).

Finally, (iv) follows from (ii) and (iii). □

Lemma 4.1.8. *Let Ω be Greenian, let $y\in\Omega$ and let W be a bounded open neighbourhood of y such that $\overline{W}\subset\Omega$. If u is a positive superharmonic function on Ω and $u\geq G_\Omega(\cdot,y)$ on ∂W, then $u\geq G_\Omega(\cdot,y)$ on $\Omega\backslash W$. In particular, $G_\Omega(\cdot,y)$ is bounded on $\Omega\backslash W$.*

Proof. Let h_y denote the greatest harmonic minorant of U_y on Ω. Then $U_y - h_y\leq u$ on ∂W. The function defined by

$$s(x) = \begin{cases} \max\{h_y(x), U_y(x)-u(x)\} & (x\in\Omega\backslash\overline{W}) \\ h_y(x) & (x\in\overline{W})\end{cases}$$

is subharmonic on Ω, by Corollary 3.2.4. Clearly $s\leq U_y$, so s is a subharmonic minorant of U_y on Ω. Hence $s\leq h_y$; that is, $U_y-u\leq h_y$ on $\Omega\backslash W$, and thus $u\geq G_\Omega(\cdot,y)$ on $\Omega\backslash W$. The particular case follows from the fact that $G_\Omega(\cdot,y)$ is bounded on ∂W. □

A glance at the Green functions in Theorems 4.1.4 – 4.1.6 reveals that they are symmetric in x and y. It will now be shown that all Green functions have this property.

Theorem 4.1.9. *Let Ω be Greenian. Then:*
(i) $G_\Omega(x,y) = G_\Omega(y,x)$ for any x and y in Ω;
(ii) the function $(x,y)\mapsto G_\Omega(x,y)$ is continuous on $\Omega\times\Omega$ (in the extended sense);
(iii) the function $(x,y)\mapsto G_\Omega(x,y)$ is superharmonic on $\Omega\times\Omega$.

Proof. Let B_0 be an open ball such that $\overline{B_0}\subset\Omega$. Further, let $(B_n) = (B(z_n,r_n))$ be a sequence of open balls in Ω such that $\overline{B_0}\subset B_1$ and $\bigcup_n B_n = \Omega$, such that $\partial B_n\cap\overline{B_0}=\emptyset$ for each n, and such that each ball in the collection $\{B_n : n\in\mathbb{N}\}$ occurs infinitely often in the sequence.

Let $u_1:\Omega\times\overline{B_0}\to\mathbb{R}$ be defined by

$$u_1(x,y) = \begin{cases} I_{U_y,z_1,r_1}(x) & (x \in B_1) \\ U_y(x) & (x \in \Omega\backslash B_1), \end{cases}$$

where $I_{f,x,r}$ is the Poisson integral introduced in Definition 1.3.2. It follows from Theorem 1.3.3(ii) and the uniform continuity of the function $(x, y) \mapsto U_y(x)$ on $\partial B_1 \times \overline{B_0}$ that u_1 is continuous on $\Omega \times \overline{B_0}$. Also, it can be seen (using the mean value property and Fubini's theorem, if $x \in B_1$) that $u_1(x, \cdot) \in \mathcal{H}(B_0)$ for each x in Ω. We now inductively define a sequence (u_n) of functions on $\Omega \times \overline{B_0}$ as follows. Given u_k, we define

$$u_{k+1}(x,y) = \begin{cases} I_{u_k(\cdot,y),z_{k+1},r_{k+1}}(x) & (x \in B_{k+1}) \\ u_k(x,y) & (x \in \Omega\backslash B_{k+1}). \end{cases}$$

By the above reasoning and the choice of (B_n), each u_n is continuous on $\Omega \times \overline{B_0}$, and $u_n(x, \cdot) \in \mathcal{H}(B_0)$ for each x in Ω.

Repeated application of Corollary 3.2.5 yields that, for each y in B_0, the functions $u_n(\cdot, y)$ are superharmonic on Ω, satisfy $u_n(\cdot, y) \le U_y$, and form a decreasing sequence. Further, if h_y denotes the greatest harmonic minorant of U_y on Ω, then $u_n(\cdot, y) \ge h_y$ on Ω. If we define $u_\infty(\cdot, y) = \lim_{n\to\infty} u_n(\cdot, y)$, then $u_\infty(\cdot, y)$ must be harmonic on each ball B_n, by Corollary 1.5.4 and the repetitious nature of (B_n). Hence $u_\infty(\cdot, y)$ is a harmonic minorant of U_y on Ω which satisfies $u_\infty(\cdot, y) \ge h_y$ and so $u_\infty(\cdot, y) = h_y$. It follows from the previous paragraph and Corollary 1.5.4 again that $(y \mapsto h_y(x)) \in \mathcal{H}(B_0)$ for each x in Ω. In fact, these functions must belong to $\mathcal{H}(\Omega)$, in view of the arbitrary choice of B_0. Since $h_y(x) \le U_y(x) = U_x(y)$, it follows that $h_y(x) \le h_x(y)$. This holds for any choice of x and y in Ω, so $h_y(x) = h_x(y)$, and hence

$$G_\Omega(x,y) = U_y(x) - h_y(x) = U_x(y) - h_x(y) = G_\Omega(y,x) \qquad (x,y \in \Omega),$$

proving (i).

The above functions $u_n(\cdot, \cdot)$ have the property that they are continuous on $B_n \times B_0$ and harmonic in each variable separately. It follows from Corollary 3.3.7 that $u_n(\cdot, \cdot)$ is harmonic on $B_n \times B_0$. Hence the limit function $(x, y) \mapsto h_y(x)$ is harmonic on $\Omega \times \Omega$, in view of the repetitions in (B_n) and the arbitrary nature of B_0. Since $G_\Omega(x,y) = U_y(x) - h_y(x)$, we conclude that $G_\Omega(\cdot, \cdot)$ is continuous on $\Omega \times \Omega$, in the extended sense. Thus (ii) is proved.

Part (iii) is an immediate consequence of (i) and Theorem 3.3.6. □

Theorem 4.1.10. *(i) If Ω_1 is an open subset of a Greenian open set Ω_2, then $G_{\Omega_1} \le G_{\Omega_2}$ on $\Omega_1 \times \Omega_1$.*
(ii) Let (Ω_n) be an increasing sequence of Greenian open sets and let $\Omega_\infty = \bigcup_n \Omega_n$. If Ω_∞ is Greenian, then $G_{\Omega_n} \to G_{\Omega_\infty}$ on $\Omega_\infty \times \Omega_\infty$; otherwise $G_{\Omega_n} \to +\infty$ on $\Omega_\infty \times \Omega_\infty$.

Proof. Let $h_{y,n}$ denote the greatest harmonic minorant of U_y on Ω_n for $n \in \mathbb{N}$. Part (i) follows from the fact that $h_{y,2} \le h_{y,1}$ on Ω_1.

To prove (ii) suppose first that Ω_∞ is Greenian. For each $y \in \Omega_\infty$, the function U_y has a greatest harmonic minorant $h_{y,\infty}$ on Ω_∞ and $h_{y,\infty} \le h_{y,n+1} \le h_{y,n}$ on Ω_n for each n. Hence $\lim_{n\to\infty} h_{y,n} = h_y$, say, is harmonic on Ω_∞ and $h_{y,\infty} \le h_y \le U_y$ there. By definition of $h_{y,\infty}$, it follows that $h_y = h_{y,\infty}$, and so $\lim_{n\to\infty} G_{\Omega_n} = G_{\Omega_\infty}$.

Next suppose that Ω_∞ is not Greenian. By Theorem 4.1.2, $N = 2$ and Ω_∞ is connected. Again $(h_{y,n})$ is decreasing. Let $h_y = \lim_{n\to\infty} h_{y,n}$. If $h_y \not\equiv -\infty$ for some $y \in \Omega_\infty$, then h_y is a harmonic minorant of U_y on Ω_∞, contrary to Theorem 4.1.2(iv). Hence $h_y \equiv -\infty$ for each $y \in \Omega_\infty$, and so $G_{\Omega_n}(\cdot, y) = U_y - h_{y,n} \to +\infty$. □

The following theorem, which will be useful in later chapters, shows the relationship between the Green function and the Kelvin transform. Notation and terminology are as in Section 1.6.

Theorem 4.1.11. *Let Ω be Greenian and let Ω^* denote its inverse with respect to $S(x_0, r)$. Then Ω^* is Greenian and*

$$G_{\Omega^*}(x,y) = \{r^{-2}\|x - x_0\|\,\|y - x_0\|\}^{2-N} G_\Omega(x^*, y^*) \qquad (x, y \in \Omega^*).$$

Proof. For each $z \in \Omega$ we can write $G_\Omega(\cdot, z)$ as $U_z - h_z$, where $h_z \in \mathcal{H}(\Omega)$. We now fix $y \in \Omega^*$ and define a function H_y on Ω^* by

$$H_y(x) = \begin{cases} h_{y^*}(x^*) - \log(r^{-2}\|x - x_0\|\,\|y - x_0\|) & (N = 2) \\ h_{y^*}(x^*)\{r^{-2}\|x - x_0\|\,\|y - x_0\|\}^{2-N} & (N \ge 3). \end{cases}$$

Noting that $x_0 \notin \Omega^*$, we see from Corollary 1.6.4 that $H_y \in \mathcal{H}(\Omega^*)$. Arguing as in (1.7.3) we find that

$$\|x - y\| = r^{-2}\|x^* - y^*\|\,\|x - x_0\|\,\|y - x_0\| \qquad (x, y \in \Omega^*), \tag{4.1.6}$$

and so

$$U_y(x) = \begin{cases} U_{y^*}(x^*) - \log(r^{-2}\|x - x_0\|\,\|y - x_0\|) & (N = 2) \\ U_{y^*}(x^*)\{r^{-2}\|x - x_0\|\,\|y - x_0\|\}^{2-N} & (N \ge 3). \end{cases}$$

In either case,

$$\begin{aligned} U_y(x) - H_y(x) &= \{r^{-2}\|x - x_0\|\,\|y - x_0\|\}^{2-N}\{U_{y^*}(x^*) - h_{y^*}(x^*)\} \\ &= \{r^{-2}\|x - x_0\|\,\|y - x_0\|\}^{2-N} G_\Omega(x^*, y^*) \qquad (4.1.7) \\ &\ge 0 \qquad (x \in \Omega^*), \end{aligned}$$

and it follows, in particular, that Ω^* is Greenian.

Now let h be any harmonic minorant of $U_y - H_y$ on Ω^*. Its image h^* under the Kelvin transform is harmonic on $\Omega\backslash\{x_0\}$, and

$$h^*(x) = r^{N-2}\|x-x_0\|^{2-N}h(x^*) \le \{r^{-1}\|y-x_0\|\}^{2-N}G_\Omega(x,y^*) \ (x \in \Omega\backslash\{x_0\})$$

by (4.1.7). If $x_0 \notin \Omega$, then $h^* \le 0$ by the definition of $G_\Omega(\cdot, y^*)$. If $x_0 \in \Omega$, then h^* is bounded above near x_0 since $y^* \ne x_0$, so the function

$$s_\varepsilon(x) = \begin{cases} h^*(x) - \varepsilon G_\Omega(x, x_0) & (x \in \Omega\backslash\{x_0\}) \\ -\infty & (x = x_0) \end{cases}$$

is a subharmonic minorant of $\{r^{-1}\|y - x_0\|\}^{2-N}G_\Omega(\cdot, y^*)$ for each $\varepsilon > 0$, and again $h^* \le 0$. In either case $h \le 0$. Hence H_y is the greatest harmonic minorant of U_y on Ω^*, and so $G_{\Omega^*}(\cdot, y) = U_y - H_y$. The result now follows from (4.1.7). □

Example 4.1.12. If $\omega = \mathbb{R}^N\backslash\overline{B(x_0,r)}$, then

$$G_\omega(x,y) = \begin{cases} \log\left(\dfrac{\|y-x_0\|}{r}\dfrac{\|x-y^*\|}{\|x-y\|}\right) & (N=2) \\ \|x-y\|^{2-N} - \left(\dfrac{r}{\|y-x_0\|}\dfrac{1}{\|x-y^*\|}\right)^{N-2} & (N \ge 3), \end{cases}$$

where y^* denotes the inverse of y with respect to $S(x_0, r)$, provided we assign the value $+\infty$ to these formulae when $x = y$. To see this, we take $\Omega = B(x_0, r)$ in Theorem 4.1.11 and use the formula for G_Ω in Theorem 4.1.5, equation (4.1.6) and the symmetry of $G_\Omega(\cdot,\cdot)$.

The above formula for G_ω can alternatively be established by an argument similar to the proof of Theorem 4.1.5.

4.2. Potentials

Definition 4.2.1. Let μ be a measure on a Greenian open set Ω. We define

$$G_\Omega\mu(x) = \int_\Omega G_\Omega(x,y)\,d\mu(y) \quad (x \in \Omega).$$

Clearly $G_\Omega\mu$ takes values in $[0, +\infty]$. The function $G_\Omega\mu$ is called a *(Green) potential* if each component of Ω contains a point at which $G_\Omega\mu$ is finite. It follows from Theorem 3.3.1 (applied to each component of Ω separately) that a potential $G_\Omega\mu$ is superharmonic on Ω. A potential on the whole of $\mathbb{R}^N$ ($N \ge 3$) is sometimes referred to as a *Newtonian potential*; it has the form

$$G\mu(x) = \int_{\mathbb{R}^N} \|x-y\|^{2-N}\,d\mu(y) \quad (x \in \mathbb{R}^N).$$

Lemma 4.2.2. *If Ω is Greenian and μ is a measure with compact support contained in Ω, then $G_\Omega\mu$ is a potential and its greatest harmonic minorant is the zero function. Further, if Ω is a ball, then $G_\Omega\mu$ has limit zero at all points of $\partial\Omega$.*

Proof. Since each component of Ω can be considered separately, it is enough to deal with the case where Ω is connected. Let K denote the support of μ, let $y_0 \in K$ and let W be a bounded connected open set such that $K \subset W$ and $\overline{W} \subset \Omega$. The functions $G_\Omega(x, \cdot)$, where $x \in \Omega\backslash W$, are positive and harmonic on W. It follows from Harnack's inequalities that there is a positive constant C such that $G_\Omega(x,y) \le CG_\Omega(x,y_0)$ whenever $x \in \Omega\backslash W$ and $y \in K$. Integration with respect to $d\mu(y)$ now yields

$$G_\Omega\mu(x) \le C\mu(K)G_\Omega(x,y_0) \qquad (x \in \Omega\backslash W). \tag{4.2.1}$$

Thus $G_\Omega\mu$ is a potential. Further, if h is a harmonic minorant of $G_\Omega\mu$ on Ω, then h is majorized by a multiple of $G_\Omega(x, y_0)$ on $\Omega\backslash W$, and hence on Ω by the minimum principle. It follows that $h \le 0$, as required. Finally, if Ω is a ball, then $G_\Omega(\cdot, y_0)$ has limit zero at all points of $\partial\Omega$, by Theorem 4.1.5. Hence $G_\Omega\mu$ has limit zero at all points of $\partial\Omega$ by (4.2.1). □

Theorem 4.2.3. *Let $G_\Omega\mu$ be a potential on a Greenian open set Ω. If W is a non-empty open subset of Ω such that $\mu(W) = 0$, then $G_\Omega\mu$ is harmonic on W.*

Proof. It is enough to show that $G_\Omega\mu$ is harmonic on any open ball B_0 which satisfies $\overline{B_0} \subset W$. Since $G_\Omega\mu$ is locally integrable, there exists x_0 in B_0 such that $G_\Omega\mu(x_0) < +\infty$. It follows from Harnack's inequalities that there is a positive constant C such that $G_\Omega(x,y) \le CG_\Omega(x_0,y)$ whenever $x \in B_0$ and $y \in \Omega\backslash W$. We can now apply Theorem 3.3.1 to see that $G_\Omega\mu$ and $-G_\Omega\mu$ both belong to $\mathcal{U}(B_0)$, and hence to $\mathcal{H}(B_0)$. □

Theorem 4.2.4. *Let μ be a measure on a connected Greenian open set Ω and let $\overline{B(z,r)} \subset \Omega$. Then $G_\Omega\mu$ is a potential if and only if*

$$\int_{\Omega\backslash B(z,r)} G_\Omega(z,y)\,d\mu(y) < +\infty. \tag{4.2.2}$$

In particular, if $\mu(\Omega) < +\infty$, then $G_\Omega\mu$ is a potential.

Proof. If $G_\Omega\mu$ is a potential, then so is $G_\Omega\left(\mu|_{\Omega\backslash B(z,r)}\right)$, and (4.2.2) then follows from Theorem 4.2.3. Conversely, if (4.2.2) holds, then $G_\Omega\left(\mu|_{\Omega\backslash B(z,r)}\right)$ is superharmonic. So also is $G_\Omega\left(\mu|_{B(z,r)}\right)$, by Lemma 4.2.2. It follows that $G_\Omega\mu$, which is the sum of the above two potentials, is superharmonic on Ω, and hence a potential. The final assertion follows using Lemma 4.1.8. □

Theorem 4.2.5. *(i) Let μ be a measure on $\mathbb{R}^N (N \ge 3)$. Then $G\mu$ is a potential if and only if*

$$\int_{\mathbb{R}^N} (1 + \|y\|)^{2-N}\,d\mu(y) < +\infty.$$

(ii) Let μ be a measure on the open unit ball B. Then $G_B\mu$ is a potential if and only if

$$\int_B (1 - \|y\|)\, d\mu(y) < +\infty.$$

(iii) Let μ be a measure on $D = \{(x_1, \ldots, x_N) : x_N > 0\}$. Then $G_D\mu$ is a potential if and only if

$$\int_D \frac{y_N}{(1 + \|y\|)^N}\, d\mu(y) < +\infty. \qquad (4.2.3)$$

Proof. Part (i) follows easily from Theorem 4.2.4 on setting $z = 0$ and $r = 1$. Part (ii) follows from Theorem 4.2.4 on setting $z = 0$, $r = 1/2$ and noting from Theorem 4.1.5 that

$$G_B(0, y) = \begin{cases} -\log \|y\| & (N = 2) \\ \|y\|^{2-N} - 1 & (N \geq 3). \end{cases}$$

In order to prove (iii) we note that G_D, as given in Theorem 4.1.6, can be rewritten as

$$G_D(x, y) = \max\{N - 2, 1\} x_N y_N \int_0^2 \{\|x - y\|^2 + 2x_N y_N t\}^{-N/2}\, dt.$$

Applying the inequality $0 \leq t \leq 2$ to the integrand above, we obtain

$$\frac{2\max\{N-2, 1\} x_N y_N}{\|\overline{x} - y\|^N} \leq G_D(x, y) \leq \frac{2\max\{N-2, 1\} x_N y_N}{\|x - y\|^N}, \qquad (4.2.4)$$

where $\overline{x} = (x_1, \ldots, x_{N-1}, -x_N)$. If we let $z = (0, \ldots, 0, 2)$, then it follows easily that (4.2.3) holds if and only if

$$\int_{D \setminus B(z,1)} G_D(z, y)\, d\mu(y) < +\infty,$$

and we can again appeal to Theorem 4.2.4. □

Theorem 4.2.6. *If $G_\Omega\mu$ is a potential on a Greenian open set Ω, then the greatest harmonic minorant of $G_\Omega\mu$ is the zero function.*

Proof. Let (K_n) be an increasing sequence of compact subsets of Ω such that $\bigcup_n K_n = \Omega$, and let h denote the greatest harmonic minorant of $G_\Omega\mu$. Clearly $h \geq 0$. Since $G_\Omega\mu = G_\Omega(\mu|_{K_n}) + G_\Omega(\mu|_{\Omega \setminus K_n})$, we see from Theorem 3.6.4 and Lemma 4.2.2 that $h \leq G_\Omega(\mu|_{\Omega \setminus K_n})$. Letting $n \to \infty$, it now follows from the monotone convergence theorem that $h \equiv 0$. □

We now briefly digress to introduce a different kind of potential on $\mathbb{R}^2$ which in some respects is analogous to a Newtonian potential.

Definition 4.2.7. If μ is a measure on $\mathbb{R}^2$ which satisfies

$$\int_{\mathbb{R}^2 \setminus B} \log \|y\|\, d\mu(y) < +\infty, \qquad (4.2.5)$$

then we call the function $U\mu$, defined by

$$U\mu(x) = -\int \log \|x - y\|\, d\mu(y) \qquad (x \in \mathbb{R}^2),$$

a *logarithmic potential.* Unlike a (Green) potential, a logarithmic potential may assume any values in the range $(-\infty, +\infty]$.

Theorem 4.2.8. *A logarithmic potential $U\mu$ is superharmonic on $\mathbb{R}^2$ and is harmonic on any open set W which satisfies $\mu(W) = 0$.*

Proof. Let B_0 be an open disc. Then there is a positive constant c_1 such that

$$-\log \|x - y\| \geq -c_1 \log(2 + \|y\|) \qquad (x \in B_0; y \in \mathbb{R}^2).$$

Since (4.2.5) holds, we see from Theorem 3.3.1 that $U\mu \in \mathcal{U}(B_0)$. Further, if $\overline{B_0} \subset W$, where W is an open set satisfying $\mu(W) = 0$, then the function $(x, y) \mapsto \log \|x - y\|$ is bounded below on $B_0 \times (\mathbb{R}^2 \setminus W)$. Since $\mu(\mathbb{R}^2) < +\infty$ by (4.2.5), we can apply Theorem 3.3.1 once more to see that $-U\mu \in \mathcal{U}(B_0)$ and hence $U\mu \in \mathcal{H}(B_0)$. The theorem now follows in view of the arbitrary nature of B_0. □

We conclude this section with some examples of potentials and logarithmic potentials.

Example 4.2.9. (i) Let $N \geq 3$ and let μ denote surface area measure on $S(0, r)$, normalized to have total mass 1. Clearly $G\mu$ is a potential, and it follows from the rotational invariance of μ that $G\mu(x)$ depends only on $\|x\|$. Further, by Theorem 4.2.3, $G\mu$ is harmonic on $\mathbb{R}^N \setminus S(0, r)$. Hence, by Theorem 1.1.2, $G\mu$ takes the constant value $G\mu(0) = r^{2-N}$ on $B(0, r)$. If $\|x\| > r$, then $G(x, \cdot)$ is harmonic on an open set containing $\overline{B(0, r)}$ and so

$$G\mu(x) = \mathcal{M}(G(x, \cdot); 0, r) = G(x, 0) = \|x\|^{2-N}.$$

Finally, by Corollary 3.2.6, $G\mu(x) = \lim_{r \to 0+} \mathcal{A}(G\mu; x, r) = r^{2-N}$ when $\|x\| = r$. Hence

$$G\mu(x) = \min\{\|x\|^{2-N}, r^{2-N}\} \qquad (x \in \mathbb{R}^N).$$

(ii) The same reasoning shows that, if $N = 2$ and μ is as above, then the logarithmic potential $U\mu$ is given by

$$U\mu(x) = \min\{-\log \|x\|, -\log r\} \qquad (x \in \mathbb{R}^2).$$

(iii) If $0 < r < R$ and μ is as above, then we can use Theorem 4.1.5 and the above reasoning to observe that

$$G_{B(0,R)}\mu(x) = \begin{cases} \min\{\log(R/\|x\|), \log(R/r)\} & (N=2) \\ \min\{\|x\|^{2-N}, r^{2-N}\} - R^{2-N} & (N \geq 3). \end{cases}$$

Example 4.2.10. Let $D = \{(x_1, \ldots, x_N) : x_N > 0\}$ and let μ denote $(N-1)$-dimensional Lebesgue measure on $\mathbb{R}^{N-1} \times \{t\}$, where $t > 0$. It follows from Theorem 4.2.5 that $G_D\mu$ is a potential, and from the translational invariance of μ that $G_D\mu(x)$ depends only on x_N. Also, $G_D\mu$ is harmonic on $\mathbb{R}^{N-1} \times (0,t)$ and $\mathbb{R}^{N-1} \times (t, +\infty)$, by Theorem 4.2.3. Hence

$$G_D\mu(x) = \begin{cases} a + bx_N & (0 < x_N < t) \\ c & (x_N = t) \\ d + fx_N & (x_N > t). \end{cases}$$

It follows from (4.2.4) and monotone convergence that

$$\frac{G_D\mu(0,\ldots,0,x_N)}{x_N \max\{N-2,1\}} \leq 2t \int_{\mathbb{R}^{N-1}} \frac{d\lambda'(y')}{\{(x_N - t)^2 + \|y'\|^2\}^{N/2}}$$
$$\to 0 \qquad (x_N \to +\infty)$$

and

$$\frac{G_D\mu(0,\ldots,0,x_N)}{x_N \max\{N-2,1\}} \to 2t \int_{\mathbb{R}^{N-1}} \frac{d\lambda'(y')}{\{t^2 + \|y'\|^2\}^{N/2}} = \sigma_N \qquad (x_N \to 0).$$

(For the last equation, see Lemma 1.7.4.) Hence $a = 0 = f$ and $b = \sigma_N \max\{N-2, 1\}$. Finally, we see from the lower semicontinuity of $G_D\mu$ that $c \leq \min\{bt, d\}$, and from Corollary 3.2.6 that $c = (bt + d)/2$. Hence $bt = c = d$, and so

$$G_D\mu(x) = \sigma_N \max\{N-2, 1\} \min\{x_N, t\} \qquad (x \in D).$$

4.3. The distributional Laplacian

We have seen that if u is a potential on a Greenian open set Ω, then u is superharmonic on Ω and the greatest harmonic minorant of u on Ω is the zero function. Here we prepare for the proof of a converse result, Theorem 4.4.1 below.

Definition 4.3.1. We use $C_0(\Omega)$ to denote the vector space of all real-valued continuous functions Ψ on $\mathbb{R}^N$ such that $\operatorname{supp}\Psi$ is a compact subset of Ω,

and use $C_0^\infty(\Omega)$ to denote the subspace of infinitely differentiable elements of $C_0(\Omega)$. If $u : \Omega \to [-\infty, +\infty]$ is locally integrable on Ω, then we define a linear functional on $C_0^\infty(\Omega)$ by

$$L_u(\Psi) = \int_\Omega u \Delta\Psi \, d\lambda \quad (\Psi \in C_0^\infty(\Omega)),$$

and call L_u the ***distributional Laplacian*** of u. If v is another locally integrable function on Ω, then clearly $L_{u+v} = L_u + L_v$ on $C_0^\infty(\Omega)$.

Theorem 4.3.2. *(i) If $u \in C^2(\Omega)$, then $L_u(\Psi) = \int_\Omega \Psi \Delta u \, d\lambda$ for each Ψ in $C_0^\infty(\Omega)$.*
(ii) If $h \in \mathcal{H}(\Omega)$, then L_h is the zero functional on $C_0^\infty(\Omega)$.
(iii) If $s \in \mathcal{S}(\Omega)$, then L_s is a positive linear functional on $C_0^\infty(\Omega)$.

Proof. Let $u \in C^2(\Omega)$, let $\Psi \in C_0^\infty(\Omega)$, and let ω be a bounded open set such that $\operatorname{supp}\Psi \subset \omega$ and $\overline{\omega} \subset \Omega$. Further, let $g \in C_0^\infty(\omega)$ be such that $g = 1$ on an open set containing $\operatorname{supp}\Psi$, and let B_0 be an open ball containing ω. If we define ug to be 0 on $\mathbb{R}^N \backslash \omega$, then Green's formula yields

$$\int_{B_0} \{(ug)\Delta\Psi - \Psi\Delta(ug)\} \, d\lambda = \int_{\partial B_0} 0 \, d\sigma,$$

and so

$$\int_\Omega u\Delta\Psi \, d\lambda = \int_\Omega \Psi \Delta u \, d\lambda. \tag{4.3.1}$$

This proves (i) and (ii).

To prove (iii), let $s \in \mathcal{S}(\Omega)$, and let Ψ, ω, g be as above. It follows from Theorem 3.3.3 that there is a decreasing sequence (s_n) in $\mathcal{S}(\omega) \cap C^\infty(\omega)$ such that $s_n \to s$ pointwise on ω. In particular, $\Delta s_n \geq 0$ on ω for each n, by Corollary 3.2.8. Thus $\int_\Omega s_n \Delta\Psi \, d\lambda \geq 0$ whenever $\Psi \geq 0$, by (4.3.1). Letting $n \to \infty$ and using the monotone convergence of $(s_n(\Delta\Psi)^+)$ and $(s_n(\Delta\Psi)^-)$, we conclude that $L_s(\Psi) \geq 0$, as required. □

Corollary 4.3.3. *If $s \in \mathcal{S}(\Omega)$, then there is a unique measure μ_s on Ω such that*

$$a_N^{-1} L_s(\Psi) = \int_\Omega \Psi \, d\mu_s \qquad (\Psi \in C_0^\infty(\Omega)),$$

where $a_N = \sigma_N \max\{1, N-2\}$.

Proof. We will make use of the smoothing functions ϕ_n defined in Section 3.3. Given any $\Psi \in C_0(\Omega)$, the functions

$$\Psi_n(x) = \int \phi_n(x - y)\Psi(y) d\lambda(y) \qquad (x \in \mathbb{R}^N)$$

belong to $C_0^\infty(\Omega)$ for all sufficiently large n, and $\Psi_n \to \Psi$ uniformly on $\mathbb{R}^N$ by the uniform continuity of Ψ. By Theorem 4.3.2, L_s is a positive linear functional on $C_0^\infty(\Omega)$. It follows easily that $(L_s(\Psi_n))$ is Cauchy, so we may define

$$L(\Psi) = \lim_{n\to\infty} a_N^{-1} L_s(\Psi_n).$$

This yields a positive linear functional on $C_0(\Omega)$ such that $L = a_N^{-1} L_s$ on $C_0^\infty(\Omega)$. Further, any positive linear functional on $C_0(\Omega)$ with this property must satisfy the equation used to define L, so L is the only such functional. The result now follows from the Riesz representation theorem (see Appendix). □

Definition 4.3.4. If $s \in \mathcal{S}(\Omega)$, then we call the measure μ_s in Corollary 4.3.3 the *Riesz measure* associated with s. If $u \in \mathcal{U}(\Omega)$, then we define the *Riesz measure* μ_u associated with u to be that associated with the subharmonic function $-u$. Thus, in all cases, the Riesz measure is a (non-negative) measure. The reason for the constant a_N in the statement of Corollary 4.3.3 will become clear in Theorem 4.3.8 below.

Theorem 4.3.5. *Let $u, v \in \mathcal{S}(\Omega)$ and suppose that $L_u = L_v$ on $C_0^\infty(\Omega)$. Then there exists h in $\mathcal{H}(\Omega)$ such that $u = v + h$ on Ω. In particular, if $u \in \mathcal{S}(\Omega)$ and L_u is the zero functional, then $u \in \mathcal{H}(\Omega)$.*

Proof. Let u and v be as in the first sentence of the theorem. Also, let ω be a bounded open set such that $\overline{\omega} \subset \Omega$ and choose n_0 in $\mathbb{N}$ such that $n_0^{-1} < \operatorname{dist}(\overline{\omega}, \mathbb{R}^N \setminus \Omega)$. (If $\Omega = \mathbb{R}^N$, then we choose $n_0 = 1$.) We define

$$T_{s,n}(x) = \int_\Omega \phi_n(x-y)s(y)\,d\lambda(y) \qquad (s \in \mathcal{S}(\Omega); n \geq n_0; x \in \omega),$$

where ϕ_n is the C^∞ function described in the introduction to Theorem 3.3.3. It follows from that theorem that $T_{s,n} \in C^\infty(\omega)$, and differentiation under the integral sign yields $\Delta T_{s,n}(x) = L_s(\phi_n(x - \cdot))$ when $x \in \omega$ since

$$\frac{\partial^2}{\partial x_i^2}\phi_n(x-y) = \frac{\partial^2}{\partial y_i^2}\phi_n(x-y) \qquad \text{for each } i.$$

Let $h_n = T_{u,n} - T_{v,n}$. Then

$$\Delta h_n(x) = L_u(\phi_n(x-\cdot)) - L_v(\phi_n(x-\cdot)) = 0 \quad (x \in \omega),$$

and so $h_n \in \mathcal{H}(\omega)$. It was shown in Theorem 3.3.3 that $T_{s,n} \downarrow s$ as $n \to \infty$. If $x \in \omega$, then it follows by monotone convergence that

$$\begin{aligned} h_n(x) = \mathcal{A}(h_n; x, r) &= \mathcal{A}(T_{u,n}; x, r) - \mathcal{A}(T_{v,n}; x, r) \\ &\to \mathcal{A}(u; x, r) - \mathcal{A}(v; x, r) \quad (n \to \infty) \qquad (4.3.2) \end{aligned}$$

for all sufficiently small values of r. Further, for fixed r, the functions $\mathcal{A}(T_{u,n};\cdot,r)$ and $\mathcal{A}(T_{v,n};\cdot,r)$ are uniformly bounded on ω since $s \leq T_{s,n} \leq T_{s,n_0}$ and s is locally integrable. Hence the function $h = \lim_{n\to\infty} h_n$ is harmonic on ω, by Theorem 1.5.8. From (4.3.2) we see that $\mathcal{A}(u; x, r) = \mathcal{A}(v; x, r) + h(x)$, and letting $r \to 0+$ we obtain $u = v + h$ on ω, by Corollary 3.2.6. The first assertion of the theorem is now established, in view of the arbitrary nature of ω. The second assertion follows from the first on letting $v \equiv 0$. □

Lemma 4.3.6. *Let $y \in \mathbb{R}^N$ and $v = U_y$. Then $L_v(\Psi) = -a_N\Psi(y)$ for each Ψ in $C_0^\infty(\Omega)$.*

Proof. Let $\Psi \in C_0^\infty(\Omega)$ and let B_0 be an open ball containing $(\operatorname{supp}\Psi) \cup \{y\}$. Then Green's formula yields

$$\begin{aligned} &\int_{B_0\setminus\overline{B(y,\varepsilon)}} U_y \Delta\Psi\,d\lambda \\ &= -\int_{S(y,\varepsilon)} \left\{ U_y(x)\langle\nabla\Psi(x), \frac{x-y}{\|x-y\|}\rangle - \Psi(x)\langle\nabla U_y(x), \frac{x-y}{\|x-y\|}\rangle \right\} d\sigma(x), \end{aligned}$$

where $\varepsilon > 0$ is small, and so

$$\begin{aligned} &\left| \int_{B_0\setminus\overline{B(y,\varepsilon)}} U_y\Delta\Psi\,d\lambda + \max\{N-2,1\}\varepsilon^{1-N}\int_{S(y,\varepsilon)} \Psi(x)\,d\sigma(x) \right| \\ &\leq \max\{\|\nabla\Psi(x)\| : x \in \overline{B(y,\varepsilon)}\} \int_{S(y,\varepsilon)} U_y(x)\,d\sigma(x) \to 0 \qquad (\varepsilon \to 0+). \end{aligned}$$

Hence

$$\begin{aligned} L_v(\Psi) &= \int_B U_y\Delta\Psi\,d\lambda \\ &= -\lim_{\varepsilon\to 0+}(a_N/\sigma_N)\varepsilon^{1-N}\int_{S(y,\varepsilon)} \Psi(x)\,d\sigma(x) \\ &= -a_N\Psi(y). \end{aligned}$$ □

Theorem 4.3.7. *Let f be a holomorphic function on a domain Ω in $\mathbb{C}$ such that $f \not\equiv 0$, and let $z_1, z_2, \ldots$ be the zeros of f with multiplicities $m_1, m_2, \ldots$. Then the Riesz measure associated with the subharmonic function $\log|f|$ is $\sum_n m_n\delta_{z_n}$, where δ_z is the unit measure concentrated at the point z.*

Proof. We can write $f(z)$ as $(z-z_n)^{m_n}g_n(z)$ on some open disc B_n centred at z_n, where g_n is holomorphic and free from zeros in B_n. Hence

$$\log|f(z)| = -m_nU_{z_n}(z) + \log|g_n(z)| \qquad (z \in B_n),$$

and $\log|g_n(z)| \in \mathcal{H}(B_n)$. It follows from Lemma 4.3.6 and Theorem 4.3.2(ii) that

$$L_{\log|f|}(\Psi) = \begin{cases} 2\pi m_n \Psi(z_n) & (\Psi \in C_0^\infty(B_n)) \\ 0 & (\Psi \in C_0^\infty(\Omega\backslash\{z_n : n \geq 1\})). \end{cases}$$

Since any $\Psi \in C_0^\infty(\Omega)$ can be written as $\Psi_0 + \cdots + \Psi_k$ for some $k \in \mathbb{N}$, where $\Psi_0 \in C_0^\infty(\Omega\backslash\{z_n : n \geq 1\})$ and $\Psi_n \in C_0^\infty(B_n)$ when $1 \leq n \leq k$, the Riesz measure associated with $\log|f|$ is as stated. □

Theorem 4.3.8. *(i) The Riesz measure associated with a potential $G_\Omega\mu$ on a Greenian open set Ω is μ.*
(ii) The Riesz measure associated with a logarithmic potential $U\mu$ on $\mathbb{R}^2$ is μ.

Proof. We write $G_\Omega(x,y)$ as $U_y(x) - h_y(x)$ and let $v_y = G_\Omega(\cdot\,,y)$. Then

$$L_{v_y}(\Psi) = L_{U_y}(\Psi) \qquad (\Psi \in C_0^\infty(\Omega)), \tag{4.3.3}$$

by Theorem 4.3.2, since $h_y \in \mathcal{H}(\Omega)$. Now let $u = G_\Omega\mu$ and $\Psi \in C_0^\infty(\Omega)$. From Fubini's theorem, (4.3.3) and Lemma 4.3.6 we obtain

$$\begin{aligned} L_u(\Psi) &= \int_\Omega \left\{ \int_\Omega G_\Omega(x,y)\,d\mu(y) \right\} \Delta\Psi(x)\,d\lambda(x) \\ &= \int_\Omega L_{v_y}(\Psi)\,d\mu(y) = -a_N \int_\Omega \Psi(y)\,d\mu(y), \end{aligned}$$

from which (i) follows.

To prove (ii) we let $\Psi \in C_0^\infty(\mathbb{R}^2)$ and choose r such that $\operatorname{supp}\Psi \subset B(0,r)$. Let $w_1 = U(\mu|_{B(0,r)})$ and $w_2 = U(\mu|_{\mathbb{R}^2\backslash B(0,r)})$. Then $U\mu = w_1 + w_2$ and $w_2 \in \mathcal{H}(B(0,r))$, so $L_{U\mu}(\Psi) = L_{w_1}(\Psi)$. Since the function $(x,y) \mapsto U_y(x)$ is bounded below on $(\operatorname{supp}\Psi) \times B(0,r)$, the argument used to prove (i) now applies also in this case. □

Corollary 4.3.9. *Let $G_\Omega\mu$ be a potential on a Greenian open set Ω and let W be an open subset of Ω. Then $G_\Omega\mu$ is harmonic on W if and only if $\mu(W) = 0$.*

Proof. Let $v = G_\Omega\mu$. If $\mu(W) = 0$, then $v \in \mathcal{H}(W)$, by Theorem 4.2.3. Conversely, if $v \in \mathcal{H}(W)$, then it follows from Theorem 4.3.2 (applied to $C_0^\infty(W)$) and Corollary 4.3.3 that $\mu_v(W) = 0$. Hence $\mu(W) = 0$, by Theorem 4.3.8. □

In the opposite direction to Corollary 4.3.3 we prove the following.

Theorem 4.3.10. *Let μ be a measure on an open set Ω. Then there exists s in $\mathcal{S}(\Omega)$ such that the Riesz measure associated with s is μ.*

Proof. Let (K_n) be a sequence of compact subsets of Ω such that $K_n \subset K_{n+1}^\circ$ for each n and such that $\bigcup_n K_n = \Omega$. Further, we choose (K_n) in such a way that, for each n, each bounded component of $\mathbb{R}^N\backslash K_n$ contains a point of $\mathbb{R}^N\backslash\Omega$. Let $\mu_1 = \mu|_{K_2}$, let $\mu_n = \mu|_{K_{n+1}\backslash K_n}$ $(n \geq 2)$ and define

$$u_n(x) = -\int U_y(x)\,d\mu_n(y) \qquad (x \in \mathbb{R}^N; n \geq 1).$$

Then $u_n \in \mathcal{S}(\mathbb{R}^N)$ $(n \geq 1)$ and $u_n \in \mathcal{H}(K_n^\circ)$ $(n \geq 2)$. It follows from Theorem 2.6.4 that, whenever $n \geq 2$, there exists h_n in $\mathcal{H}(\Omega)$ such that $|u_n - h_n| < 2^{-n}$ on K_{n-1}. We now define

$$s = u_1 + \sum_{m=2}^\infty (u_m - h_m)$$

on Ω, and observe that, if $n \geq 2$, the series

$$\sum_{m=n}^\infty (u_m - h_m)$$

converges uniformly on K_n° to a harmonic function. Hence $s \in \mathcal{S}(\Omega)$.

Let $\Psi \in C_0^\infty(\Omega)$, and choose $n \geq 3$ such that $\operatorname{supp}\Psi \in K_n^\circ$. Then

$$s = -\int_{K_n} U_y\,d\mu(y) - \sum_{m=2}^{n-1} h_m + \sum_{m=n}^\infty (u_m - h_m)$$

on Ω. Since the second and third terms on the right-hand side of this equation are harmonic on K_n°, it follows from Theorem 4.3.8 that

$$a_N^{-1} L_s(\Psi) = \int_{K_n} \Psi\,d\mu = \int_\Omega \Psi\,d\mu.$$

Hence the Riesz measure associated with s is μ. □

4.4. The Riesz decomposition

Theorem 4.4.1. (Riesz decomposition theorem) *Let u be superharmonic on a Greenian open set Ω, let μ_u denote its associated Riesz measure and suppose that u has a subharmonic minorant on Ω. Then $G_\Omega\mu_u$ is a potential on Ω and $u = G_\Omega\mu_u + h$, where h is the greatest harmonic minorant of u on Ω.*

Proof. Let (K_n) be a sequence of compact subsets of Ω such that $K_n \subset K_{n+1}^\circ$ for each n, and such that $\bigcup_n K_n = \Omega$. Further, let $\mu_u^{(n)}$ denote the restriction of μ_u to K_n. It follows from Theorem 4.3.8(i) that the distributional Laplacian of $G_\Omega\mu_u^{(n)}$ is equal to L_u on $C_0^\infty(K_n^\circ)$. Hence, by Theorem 4.3.5, there exists v_n

in $\mathcal{H}(K_n^\circ)$ such that $u = v_n + G_\Omega \mu_u^{(n)}$ on K_n°. It follows that $v_{n+1} + G_\Omega \mu_u^{(n+1)} = v_n + G_\Omega \mu_u^{(n)}$ on K_n°. Let

$$v_n'(x) = v_{n+1}(x) + \int_{K_{n+1}\backslash K_n} G_\Omega(x,y)\, d\mu_u(y) \qquad (x \in K_{n+1}^\circ).$$

Since potentials are finite almost everywhere (λ), we see that $v_n' = v_n$ almost everywhere (λ) on K_n° , and hence everywhere on K_n°, by continuity. Also, $u = v_n' + G_\Omega \mu_u^{(n)}$ on K_{n+1}° and $v_n' \in \mathcal{U}(K_{n+1}^\circ) \cap \mathcal{H}(K_n^\circ)$. Hence the function

$$v_n''(x) = \begin{cases} v_n'(x) & (x \in K_{n+1}^\circ) \\ u(x) - G_\Omega \mu_u^{(n)}(x) & (x \in \Omega \backslash K_n) \end{cases}$$

is well-defined, belongs to $\mathcal{U}(\Omega) \cap \mathcal{H}(K_n^\circ)$, and satisfies $u = v_n'' + G_\Omega \mu_u^{(n)}$ on Ω.

Let h be the greatest harmonic minorant of u on Ω (this exists by Theorem 3.6.3). Then $v_n'' - h \geq -G_\Omega \mu_u^{(n)}$, so the greatest harmonic minorant, h' say, of $v_n'' - h$ satisfies $G_\Omega \mu_u^{(n)} \geq -h'$, and hence $h' \geq 0$. It follows that

$$u - h = v_n'' - h + G_\Omega \mu_u^{(n)} \geq h' + G_\Omega \mu_u^{(n)} \geq G_\Omega \mu_u^{(n)}.$$

Letting $n \to \infty$, we see from the monotone convergence theorem that $u - h \geq G_\Omega \mu_u$, and so $G_\Omega \mu_u$ is a potential. Also, by Theorems 4.3.5 and 4.3.8, there is a harmonic function h'' on Ω such that $u = G_\Omega \mu_u + h''$. Hence $u \geq h'' \geq h$, and so $h'' = h$, by the definition of h, which proves the result. □

If the superharmonic function u in the above result is non-negative, then its greatest harmonic minorant h is certainly non-negative. This leads to the following representation result for the case where Ω is all of $\mathbb{R}^N$, a ball or a half-space.

Corollary 4.4.2. *(i) If $u \in \mathcal{U}(\mathbb{R}^N)$ $(N \geq 3)$ and $u \geq 0$, then $u = \int U_y\, d\mu_u(y) + c$, where c is a non-negative constant.*
(ii) If B_0 is an open ball, $u \in \mathcal{U}(B_0)$ and $u \geq 0$, then $u = G_{B_0}\mu_u + I_\nu$, where I_ν denotes the Poisson integral of a measure ν on ∂B_0.
(iii) If $u \in \mathcal{U}(D)$, where $D = \{(x_1, \ldots, x_N) : x_N > 0\}$ and $u \geq 0$, then

$$u(x) = G_D \mu_u(x) + \mathcal{I}_\nu(x) + c x_N \quad (x \in D),$$

where $\mathcal{I}_\nu$ denotes the Poisson integral of a measure ν on ∂D satisfying (1.7.1) and $c \geq 0$.

Proof. This follows by combining Theorem 4.4.1 with earlier results for non-negative harmonic functions on these particular open sets: see Theorem 1.2.6 for (i), Theorem 1.3.8 for (ii), and Theorem 1.7.3 for (iii). □

Corollary 4.4.3. *Let $u \in \mathcal{U}(\Omega)$ and let W be a bounded open set such that $\overline{W} \subset \Omega$. Then there exists h in $\mathcal{H}(W)$ such that $u = \int_W U_y\, d\mu_u(y) + h$ on W.*

Proof. Since W is Greenian and u is bounded below on W, the Riesz representation theorem shows that $u = G_W(\mu_u|_W) + h'$ for some h' in $\mathcal{H}(W)$. We write $G_W(\cdot, y) = U_y - h_y$ and note that the function $(x,y) \mapsto h_y(x)$ is symmetric on $W \times W$. Hence $y \mapsto h_y(x)$ is harmonic on W for each $x \in W$. Also, if V is any bounded open set satisfying $\overline{V} \subset W$, then the function $(x,y) \mapsto U_y(x)$ is bounded on $\overline{V} \times \partial W$, and it follows by applying the minimum principle on W that the function $(x,y) \mapsto h_y(x)$ is bounded on $\overline{V} \times W$. Noting that $\mu_u(W) < +\infty$, we obtain

$$u(x) = \int_W U_y(x)\, d\mu_u(y) - \int_W h_y(x)\, d\mu_u(y) + h'(x) \qquad (x \in W).$$

Since the second integral in this equation belongs to $\mathcal{H}(W)$ (see Theorem 3.3.1 and use the above boundedness property of $h_y(x)$), the result is proved. □

Corollary 4.4.4. *Let u be superharmonic on an open set which contains $\overline{B(z,r)}$. Then*

$$u(z) = \mathcal{M}(u; z, r) + \max\{N-2, 1\} \int_0^r t^{1-N} \mu_u(B(z,t))\, dt \tag{4.4.1}$$

and

$$\mathcal{M}(u; z, \varepsilon)/V_N(\varepsilon) \to \mu_u(\{z\}) \qquad (\varepsilon \to 0+), \tag{4.4.2}$$

where $V_N(t) = t^{2-N}$ if $N \geq 3$, and $V_2(t) = \log(1/t)$.

Proof. It follows from Corollary 4.4.3 that there exists h in $\mathcal{H}(B(z,r))$ such that $u = \int_{B(z,r)} U_y\, d\mu_u(y) + h$ on $B(z,r)$. Hence, by Fubini's theorem and the mean value property of harmonic functions,

$$\mathcal{M}(u; z, \rho) = \int_{B(z,r)} \mathcal{M}(U_y; z, \rho)\, d\mu_u(y) + h(z) \qquad (0 < \rho < r). \tag{4.4.3}$$

It follows from parts (i) and (ii) of Example 4.2.9 that

$$\mathcal{M}(U_y; z, \rho) = \min\{U_z(y), V_N(\rho)\}. \tag{4.4.4}$$

Let $\alpha(t) = \mu_u(B(z,t))$. If $0 < \varepsilon < \rho < r$, then we see from (4.4.3) and (4.4.4) that

$$\begin{aligned} &\mathcal{M}(u; z, \varepsilon) - \mathcal{M}(u; z, \rho) \\ &\quad = \int_\varepsilon^r V_N(t)\, d\alpha(t) + V_N(\varepsilon)\alpha(\varepsilon) - \int_\rho^r V_N(t)\, d\alpha(t) - V_N(\rho)\alpha(\rho) \\ &\quad = \int_\varepsilon^\rho V_N(t)\, d\alpha(t) + V_N(\varepsilon)\alpha(\varepsilon) - V_N(\rho)\alpha(\rho) \qquad (4.4.5) \\ &\quad = -\int_\varepsilon^\rho \alpha(t)\, dV_N(t), \end{aligned}$$

using integration by parts for Riemann–Stieltjes integrals. Letting $\rho \to r-$ and $\varepsilon \to 0+$ we obtain (4.4.1), in view of Corollary 3.2.6 and the continuity of $\mathcal{M}(u;z,\cdot)$ on $(0,r]$. Further, (4.4.5) and the monotonicity of V_N yield

$$\limsup_{\varepsilon\to 0+}\left|\frac{\mathcal{M}(u;z,\varepsilon)}{V_N(\varepsilon)}-\mu_u(\{z\})\right| \le \limsup_{\varepsilon\to 0+}\left\{\int_\varepsilon^\rho d\alpha(t)+\big(\alpha(\varepsilon)-\mu_u(\{z\})\big)\right\}$$
$$=\alpha(\rho)-\mu_u(\{z\}),$$

and since ρ can be arbitrarily small, (4.4.2) follows. □

Corollary 4.4.5. (Jensen's formula) *Let f be holomorphic on a disc $B(z,R)$, let $0<r<R$, and suppose that $f\not\equiv 0$. Then*

$$\log\big|f(z)\big| = \mathcal{M}(\log|f|;z,r)-\sum_{k=1}^{n} m_k\log(r/|z_k|),$$

where $z_1,\dots,z_n$ are the zeros of f in $B(z,r)$ with multiplicities $m_1,\dots,m_n$.

Proof. This is the special case of the previous corollary where $u=-\log|f|$: by Theorem 4.3.7,

$$\int_0^r t^{-1}\mu_u(B(z,t))dt=\sum_{k=1}^n m_k\int_{|z_k|}^r t^{-1}dt=\sum_{k=1}^n m_k\log(r/|z_k|).$$

□

Corollary 4.4.6. *Let $u\in\mathcal{U}(\Omega)$, where Ω is Greenian. Then u has a harmonic minorant if and only if $G_\Omega\mu_u$ is a potential.*

Proof. If u has a harmonic minorant, then $G_\Omega\mu_u$ is a potential, by the Riesz decomposition theorem. Conversely, if $G_\Omega\mu_u$ is a potential, then Theorems 4.3.5 and 4.3.8 show that there exists h in $\mathcal{H}(\Omega)$ such that $u=G_\Omega\mu_u+h$, and so h is a harmonic minorant of u. □

Corollary 4.4.7. *Let $u\in\mathcal{U}(\Omega)$, where $u\ge 0$ and Ω is Greenian. Then u is a potential if and only if the greatest harmonic minorant of u is the zero function.*

Proof. The Riesz decomposition theorem asserts that $u=G_\Omega\mu_u+h$, where h is the greatest harmonic minorant of u on Ω. If $h\equiv 0$, then u is clearly a potential. Conversely, if $u=G_\Omega\mu$ for some measure μ, then $\mu=\mu_u$ by Theorem 4.3.8, and so $h\equiv 0$. □

Corollary 4.4.8. *Let f be a holomorphic function on the unit disc B such that $f\not\equiv 0$, and let $z_1,z_2,\dots$ be the zeros of f with multiplicities $m_1,m_2,\dots$. The following are equivalent:*

(a) $\mathcal{M}(\log|f|;0,\cdot)$ *is bounded above on* $(0,1)$*;*
(b) $\log|f|$ *has a harmonic majorant on* B*;*
(c) $\sum_n m_n(1-|z_n|)<+\infty$*;*
(d) $\log|f(z)|=-\sum_n m_nG_B(z,z_n)+h(z)$ *for all* $z\in B$*, where* $h\in\mathcal{H}(B)$ *and* G_B *is as given in Theorem 4.1.5(i).*

Proof. We know from Corollary 3.2.9 that $\log|f|$ is subharmonic on B and harmonic on $B\backslash\{z_n:n\ge 1\}$. Further, it was shown in Theorem 4.3.7 that the Riesz measure associated with $\log|f|$ is the sum of point measures of mass m_n located at the points z_n. The equivalence of (b) and (c) now follows from Corollary 4.4.6 and Theorem 4.2.5(ii). The Riesz decomposition theorem shows that (b) implies (d), and the converse also clearly holds. Finally, (a) and (b) are equivalent by Theorem 3.6.6. □

4.5. Continuity and smoothness properties

Theorem 4.5.1. *Let $u\in\mathcal{U}(\Omega)$ and $E=(\operatorname{supp}\mu_u)\cap\Omega$. If $u|_E$ is continuous in the extended sense at a point z in E, then u is continuous at z.*

Proof. Let $z\in E$, choose ε_0 such that $\overline{B(z,\varepsilon_0)}\subset\Omega$, and define $E'=E\cap\overline{B(z,\varepsilon_0)}$. If $u(z)=+\infty$, then u has limit $+\infty$ at z by lower semicontinuity. We may therefore assume that $u(z)<+\infty$. If $0<\varepsilon<\varepsilon_0$, then by Corollary 4.4.3 there exists h_ε in $\mathcal{H}(B(z,\varepsilon))$ such that $u=v_\varepsilon+h_\varepsilon$ on $B(z,\varepsilon)$, where

$$v_\varepsilon(x)=\int_{B(z,\varepsilon)}U_y(x)\,d\mu_u(y)\quad(x\in\mathbb{R}^N).$$

We note that $\mu_u(\{z\})=0$ since $u(z)<+\infty$. Suppose that $u|_E$ is continuous at z, and let (x_n) be any sequence of points in $\Omega\backslash\{z\}$ such that $x_n\to z$. For each n we choose $x_n'\in E'$ such that $\|x_n-x_n'\|=\operatorname{dist}(x_n,E')$. If $y\in E'$, then $\|x_n-y\|\ge\|x_n-x_n'\|$, so

$$\|x_n'-y\|\le\|x_n'-x_n\|+\|x_n-y\|\le 2\|x_n-y\|.$$

Hence

$$v_\varepsilon(x_n)\le\begin{cases}v_\varepsilon(x_n')+\mu_u\big(B(z,\varepsilon)\big)\log 2 & (N=2)\\ 2^{N-2}v_\varepsilon(x_n') & (N\ge 3)\end{cases}\tag{4.5.1}$$

and $x_n'\to z$. Thus

$$0\le\liminf_{n\to\infty}u(x_n)-u(z)\le\limsup_{n\to\infty}u(x_n)-u(z)$$
$$=\limsup_{n\to\infty}v_\varepsilon(x_n)-v_\varepsilon(z)$$
$$\le\begin{cases}\mu_u\big(B(z,\varepsilon)\big)\log 2 & (N=2)\\ (2^{N-2}-1)v_\varepsilon(z) & (N\ge 3),\end{cases}$$

using (4.5.1) and the continuity of $v_\varepsilon|_E$ at z. Since this holds for arbitrarily small ε it follows that $u(x_n) \to u(z)$, as required. □

Corollary 4.5.2. *Let Ω be Greenian and let μ be a measure with support K, where K is a compact subset of Ω. If $G_\Omega\mu$ is finite-valued then, for each positive number ε, there is a compact subset L of K such that $\mu(K\backslash L) < \varepsilon$ and $G_\Omega(\mu|_L)$ is continuous.*

Proof. Let $\varepsilon > 0$ and $v = G_\Omega\mu$. It follows from Lusin's theorem (see Appendix) that there is a compact subset L of K such that $\mu(K\backslash L) < \varepsilon$ and $v|_L$ is continuous. Let $v_1 = G_\Omega(\mu|_L)$ and $v_2 = G_\Omega(\mu|_{K\backslash L})$, so that $v = v_1 + v_2$. Also, if z is a limit point of L, then using the lower semicontinuity of v_k $(k = 1, 2)$ and the continuity of $v|_L$, we obtain

$$\begin{aligned} 0 &\le \liminf_{x\to z, x\in L} v_1(x) - v_1(z) \\ &\le \limsup_{x\to z, x\in L} v_1(x) - v_1(z) \\ &= \limsup_{x\to z, x\in L} \{-v_2(x)\} + v_2(z) \le 0. \end{aligned}$$

Hence $v_1|_L$ is continuous. It follows from Theorem 4.5.1 that v_1 is continuous. □

Theorem 4.5.3. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a bounded Lebesgue integrable function with compact support. Then the function*

$$u(x) = \int U_y(x) f(y)\, d\lambda(y) \qquad (x \in \mathbb{R}^N) \tag{4.5.2}$$

belongs to $C^1(\mathbb{R}^N)$ and, for any $i \in \{1, \ldots, N\}$,

$$\frac{\partial u}{\partial x_i}(x) = \int \left\{\frac{\partial}{\partial x_i} U_y(x)\right\} f(y)\, d\lambda(y) \qquad (x = (x_1, x_2, \ldots, x_N)). \tag{4.5.3}$$

Proof. Let $z \in \mathbb{R}^N$ and $0 < \varepsilon < 1$, and define

$$u_\varepsilon(x) = \int_{B(z,\varepsilon)} U_y(x) f(y)\, d\lambda(y) \quad (x \in \mathbb{R}^N),$$

$$v_\varepsilon(x) = \int_{\mathbb{R}^N\backslash B(z,\varepsilon)} U_y(x) f(y)\, d\lambda(y) \quad (x \in \mathbb{R}^N).$$

Then $u = u_\varepsilon + v_\varepsilon$, and v_ε is harmonic on $B(z, \varepsilon)$. Thus

$$\begin{aligned} \limsup_{x\to z} |u(x) - u(z)| &\le \limsup_{x\to z} |u_\varepsilon(x) - u_\varepsilon(z)| \\ &\le 2(\sup|f|) \int_{B(z,\varepsilon)} U_y(z) d\lambda(y) \\ &\to 0 \quad (\varepsilon \to 0+), \end{aligned}$$

proving the continuity of u. Similar reasoning shows that, if w_i denotes the right-hand side of (4.5.3) (where $i = 1, 2, \ldots, N$), then w_i is continuous. Let $p_1, p_2, \ldots, p_N$ denote the standard unit basis vectors for $\mathbb{R}^N$. If $\|z - y\| \ge 2\varepsilon$, then in the case $N \ge 3$,

$$\begin{aligned} |U_y(z + \varepsilon p_i) - U_y(z)| &\le (\|z - y\| - \varepsilon)^{2-N} - \|z - y\|^{2-N} \\ &\le (N-2)\varepsilon(\|z - y\| - \varepsilon)^{1-N} \\ &\le 2^{N-1}(N-2)\varepsilon\|z - y\|^{1-N}, \end{aligned}$$

and in the case $N = 2$, similar estimates yield

$$|U_y(z + \varepsilon p_i) - U_y(z)| \le 2\varepsilon\|z - y\|^{-1}.$$

Thus

$$\frac{v_{2\varepsilon}(z + \varepsilon p_i) - v_{2\varepsilon}(z)}{\varepsilon} \to w_i(z) \qquad (\varepsilon \to 0+; i = 1, \ldots, N),$$

by dominated convergence. Also,

$$\begin{aligned} \limsup_{\varepsilon\to 0+} \left|\frac{u_{2\varepsilon}(z + \varepsilon p_i) - u_{2\varepsilon}(z)}{\varepsilon}\right| &\le \limsup_{\varepsilon\to 0+} \frac{2\sup|f|}{\varepsilon} \int_{B(z,2\varepsilon)} U_y(z)\, d\lambda(y) \\ &= 0. \end{aligned}$$

Hence $\{u(z+\varepsilon p_i) - u(z)\}/\varepsilon$ has limit $w_i(z)$ as $\varepsilon \to 0$. This shows that (4.5.3) holds and that $u \in C^1(\mathbb{R}^N)$ since w_i is continuous. □

Corollary 4.5.4. *Let f be a function in $C^k(\mathbb{R}^N)$ $(k = 1, 2, \ldots)$ with compact support. Then the function u defined by (4.5.2) is in $C^{k+1}(\mathbb{R}^N)$.*

Proof. Suppose that $k = 1$. Using the symmetry of the function $(x, y) \mapsto U_y(x)$ and integration by parts, we see from Theorem 4.5.3 that

$$\frac{\partial u}{\partial x_i}(x) = -\int \left\{\frac{\partial}{\partial y_i} U_y(x)\right\} f(y)\, d\lambda(y) = \int U_y(x) \frac{\partial f}{\partial y_i}(y) d\lambda(y)$$

for any $x \in \mathbb{R}^N$. Appealing to Theorem 4.5.3 once more, we see that $\partial u/\partial x_i \in C^1(\mathbb{R}^N)$ and

$$\frac{\partial^2 u}{\partial x_j \partial x_i}(x) = \int \left\{\frac{\partial}{\partial x_j} U_y(x)\right\} \frac{\partial f}{\partial y_i}(y)\, d\lambda(y) \qquad (i, j = 1, 2, \ldots, N).$$

In particular, $u \in C^2(\mathbb{R}^N)$. An induction argument deals with general values of k. □

The equation $\Delta u = -a_N f$ is known as Poisson's equation. (Here $a_N = \sigma_N \max\{1, N-2\}$, as before.)

Corollary 4.5.5. *Let f be a function in $C^1(\mathbb{R}^N)$ with compact support. Then (4.5.2) defines a solution of Poisson's equation.*

Proof. It follows from the preceding result that u, as defined by (4.5.2), belongs to $C^2(\mathbb{R}^N)$. Hence, by Theorem 4.3.2(i),

$$L_u(\Psi) = \int \Psi \Delta u \, d\lambda \quad (\Psi \in C_0^\infty(\mathbb{R}^N)). \tag{4.5.4}$$

Also, writing u as the difference of two Newtonian potentials (or logarithmic potentials, if $N = 2$) corresponding to the measures $f^+ d\lambda$ and $f^- d\lambda$, we see from Theorem 4.3.8 that

$$\begin{aligned} L_u(\Psi) &= -a_N \left(\int \Psi f^+ \, d\lambda - \int \Psi f^- \, d\lambda \right) \\ &= -a_N \int \Psi f \, d\lambda \quad (\Psi \in C_0^\infty(\mathbb{R}^N)). \end{aligned} \tag{4.5.5}$$

Since $\Delta u + a_N f \in C(\mathbb{R}^N)$, it follows from (4.5.4) and (4.5.5) that this function must vanish identically. □

4.6. Classical boundary limit theorems

The purpose of this section is to study the boundary behaviour of superharmonic functions on the unit ball B. We noted in Lemma 4.2.2 that, if μ has compact support contained in B, then the potential $G_B\mu$ has limit zero at all points of S. Our first step will be to show that every potential on B has zero radial limit σ-almost everywhere on S.

We begin with a covering lemma.

Lemma 4.6.1. *Let $\{B(x_\alpha, r_\alpha) : \alpha \in I\}$ be a collection of balls and let $E \subseteq \bigcup_\alpha B(x_\alpha, r_\alpha)$, where E is a bounded set and $\sup_\alpha r_\alpha < +\infty$. Then there is a countable disjoint subcollection $\{B(x_{\alpha_k}, r_{\alpha_k}) : k \geq 1\}$ of these balls such that $E \subseteq \bigcup_k B(x_{\alpha_k}, 5r_{\alpha_k})$.*

Proof. We may assume, by deleting redundant balls, that $E \cap B(x_\alpha, r_\alpha) \neq \emptyset$ for all α. We choose α_1 such that $r_{\alpha_1} \geq 2^{-1} \sup_\alpha r_\alpha$, and proceed inductively as follows. Given $\alpha_1, \alpha_2, \ldots, \alpha_k$ we choose (if possible) α_{k+1} such that

$$r_{\alpha_{k+1}} \geq 2^{-1} \sup\{r_\alpha : B(x_\alpha, r_\alpha) \cap B(x_{\alpha_j}, r_{\alpha_j}) = \emptyset \text{ for } j = 1, 2, \ldots, k\}.$$

If, for some k, there is no ball $B(x_\alpha, r_\alpha)$ which is disjoint from the balls $B(x_{\alpha_1}, r_{\alpha_1}), \ldots, B(x_{\alpha_k}, r_{\alpha_k})$, then the construction terminates at this stage, leaving us with a finite disjoint subcollection of balls. Otherwise we obtain a countably infinite disjoint subcollection. In the latter case it is clear that $r_{\alpha_k} \to 0$ as $k \to \infty$, since E is bounded.

We now fix α' such that $\alpha' \notin \{\alpha_k : k \geq 1\}$. (If this is not possible, then there is nothing further to prove.) It is clear from our choice of α_1 that $r_{\alpha_1} \geq r_{\alpha'}/2$. If $r_{\alpha_k} \geq r_{\alpha'}/2$ for all k, then our subcollection is finite and

$$B(x_{\alpha'}, r_{\alpha'}) \cap B(x_{\alpha_j}, r_{\alpha_j}) \neq \emptyset \tag{4.6.1}$$

for some j, whence

$$B(x_{\alpha'}, r_{\alpha'}) \subseteq B(x_{\alpha_j}, 5r_{\alpha_j}). \tag{4.6.2}$$

If there is a value of k for which $r_{\alpha_{k+1}} < r_{\alpha'}/2$, then we choose k_0 to be the least such value. It follows that (4.6.1) holds for some j in $\{1, 2, \ldots, k_0\}$; for, if that were not the case, then α' should have been chosen in preference to α_{k_0+1} in the construction of the subcollection. Since $r_{\alpha_j} \geq r_{\alpha'}/2$, we again obtain (4.6.2).

Finally, in view of the arbitrary nature of α', we conclude that

$$E \subseteq \bigcup_\alpha B(x_\alpha, r_\alpha) \subseteq \bigcup_k B(x_{\alpha_k}, 5r_{\alpha_k}).$$

□

Lemma 4.6.2. *The Green function for B satisfies*

$$G_B(x,y) \leq 2 \max\{1, N-2\} \frac{(1 - \|x\|)(1 - \|y\|)}{\|x - y\|^N} \quad (x, y \in B). \tag{4.6.3}$$

Further, if $N = 2$, then

$$G_B(x,y) \leq \log\left(\frac{5(1 - \|x\|)}{2\|x - y\|} \right) \quad (y \in B(x, (1 - \|x\|)/2)). \tag{4.6.4}$$

Proof. Since $\log(1+t) \leq t$ and $1 - (1+t)^{1-N/2} \leq (N/2 - 1)t$ when $t > 0$, we see from Theorem 4.1.5 (the formula involving ϕ) that

$$G_B(x,y) \leq 2^{-1} \max\{1, N-2\} \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{\|x - y\|^N}$$

and so (4.6.3) holds. Secondly, if $\|y - x\| \leq (1 - \|x\|)/2$, then $1 - \|y\| \leq 3(1 - \|x\|)/2$ and Theorem 4.1.5(i) yields

$$G_B(x,y) \leq 2^{-1} \log\left\{ 1 + 6 \frac{(1 - \|x\|)^2}{\|x - y\|^2} \right\} \leq 2^{-1} \log\left\{ \frac{25}{4} \frac{(1 - \|x\|)^2}{\|x - y\|^2} \right\}$$

when $N = 2$. Thus (4.6.4) also holds. □

Lemma 4.6.3. *If u is a positive superharmonic function on B and $a > 0$, then there is a sequence $(B(x_k, r_k))$ of balls such that*

$$\{x \in B : u(x) > a\} \subseteq \bigcup_k B(x_k, r_k) \tag{4.6.5}$$

and

$$\sum_k r_k^{N-1} \leq \frac{C}{a} u(0), \tag{4.6.6}$$

where C is a positive constant depending only on N. Further, if u is harmonic on B, then $r_k > 5(1 - \|x_k\|)$ for each k.

Proof. By the Riesz–Herglotz theorem and Riesz decomposition theorem, we can write u as the sum of a potential and a Poisson integral: $G_B\mu + I_\nu$. For each $x \in B$ we define

$$B_x = B\left(x, \frac{1 - \|x\|}{2}\right), \qquad u_1(x) = G_B(\mu|_{B_x})(x)$$

and

$$u_2(x) = G_B(\mu|_{B \setminus B_x})(x) + I_\nu(x).$$

In view of Theorem 4.2.5(ii), we obtain a finite measure μ' on $\mathbb{R}^N$ by writing

$$\mu'(E) = \max\{1, N-2\} \int_{E \cap B} (1 - \|y\|) d\mu(y) + \sigma_N^{-1} \nu(E \cap S)$$

for each Borel set E. We define $m_x(t) = \mu'(B(x,t))$.

Now suppose that $x \in B$ is such that

$$m_x(t) \leq \frac{a}{6N} t^{N-1} \qquad (t > 0). \tag{4.6.7}$$

If $N \geq 3$, then $G_B(x,y) \leq \|x - y\|^{2-N}$, so we may use integration by parts and (4.6.7) to obtain

$$\begin{aligned} u_1(x) &\leq \frac{1}{N-2} \int_{B_x} \|x-y\|^{2-N} (1 - \|y\|)^{-1} d\mu'(y) \\ &\leq \frac{2}{(N-2)(1-\|x\|)} \int_{B_x} \|x-y\|^{2-N} d\mu'(y) \\ &= \frac{2}{(N-2)(1-\|x\|)} \int_0^{(1-\|x\|)/2} t^{2-N} dm_x(t) \\ &\leq \frac{2}{1-\|x\|} \int_0^{(1-\|x\|)/2} t^{1-N} m_x(t) dt \\ &\quad + \frac{1}{N-2} \left(\frac{1-\|x\|}{2}\right)^{1-N} m_x\left(\frac{1-\|x\|}{2}\right) \\ &\leq \frac{a}{3N} < \frac{a}{3}. \end{aligned} \tag{4.6.8}$$

If $N = 2$, then we similarly use (4.6.4) to obtain

$$\begin{aligned} u_1(x) &\leq \frac{2}{1-\|x\|} \int_0^{(1-\|x\|)/2} \log\left(\frac{5(1-\|x\|)}{2t}\right) dm_x(t) \\ &\leq \frac{2}{1-\|x\|} \int_0^{(1-\|x\|)/2} t^{-1} m_x(t) dt + \frac{2\log 5}{1-\|x\|} m_x\left(\frac{1-\|x\|}{2}\right) \\ &\leq \frac{1+\log 5}{12} a < \frac{a}{3}. \end{aligned} \tag{4.6.9}$$

Also, when $N \geq 2$, we can use (1.3.1), (4.6.3) and (4.6.7) to obtain

$$\begin{aligned} u_2(x) &\leq 2(1-\|x\|) \int_{\overline{B} \setminus B_x} \|x-y\|^{-N} d\mu'(y) \\ &= 2(1-\|x\|) \int_{(1-\|x\|)/2}^{+\infty} t^{-N} dm_x(t) \\ &\leq 2N(1-\|x\|) \int_{(1-\|x\|)/2}^{+\infty} t^{-N-1} m_x(t) dt \\ &\leq \frac{a(1-\|x\|)}{3} \int_{(1-\|x\|)/2}^{+\infty} t^{-2} dt = \frac{2a}{3}. \end{aligned} \tag{4.6.10}$$

If we combine (4.6.8) – (4.6.10), we see that $u(x) = u_1(x) + u_2(x) \leq a$ whenever (4.6.7) holds.

Let $E_a = \{x \in B : u(x) > a\}$. We have shown that, if $x \in E_a$, then (4.6.7) fails and so there exists $t_x \in (0, 2)$ such that $m_x(t_x) > a t_x^{N-1}/6N$. Since $E_a \subseteq \bigcup_{x \in E_a} B(x, t_x)$, we see from Lemma 4.6.1 that there is a countable disjoint subcollection $\{B(x_k, t_{x_k}) : k \geq 1\}$ such that $E_a \subseteq \bigcup_k B(x_k, r_k)$ where $r_k = 5t_{x_k}$. Hence (4.6.5) holds. Also, it follows easily from Theorem 4.1.5 that

$$G_B(0, x) \geq \max\{1, N-2\}(1 - \|x\|) \qquad (x \in B).$$

Thus

$$\begin{aligned} u(0) &\geq \max\{1, N-2\} \int_B (1-\|x\|) d\mu(x) + \sigma_N^{-1} \nu(S) \\ &= \mu'(\overline{B}) \geq \sum_k \mu'(B(x_k, t_{x_k})) > \frac{5^{1-N}}{6N} a \sum_k r_k^{N-1}, \end{aligned}$$

and so (4.6.6) holds with $C = 5^{N-1} 6N$. Finally, if u is harmonic on B, then $\mu = 0$, so $t_x > 1 - \|x\|$ for each $x \in E_a$ and hence $r_k = 5t_{x_k} > 5(1 - \|x_k\|)$ for each k. □

Theorem 4.6.4. (Littlewood) *If u is a potential on B, then $u(rz) \to 0$ as $r \to 1-$ for σ-almost every z in S.*

Proof. We can write u as $G_B\mu$. Let $n \geq 2$ and $a > 0$, and define

$$v_n = G_B(\mu|_{B(0,1-n^{-1})}) \quad \text{and} \quad u_n = G_B(\mu|_{B\setminus B(0,1-n^{-1})}).$$

Thus $u = u_n + v_n$. By Lemma 4.2.2, v_n has limit 0 at all points of S. Let $(B(x_k, r_k))$ be a sequence of balls as in Lemma 4.6.3 for the superharmonic function u_n. By (4.6.5),

$$\limsup_{r\to 1-} u(rz) = \limsup_{r\to 1-} u_n(rz) \leq a$$

for any z in S such that the radius from 0 to z does not intersect the set $B(x_k, r_k)\setminus B(0,1/2)$ for any k. Since $u_n(0) \to 0$ as $n \to \infty$ by monotone convergence, we see from (4.6.6) that $\limsup_{r\to 1-} u(rz) \leq a$ for σ-almost every z in S. The result now follows since a can be arbitrarily small. □

In order to study the boundary behaviour of more general superharmonic functions, we now examine boundary limit properties of harmonic functions.

Definition 4.6.5. (i) If $z \in S$ and $0 < \alpha < \pi/2$, then we define the cone

$$\Gamma_{z,\alpha} = \{x \in B : \langle z - x, z\rangle > \|x - z\| \cos\alpha\}.$$

(ii) A function $f : B \to \mathbb{R}$ is said to have *non-tangential limit* l at $z \in S$ if

$$\lim_{x\to z, x\in C_{z,\alpha}} f(x) = l$$

for every α in $(0, \pi/2)$.

Theorem 4.6.6. *If f is a σ-integrable function on S, then I_f has non-tangential limit $f(z)$ at σ-almost every point z of S.*

Proof. Let $\varepsilon > 0$, $a > 0$ and $0 < \alpha < \pi/2$. It follows easily from Lusin's theorem (see Appendix) that there is a continuous function g on S such that $\int |f - g| d\sigma < \sigma_N \varepsilon$. By Lemma 4.6.3 there is a sequence $(B(x_k, r_k))$ such that $r_k > 5(1 - \|x_k\|)$,

$$\{x \in B : I_{|f-g|}(x) > a\} \subseteq \bigcup_k B(x_k, r_k)$$

and

$$\sum_k r_k^{N-1} \leq \frac{C}{a} I_{|f-g|}(0) < \frac{C\varepsilon}{a}.$$

When ε is small so are all the radii r_k, and it is not hard to see that

$$\sigma\left(\left\{z \in S : \limsup_{x\to z, x\in C_{z,\alpha}} I_{|f-g|}(x) \geq a\right\}\right) \leq C(N,\alpha) \sum_k r_k^{N-1} \leq \frac{C(N,\alpha)\varepsilon}{a}, \tag{4.6.11}$$

where $C(N,\alpha)$ is a constant depending only on N and α, not necessarily the same on any two occurrences. We also have the simple estimate

$$\sigma(\{z \in S : |f - g|(z) \geq a\}) \leq \frac{1}{a} \int_{\{|f-g|\geq a\}} |f - g| d\sigma \leq \frac{\sigma_N \varepsilon}{a}. \tag{4.6.12}$$

Further, by (1.3.4),

$$\begin{aligned}\limsup_{x\to z, x\in C_{z,\alpha}} |I_f(x) - f(z)| &= \limsup_{x\to z, x\in C_{z,\alpha}} |I_{f-g}(x) - (f-g)(z) + I_g(x) - g(z)| \\ &= \limsup_{x\to z, x\in C_{z,\alpha}} |I_{f-g}(x) - (f-g)(z)| \\ &\leq \limsup_{x\to z, x\in C_{z,\alpha}} I_{|f-g|}(x) + |f-g|(z).\end{aligned}$$

Hence, by (4.6.11) and (4.6.12),

$$\sigma\left(\left\{z \in S : \limsup_{x\to z, x\in C_{z,\alpha}} |I_f(x) - f(z)| \geq 2a\right\}\right) \leq \frac{C(N,\alpha)\varepsilon}{a}.$$

Letting $\varepsilon \to 0$, we see that

$$\limsup_{x\to z, x\in C_{z,\alpha}} |I_f(x) - f(z)| < 2a \quad \text{for } \sigma\text{-almost every } z \in S.$$

Since a can be arbitrarily small, and α is arbitrary in $(0, \pi/2)$, the theorem follows. □

We know from Corollary 1.3.10 that any bounded harmonic function on B can be expressed as the Poisson integral of a σ-integrable boundary function f. The above theorem shows that such a harmonic function has finite non-tangential limits almost everywhere on S, and identifies the limits with the values of f almost everywhere. In view of the Riesz–Herglotz theorem the next result shows that any positive harmonic function on B also has finite non-tangential limits almost everywhere on S. We refer to the Appendix for the terminology of the next result, which arises out of the Radon–Nikodým theorem.

Theorem 4.6.7. (Fatou) *Let $h = I_\mu$, where μ is a measure on S, and let f denote the Radon–Nikodým derivative of the absolutely continuous component of μ with respect to σ. Then h has non-tangential limit $f(z)$ at σ-almost every z in S.*

Proof. By the Lebesgue decomposition theorem and the Radon–Nikodým theorem we can write $d\mu$ as $f d\sigma + d\nu$, where ν is singular with respect to σ. Let u denote the non-negative superharmonic function $\min\{1, I_\nu\}$. By Corollary 1.3.10 and Theorem 1.3.8 we can write the greatest harmonic minorant of u on B as I_g for some non-negative σ-integrable function g. Then

$I_{\nu-g\sigma} = I_\nu - I_g \geq 0$, so $\nu - g\sigma \geq 0$, and it follows that $g = 0$ almost everywhere (σ). Thus u is a potential on B. By Littlewood's theorem u, and hence I_ν, has radial limit 0 almost everywhere (σ) on S. By Harnack's inequalities it follows easily that I_ν has non-tangential limit 0 at the same points of S. The result is now established, in view of Theorem 4.6.6. □

Corollary 4.6.8. *If u is a positive superharmonic function on B, then $u(rz)$ has a finite limit as $r \to 1-$ for σ-almost every z in S.*

Proof. This follows immediately from Fatou's theorem and Littlewood's theorem in view of the Riesz decomposition theorem. □

4.7. Exercises

Exercise 4.1. Let $B_+ = \{x \in B : x_N > 0\}$. Write down the Green function for B_+ in terms of the Green function for B.

Exercise 4.2. Let $D = \mathbb{R}^{N-1} \times (0, +\infty)$ and $0 < p \leq (N-1)/N$. Use the inequalities (4.2.4) to show that for each $y \in D$ the function

$$t \mapsto \int_{\mathbb{R}^{N-1}} (G_D((x',t),y))^p d\lambda'(x')$$

is identically $+\infty$ on $(0,+\infty)$. Using Lemma 4.1.8, deduce that if u is positive and superharmonic on D, then

$$\int_{\mathbb{R}^{N-1}} (u(x',t))^p \, d\lambda'(x') = +\infty \qquad (t > 0).$$

Exercise 4.3. Evaluate the distributional Laplacian of the subharmonic function $s(x) = e^{|x_1|} \sin x_2$ on $\mathbb{R} \times (0, \pi)$.

Exercise 4.4. Let $u \in \mathcal{U}(B)$, where $u \geq 0$. Show that u is a potential on B if and only if $\mathcal{M}(u; 0, r) \to 0$ as $r \to 1-$.

Exercise 4.5. Let (u_n) be an increasing sequence of potentials on B. Show that $\lim u_n$ is a potential if and only if $\mathcal{M}(u_n; 0, r) \to 0$ as $r \to 1-$ uniformly in n. Write down an increasing sequence of potentials (v_n) on B such that $\lim v_n \equiv 1$.

Exercise 4.6. Let $u = K(\cdot, y)$, where K is the Poisson kernel of B and $y \in S$. Show that if $a > 0$, then $\min\{u, a\}$ is a potential on B.

Exercise 4.7. Let u be a potential on a Greenian open set Ω. By considering subharmonic minorants, show that if $0 < p < 1$, then u^p is a potential on Ω.

Exercise 4.8. Show that every positive superharmonic function on a Greenian open set Ω is the limit of an increasing sequence of potentials on Ω.

Exercise 4.9. Let Ω be a Greenian open set and let μ be a measure with compact support in Ω. Show that there exists an increasing sequence (u_n) of potentials belonging to $C^\infty(\Omega)$ such that $u_n \to G_\Omega\mu$ on Ω as $n \to \infty$.

Exercise 4.10. Show that if Ω is a Greenian open subset of $\mathbb{R}^2$, then the subharmonic function $\log^+ \|x\|$ has a harmonic majorant on Ω.

Exercise 4.11. Let f be a non-constant holomorphic function on the unit disc B. Use the result of Exercise 4.10 to show that if $f(B)$ is Greenian, then f belongs to the Nevanlinna class; that is,

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta < +\infty.$$

Exercise 4.12. Show that if u is a positive superharmonic function on $\mathbb{R}^N$ $(N \geq 3)$ with associated Riesz measure μ_u, then $r^{2-N}\mu_u(B(0,r)) \to 0$ as $r \to +\infty$. (Hint : use Corollary 4.4.4.)

Exercise 4.13. Let (a_n) be a sequence of complex numbers such that $0 < |a_1| \leq |a_2| \leq \ldots < 1$ and $\sum(1 - |a_n|)$ converges. Let f be the "Blaschke product" defined by $f = \prod_{n=1}^{\infty} f_n$, where

$$f_n(z) = \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z} \qquad (|z| < 1).$$

Show that the product is locally uniformly convergent on the unit disc B, that $|f| < 1$ on B, and that $-\log|f|$ is a potential on B. (Hint: show that if $|z| \leq R < 1$, then

$$\left|1 - \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}\right| \leq \frac{2(1 - |a_n|)}{1 - R}$$

and use the fact that $\prod f_n$ converges locally uniformly on B if $\sum |1 - f_n|$ converges locally uniformly on B.)

Exercise 4.14. Let $h \in \mathcal{H}(B)$. Show that the following are equivalent:
(a) there exist $h_1, h_2 \in \mathcal{H}_+(B)$ such that $h = h_1 - h_2$;
(b) $\displaystyle\int_{\{|h|<1\}} (1 - \|x\|)\|\nabla h(x)\|^2 \, d\lambda(x) < +\infty$;
(c) $\displaystyle\int_B (1 - \|x\|)\|\nabla h(x)\|^2 (1 + |h(x)|)^{-3} d\lambda(x) < +\infty$.
(Hint: compute $\Delta(1 + h^2)^{1/2}$ and $\Delta(\phi \circ h)$, where $\phi \in C^2(\mathbb{R})$ is a convex function such that $\phi(t) = |t|$ when $|t| \geq 1$. Then use Theorem 4.2.5(ii) and the subharmonic form of Corollary 4.4.6.)

Exercise 4.15. Let $h \in \mathcal{H}(B)$ and let $\psi : [0, +\infty) \to (0, +\infty)$ be a decreasing continuous function such that

$$\int_0^{+\infty} \psi(t)\, dt = +\infty.$$

Show that if

$$\int_B (1 - \|x\|)\|\nabla h(x)\|^2 (\psi \circ |h|)(x)\, d\lambda(x) < +\infty,$$

then h is the Poisson integral of some integrable function on S. (Hint: define

$$\phi(t) = \int_0^{|t|} \int_0^{\tau} \psi(r)\, dr\, d\tau,$$

compute $\Delta(\phi \circ h)$, and use Corollary 4.4.6 and Theorem 1.3.9.)

Exercise 4.16. Let u be a positive superharmonic function on $\mathbb{R}^N$ $(N \geq 3)$ which is harmonic on B. Show that

$$(1 + \|x\|)^{2-N} u(0) \leq u(x) \leq (1 - \|x\|)^{2-N} u(0) \qquad (x \in B)$$

and deduce that

$$\|\nabla u(0)\| \leq (N - 2) u(0).$$

(Compare the Harnack inequalities in Section 1.4.)

Exercise 4.17. Suppose that $0 < \alpha < N$ and let μ be a measure with compact support in $\mathbb{R}^N$. Define

$$U_{\alpha,\mu}(x) = \int_{\mathbb{R}^N} \|x - y\|^{\alpha - N} d\mu(y) \qquad (x \in \mathbb{R}^N).$$

Show that $U_{\alpha,\mu} \in \mathcal{U}(\mathbb{R}^N)$ if $\alpha > 2$ and $U_{\alpha,\mu} \in \mathcal{S}(\mathbb{R}^N \setminus \operatorname{supp} \mu)$ if $\alpha < 2$.

Exercise 4.18. Let u be a positive superharmonic function on $D = \mathbb{R}^{N-1} \times (0, +\infty)$ and let $a > 0$. Show that there is a sequence $(B(x_k, r_k))$ of balls such that

$$\{x \in B \cap D : u(x) > a\} \subseteq \bigcup_k B(x_k, r_k)$$

and

$$\sum_k r_k^{N-1} \leq \frac{C}{a} u(0, \ldots, 0, 1),$$

where C is a positive constant depending only on N. Show further that, if u is harmonic on D, then it can be arranged that $r_k > 5(1 - \|x_k\|)$ for each k.

Exercise 4.19. Show that, if u is a potential on D, then

$$u(x', x_N) \to 0 \qquad (x_N \to 0+)$$

for λ'-almost every $(x', 0)$ in ∂D. (Recall that λ' denotes $(N-1)$-dimensional Lebesgue measure on ∂D.)

Exercise 4.20. A function $g : D \to [-\infty, +\infty]$ is said to have *non-tangential limit* l at $y \in \partial D$ if

$$\lim_{x \to y,\, x \in \Gamma'_{y,a}} g(x) = l$$

for every $a > 0$, where

$$\Gamma'_{y,a} = \{(x', x_N) : x_N > a\|x' - y'\|\} \qquad (y = (y', 0)).$$

Prove that, if $f : \partial D \to [-\infty, +\infty]$ satisfies

$$\int_{\partial D} \frac{|f(y)|}{1 + \|y\|^N} d\lambda'(y) < +\infty,$$

then the half-space Poisson integral $\mathcal{I}_f$ has non-tangential limit $f(y)$ at λ'-almost every $y \in \partial D$.

Chapter 5. Polar Sets and Capacity

5.1. Polar sets

Sets on which a superharmonic function can have the value $+\infty$ are called polar. Since superharmonic functions are locally integrable, such sets must be of Lebesgue measure zero. Indeed, polar sets are the negligible sets of potential theory and will be seen to play a role reminiscent of that played by sets of measure zero in integration. A useful result proved in Section 5.2 is that closed polar sets are removable singularities for lower-bounded superharmonic functions and for bounded harmonic functions. In Section 5.3 we will introduce the notion of reduced functions. Given a positive superharmonic function u on a Greenian open set Ω and $E \subseteq \Omega$, we consider the collection of all non-negative superharmonic functions v on Ω which satisfy $v \geq u$ on E. The infimum of this collection is called the reduced function of u relative to E in Ω. Some basic properties of reduced functions will be observed, including the fact that they are "almost" superharmonic. Later, in Section 5.7, deeper properties will be proved via an important result known as the fundamental convergence theorem of potential theory. Before that, however, we will develop the notion of the capacity of a set, beginning with compact sets. Taking $u \equiv 1$ and E to be compact, the above reduced function is almost everywhere equal to a potential on Ω, and the total mass of the associated Riesz measure is called the capacity of E. For arbitrary sets E, we will define inner and outer capacity and, if these are equal, will term E capacitable. It will take some effort to establish that most reasonable sets (including all Borel sets) are capacitable. In Section 5.8 we will study the related notion of logarithmic capacity for plane sets, which can be used to get around the fact that $\mathbb{R}^2$ is not Greenian. Finally, the metric size of polar sets will be studied using the notion of Hausdorff measure. Polar sets in $\mathbb{R}^N$ will be shown to have Hausdorff dimension at most $N-2$, and a result in the opposite direction will also be given.

Definition 5.1.1. A set E in $\mathbb{R}^N$ is called *polar* if there is a superharmonic function u on some open set ω such that $E \subseteq \{x \in \omega : u(x) = +\infty\}$.

It is clear from the local integrability of superharmonic functions that polar sets have zero λ-measure. Also, any subset of a polar set is polar.

Example 5.1.2. (i) Any singleton $\{y\}$ is a polar set, since U_y is a superharmonic function on $\mathbb{R}^N$ valued $+\infty$ at y. In fact, any countable set $\{y_k: k \in \mathbb{N}\}$ is polar. To see this, we note that when $N \geq 3$ the function

$$u(x) = \sum_k 2^{-k} U_{y_k}(x) \qquad (x \in \mathbb{R}^N)$$

is a potential (see Theorem 4.2.5), and when $N = 2$ the function

$$u(x) = \sum_k 2^{-k}(1 + \log^+ ||y_k||)^{-1} U_{y_k}(x) \qquad (x \in \mathbb{R}^2)$$

is a logarithmic potential (see (4.2.5)).
(ii) If $N \geq 3$, then the set $E = \{0\}^2 \times \mathbb{R}^{N-2}$ is polar, since the function

$$u(x_1, \ldots, x_N) = -\log(x_1^2 + x_2^2)$$

is superharmonic on $\mathbb{R}^N$. (Clearly u is continuous in the extended sense on $\mathbb{R}^N$, harmonic on $\mathbb{R}^N \backslash E$, and satisfies the superharmonic mean value inequality at points of E.)

In each of the above examples the function u is superharmonic on all of $\mathbb{R}^N$. The following result shows, in particular, that the function u in Definition 5.1.1 can always be chosen to have this property.

Theorem 5.1.3. *Let E be a polar set such that $E \subset \Omega$, and let $z \in \Omega \backslash E$.*
(i) If Ω is Greenian, then there is a potential $G_\Omega\mu$ valued $+\infty$ on E such that $G_\Omega\mu(z) < +\infty$ and $\mu(\Omega) < +\infty$.
(ii) If $\Omega = \mathbb{R}^2$, then there is a logarithmic potential $U\mu$ valued $+\infty$ on E such that $U\mu(z) < +\infty$.

Proof. Since E is polar, there exist an open set ω and a superharmonic function u as in the definition. We may assume that $\omega \subseteq \Omega$ and $z = 0 \notin \omega$. Let (B_k) be a sequence of open balls such that $\overline{B}_k \subset \omega$ for each k and $\bigcup_k B_k = \omega$. For each k we define a measure ν_k by

$$\nu_k(A) = \frac{\mu_u(A \cap B_k)}{\mu_u(B_k) + 1} \quad \text{for any Borel set } A,$$

where μ_u is the Riesz measure associated with u. It follows from Corollary 4.4.3 that the function $u_k = \int U_y d\nu_k(y)$ is valued $+\infty$ on $E \cap B_k$. Also, $u_k(0) < +\infty$, since $0 \notin \overline{B}_k$.
If $N \geq 3$, let

$$\mu = \sum_k 2^{-k} \frac{\nu_k}{1 + u_k(0)}.$$

Then $\mu(\Omega) \leq 1$ and $G_\Omega\mu$ is a potential on Ω (see Theorem 4.2.4) which is valued $+\infty$ on E but is finite at 0. Thus we obtain the higher dimensional case of (i).

If $N = 2$, let

$$\mu = \sum_k 2^{-k} \left\{1 + \int_{B_k} |\log ||y|| \,| d\nu_k(y)\right\}^{-1} \nu_k.$$

Then (4.2.5) holds, so the function $U\mu$ is a logarithmic potential which is valued $+\infty$ on E but is finite at 0. This proves (ii). Further, $\mu(\mathbb{R}^2) \leq 1$, so if Ω is Greenian, then $G_\Omega\mu$ is a potential by Theorem 4.2.4. Hence (i) holds, since $G_\Omega\mu$ differs from $U\mu$ by a harmonic function (see Theorem 4.3.5). □

Corollary 5.1.4. *A countable union of polar sets is polar.*

Proof. Let $\{E_k: k \in \mathbb{N}\}$ be a countable collection of polar sets. If $N \geq 3$, then for each k we can choose a potential $G\mu_k$ on $\mathbb{R}^N$ such that $G\mu_k = +\infty$ on E_k and $\mu_k(\mathbb{R}^N) \leq 1$. If $N = 2$, then for each k we can choose a logarithmic potential $U\mu_k$ such that $U\mu_k = +\infty$ on E_k and $\int \log(2 + ||x||) d\mu_k(x) \leq 1$ (see (4.2.5)). In either case we define $\mu = \sum_k 2^{-k}\mu_k$ and observe that the function $v = \int U_y d\mu(y)$ is a potential (or logarithmic potential) on $\mathbb{R}^N$ which is valued $+\infty$ on $\bigcup_k E_k$. □

Corollary 5.1.5. *(i) If Ω is connected and E is a relatively closed polar subset of Ω, then $\Omega \backslash E$ is connected.*
(ii) If Ω is a non-empty open set such that $\partial\Omega$ is polar, then Ω is connected and $\overline{\Omega} = \mathbb{R}^N$.

Proof. To prove (i), let ω be a component of $\Omega \backslash E$, let u be a superharmonic function on $\mathbb{R}^N$ which is valued $+\infty$ on E, and define

$$v(x) = \begin{cases} u(x) & (x \in \omega) \\ +\infty & (x \in \Omega \backslash \omega). \end{cases}$$

Then v is lower semicontinuous and $v(x) \geq \mathcal{M}(v; x, r)$ when $x \in \Omega$ and r is small enough. Also, $v \not\equiv +\infty$, so $v \in \mathcal{U}(\Omega)$ and, in particular, v is locally integrable on Ω. Hence $\Omega \backslash E$ can contain no component other than ω and (i) is proved.
To prove (ii), suppose that $\partial\Omega$ is polar. Then $\mathbb{R}^N \backslash \partial\Omega$ is connected, by (i). Hence $\mathbb{R}^N \backslash \overline{\Omega} = \emptyset$ and Ω is connected. □

Example 5.1.6. In view of Corollary 5.1.5(i), a non-empty relatively open subset of a hyperplane cannot be polar. In particular, any line segment in $\mathbb{R}^2$ is non-polar.

Theorem 5.1.7. *Let E be a polar set. Then:*
(i) E is contained in a G_δ polar set;
(ii) $\sigma(E \cap S(x,r)) = 0$ for any sphere $S(x,r)$;
(iii) the inverse of E with respect to any sphere is polar.

Proof. If ω and u are as in Definition 5.1.1, then

$$E \subseteq \bigcap_{n=1}^{\infty} \{x \in \omega : u(x) > n\},$$

so (i) holds. There is a superharmonic function v on $\mathbb{R}^N$ which is valued $+\infty$ on E. Since v is integrable over every sphere, (ii) holds. Finally, (iii) follows from the fact that the Kelvin transform preserves superharmonicity (see Corollary 3.3.5). □

Theorem 5.1.8. (Reciprocity theorem) *If μ and ν are measures on a Greenian set Ω, then $\int G_\Omega \mu d\nu = \int G_\Omega \nu d\mu$.*

Proof. This follows by a change in order of integration, in view of the symmetry and joint continuity of the Green function. □

We saw in Example 4.2.9 that it is possible to put a non-zero measure on a sphere in such a way that the resulting potential (or logarithmic potential) is bounded above. It is not possible to achieve this upper boundedness if the measure is to be placed on a polar set, as the following theorem shows.

Theorem 5.1.9. *Let u be a locally bounded superharmonic function on Ω. Then the associated Riesz measure μ_u has the property that $\mu_u(E) = 0$ for each Borel polar subset E of Ω.*

Proof. Let E be a Borel subset of Ω which is also polar, let U be a bounded open set such that $\overline{U} \subset \Omega$, and let $a = \inf_U u$. There is a potential $G_U \nu$ on U such that $G_U \nu = +\infty$ on $U \cap E$ and $\nu(U) < +\infty$. The Riesz decomposition theorem shows that $u - a \geq G_U \mu_u$ on U. Hence

$$(+\infty)\mu_u(U \cap E) \leq \int_U G_U \nu d\mu_u = \int_U G_U \mu_u d\nu \leq \nu(U) \sup_U (u - a) < +\infty,$$

by the reciprocity theorem, and so $\mu_u(U \cap E) = 0$. Since this is true for all such open sets U, we see that $\mu_u(E) = 0$. □

Definition 5.1.10. If a proposition $P(x)$, concerning a point x in a set A, is true for all x in A apart from a polar set, then $P(x)$ is said to hold *quasi-everywhere (q.e.)* on A, or for *quasi-every* point x of A.

Theorem 5.1.11. (Maria–Frostman domination principle) *Let Ω be Greenian and let $G_\Omega \mu$ be a finite-valued potential on Ω. If u is a positive superharmonic function on Ω and $u \geq G_\Omega \mu$ quasi-everywhere on $\Omega \cap \operatorname{supp} \mu$, then $u \geq G_\Omega \mu$ on Ω.*

Proof. There is a polar subset F of $\Omega \cap \operatorname{supp} \mu$ such that $u \geq G_\Omega \mu$ on $(\operatorname{supp} \mu) \backslash F$. Let w be a potential on Ω such that $w = +\infty$ on F and let $\varepsilon > 0$. Then $u + \varepsilon w \geq G_\Omega \mu$ on $\operatorname{supp} \mu$. Let (K_n) be an increasing sequence of compact subsets of Ω such that $\bigcup_n K_n = \Omega$, and let μ_n denote the restriction of μ to K_n. By Corollary 4.5.2 there is a compact subset $L_{n,\varepsilon}$ of $K_n \cap \operatorname{supp} \mu$ such that $\mu_n(K_n \backslash L_{n,\varepsilon}) < \varepsilon$ and such that the function $v_{n,\varepsilon} = G_\Omega (\mu_n|_{L_{n,\varepsilon}})$ is continuous on Ω. Let

$$s(x) = \begin{cases} (v_{n,\varepsilon}(x) - u(x) - \varepsilon w(x))^+ & (x \in \Omega \backslash L_{n,\varepsilon}) \\ 0 & (x \in L_{n,\varepsilon}). \end{cases}$$

Since $u + \varepsilon w \geq v_{n,\varepsilon}$ on $L_{n,\varepsilon}$, the function s is subharmonic on Ω (see Corollary 3.2.4). Further, s is a non-negative minorant of the potential $v_{n,\varepsilon}$ on Ω. Hence $s \equiv 0$, and so $u + \varepsilon w \geq v_{n,\varepsilon}$ on all of Ω. If $x \in \Omega \backslash (K_n \cap \operatorname{supp} \mu)$, we thus have

$$G_\Omega \mu_n(x) \leq u(x) + \varepsilon w(x) + \varepsilon \sup\{G_\Omega(x, y) : y \in K_n \cap \operatorname{supp} \mu\}.$$

Since ε can be arbitrarily small, we conclude that $G_\Omega \mu_n \leq u$ quasi-everywhere and hence, by (3.2.2), everywhere on Ω. Finally, if we let $n \to \infty$ and appeal to the monotone convergence theorem, we obtain $u \geq G_\Omega \mu$ on Ω. □

5.2. Removable singularity theorems

The purpose of this section is to show that polar sets are removable singularities for several classes of functions. Let f be a function on $\Omega \backslash E$ where $E^\circ = \emptyset$ and $E \subseteq \Omega$. We will say that f satisfies a given condition *near points of* E if each point of E has a neighbourhood V such that the function f satisfies the condition on $V \backslash E$.

Theorem 5.2.1. *Let E be a polar subset of Ω, where Ω is connected, and let $u: \Omega \backslash E \to (-\infty, +\infty]$ be a lower semicontinuous function which is bounded below near points of E. Suppose that $u \not\equiv +\infty$ and that, for each x in $\Omega \backslash E$, there exists $r_x > 0$ such that $u(x) \geq \mathcal{M}(u; x, r)$ whenever $0 < r < r_x$. Then u has a unique superharmonic extension $\overline{u}$ to Ω. Further,*

$$\overline{u}(x) = \liminf_{y \to x, y \in \Omega \backslash E} u(y) \qquad (x \in E). \tag{5.2.1}$$

Proof. The uniqueness is immediate from the fact that any two such extensions would be equal almost everywhere (λ). To prove existence, it is enough to deal with the case where Ω is a ball and $u > 0$. Let $z \in \Omega \backslash E$. Then, by Theorem 5.1.3, there is a positive superharmonic function v on Ω such that $v = +\infty$ on E and $v(z) < +\infty$. For each n in $\mathbb{N}$ we define $u_n = u + v/n$ on $\Omega \backslash E$ and $u_n = +\infty$ on E. Each function u_n has limit $+\infty$ at every point of E, and it follows easily from the hypotheses on u that $u_n \in \mathcal{U}(\Omega)$. Let $w = \inf_n u_n$. Clearly the lower semicontinuous regularization $\hat{w}$ of w satisfies

$u \le \hat{w} \le w$ on $\Omega\backslash E$. Theorem 3.7.5 shows that $\hat{w} \in \mathcal{U}(\Omega)$, that $\hat{w} = w = u$ almost everywhere (λ), and that

$$\hat{w}(x) = \liminf_{y \to x} w(y) \qquad (x \in \Omega). \tag{5.2.2}$$

Clearly $w(z) = u(z)$, so $\hat{w}(z) = u(z)$. If we replace z by another point z_1 in $\Omega\backslash E$ in the above argument, we obtain another function $\hat{w}_1$ in $\mathcal{U}(\Omega)$ such that $\hat{w}_1 = u = \hat{w}$ almost everywhere (λ), and hence $\hat{w}_1 = \hat{w}$ everywhere, on Ω. In particular, $\hat{w}(z_1) = \hat{w}_1(z_1) = u(z_1)$. Thus $\hat{w}$ is a superharmonic extension of u to Ω. Finally, since $w = +\infty$ on E, we see from (5.2.2) and the lower semicontinuity of $\hat{w}$ that

$$\hat{w}(x) = \liminf_{y \to x, y \in \Omega\backslash E} w(y) \ge \liminf_{y \to x, y \in \Omega\backslash E} \hat{w}(y) \ge \hat{w}(x) \qquad (x \in \Omega).$$

Since $\hat{w} = u$ on $\Omega\backslash E$, this yields (5.2.1) with $\hat{w}$ in place of $\overline{u}$. □

Corollary 5.2.2. *Let E be a relatively closed polar subset of Ω. If $u \in \mathcal{U}(\Omega\backslash E)$ and u is bounded below near points of E, then u has a unique superharmonic extension to Ω.*

Proof. It is enough to deal with the case where Ω is connected, in which case the hypotheses of Theorem 5.2.1 are satisfied. □

In the above result, and the next, the hypothesis that E is polar cannot be relaxed, as will be seen in Theorem 5.3.7.

Corollary 5.2.3. *Let E be a relatively closed polar subset of Ω. If $h \in \mathcal{H}(\Omega\backslash E)$ and h is bounded near points of E, then h has a unique harmonic extension to Ω.*

Proof. It follows from Corollary 5.2.2 that h (respectively $-h$) has a unique superharmonic extension u_1 (respectively u_2) to Ω. Since $u_1 = -u_2$ almost everywhere (λ), we obtain $u_1 = -u_2$ on all of Ω and hence $u_1 \in \mathcal{H}(\Omega)$. □

Corollary 5.2.4. *Let E be a relatively closed polar subset of a Greenian set Ω. If $s \in \mathcal{S}(\Omega\backslash E)$ and s^+ has a harmonic majorant on $\Omega\backslash E$, then there exist $h_1, h_2 \in \mathcal{H}_+(\Omega)$ and measures μ_1, μ_2 on Ω, where $\mu_1(\Omega\backslash E) = 0$, such that $s = h_1 - h_2 + G_\Omega\mu_1 - G_\Omega\mu_2$ on $\Omega\backslash E$.*

Proof. Let u_1 be a harmonic majorant of s^+ and let $u_2 = u_1 - s$. By Corollary 5.2.2, there is a non-negative superharmonic extension $\overline{u}_k$ of u_k to Ω ($k = 1, 2$). Since $s = u_1 - u_2$ on $\Omega\backslash E$, the result now follows by applying the Riesz decomposition separately to $\overline{u}_1$ and $\overline{u}_2$. □

Corollary 5.2.5. *Let E be a relatively closed polar subset of Ω. If $\Omega\backslash E$ is Greenian, then so also is Ω and $G_{\Omega\backslash E}(\cdot,\cdot) = G_\Omega(\cdot,\cdot)$ on $(\Omega\backslash E) \times (\Omega\backslash E)$.*

Proof. Let $y \in \Omega\backslash E$. Any subharmonic minorant of U_y on $\Omega\backslash E$ has an extension to a subharmonic minorant of U_y on Ω by Corollary 5.2.2, so Ω is Greenian by Theorem 4.1.2(iv). Further, it follows that the greatest subharmonic minorant of U_y on $\Omega\backslash E$ coincides (on $\Omega\backslash E$) with the greatest subharmonic minorant of U_y on Ω. Hence $G_{\Omega\backslash E}(\cdot, y) = G_\Omega(\cdot, y)$ on $\Omega\backslash E$ whenever $y \in \Omega\backslash E$. □

Finally, we can give the following improved version of the maximum principle for subharmonic functions.

Theorem 5.2.6. *Let $s \in \mathcal{S}(\Omega)$, where Ω is Greenian and s is bounded above, and suppose that $\limsup_{x\to y} s(x) \le 0$ for quasi-every point y in $\partial\Omega$.*
(i) If either $N = 2$ or Ω is bounded, then $s \le 0$ on Ω.
(ii) If $N \ge 3$, Ω is unbounded and $\limsup_{x\to\infty} s(x) \le 0$, then $s \le 0$ on Ω.

Proof. Let E be the polar set of points y in $\partial\Omega$ where $\limsup_{x\to y} s(x) > 0$, and let

$$w(x) = \begin{cases} s^+(x) & (x \in \Omega) \\ 0 & (x \in \mathbb{R}^N\backslash(\Omega \cup E)). \end{cases}$$

It follows from Theorem 5.2.1 (applied to $-w$) that w has a subharmonic extension $\overline{w}$ to $\mathbb{R}^N$. If $N = 2$, then because $\overline{w}$ is bounded it is constant (see Corollary 3.5.4). Further, $\mathbb{R}^2\backslash\Omega$ is not polar in view of Corollary 5.2.5, since $\mathbb{R}^2$ is not Greenian. Hence w attains the value 0 and so $w \equiv 0$. If $N \ge 3$, then $w(x) \to 0$ as $x \to \infty$ whether Ω is bounded or not, so again $w \equiv 0$ by the subharmonic mean value inequality applied to $\overline{w}$ on large spheres. In either case we conclude that $s \le 0$ on Ω. □

5.3. Reduced functions

Throughout Sections 5.3–5.6

Ω denotes a fixed Greenian open set.

We denote by $\mathcal{U}_+(\Omega)$ the collection of all non-negative superharmonic functions on Ω.

Definition 5.3.1. If $u \in \mathcal{U}_+(\Omega)$ and $E \subseteq \Omega$, then the *reduced function* (or *réduite*) of u relative to E in Ω is defined by

$$R_u^E(x) = \inf\{v(x) : v \in \mathcal{U}_+(\Omega) \text{ and } v \ge u \text{ on } E\} \qquad (x \in \Omega).$$

It follows from Theorem 3.7.5 that the lower semicontinuous regularization $\widehat{R}_u^E$ is superharmonic on Ω, that $\widehat{R}_u^E = R_u^E$ almost everywhere (λ), and that

$$\widehat{R}_u^E(x) = \liminf_{y\to x} R_u^E(y) \qquad (x \in \Omega).$$

We call $\widehat{R}_u^E$ the *regularized reduced function* (or *balayage*) of u relative to E in Ω.

It is immediately obvious that $u \geq R_u^E \geq \widehat{R}_u^E \geq 0$ on Ω, that $u = R_u^E$ on E and that $\widehat{R}_u^E = R_u^E$ on E°.

Example 5.3.2. (i) Let $N = 2$, $\Omega = B$, $u \equiv 1$ and $E = \overline{B(0, e^{-1})}$. Then $R_u^E = \widehat{R}_u^E = w$, where $w(x) = \min\{-\log\|x\|, 1\}$ (see Figure 5.1).

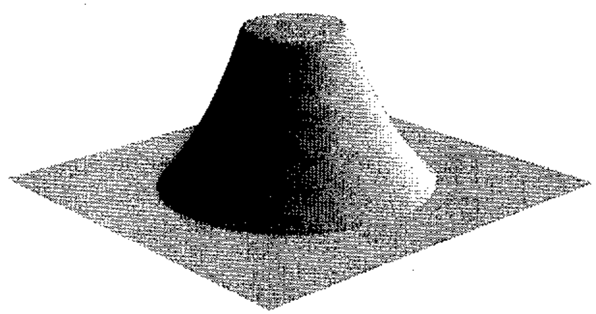

Figure 5.1.

To see this, we note that $w \in \mathcal{U}_+(B)$, since w is the minimum of two positive superharmonic functions on B, and clearly $w \geq u$ on E. Further, if $v \in \mathcal{U}_+(B)$ and $v \geq u$ on E, then $v \geq w$ on B by the minimum principle applied to $v - w$ on $B \backslash E$. Hence $R_u^E = w$ and, since R_u^E is continuous, $\widehat{R}_u^E = R_u^E$.
(ii) Let $N = 2$, $\Omega = D = \mathbb{R} \times (0, +\infty)$, $u \equiv 1$ and $E = \overline{B((0, 5/4), 3/4)}$. Then

$$R_u^E(x) = \widehat{R}_u^E(x) = \min\left\{\frac{1}{2}\log_3\left(\frac{x_1^2 + (x_2+1)^2}{x_1^2 + (x_2-1)^2}\right), 1\right\} \qquad (x \in D)$$

(see Figure 5.2), by the same reasoning as was used in (i).

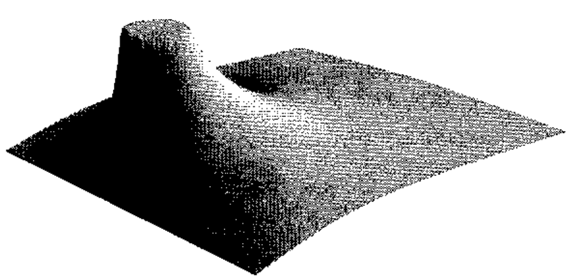

Figure 5.2.

(iii) If E is a polar set and $u \in \mathcal{U}_+(\Omega)$, then

$$R_u^E(x) = \begin{cases} u(x) & (x \in E) \\ 0 & (x \in \Omega \backslash E), \end{cases}$$

and hence $\widehat{R}_u^E \equiv 0$. To see this, let $x_0 \in \Omega \backslash E$. We can use Theorem 5.1.3 to obtain v in $\mathcal{U}_+(\Omega)$ such that $v = +\infty$ on E and $v(x_0) < +\infty$. Thus $v/n \geq u$ on E for each n in $\mathbb{N}$ and so $R_u^E(x_0) = 0$. Hence $R_u^E = 0$ on $\Omega \backslash E$, and it follows that $\widehat{R}_u^E = 0$, since $E^\circ = \emptyset$.
(iv) If $E \subseteq \Omega$ and $y \in E^\circ$, then $\widehat{R}^E_{G_\Omega(\cdot,y)} = R^E_{G_\Omega(\cdot,y)} = G_\Omega(\cdot, y)$. To see this, suppose that $v \in \mathcal{U}_+(\Omega)$ and $v \geq G_\Omega(\cdot, y)$ on E, and let B_0 be an open ball containing y such that $\overline{B_0} \subset E^\circ$. Since $v \geq G_\Omega(\cdot, y)$ on $\overline{B_0}$, this inequality holds on all of Ω by Lemma 4.1.8. Hence $\widehat{R}^E_{G_\Omega(\cdot,y)} = R^E_{G_\Omega(\cdot,y)} = G_\Omega(\cdot, y)$ on Ω.

Lemma 5.3.3. *Let $E \subseteq \Omega$. The following are equivalent:*
(a) E is polar;
(b) there is a positive superharmonic function u on Ω such that $\widehat{R}_u^E \equiv 0$;
(c) $\widehat{R}_u^E \equiv 0$ for every u in $\mathcal{U}_+(\Omega)$.

Proof. If (a) holds, then (c) follows from Example 5.3.2(iii). Clearly (c) implies (b). If (b) holds and ω is a component of Ω, then $R_u^E = 0$ almost everywhere (λ) on ω, so in particular there exists x_0 in ω such that $R_u^E(x_0) = 0$. For each n in $\mathbb{N}$ we may thus choose v_n in $\mathcal{U}_+(\Omega)$ such that $v_n \geq u$ on E and $v_n(x_0) \leq 2^{-n}$. Let $v = \sum_n v_n$. Then $v(x_0) \leq 1$, so $v \in \mathcal{U}_+(\omega)$. Since $v = +\infty$ on $E \cap \omega$, this set is polar. Hence (a) holds, in view of the arbitrary nature of ω. □

Some elementary properties of reduced functions are summarized below.

Theorem 5.3.4. *(i) If $u \leq v$ on E, then $R_u^E \leq R_v^E$ and $\widehat{R}_u^E \leq \widehat{R}_v^E$.*
(ii) If $E \subseteq F$, then $R_u^E \leq R_u^F$ and $\widehat{R}_u^E \leq \widehat{R}_u^F$.
(iii) The functions R_u^E and $\widehat{R}_u^E$ are equal and harmonic on $\Omega \backslash \overline{E}$.
(iv) If F is a polar set, then $\widehat{R}_u^{E \cup F} = \widehat{R}_u^E$.
(v) If ω is an open set, then $\widehat{R}_u^\omega = R_u^\omega$.
(vi) If u is finite-valued and continuous on an open set containing E, then

$$R_u^E = \inf\{R_u^A : A \text{ is an open set which contains } E\}. \tag{5.3.1}$$

Proof. Assertions (i) and (ii) are immediate from Definition 5.3.1.
Let

$$\mathcal{F} = \{v|_{\Omega \backslash \overline{E}} : v \in \mathcal{U}_+(\Omega) \text{ and } v \geq u \text{ on } E\}.$$

Then $\mathcal{F}$ is a saturated family and hence $R_u^E \in \mathcal{H}(\Omega\backslash\overline{E})$, by Theorem 3.6.2, so $\widehat{R}_u^E = R_u^E$ on $\Omega\backslash\overline{E}$ and (iii) holds.

If F is a polar set, then we can choose w in $\mathcal{U}_+(\Omega)$ such that $w = +\infty$ on F. If $v \in \mathcal{U}_+(\Omega)$ and $v \geq u$ on E, then $v + w/n \geq u$ on $E \cup F$ and hence $v + w/n \geq R_u^{E\cup F}$ for each n in $\mathbb{N}$. If we let $n \to \infty$ and take the infimum over all possible choices of v, we see that $R_u^E \geq R_u^{E\cup F}$ on the set where $w < +\infty$. In view of (ii) we conclude that $R_u^E = R_u^{E\cup F}$ almost everywhere (λ). Hence $\widehat{R}_u^E = \widehat{R}_u^{E\cup F}$ almost everywhere (λ) and so everywhere on Ω, proving (iv).

If ω is an open set, then $\widehat{R}_u^\omega = R_u^\omega = u$ on ω. Since $\widehat{R}_u^\omega$ is superharmonic, it follows that $\widehat{R}_u^\omega \geq R_u^\omega$. The reverse inequality is always true, so (v) holds.

Finally, if $v \in \mathcal{U}_+(\Omega)$ and $v \geq u$ on E, and if $n \in \mathbb{N}$, then $v + 1/n > u$ on some open set A which contains E. Hence $v + 1/n$ majorizes the function on the right-hand side of (5.3.1). If we take the infimum over all possible choices of v and n, we see that R_u^E also majorizes the right-hand side of (5.3.1). The reverse inequality follows from (ii). □

Theorem 5.3.5. *If $\overline{E}$ is a compact subset of Ω and $u \in \mathcal{U}_+(\Omega)$, then $\widehat{R}_u^E$ is a potential.*

Proof. We may assume that Ω is connected. Let K and L be compact sets such that $\overline{E} \subset K^\circ, K \subset L^\circ$ and $L \subset \Omega$, let $y \in \Omega$ and let $v = \widehat{R}_u^K$. Since $v \in \mathcal{H}(\Omega\backslash K)$, we can choose a positive constant a such that $aG_\Omega(\cdot, y) \geq v$ on ∂L. If we define

$$w(x) = \begin{cases} v(x) & (x \in L) \\ \min\{aG_\Omega(x,y), v(x)\} & (x \in \Omega\backslash L), \end{cases}$$

then $w \in \mathcal{U}_+(\Omega)$ by Corollary 3.2.4, and $w = v \geq u$ on E. Hence $\widehat{R}_u^E \leq w$. Any harmonic minorant h of $\widehat{R}_u^E$ in Ω must thus satisfy $h \leq aG_\Omega(\cdot, y)$ on $\Omega\backslash L$, and so on Ω, by the maximum principle. Hence $h \leq 0$ and $\widehat{R}_u^E$ is a potential on Ω. □

Lemma 5.3.6. *(i) Let $u \in \mathcal{U}_+(\Omega)$. If (K_n) is an increasing sequence of compact sets and the set $\omega = \bigcup_n K_n$ is open, then $\widehat{R}_u^{K_n} \to R_u^\omega$.*
(ii) Let $u: \Omega \to (0, +\infty)$ be continuous and superharmonic. If (K_n) is a decreasing sequence of compact sets and $K = \bigcap_n K_n$, then $R_u^{K_n} \to R_u^K$.

Proof. (i) Clearly the sequence $(\widehat{R}_u^{K_n})$ is increasing, so the limit function v is superharmonic and satisfies $v \leq R_u^\omega$. Since $\widehat{R}_u^{K_n} = R_u^{K_n} = u$ almost everywhere (λ) on K_n for each n, it follows that $v = u$ almost everywhere (λ) on ω, and hence everywhere on ω. Thus $v \geq R_u^\omega$ and (i) is established.

(ii) This follows easily from Theorem 5.3.4(vi). □

We will now show that we cannot relax the requirement that E be polar in Corollaries 5.2.2 and 5.2.3.

Theorem 5.3.7. *(i) Let E be a relatively closed non-polar subset of Ω. Then there is a bounded continuous potential $G_\Omega\mu$ on Ω, where $\mu \not\equiv 0$ and $\operatorname{supp}\mu$ is a compact subset of E. In particular, if E is compact, then there is a bounded harmonic function on $\Omega\backslash E$ which does not have a superharmonic extension to Ω.*
(ii) Let E be a closed non-polar set in $\mathbb{R}^2$. Then there is an upper-bounded continuous logarithmic potential $U\mu$, where $\mu \not\equiv 0$ and $\operatorname{supp}\mu$ is a compact subset of E.

Proof. We can choose a compact non-polar subset K of E, in view of Corollary 5.1.4. To prove (i), let 1 denote the constant function of that value on Ω. We note that $\widehat{R}_1^K$ is a non-zero bounded potential (see Lemma 5.3.3 and Theorem 5.3.5) which is harmonic on $\Omega\backslash K$. Thus we can write it as $G_\Omega\nu$, where $\operatorname{supp}\nu \subseteq K$ by Corollary 4.3.9. By Corollary 4.5.2 there is a compact subset L of K such that the function $u = G_\Omega(\nu|_L)$ is a non-zero continuous potential on Ω. The particular case follows by considering the function $-u$: it is bounded and harmonic on $\Omega\backslash E$. Further, if v is a superharmonic extension of $-u$ to Ω, then $v + u$ is superharmonic on Ω and valued 0 outside E, so $v + u \equiv 0$ by the minimum principle. This yields the contradictory conclusion that $u \in \mathcal{S}(\Omega)$. This proves (i).

To prove (ii), let Ω be an open disc such that $K \subset \Omega$, and let $G_\Omega\mu$ be the potential in (i). Then $U\mu = G_\Omega\mu + h$ on Ω for some $h \in \mathcal{H}(\Omega)$ by Theorem 4.3.5, so (ii) follows. □

Theorem 5.3.8. (Myrberg) *Let Ω_0 be a non-empty open set in $\mathbb{R}^2$. The following are equivalent:*
(a) Ω_0 is Greenian;
(b) $\mathbb{R}^2\backslash\Omega_0$ is non-polar;
(c) $\log^+\|x\|$ has a harmonic majorant on Ω_0;
(d) $\mathcal{U}_+(\Omega_0)$ contains a non-constant function.

Proof. All four assertions clearly hold if Ω_0 is bounded, so we assume the opposite. If (a) holds, then (b) follows from Corollary 5.2.5. If (b) holds then, by the preceding result, there is a non-zero measure μ with compact support K contained in $\mathbb{R}^2\backslash\Omega_0$ such that $U\mu$ is bounded above, by a say. Clearly

$$U\mu(x) + \mu(K)\log\|x\| \to 0 \qquad (x \to \infty).$$

Hence $a + 1 - U\mu$ is a positive harmonic function on Ω_0 which majorizes a positive multiple of $\log^+\|x\|$, and so (c) holds. Clearly (c) implies (d).

It remains to show that (d) implies (a). In view of Theorem 4.1.2(v) we may assume that Ω_0 is connected. Let u be a non-constant member of $\mathcal{U}_+(\Omega_0)$. We may assume (replacing u by $\min\{u, b\}$ for a suitable constant b, if necessary) that $u \notin \mathcal{H}(\Omega_0)$. Thus we can find a compact subset K of Ω_0 such that $\mu_u(K) > 0$. Let $y \in K$, let $\omega(n) = B(y, n) \cap \Omega_0$, and let m be large enough so that $K \subset \omega(m)$. It follows from the Riesz decomposition theorem that $u \geq G_{\omega(n)}(\mu_u|_K)$ on $\omega(n)$ whenever $n > m$. Thus (a) holds in view of Theorem 4.1.10(ii). □

5.4. The capacity of a compact set

We continue to use Ω to denote a fixed Greenian open set, and denote by 1 the constant function valued 1 on Ω.

Definition 5.4.1. Let K be a compact subset of Ω. It follows from Theorems 5.3.4(iii) and 5.3.5 that $\widehat{R}_1^K$ is a potential on Ω which is harmonic on $\Omega\backslash\partial K$. We call this function the *capacitary potential* of K. The associated Riesz measure ν_K, for which $\widehat{R}_1^K = G_\Omega\nu_K$, is called the *capacitary distribution* of K. Clearly supp $\nu_K \subseteq \partial K$. The *(Green) capacity* $\mathcal{C}(K)$ of K is defined by $\mathcal{C}(K) = \nu_K(\Omega)$. All these definitions are relative to the fixed Greenian set Ω. In the case where $\Omega = \mathbb{R}^N\,(N \geq 3)$, we also refer to $\mathcal{C}(K)$ as the *Newtonian capacity* of K. It is clear from Lemma 5.3.3 that a compact subset of Ω has zero capacity if and only if it is polar.

Lemma 5.4.2. *Let K be a compact subset of Ω. Then:*
(i) $\mathcal{C}(K) = \mathcal{C}(\partial K) = \mathcal{C}(\partial\tilde{K})$, where $\tilde{K}$ denotes the union of K with the bounded components ω of $\Omega\backslash K$ such that $\overline{\omega} \subset \Omega$;
(ii) $\mu_1(\Omega) \leq \mu_2(\Omega)$ for any measures μ_1 and μ_2 on Ω such that supp $\mu_1 \subseteq K$ *and $G_\Omega\mu_1 \leq G_\Omega\mu_2$;*
(iii) $\mathcal{C}(J) \leq \mathcal{C}(K)$ for every compact subset J of K;
(iv) $\mu_n(\Omega) \to \mu(\Omega)$ for any increasing or decreasing sequence $(G\mu_n)$ of potentials converging on $\Omega\backslash K$ to a potential $G_\Omega\mu$, where supp $\mu_n \subseteq K$ *for each n;*
(v) $\mathcal{C}(K_n) \to \mathcal{C}(K)$ for any decreasing sequence (K_n) of compact sets such that $\bigcap_n K_n = K$.

Proof. It follows from the minimum principle that, if $v \in \mathcal{U}_+(\Omega)$ and $v \geq 1$ on $\partial\tilde{K}$, then $v \geq 1$ on $\tilde{K}$ and hence on K. Thus $R_1^{\partial\tilde{K}} \geq R_1^K$. Since $\partial\tilde{K} \subseteq \partial K \subseteq K$, we obtain $\widehat{R}_1^{\partial\tilde{K}} = \widehat{R}_1^{\partial K} = \widehat{R}_1^K$, and (i) follows.

To prove (ii), let L be a compact subset of Ω such that $K \subset L^\circ$. Then $G_\Omega\nu_L = 1$ on K, so

$$\begin{aligned}\mu_1(\Omega) &= \int G_\Omega\nu_L d\mu_1 = \int G_\Omega\mu_1 d\nu_L \\ &\leq \int G_\Omega\mu_2 d\nu_L = \int G_\Omega\nu_L d\mu_2 \leq \mu_2(\Omega)\end{aligned}$$

by the reciprocity theorem. Thus (ii) holds.

If $J \subseteq K$, then $\widehat{R}_1^J \leq \widehat{R}_1^K$, so (iii) follows from (ii).

To prove (iv), let L again be a compact subset of Ω such that $K \subset L^\circ$. Since $G_\Omega\mu_n \in \mathcal{H}(\Omega\backslash K)$ for each n, we have $G_\Omega\mu \in \mathcal{H}(\Omega\backslash K)$, so supp $\mu \subseteq K$. Also, $G_\Omega\nu_L = 1$ on K and supp $\nu_L \subseteq \partial L$. Thus

$$\mu_n(\Omega) = \int G\nu_L d\mu_n = \int G\mu_n d\nu_L \to \int G\mu\, d\nu_L = \int G\nu_L d\mu = \mu(\Omega)$$

by monotone convergence and the reciprocity theorem. This proves (iv).

Finally, in (v), we know from Lemma 5.3.6(ii) that $\widehat{R}_1^{K_n} \downarrow \widehat{R}_1^K$. Hence $\mathcal{C}(K_n) \to \mathcal{C}(K)$, by (iv). □

Example 5.4.3. (i) If $\Omega = \mathbb{R}^N\,(N \geq 3)$, then

$$\mathcal{C}(\overline{B(0,r)}) = \mathcal{C}(S(0,r)) = r^{N-2}.$$

To see this, we observe from the minimum principle that, if $v \in \mathcal{U}_+(\Omega)$ and $v \geq 1$ on $\overline{B(0,r)}$, then $v(x) \geq (r/\|x\|)^{N-2}$ on $\mathbb{R}^N\backslash\overline{B(0,r)}$. Hence

$$R_1^{\overline{B(0,r)}}(x) = r^{N-2}\min\{\|x\|^{2-N}, r^{2-N}\},$$

and it follows from Example 4.2.9(i) that $\mathcal{C}(\overline{B(0,r)}) = r^{N-2}$. Thus also $\mathcal{C}(S(0,r)) = r^{N-2}$, by Lemma 5.4.2(i).
(ii) Similarly, if $\Omega = B(0,\rho)$ and $0 < r < \rho$, then it follows from Example 4.2.9(iii) that

$$\mathcal{C}(\overline{B(0,r)}) = \mathcal{C}(S(0,r)) = \begin{cases}\{\log(\rho/r)\}^{-1} & (N=2)\\ (r^{2-N}-\rho^{2-N})^{-1} & (N\geq 3).\end{cases}$$

(iii) It is clear that Newtonian capacity is invariant under translations and rotations. To see the effect of dilations, let K be a compact set in $\mathbb{R}^N\,(N \geq 3)$, let $K_a = \{ax : x \in K\}$ for each positive number a, and let ν_K denote the capacitary distribution for K. Then

$$\|x\|^{N-2}\widehat{R}_1^K(x) = \|x\|^{N-2}G\nu_K(x) \to \mathcal{C}(K) \qquad (x \to \infty)$$

by dominated convergence. It is clear from the definition of reduced functions that $R_1^{K_a}(x) = R_1^K(a^{-1}x)$. Hence

$$\mathcal{C}(K_a) = \lim_{x\to\infty}\|x\|^{N-2}\widehat{R}_1^{K_a}(x) = \lim_{x\to\infty}\|x\|^{N-2}\widehat{R}_1^K(a^{-1}x) = a^{N-2}\mathcal{C}(K).$$

(iv) Let K be a compact subset of B in $\mathbb{R}^2$, let $a > 0$ and let K_a be as in (iii). Then ${}^1\mathcal{C}(K) = {}^a\mathcal{C}(K_a)$, where ${}^a\mathcal{C}(\cdot)$ denotes capacity relative to $\Omega = B(0,a)$. To see this, let u_a denote the capacitary potential of K_a in $B(0,a)$ and let ν_a be the corresponding capacitary distribution. Clearly $u_1(x/a) = u_a(x)$ on $B(0,a)$ so, using the explicit formula for the Green function of a disc (see Theorem 4.1.5(i)), we obtain

$$\begin{aligned} u_1(x/a) &= \int G_{B(0,a)}(x,y)\,d\nu_a(y) \\ &= \int \log\left(\|y/a\|\frac{\|x/a-(y/a)^*\|}{\|x/a-y/a\|}\right)d\nu_a(y) \\ &= \int G_B(x/a,y/a)\,d\nu_a(y), \end{aligned}$$

where $y^* = \|y\|^{-2}y$, and hence

$$^1\mathcal{C}(K) = \nu_a(B(0,a)) = {}^a\mathcal{C}(K_a).$$

Theorem 5.4.4. *If K_1 and K_2 are compact subsets of Ω, then*

$$\widehat{R}_1^{K_1\cup K_2} + \widehat{R}_1^{K_1\cap K_2} \le \widehat{R}_1^{K_1} + \widehat{R}_1^{K_2}. \tag{5.4.1}$$

Proof. Let $v_k \in \mathcal{U}_+(\Omega)$ be such that $v_k \ge 1$ on K_k $(k=1,2)$. Also, let

$$u = \widehat{R}_1^{K_1\cup K_2} + \widehat{R}_1^{K_1\cap K_2},$$

which is a potential on Ω and is harmonic on $\Omega\backslash(K_1\cup K_2)$. Then

$$v_1 + v_2 \ge 2 \ge u \quad \text{on} \quad K_1\cap K_2.$$

On $K_1\backslash K_2$, we have

$$v_1 \ge 1 \ge \widehat{R}_1^{K_1\cup K_2} \quad \text{and} \quad v_2 \ge \widehat{R}_1^{K_1\cap K_2},$$

so $v_1 + v_2 \ge u$. This last inequality can similarly be shown to hold on $K_2\backslash K_1$, so it holds on all of $K_1\cup K_2$. It follows from the Maria–Frostman domination principle that $v_1+v_2 \ge u$ on Ω. Taking infima over all possible choices of v_1 and v_2, we see that

$$\widehat{R}_1^{K_1\cup K_2} + \widehat{R}_1^{K_1\cap K_2} \le R_1^{K_1} + R_1^{K_2}. \tag{5.4.2}$$

Finally, since $\widehat{R}_1^{K_1} = R_1^{K_1}$ almost everywhere (λ), we obtain (5.4.1) by taking means over a ball in (5.4.2) and letting the radius shrink to 0 (see (3.2.2)). □

Corollary 5.4.5. *If K_1 and K_2 are compact subsets of Ω, then*

$$\mathcal{C}(K_1\cup K_2) + \mathcal{C}(K_1\cap K_2) \le \mathcal{C}(K_1) + \mathcal{C}(K_2).$$

Proof. This follows from the above theorem and Lemma 5.4.2(ii). □

The property of capacity established above is called *strong subadditivity*.

5.5. Inner and outer capacity

As before, Ω denotes a fixed Greenian open set; all other sets are contained in Ω. We will now develop notions of capacity for arbitrary subsets of Ω.

Definition 5.5.1. If $E \subseteq \Omega$, then we define the *inner capacity* of E by

$$\mathcal{C}_*(E) = \sup\{\mathcal{C}(K) : K \text{ is a compact subset of } E\}$$

and the *outer capacity* of E by

$$\mathcal{C}^*(E) = \inf\{\mathcal{C}_*(\omega) : \omega \text{ is an open set containing } E\}.$$

These set functions take values in $[0,+\infty]$. If $E \subseteq F \subseteq \Omega$, then clearly $\mathcal{C}_*(E) \le \mathcal{C}_*(F)$ and $\mathcal{C}^*(E) \le \mathcal{C}^*(F)$. Further, it is easy to see from Lemma 5.4.2(iii) that $\mathcal{C}_*(E) \le \mathcal{C}^*(E)$. A set E is called *capacitable* if $\mathcal{C}_*(E) = \mathcal{C}^*(E)$. Clearly any open set is capacitable. All these definitions are relative to the fixed Greenian set Ω.

Lemma 5.5.2. *(i) Any compact set K is capacitable and $\mathcal{C}_*(K) = \mathcal{C}^*(K) = \mathcal{C}(K)$.*
(ii) If ω is a bounded open set such that $\overline{\omega} \subset \Omega$, then $\mathcal{C}_*(\omega) = \nu_\omega(\Omega)$, where ν_ω is the Riesz measure associated with the potential R_1^ω.

Proof. To prove (i), let K be compact and (K_n) be a decreasing sequence of compact sets such that $K \subset K_n^\circ$ for all n and $\bigcap_n K_n = K$. Then

$$\mathcal{C}_*(K) \le \mathcal{C}^*(K) \le \mathcal{C}_*(K_n^\circ) \le \mathcal{C}_*(K_n) = \mathcal{C}(K_n) \to \mathcal{C}(K) = \mathcal{C}_*(K)$$

by Lemma 5.4.2(v), so (i) holds.

To prove (ii), we first note from Theorems 5.3.4(v) and 5.3.5 that R_1^ω is a potential. Let (K_n) be an increasing sequence of compact sets such that $K_n \subset K_{n+1}^\circ$ for all n and $\bigcup_n K_n = \omega$. Then $\widehat{R}_1^{K_n} \uparrow R_1^\omega$, by Lemma 5.3.6(i), so

$$\mathcal{C}_*(\omega) = \lim_{n\to\infty} \mathcal{C}(K_n) = \lim_{n\to\infty} \nu_{K_n}(\Omega) = \nu_\omega(\Omega)$$

by Lemma 5.4.2(iv). □

Definition 5.5.3. If E is capacitable, then we write $\mathcal{C}(E)$ for the common value of $\mathcal{C}_*(E)$ and $\mathcal{C}^*(E)$, and call this the *capacity* of E. In view of part (i) of the above lemma, this is consistent with the earlier definition of $\mathcal{C}(K)$ for compact sets K.

Lemma 5.5.4. *(i) If (ω_n) is an increasing sequence of open sets, then*

$$\mathcal{C}(\omega_n) \to \mathcal{C}\Big(\bigcup_n \omega_n\Big).$$

(ii) If ω_1 and ω_2 are open sets, then

$$\mathcal{C}(\omega_1 \cup \omega_2) + \mathcal{C}(\omega_1 \cap \omega_2) \leq \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2). \tag{5.5.1}$$

(iii) If $\{\omega_n : n \in I\}$ is a countable collection of open sets, then

$$\mathcal{C}(\bigcup_n \omega_n) \leq \sum_n \mathcal{C}(\omega_n).$$

Proof. To prove (i), we note that $(\mathcal{C}(\omega_n))$ is increasing and that $\lim \mathcal{C}(\omega_n) \leq \mathcal{C}(\bigcup_n \omega_n)$. If K is a compact subset of $\bigcup_n \omega_n$, then $K \subset \omega_m$ for some m, and so $\mathcal{C}(K) \leq \lim \mathcal{C}(\omega_n)$. If we take the supremum over all possible choices of K, we obtain $\mathcal{C}(\bigcup_n \omega_n) \leq \lim \mathcal{C}(\omega_n)$, and (i) is proved.

To prove (ii), let K be a compact subset of $\omega_1 \cap \omega_2$ and L be a compact subset of $\omega_1 \cup \omega_2$. We choose disjoint open sets W_1 and W_2 such that

$$L \backslash \omega_2 \subset W_1 \subset \omega_1 \text{ and } L \backslash \omega_1 \subset W_2 \subset \omega_2$$

and define $L_1 = L \backslash W_2$ and $L_2 = L \backslash W_1$. Then $L_1 \subset \omega_1, L_2 \subset \omega_2$ and $L_1 \cup L_2 = L \backslash (W_1 \cap W_2) = L$. It follows from Corollary 5.4.5 that

$$\mathcal{C}(K \cup L) + \mathcal{C}((K \cup L_1) \cap (K \cup L_2)) \leq \mathcal{C}(K \cup L_1) + \mathcal{C}(K \cup L_2).$$

Since $L \subseteq K \cup L$ and $K \subseteq (K \cup L_1) \cap (K \cup L_2)$, we obtain

$$\mathcal{C}(L) + \mathcal{C}(K) \leq \mathcal{C}(K \cup L_1) + \mathcal{C}(K \cup L_2) \leq \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2).$$

If we take the supremum over all possible choices of K and L, we obtain (5.5.1).

It follows from (ii) that $\mathcal{C}(\omega_1 \cup \omega_2) \leq \mathcal{C}(\omega_1) + \mathcal{C}(\omega_2)$ and hence, by induction, that

$$\mathcal{C}\left(\bigcup_{n=1}^{m} \omega_n\right) \leq \sum_{n=1}^{m} \mathcal{C}(\omega_n).$$

A countably infinite union of open sets is dealt with by letting $m \to \infty$ and using (i). □

The following alternative characterizations of inner and outer capacity are sometimes useful.

Theorem 5.5.5. *If $\overline{E}$ is a compact subset of Ω, then:*
(i) $\mathcal{C}_*(E) = \sup\{\mu(\Omega)$: supp $\mu \subseteq E$ and $G_\Omega \mu \leq 1$ on $\Omega\}$;
(ii) $\mathcal{C}^*(E) = \inf\{\mu(\Omega) : G_\Omega \mu \geq 1$ quasi-everywhere on $E\}$.

Proof. To prove (i), let μ be a measure on Ω such that supp $\mu \subseteq E$ and $G_\Omega \mu \leq 1$, and let $K = \text{supp } \mu$. Further, let $v \in \mathcal{U}_+(\Omega)$ be such that $v \geq 1$ on K. It follows from the Maria–Frostman domination principle that $v \geq G_\Omega \mu$ on Ω. Hence $R_1^K \geq G_\Omega \mu$, and so also $\widehat{R}_1^K \geq G_\Omega \mu$. It follows from Lemma 5.4.2(ii) that $\mu(\Omega) \leq \nu_K(\Omega)$. Thus the supremum in (i) is equal to

$$\sup\{\nu_K(\Omega) : K \text{ a compact subset of } E\}$$

which, by definition, is $\mathcal{C}_*(E)$. This proves (i).

To prove (ii), let μ be a measure on Ω such that $G_\Omega \mu \geq 1$ on $E \backslash F$, where F is some polar set, and let $\varepsilon > 0$. By Theorem 5.1.3(i) there is a measure μ_1 on Ω such that $G_\Omega \mu_1 = +\infty$ on F and $\mu_1(\Omega) < \varepsilon$. Further, we can arrange that $G_\Omega \mu_1 > 0$ on each component of Ω. We can choose a bounded open set ω such that $E \subseteq \omega$ and $\overline{\omega} \subset \Omega$ and such that $G_\Omega(\mu + \mu_1) > 1$ on ω. Hence $G_\Omega(\mu + \mu_1) \geq R_1^\omega = G_\Omega \nu_\omega$, and it follows from Lemmas 5.4.2(ii) and 5.5.2(ii) that

$$\mathcal{C}(\omega) = \nu_\omega(\Omega) \leq \mu(\Omega) + \mu_1(\Omega) < \mu(\Omega) + \varepsilon.$$

Thus the infimum in (ii) is equal to

$$\inf\{\mathcal{C}(\omega) : \omega \text{ is open and } E \subseteq \omega\},$$

which, by definition, is $\mathcal{C}^*(E)$. This proves (ii). □

We now generalize Lemmas 5.5.2(ii) and 5.5.4 to deal with arbitrary subsets of Ω. Property (iii) below is referred to as *countable subadditivity.*

Theorem 5.5.6. *(i) If (E_n) is an increasing sequence of sets, then*

$$\mathcal{C}^*(E_n) \to \mathcal{C}^*(\bigcup_n E_n).$$

(ii) If E_1 and E_2 are sets, then

$$\mathcal{C}^*(E_1 \cup E_2) + \mathcal{C}^*(E_1 \cap E_2) \leq \mathcal{C}^*(E_1) + \mathcal{C}^*(E_2). \tag{5.5.2}$$

(iii) If $\{E_n : n \in I\}$ is a countable collection of sets, then

$$\mathcal{C}^*(\bigcup_n E_n) \leq \sum_n \mathcal{C}^*(E_n).$$

(iv) If $\overline{E}$ is a compact subset of Ω, then $\mathcal{C}^*(E) = \nu_E(\Omega)$, where ν_E is the Riesz measure associated with the potential $\widehat{R}_1^E$.

Proof. In proving (i) we may assume that $\mathcal{C}^*(E_n) < +\infty$ for each n, for otherwise the conclusion is trivial. Let $\varepsilon > 0$. For each n we choose an open set ω_n such that $E_n \subseteq \omega_n$ and $\mathcal{C}(\omega_n) < \mathcal{C}^*(E_n) + 2^{-n}\varepsilon$. We claim that

$$\mathcal{C}\left(\bigcup_{n=1}^{m} \omega_n\right) < \mathcal{C}^*(E_m) + (1 - 2^{-m})\varepsilon \tag{5.5.3}$$

for each m in $\mathbb{N}$. This inequality is obvious when $m = 1$. If it is true when $m = k$, then we use Lemma 5.5.4(ii) to obtain

$$\mathcal{C}\left(\bigcup_{n=1}^{k+1}\omega_n\right)+\mathcal{C}\left(\left(\bigcup_{n=1}^{k}\omega_n\right)\cap\omega_{k+1}\right)\le\mathcal{C}\left(\bigcup_{n=1}^{k}\omega_n\right)+\mathcal{C}(\omega_{k+1})$$
$$<\mathcal{C}^*(E_k)+\mathcal{C}^*(E_{k+1})+(1-2^{-k-1})\varepsilon. \qquad (5.5.4)$$

Since $E_k \subseteq E_{k+1}$, we see that

$$E_k\subseteq\left(\bigcup_{n=1}^{k}\omega_n\right)\cap\omega_{k+1},$$

and it follows easily from (5.5.4) that (5.5.3) holds when $m = k+1$. By induction (5.5.3) holds for all m in $\mathbb{N}$. We let $m \to \infty$ and use Lemma 5.5.4(i) to obtain

$$\mathcal{C}^*\left(\bigcup_{n=1}^{\infty}E_n\right)\le\mathcal{C}\left(\bigcup_{n=1}^{\infty}\omega_n\right)\le\lim_{n\to\infty}\mathcal{C}^*(E_n)+\varepsilon.$$

Since ε can be arbitrarily small we obtain

$$\mathcal{C}^*(\bigcup_{n=1}^{\infty}E_n)\le\lim_{n\to\infty}\mathcal{C}^*(E_n).$$

The reverse inequality is immediate, so (i) is proved.

To prove (ii), let ω_1 and ω_2 be open sets such that $E_1 \subseteq \omega_1$ and $E_2 \subseteq \omega_2$. Then

$$\mathcal{C}^*(E_1\cup E_2)+\mathcal{C}^*(E_1\cap E_2)\le\mathcal{C}(\omega_1\cup\omega_2)+\mathcal{C}(\omega_1\cap\omega_2)\le\mathcal{C}(\omega_1)+\mathcal{C}(\omega_2),$$

by Lemma 5.5.4(ii). If we take infima over all possible choices of ω_1 and ω_2 we obtain (5.5.2).

It follows from (ii) that $\mathcal{C}^*(E_1 \cup E_2) \le \mathcal{C}^*(E_1) + \mathcal{C}^*(E_2)$, and induction yields

$$\mathcal{C}^*\left(\bigcup_{n=1}^{m}E_n\right)\le\sum_{n=1}^{m}\mathcal{C}^*(E_n)\qquad(m\in\mathbb{N}).$$

The case of a countably infinite union of sets is dealt with by letting $m \to \infty$ and using (i). This proves (iii).

It remains to establish (iv). Let $0 < \varepsilon < 1$ and let ω be a bounded open set satisfying $E \subseteq \omega$, $\overline{\omega} \subset \Omega$ and $\mathcal{C}(\omega) < \mathcal{C}^*(E) + \varepsilon$. In view of Lemma 3.7.4, there is a sequence (u_n) in $\mathcal{U}_+(\Omega)$ such that $u_n \ge 1$ on E and such that the lower regularization of $\inf_n u_n$ is $\widehat{R}_1^E$. Further, we may choose u_1 to be the constant function 1. Let

$$v_n=\min\{u_1,u_2,\dots,u_n\},\quad w_n=R_{v_n}^{\omega}\text{ and }\omega_n=\{x\in\omega\colon w_n>1-\varepsilon\}.$$

Then ω_n is an open set containing E and $w_n \in \mathcal{H}(\Omega\backslash\overline{\omega})$ for each n. Since $v_n \ge w_n \ge \widehat{R}_1^E$ and (w_n) is decreasing, it follows that $\inf_n u_n \ge \lim w_n \ge \widehat{R}_1^E$ on Ω and so $\lim w_n = \widehat{R}_1^E$ on $\Omega\backslash\overline{\omega}$. Also, $w_n \ge (1-\varepsilon)R_1^{\omega_n}$. From parts (ii) and (iv) of Lemma 5.4.2 we obtain $\nu_E(\Omega) \ge (1-\varepsilon)\lim \nu_{\omega_n}(\Omega)$, and $\nu_\omega(\Omega) \ge \nu_E(\Omega)$ since $R_1^\omega \ge \widehat{R}_1^E$. Thus, from Lemma 5.5.2(ii) we see that

$$\mathcal{C}^*(E)+\varepsilon>\mathcal{C}(\omega)\ge\nu_E(\Omega)\ge(1-\varepsilon)\lim_{n\to\infty}\mathcal{C}(\omega_n)\ge(1-\varepsilon)\mathcal{C}^*(E).$$

Since ε can be arbitrarily small, we obtain $\nu_E(\Omega) = \mathcal{C}^*(E)$, as required. □

Corollary 5.5.7. (Cartan) *Let $E \subseteq \Omega$. Then E is polar if and only if $\mathcal{C}^*(E) = 0$.*

Proof. Let U be a bounded open set such that $\overline{U} \subset \Omega$. It follows from Theorem 5.5.6(iv) and Lemma 5.3.3 that

$$E\cap U\text{ is polar}\Leftrightarrow\widehat{R}_1^{E\cap U}\equiv0\Leftrightarrow\mathcal{C}^*(E\cap U)=0.$$

Since Ω can be written as a countable union of such open sets U, the "if" part of the result follows from the fact that a countable union of polar sets is polar, while the "only if" part follows from Theorem 5.5.6(iii). □

Theorem 5.5.8. *Let $u \in \mathcal{U}(\Omega)$. For each positive number ε there is an open subset W_ε of Ω such that $\mathcal{C}(W_\varepsilon) < \varepsilon$ and $u|_{\Omega\backslash W_\varepsilon}$ is continuous.*

Proof. Let U be a bounded open set such that $\overline{U} \subset \Omega$. If we can show that there is an open subset W_ε of U such that $\mathcal{C}(W_\varepsilon) < \varepsilon$ and $u|_{U\backslash W_\varepsilon}$ is continuous, then the general result will follow easily from Lemma 5.5.4(iii).

Let K be a compact set such that $\overline{U} \subset K^\circ$ and $K \subset \Omega$, and let μ denote the restriction to K° of the Riesz measure associated with u. In view of Theorems 4.3.5 and 4.3.8(i), there exists h in $\mathcal{H}(K^\circ)$ such that $u = h + G_\Omega\mu$ on K. We choose n_0 in $\mathbb{N}$ such that $n_0 > 2\mu(\Omega)/\varepsilon$ and define $\omega = \{x \in U\colon G_\Omega\mu(x) > n_0\}$. Then $\mathcal{C}(\omega) \le \mu(\Omega)/n_0 < \varepsilon/2$ by Theorem 5.5.5(ii). The function $\min\{G_\Omega\mu, n_0\}$ is a bounded potential: we write it as $G_\Omega\mu'$ and note that $G_\Omega\mu' = G_\Omega\mu$ on $U\backslash\omega$.

For each n in $\mathbb{N}$ we use Corollary 4.5.2 to obtain a compact subset K_n of K such that $\mu'(K\backslash K_n) < 4^{-n}\varepsilon$ and $G_\Omega(\mu'|_{K_n})$ is continuous. If

$$\omega_n=\left\{x\in U\colon G_\Omega\left(\mu'|_{K\backslash K_n}\right)>2^{1-n}\right\}\qquad(n\in\mathbb{N}),$$

then $\mathcal{C}(\omega_n) < 2^{n-1}4^{-n}\varepsilon = 2^{-n-1}\varepsilon$, and it follows that the open set $W_\varepsilon = \omega \cup (\bigcup_n \omega_n)$ satisfies $\mathcal{C}(W_\varepsilon) < \varepsilon$. Further, if $y \in U\backslash W_\varepsilon$, then

$$\limsup_{x\to y,x\in U\backslash W_\varepsilon}|u(x)-u(y)|=\limsup_{x\to y,x\in U\backslash W_\varepsilon}|G_\Omega\mu(x)-G_\Omega\mu(y)|$$
$$=\limsup_{x\to y,x\in U\backslash W_\varepsilon}\left|G_\Omega\left(\mu'|_{K\backslash K_n}\right)(x)-G_\Omega\left(\mu'|_{K\backslash K_n}\right)(y)\right|$$
$$\le2^{2-n}\qquad(n\in\mathbb{N}).$$

Hence $u|_{\Omega\setminus W_\varepsilon}$ is continuous. □

We call $f:\mathbb{R}^N \to \mathbb{R}^N$ a *contraction* if $\|f(x)-f(y)\| \le \|x-y\|$ for all x and y.

Theorem 5.5.9. *If $E \subseteq \mathbb{R}^N$ $(N \ge 3)$ and $f:\mathbb{R}^N \to \mathbb{R}^N$ is a contraction, then the Newtonian outer capacity of $f(E)$ satisfies $\mathcal{C}^*(f(E)) \le \mathcal{C}^*(E)$.*

Proof. We begin by considering the case where E is compact. Let L be a compact set such that $E \subset L^\circ$ and let $0<\varepsilon<1$. Then $G\nu_L = 1$ on E. Further, since $(x,y)\mapsto \|x-y\|^{2-N}$ is uniformly continuous on the compact set $E\times\partial L$, we can choose points $y_1, y_2, \ldots, y_n$ in ∂L and non-negative constants $a_1, a_2, \ldots, a_n$ such that

$$\sum_{k=1}^n a_k\|x-y_k\|^{2-N} \ge 1-\varepsilon \qquad (x\in E)$$

and

$$\sum_k a_k = \nu_L(\partial L) = \mathcal{C}(L).$$

If we define

$$u(x) = \sum_{k=1}^n a_k\|x-f(y_k)\|^{2-N} \qquad (x\in\mathbb{R}^N),$$

then $u(f(x)) \ge 1-\varepsilon$ when $x\in E$ since f is a contraction. Hence $u \ge 1-\varepsilon$ on the compact set $f(E)$, so $(1-\varepsilon)^{-1}u \ge \widehat{R}_1^{f(E)}$, and it follows from Lemma 5.4.2(ii) that

$$\mathcal{C}(f(E)) \le \mathcal{C}(L)/(1-\varepsilon) \to \mathcal{C}(L) \qquad (\varepsilon\to 0).$$

If we replace L by L_n, where (L_n) is a decreasing sequence of such compact sets satisfying $\bigcap_n L_n = E$, then we obtain $\mathcal{C}(f(E)) \le \mathcal{C}(E)$, in view of Lemma 5.4.2(v).

Now let E be an arbitrary set and, ignoring the trivial case where $\mathcal{C}^*(E)$ is infinite, let ω be an open set such that $E\subseteq\omega$ and $\mathcal{C}(\omega) < \mathcal{C}^*(E)+\varepsilon$. Let (K_n) be an increasing sequence of compact sets such that $\bigcup_n K_n = \omega$. Then, by Theorem 5.5.6(i) and the special case established above,

$$\begin{aligned}\mathcal{C}^*(f(E)) &\le \mathcal{C}^*(f(\omega))\\ &= \lim_{n\to\infty}\mathcal{C}(f(K_n))\\ &\le \lim_{n\to\infty}\mathcal{C}(K_n)\\ &= \mathcal{C}(\omega)\\ &< \mathcal{C}^*(E)+\varepsilon.\end{aligned}$$

Since ε is arbitrary, the result follows. □

5.6. Capacitable sets

So far we have shown that all open subsets and all compact subsets of a fixed Greenian set Ω are capacitable. The purpose of this section is to show that the class of capacitable sets is very large, and includes all Borel subsets of Ω.

Definition 5.6.1. Let $\mathcal{K}$ denote the collection of all compact subsets of $\mathbb{R}^N$. The following formulation involves both $\mathbb{N}^{\mathbb{N}}$, that is, the collection of all (infinite) sequences of natural numbers, and also $\bigcup_k \mathbb{N}^k$, that is, the collection of all finite sequences of natural numbers. A subset A of $\mathbb{R}^N$ is called *analytic* if there exists a map $K:(\bigcup_k \mathbb{N}^k)\to\mathcal{K}$ such that

$$A = \bigcup_{(m_n)\in\mathbb{N}^{\mathbb{N}}} \left(K(m_1)\cap K(m_1,m_2)\cap K(m_1,m_2,m_3)\cap\ldots\right). \tag{5.6.1}$$

The class of all analytic sets will be denoted by $\mathcal{A}$.

Lemma 5.6.2. *(i) If (A_n) is a sequence of analytic sets, then $\bigcap_n A_n$ and $\bigcup_n A_n$ are analytic.*
(ii) Every Borel set in $\mathbb{R}^N$ is analytic.
(iii) If A is analytic and $A\subseteq\Omega$, then the compact sets $K(m_1,\ldots,m_n)$ in (5.6.1) can be chosen to be subsets of Ω.

Proof. To prove (i), let (A_l) be a sequence of analytic sets. Then, for each l, there is a map $K_l:(\bigcup_k\mathbb{N}^k)\to\mathcal{K}$ such that

$$A_l = \bigcup_{(m_n)\in\mathbb{N}^{\mathbb{N}}}\left(K_l(m_1)\cap K_l(m_1,m_2)\cap K_l(m_1,m_2,m_3)\cap\ldots\right).$$

Let $n\mapsto(a(n),b(n))$ be a bijective map from $\mathbb{N}$ to $\mathbb{N}^2$. If we define $K:(\bigcup_k\mathbb{N}^k)\to\mathcal{K}$ by

$$K(m_1,\ldots,m_n) = K_{a(m_1)}(b(m_1),m_2,m_3,\ldots,m_n),$$

then

$$A_l = \bigcup_{\{(m_n):a(m_1)=l\}}\left(K(m_1)\cap K(m_1,m_2)\cap K(m_1,m_2,m_3)\cap\ldots\right).$$

Hence $\bigcup_l A_l$ is equal to the right-hand side of (5.6.1) and so is analytic.

We will next show that $\bigcap_l A_l$ is also analytic. To each (m_n) in $\mathbb{N}^{\mathbb{N}}$ there corresponds a function $f:\mathbb{N}^2\to\mathbb{N}$ defined by equating $f(l,p)$ to the (l,p)-entry of the infinite matrix

$$\begin{matrix} m_1 & m_2 & m_4 & m_7 & \cdot & \cdot & \cdot \\ m_3 & m_5 & m_8 & \cdot & \cdot & \cdot & \cdot \\ m_6 & m_9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{10} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \tag{5.6.2}$$

We then define $K(m_1,\ldots,m_n)$ to be the entry in the infinite matrix

$$\begin{array}{llll} K_1(f(1,1)) & K_1(f(1,1),f(1,2)) & K_1(f(1,1),f(1,2),f(1,3)) & \cdots \\ K_2(f(2,1)) & K_2(f(2,1),f(2,2)) & \cdot & \cdots \\ K_3(f(3,1)) & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}$$

which corresponds to the position of m_n in (5.6.2). Thus, for example, $K(m_1,\ldots,m_9)=K_3(f(3,1),f(3,2))$. Now

$$\begin{aligned} & x\in\bigcap_l A_l \\ \Leftrightarrow\ & \forall l\in\mathbb{N}\quad \exists(m_n^{(l)})\in\mathbb{N}^{\mathbb{N}} \text{ such that } x\in K_l(m_1^{(l)})\cap K_l(m_1^{(l)},m_2^{(l)})\cap\ldots \\ \Leftrightarrow\ & \exists f:\mathbb{N}^2\to\mathbb{N} \text{ such that} \\ & x\in\bigcap_{l\in\mathbb{N}}(K_l(f(l,1))\cap K_l(f(l,1),f(l,2))\cap K_l(f(l,1),f(l,2),f(l,3))\cap\ldots). \end{aligned}$$

Hence $\bigcap_l A_l$ equals the right-hand side of (5.6.1) with the map K as defined above in this paragraph, and so is analytic. This completes the proof of (i).

We now prove (ii). Any compact set E is analytic, as can be seen from defining $K(m_1,\ldots,m_n)=E$ for any choice of natural numbers $m_1,\ldots,m_n$. Since any open set and any closed set can be written as a countable union of compact sets, it follows from (i) that such sets are also analytic. Now let $\mathcal{F}$ be the collection of analytic sets A for which $\mathbb{R}^N\backslash A$ is also analytic. If (A_n) is a sequence of sets in $\mathcal{F}$, then (i) shows that $\bigcup_n A_n\in\mathcal{A}$ and

$$\mathbb{R}^N\backslash(\bigcup_n A_n)=\bigcap_n(\mathbb{R}^N\backslash A_n)\in\mathcal{A}$$

and so $\bigcup_n A_n\in\mathcal{F}$. Thus $\mathcal{F}$ is a σ-algebra which contains the open sets. It follows that $\mathcal{F}$, and hence $\mathcal{A}$, contains the Borel sets. This proves (ii).

Finally, suppose that A is analytic and that $A\subseteq\Omega$. Then there is a mapping K': $(\bigcup_k\mathbb{N}^k)\to\mathcal{K}$ such that

$$A=\bigcup_{(m_n)\in\mathbb{N}^{\mathbb{N}}}(K'(m_1)\cap K'(m_1,m_2)\cap K'(m_1,m_2,m_3)\cap\ldots).$$

Let (L_n) be an increasing sequence of compact sets such that $\bigcup_n L_n=\Omega$. If we define

$$K(m_1,\ldots,m_n)=\begin{cases} L_{m_1} & (n=1)\\ L_{m_1}\cap K'(m_2,\ldots,m_n) & (n\geq 2),\end{cases}$$

then each compact set $K(m_1,\ldots,m_n)$ is contained in Ω and (5.6.1) holds. □

Lemma 5.6.3. *Suppose that A is given by (5.6.1), let $(k_n)\in\mathbb{N}^{\mathbb{N}}$, and define*

$$E_l=\bigcup_{\{(m_n)\in\mathbb{N}^{\mathbb{N}}:m_n\leq k_n \text{ when } n\leq l\}}(K(m_1)\cap K(m_1,m_2)\cap\ldots)\quad (l\in\mathbb{N}),$$

$$F_l=\bigcup_{\{(m_n)\in\mathbb{N}^{l}:m_n\leq k_n \text{ when } n\leq l\}}(K(m_1)\cap\ldots\cap K(m_1,\ldots,m_l))\quad (l\in\mathbb{N})$$

and $F=\bigcap_l F_l$. Then:
(i) $E_l\subseteq A$ and $E_l\subseteq F_l$ for each l, and (E_l) is a decreasing sequence of sets;
(ii) (F_l) is a decreasing sequence of compact sets, and $F\subseteq A$.

Proof. It is clear that (i) holds. Further, each set F_l is a finite union of compact sets, and so is compact, and the sequence (F_l) is obviously decreasing. It remains to show that $F\subseteq A$.

Let $x\in F$. Then, for any choice of l, there is an l-tuple $(m_1^{(l)},m_2^{(l)},\ldots,m_l^{(l)})$ such that $m_n^{(l)}\leq k_n$ for each n in $\{1,2,\ldots,l\}$, and such that

$$x\in K(m_1^{(l)})\cap K(m_1^{(l)},m_2^{(l)})\cap\ldots\cap K(m_1^{(l)},\ldots,m_l^{(l)}).$$

Since $m_1^{(l)}\in\{1,\ldots,k_1\}$ for each l, there exists m_1' in $\{1,\ldots,k_1\}$ such that $m_1^{(l)}=m_1'$ for infinitely many l. Similarly, there exists m_2' in $\{1,\ldots,k_2\}$ such that $(m_1^{(l)},m_2^{(l)})=(m_1',m_2')$ for infinitely many l. Proceeding in this manner, we obtain a sequence (m_n') such that

$$x\in K(m_1')\cap K(m_1',m_2')\cap K(m_1',m_2',m_3')\cap\ldots\quad\subseteq A.$$

Hence $F\subseteq A$, and (ii) is proved. □

Theorem 5.6.4. (Choquet) *Every analytic subset of a Greenian set Ω is capacitable.*

Proof. Suppose that $A\subseteq\Omega$ and $A\in\mathcal{A}$. Then we can write A as in (5.6.1), where each compact set $K(m_1,\ldots,m_n)$ is contained in Ω (see Lemma 5.6.2 (iii)). Let $a<\mathcal{C}^*(A)$. We inductively define a sequence (k_n) of natural numbers as follows. In view of Theorem 5.5.6(i), a sufficiently large choice of k_1 will ensure that the set E_1 of Lemma 5.6.3 satisfies $\mathcal{C}^*(E_1)>a$. Given $k_1,k_2,\ldots,k_{n-1}$ such that $\mathcal{C}^*(E_{n-1})>a$, we can similarly choose k_n large enough such that $\mathcal{C}^*(E_n)>a$.

Now that (k_n) has been defined, we see from Lemmas 5.4.2(v) and 5.6.3 that

$$\mathcal{C}_*(A)\geq\mathcal{C}(F)=\lim_{n\to\infty}\mathcal{C}(F_n)\geq\lim_{n\to\infty}\mathcal{C}^*(E_n)\geq a.$$

If we let $a\to\mathcal{C}^*(A)$ we obtain $\mathcal{C}_*(A)\geq\mathcal{C}^*(A)$, and the reverse inequality is always true. Hence A is capacitable. □

5.7. The fundamental convergence theorem

The results of the previous section, when combined with Corollary 5.5.7, reveal that any Borel set of inner capacity zero (for some Greenian set Ω) is polar. It is this crucial fact that will now enable us to improve Theorem 3.7.5 by showing that, if u is the infimum of a family of superharmonic functions on Ω which is locally uniformly bounded below, then the exceptional set where $\hat{u} \neq u$ is not just of λ-measure zero, but actually polar. We no longer assume that Ω is Greenian, unless this is explicitly stated. The following result is known as the ***fundamental convergence theorem*** of potential theory.

Theorem 5.7.1. *Let $\mathcal{F}$ be a family in $\mathcal{U}(\Omega)$ and let $u = \inf \mathcal{F}$. If $\mathcal{F}$ is locally uniformly bounded below, then*
(i) $\hat{u} \in \mathcal{U}(\Omega)$;
(ii) $\hat{u} = u$ quasi-everywhere;
(iii) $\hat{u}(x) = \liminf_{y \to x} u(y) \qquad (x \in \Omega)$.

Proof. We know from Theorem 3.7.5 that (i) and (iii) hold, and that $\hat{u} = u$ almost everywhere (λ). By Lemma 3.7.4 there exists a sequence (u_n) of functions in $\mathcal{F}$ such that $\hat{v} = \hat{u}$, where $v = \inf_n u_n$. Let $v_n = \min\{u_1, \ldots, u_n\}$. Then (v_n) is a decreasing sequence in $\mathcal{U}(\Omega)$ with limit v.

Let U and V be open balls such that $\overline{U} \subset V$ and $\overline{V} \subset \Omega$, and let μ_n denote the restriction to $\overline{U}$ of the Riesz measure associated with v_n. By adding a suitable constant to all the functions we may assume that $v_n > 0$ on V for each n. The Riesz decomposition theorem and Theorem 4.2.3 together yield that $v_n = h_n + G_V\mu_n$ on U, where $h_n \in \mathcal{H}_+(U)$. By choosing a suitable subsequence, if necessary, we may assume that (h_n) converges locally uniformly on U to a harmonic function h (see Theorem 1.5.11). Hence $(G_V\mu_n)$ also converges on U.

Let a denote the infimum of $G_V(\cdot,\cdot)$ on $U \times U$. Then $a > 0$ and

$$+\infty > \inf_U v_1 \geq \inf_U v_n \geq \inf_U G_V\mu_n \geq a\mu_n(\overline{U}) \qquad (n \in \mathbb{N}).$$

Hence (see Appendix, Theorem A.10) there is a subsequence (μ_{n_k}) of (μ_n) which is w^*-convergent to some measure μ on $\overline{U}$. It follows that

$$\lim_{k\to\infty} G_V\mu_{n_k}(x) \geq \lim_{k\to\infty} \int \min\{G_V(x,y), m\}\, d\mu_{n_k}(y) \qquad (m \in \mathbb{N}; x \in U)$$
$$= \int \min\{G_V(x,y), m\}\, d\mu(y) \to G_V\mu(x) \qquad (m \to \infty).$$

Let

$$E = \left\{x \in U\colon \lim_{k\to\infty} G_V\mu_{n_k}(x) > G_V\mu(x)\right\},$$

and suppose that $C_*(E) > 0$. Then there is a compact subset K of E such that $C(K) > 0$. It follows from Theorem 5.3.7(i) that there is a bounded

continuous potential $G_V\nu$ on V such that $\nu \not\equiv 0$ and $\operatorname{supp} \nu \subseteq K$. Hence, by the reciprocity theorem and Fatou's lemma,

$$\int_K G_V\mu\, d\nu = \int G_V\nu\, d\mu = \lim_{k\to\infty} \int G_V\nu\, d\mu_{n_k} = \lim_{k\to\infty} \int_K G_V\mu_{n_k}\, d\nu$$
$$\geq \int_K \lim_{k\to\infty} G_V\mu_{n_k}\, d\nu > \int_K G_V\mu\, d\nu,$$

which yields a contradiction. Thus $C_*(E) = 0$ and, since E is a Borel set, it follows that $C^*(E) = 0$, whence E is polar. We have now established that

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} (h_n + G_V\mu_n) = h + G_V\mu \quad \text{q.e. on } U.$$

Since $v = \hat{v} = \hat{u}$ almost everywhere (λ), it follows that $\hat{u} = h + G_V\mu$ on U, and so $v = \hat{u}$ quasi-everywhere on U. Since $\hat{u} \leq u \leq v$, we obtain $\hat{u} = u$ quasi-everywhere on U. This proves (ii) in view of the arbitrary nature of U. □

Corollary 5.7.2. *Let (u_n) be a sequence in $\mathcal{U}(\Omega)$, where Ω is connected. Suppose that (u_n) is locally uniformly bounded below and let $u = \liminf_{n\to\infty} u_n$. If $u \not\equiv +\infty$, then $\hat{u} \in \mathcal{U}(\Omega)$ and $\hat{u} = u$ quasi-everywhere Further, if $u \geq M$ on Ω, then for any compact subset K of Ω and $\varepsilon > 0$, there exists n_0 such that*

$$u_n(x) \geq M - \varepsilon \qquad (n \geq n_0; x \in K).$$

Proof. Let $v_n = \inf\{u_k\colon k \geq n\}$. Then $\hat{v}_n \in \mathcal{U}(\Omega)$ and $\hat{v}_n = v_n$ quasi-everywhere by Theorem 5.7.1. Since $(\hat{v}_n)$ is increasing, the function $v = \lim_{n\to\infty} \hat{v}_n$ is either superharmonic or identically $+\infty$ on Ω, and $v = u$ quasi-everywhere. Also $v \leq \hat{u} \leq u$, so $v = \hat{u}$ on Ω. Hence $\hat{u} = u$ quasi-everywhere and if $u \not\equiv +\infty$, then $\hat{u} \in \mathcal{U}(\Omega)$.

Now we recall Dini's theorem which says that, if (f_n) is a decreasing sequence of upper semicontinuous functions on a compact set K and $f_n \to 0$ pointwise, then the convergence is uniform on K. (This follows easily from the observation that, for any $\varepsilon > 0$, the open sets $\{x : f_n(x) < \varepsilon\}$ cover K.) Let M, K, ε be as stated. Then $((M - \hat{v}_n)^+)$ is a decreasing sequence of upper semicontinuous functions on Ω with limit 0, so there is an n_0 such that

$$u_n(x) \geq v_n(x) \geq \hat{v}_n(x) \geq M - \varepsilon \qquad (n \geq n_0; x \in K).$$ □

The fundamental convergence theorem allows us to establish some further properties of reduced functions.

Theorem 5.7.3. *Let Ω be Greenian. All reduced functions below are of functions in $\mathcal{U}_+(\Omega)$ relative to subsets of Ω.*
(i) $\widehat{R}_u^E = R_u^E$ quasi-everywhere on Ω.

(ii) $\widehat{R}_u^E = \inf\{v \in \mathcal{U}_+(\Omega): v \geq u$ quasi-everywhere on $E\}$.
(iii) $\widehat{R}_u^E = R_u^E$ *on* $(\Omega \backslash E) \cup E^\circ$.
(iv) If (E_n) *is an increasing sequence of sets and* $E = \bigcup_n E_n$*, then* $R_u^{E_n} \to R_u^E$ *and* $\widehat{R}_u^{E_n} \to \widehat{R}_u^E$.
(v) If (u_n) *is an increasing sequence from* $\mathcal{U}_+(\Omega)$ *and* $\lim u_n$ *is superharmonic on* Ω*, then* $\widehat{R}_{u_n}^E \uparrow \widehat{R}_{\lim u_n}^E$.

Proof. Part (i) follows from the fundamental convergence theorem, since R_u^E is the infimum of a family of non-negative superharmonic functions.

Since $\widehat{R}_u^E = R_u^E = u$ quasi-everywhere on E, the non-negative superharmonic function $\widehat{R}_u^E$ certainly majorizes u quasi-everywhere on E. If v is another non-negative superharmonic function with this property, then there is a polar subset F of E such that $v \geq u$ on $E \backslash F$. Hence $v \geq \widehat{R}_u^{E \backslash F} = \widehat{R}_u^E$ by Theorem 5.3.4(iv), proving (ii).

We know that $\widehat{R}_u^E = R_u^E$ on E°. To prove equality at any point x_0 of $\Omega \backslash E$, let F be the polar subset of E where $\widehat{R}_u^E < R_u^E$, and choose v in $\mathcal{U}_+(\Omega)$ such that $v = +\infty$ on F and $v(x_0) < +\infty$ (see Theorem 5.1.3(i)). Then $\widehat{R}_u^E + v/n \geq u$ on E, so $\widehat{R}_u^E + v/n \geq R_u^E$ for each n in $\mathbb{N}$. If we let $n \to \infty$, we obtain $\widehat{R}_u^E(x_0) \geq R_u^E(x_0)$, whence $\widehat{R}_u^E(x_0) = R_u^E(x_0)$ and (iii) is proved.

If (E_n) is an increasing sequence of sets, then $(R_u^{E_n})$ and $(\widehat{R}_u^{E_n})$ are increasing. Thus the function $v = \lim_{n\to\infty} \widehat{R}_u^{E_n}$ is non-negative and superharmonic, and clearly $v \leq \widehat{R}_u^E$. From (i) v majorizes u quasi-everywhere on each set E_n and so quasi-everywhere on E, whence $v \geq \widehat{R}_u^E$ by (ii). Thus $v = \widehat{R}_u^E$. Also, by (iii),

$$\lim_{n\to\infty} R_u^{E_n}(x) = \begin{cases} u(x) = R_u^E(x) & (x \in E) \\ \lim\limits_{n\to\infty} \widehat{R}_u^{E_n}(x) = \widehat{R}_u^E(x) = R_u^E(x) & (x \in \Omega \backslash E). \end{cases}$$

Hence (iv) holds.

Finally, if (u_n) is increasing, then $(\widehat{R}_{u_n}^E)$ is increasing, so the function $v = \lim_{n\to\infty} \widehat{R}_{u_n}^E$ is in $\mathcal{U}_+(\Omega)$. Also, $v = \lim u_n$ quasi-everywhere on E since $\widehat{R}_{u_n}^E = u_n$ quasi-everywhere on E. Thus $\widehat{R}_{\lim u_n}^E \leq v$ by (ii), and the reverse inequality is obvious, so (v) holds. □

Theorem 5.7.4. *Let* ω *be an open subset of a Greenian set* Ω *and* μ *be a measure on* Ω*. Then:*
(i) for each y *in* Ω*, the function* $G_\Omega(\cdot, y)$ *has limit* 0 *at quasi-every point of* $\partial\Omega$ *(and also at* ∞ *if* Ω *is unbounded and* $N \geq 3$*);*
(ii) $G_\Omega\mu$ *has the same limiting behaviour if* $\operatorname{supp}\mu$ *is a compact subset of* Ω*;*
(iii) $G_\omega(x,y) = G_\Omega(x,y) - R_{G_\Omega(\cdot,y)}^{\Omega\backslash\omega}(x)$ *when* $x, y \in \omega$.

Proof. Let $y \in \omega$, let h_y be the greatest harmonic minorant of U_y on Ω and let $v_y = \widehat{R}_{G_\Omega(\cdot,y)}^{\Omega\backslash\omega}(x)$. Then $G_\Omega(\cdot,y) = U_y - h_y$ and $G_\Omega(\cdot,y) \geq v_y$ so $h_y + v_y$ is a harmonic minorant of U_y on ω. Hence

$$G_\omega(x,y) \leq U_y(x) - \{h_y(x) + v_y(x)\} = G_\Omega(x,y) - v_y(x) \quad (x, y \in \omega),$$

so

$$0 \leq \limsup_{x\to z, x\in\omega} G_\omega(x,y) \leq G_\Omega(z,y) - v_y(z) \qquad (z \in \Omega \cap \partial\omega)$$

by the lower semicontinuity of v_y, and thus $G_\omega(\cdot,y)$ has limit 0 quasi-everywhere on $\Omega \cap \partial\omega$ by Theorem 5.7.3(i).

If $N \geq 3$, then we can apply the preceding paragraph with the pair $\mathbb{R}^N, \Omega$ in place of the pair Ω, ω to obtain (i). (Clearly $G_\Omega(\cdot,y)$ has limit 0 at ∞ if Ω is unbounded, since $G_\Omega(\cdot,y) \leq U_y$.) If $N = 2$, then $\mathbb{R}^2 \backslash \Omega$ is non-polar. Thus, if $z \in \partial\Omega$ and ε is sufficiently small, $\mathbb{R}^2 \backslash (\Omega \cup B(z,\varepsilon))$ is non-polar. We can therefore apply the preceding paragraph with the pair $\Omega \cup B(z,\varepsilon), \Omega$ in place of Ω, ω to see that $G_\Omega(\cdot,y)$ has limit 0 quasi-everywhere on $B(z,\varepsilon) \cap \partial\Omega$. In view of the arbitrary choice of z, (i) follows.

In proving (ii) we may assume that Ω is connected. Let $z \in \operatorname{supp}\mu$ and let U be a bounded connected open set such that $\operatorname{supp}\mu \subset U$ and $\overline{U} \subset \Omega$. By Harnack's inequalities applied to the functions $G_\Omega(x,\cdot)$, there is a positive constant c such that

$$G_\Omega(x,y) \leq cG_\Omega(x,z) \qquad (x \in \Omega \backslash U; y \in \operatorname{supp}\mu).$$

Integration with respect to $d\mu(y)$ yields

$$G_\Omega\mu(x) \leq c\mu(\Omega)G_\Omega(x,z) \qquad (x \in \Omega \backslash U),$$

and so (ii) holds.

Finally, if u is a harmonic minorant of $G_\Omega(\cdot,y) - v_y$ on ω, then we see that $\limsup_{x\to z} u(x) \leq 0$ for quasi-every z in $\Omega \cap \partial\omega$ by the first paragraph of the proof, and for quasi-every z in $\partial\Omega \cap \partial\omega$ by (i). It follows from Theorem 5.2.6 that $u \leq 0$ on ω and this establishes (iii). □

Corollary 5.7.5. *Let* Ω *be Greenian and* $y \in \Omega$*. Then* $G_\Omega(\cdot,y)$ *has a subharmonic extension to* $\mathbb{R}^N \backslash \{y\}$ *which is valued* 0 *quasi-everywhere on* $\partial\Omega$ *and everywhere on* $\mathbb{R}^N \backslash \overline{\Omega}$.

Proof. From part (i) of the above theorem there is a polar subset F of $\partial\Omega$ such that $G_\Omega(\cdot,y)$ has limit 0 on $\partial\Omega \backslash F$. If we define

$$u(x) = \begin{cases} G_\Omega(x,y) & (x \in \Omega \backslash \{y\}) \\ 0 & (x \in \mathbb{R}^N \backslash (\Omega \cup F)), \end{cases}$$

then it follows from Theorem 5.2.1 that u has a subharmonic extension to $\mathbb{R}^N \backslash \{y\}$. □

Theorem 5.7.6. *Let Ω be Greenian and $E \subseteq \Omega$. Then*

$$\widehat{R}^E_{G_\Omega(x,\cdot)}(y) = \widehat{R}^E_{G_\Omega(\cdot,y)}(x) \qquad (x, y \in \Omega). \tag{5.7.1}$$

Proof. Let $x, y \in \Omega$ and $n \in \mathbb{N}$, and let $E(n) = E \backslash (B(x, 1/n) \cup B(y, 1/n))$. If E is a relatively closed subset of Ω, then

$$\widehat{R}^{E(n)}_{G_\Omega(x,\cdot)}(y) = \widehat{R}^{E(n)}_{G_\Omega(\cdot,y)}(x) \tag{5.7.2}$$

by Theorem 5.7.4(iii) and the symmetry of $G_\Omega(\cdot,\cdot)$ and $G_{\Omega\backslash E(n)}(\cdot,\cdot)$. If we let $n \to \infty$ and use Theorem 5.7.3(iv), we obtain (5.7.1), since $\widehat{R}^{E\backslash\{x,y\}}_u = \widehat{R}^E_u$ for any u in $\mathcal{U}_+(\Omega)$.

If E is open, we let $(K(n))$ be an increasing sequence of compact sets with union E and apply the previous paragraph to obtain

$$\widehat{R}^E_{G_\Omega(x,\cdot)}(y) = \lim_{n\to\infty} \widehat{R}^{K(n)}_{G_\Omega(x,\cdot)}(y) = \lim_{n\to\infty} \widehat{R}^{K(n)}_{G_\Omega(\cdot,y)}(x) = \widehat{R}^E_{G_\Omega(\cdot,y)}(x).$$

Finally, if E is an arbitrary subset of Ω, then Theorem 5.3.4(vi) together with the conclusion of the preceding paragraph yields (5.7.2), and (5.7.1) again follows on letting $n \to \infty$. □

5.8. Logarithmic capacity

We cannot define (Green) capacity relative to $\mathbb{R}^2$ since this set is not Greenian. In this section we describe a set function, called logarithmic capacity, which is defined on subsets of $\mathbb{R}^2$ and shares some of the properties of Green capacity. Throughout this section we assume that $N = 2$.

Lemma 5.8.1. *Let $\Omega = \mathbb{R}^2 \backslash K$, where K is a compact non-polar set. There is a unique non-negative harmonic function h_K on Ω such that*
(i) h_K has limit 0 quasi-everywhere on ∂K and is bounded near ∂K;
(ii) $h_K(x) - \log\|x\|$ has a finite limit l as $x \to \infty$.
Further, there is a unit measure μ on ∂K such that

$$h_K(x) = \lim_{y\to\infty} G_\Omega(x,y) = l - U\mu(x) \qquad (x \in \Omega)$$

and $U\mu = l$ quasi-everywhere on K.

Proof. The uniqueness follows from Theorem 5.2.6. We assume, without loss of generality, that $0 \in K$. Let x^* denote the inverse of x with respect to the unit sphere and let $W = \{x^* : x \in \Omega\} \cup \{0\}$. We know from Myrberg's theorem that Ω is Greenian, and hence from Theorem 4.1.11 and Corollary 5.2.5 that W is Greenian and $G_\Omega(x,y) = G_W(x^*, y^*)$ when $x, y \in \Omega$. Thus we can define

$$h_K(x) = \lim_{y\to\infty} G_\Omega(x,y) = G_W(x^*, 0).$$

Since the Kelvin transform preserves superharmonicity (Corollary 3.3.5), the image of a polar set under inversion is polar, and so we can see from Theorem 5.7.4 that (i) holds. Clearly

$$h_K(x) - \log\|x\| = G_W(x^*, 0) - U_0(x^*),$$

and the right-hand side of the above equation has a finite limit l as $x^* \to 0$, by the definition of $G_W(\cdot, 0)$. Thus (ii) holds. Finally, h_K has a subharmonic extension to $\mathbb{R}^2$ valued 0 on K° (see Corollary 5.7.5). Let μ denote the Riesz measure associated with this extension. Then $h_K + U\mu \in \mathcal{H}(\mathbb{R}^2)$ and $\operatorname{supp}\mu \subseteq \partial K$. Also,

$$h_K(x) + U\mu(x) \to \begin{cases} +\infty & \text{if } \mu(\partial K) < 1 \\ l & \text{if } \mu(\partial K) = 1 \\ -\infty & \text{if } \mu(\partial K) > 1 \end{cases} \qquad \text{as } x \to \infty.$$

It follows from the mean value property of harmonic functions that $\mu(\partial K) = 1$ and $h_K + U\mu \equiv l$, so the lemma is established. □

Definition 5.8.2. Let K, Ω and h_K be as in the above lemma. Then h_K is called the *Green function for Ω with pole at* ∞. The limit l in (ii) is denoted by $r(K)$ and called the *Robin constant* of K. The *logarithmic capacity* of any compact set K is defined by

$$c(K) = \begin{cases} e^{-r(K)} & \text{if } K \text{ is non-polar} \\ 0 & \text{if } K \text{ is polar.} \end{cases}$$

We note that, if $J \subseteq K$, then $r(J) \geq r(K)$ by Lemma 5.8.1 and Theorem 4.1.10(i), and so $c(J) \leq c(K)$. The measure μ in Lemma 5.8.1, which is uniquely determined since $U\mu$ is, is called the ***equilibrium measure*** of K.

We will use ${}^R\mathcal{C}(\cdot)$ to denote Green capacity relative to the disc $B(0,R)$.

Example 5.8.3. (i) If K is $\overline{B(0,r)}$ or $S(0,r)$, then clearly $h_K(x) = \log\|x\| - \log r$, so $r(K) = -\log r$ and $c(K) = r$.
(ii) If K is a line segment of length l, then $c(K) = l/4$. To see this, we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way and let $K = [-l/2, l/2]$. The function $\psi(z) = (l/4)(z + z^{-1})$ maps $\{z : |z| > 1\}$ bijectively to $\mathbb{C}\backslash K$, and it follows from the characterization of h_K in Lemma 5.8.1 that

$$(h_K \circ \psi)(z) = h_{\overline{B}}(z) = \log|z|.$$

Hence

$$r(K) = \lim_{z\to\infty} ((h_K \circ \psi)(z) - \log|\psi(z)|) = \lim_{z\to\infty} \log\left(\frac{|z|}{|\psi(z)|}\right) = -\log\frac{l}{4},$$

and so $c(K) = l/4$.

(iii) It is easy to see that $r(K)$, and hence $c(K)$, is invariant under translation and rotation. To see the effect of dilations, let K be a compact set in $\mathbb{R}^2$ and let $K_a = \{ax: x \in K\}$ for each positive number a. It is clear from Lemma 5.8.1 that $h_{K_a}(x) = h_K(a^{-1}x)$ and so

$$\begin{aligned} r(K_a) &= \lim_{x\to\infty} \left(h_K(a^{-1}x) - \log\|x\|\right) \\ &= \lim_{x\to\infty} \left(h_K(a^{-1}x) - \log\|a^{-1}x\|\right) - \log a = r(K) - \log a. \end{aligned}$$

Hence $c(K_a) = ac(K)$.

Theorem 5.8.4. *If $R_0 > 0$, then*

$$R\exp\left(-\frac{1}{{}^R\mathcal{C}(\cdot)}\right) \to c(\cdot) \qquad (R \to +\infty) \tag{5.8.1}$$

uniformly on the compact subsets of $B(0, R_0)$. (We interpret $e^{-1/a}$ as 0 when $a = 0$.)

Proof. If K is a polar compact subset of $B(0, R_0)$, then ${}^R\mathcal{C}(K) = 0$ when $R \ge R_0$, and the convergence in (5.8.1) clearly holds. Now suppose that K is a non-polar compact subset of $B(0, R_0)$ and let $\nu_{K,R}$ denote the capacitary distribution of K relative to $B(0, R)$ when $R \ge R_0$. From Theorem 4.1.5,

$$G_{B(0,R)}(x,y) = \begin{cases} \log\left(\dfrac{\|y\|}{R}\dfrac{\|y^* - x\|}{\|y - x\|}\right) & (y \ne 0) \\ \log(R/\|x\|) & (y = 0), \end{cases}$$

where $y^* = (R/\|y\|)^2 y$, so

$$G_{B(0,R)}(x,y) - \log R = U_y(x) + f_R(x,y), \tag{5.8.2}$$

where

$$f_R(x,y) = \begin{cases} \log\left\|\dfrac{y}{\|y\|} - \dfrac{\|y\|x}{R^2}\right\| & (y \ne 0) \\ 0 & (y = 0). \end{cases}$$

The functions $f_R(\cdot,\cdot)$ converge to 0 locally uniformly on $\mathbb{R}^2 \times \mathbb{R}^2$ as $R \to +\infty$. Now let $(R(n))$ be any sequence in $(R_0, +\infty)$ such that $R(n) \to +\infty$ and define

$$u_n(x) = \int_K \left(G_{B(0,R(n))}(x,y) - \log R(n)\right) \frac{d\nu_{K,R(n)}(y)}{{}^{R(n)}\mathcal{C}(K)} \quad (x \in B(0, R(n)))$$

and $u_n = 0$ on $\mathbb{R}^2 \backslash B(0, R(n))$. Then $u_n \le \{{}^{R(n)}\mathcal{C}(K)\}^{-1} - \log R(n)$ on $B(0, R(n))$, with equality quasi-everywhere on K. It follows from (5.8.2) and the convergence to 0 of $f_R(\cdot,\cdot)$ that (u_n) is locally uniformly bounded below on $\mathbb{R}^2$ and locally uniformly bounded on $\mathbb{R}^2\backslash K$. Since u_n is harmonic on $B(0, R(n))\backslash K$, it follows from Theorem 1.5.11 that we can choose a subsequence (u_{n_k}) which converges locally uniformly to a harmonic function on $\mathbb{R}^2\backslash K$. Let $v = \liminf_{k\to\infty} u_{n_k}$ on $\mathbb{R}^2$. Then $\hat{v} \in \mathcal{U}(\mathbb{R}^2) \cap \mathcal{H}(\mathbb{R}^2\backslash K)$ by Corollary 5.7.2. On $\mathbb{R}^2$ we have

$$\hat{v}(x) \le \liminf_{k\to\infty}\left(\left\{{}^{R(n_k)}\mathcal{C}(K)\right\}^{-1} - \log R(n_k)\right) = a, \text{ say,} \tag{5.8.3}$$

with equality quasi-everywhere on K. Since K is non-polar, $a < +\infty$. Also, from (5.8.2) and the local uniform convergence to 0 of $f_R(\cdot,\cdot)$,

$$\log\frac{1}{\|x\| + R_0} \le \hat{v}(x) \le \log\frac{1}{\|x\| - R_0} \quad (\|x\| > R_0). \tag{5.8.4}$$

It now follows from Lemma 5.8.1 that $a - \hat{v} = h_K$, and that $r(K) = a$. Thus a is independent of the choice of $(R(n))$, so $\{{}^R\mathcal{C}(K)\}^{-1} - \log R \to r(K)$ as $R \to +\infty$, by (5.8.3), and (5.8.1) holds. Further, from Lemma 5.8.1, the Riesz measure $\mu_{\hat{v}}$ associated with $\hat{v}$ satisfies $\mu_{\hat{v}}(\mathbb{R}^2) = 1$.

To prove that this convergence holds uniformly over the collection of all compact subsets K of $B(0, R_0)$, let h denote the greatest harmonic minorant of $\hat{v}$ on $B(0, R)$. Then it follows from (5.8.4) that

$$\log\frac{1}{R + R_0} \le h \le \log\frac{1}{R - R_0} \quad \text{on } B(0,R),$$

so $a + \log(R - R_0) \le \hat{v} - h$ quasi-everywhere on K and $\hat{v} - h \le a + \log(R + R_0)$ on $B(0, R)$ (see (5.8.3)). Since $\hat{v} - h$ is a potential on $B(0, R)$ by Corollary 4.4.7, it follows from Theorem 5.5.5 that

$$\frac{1}{r(K) + \log(R + R_0)} \le {}^R\mathcal{C}(K) \le \frac{1}{r(K) + \log(R - R_0)}, \tag{5.8.5}$$

whence

$$\log\left(1 - \frac{R_0}{R}\right) \le \left(\frac{1}{{}^R\mathcal{C}(K)} - \log R\right) - r(K) \le \log\left(1 + \frac{R_0}{R}\right).$$

This establishes that the convergence in (5.8.1) is uniform over the collection of all compact subsets of $B(0, R_0)$. □

Definition 5.8.5. If $E \subseteq \mathbb{R}^2$, then we define the *inner logarithmic capacity* of E by

$$c_*(E) = \sup\{c(K)\colon K \text{ is a compact subset of } E\}$$

and the *outer logarithmic capacity* of E by

$$c^*(E) = \inf\{c_*(\omega)\colon \omega \text{ is an open set containing } E\}.$$

These set functions take values in $[0, +\infty]$. We note that, if $E \subseteq F \subseteq \Omega$, then $c_*(E) \le c_*(F)$, $c^*(E) \le c^*(F)$ and $c_*(E) \le c^*(E)$. A set E is called *log-capacitable* if $c_*(E) = c^*(E)$. As usual, if E is a log-capacitable set, then we write $c(E)$ for the common value of $c_*(E)$ and $c^*(E)$.

Theorem 5.8.6. *If E is a bounded set, then*

$$c_*(E) = \lim_{R\to+\infty} R\exp\left(-\frac{1}{{}^R C_*(E)}\right) \tag{5.8.6}$$

and

$$c^*(E) = \lim_{R\to+\infty} R\exp\left(-\frac{1}{{}^R C^*(E)}\right). \tag{5.8.7}$$

Proof. Let R_0 be such that $E \subseteq B(0, R_0)$ and let $\varepsilon > 0$. It follows from Theorem 5.8.4 that there exists R_1 such that

$$R\exp\left(-\frac{1}{{}^R C(K)}\right) - \varepsilon < c(K) < R\exp\left(-\frac{1}{{}^R C(K)}\right) + \varepsilon \quad (R \geq R_1)$$

for every compact subset K of E. If we take the supremum over all such K, we obtain

$$R\exp\left(-\frac{1}{{}^R C_*(E)}\right) - \varepsilon \leq c_*(E) \leq R\exp\left(-\frac{1}{{}^R C_*(E)}\right) + \varepsilon \quad (R \geq R_1).$$

Hence (5.8.6) holds. Further, if we replace E in the above inequality by an open set ω satisfying $E \subseteq \omega \subseteq B(0, R_0)$, and take the infimum over all such ω, then (5.8.7) is seen to hold. □

Corollary 5.8.7. *(i) Any bounded analytic set is log-capacitable.*
(ii) A bounded set E is polar if and only if $c^*(E) = 0$.

Proof. Part (i) follows from Theorems 5.8.6 and 5.6.4.

To prove (ii), let E be a bounded polar set. Then ${}^R C^*(E) = 0$ for all large R, and so $c^*(E) = 0$. Conversely, if $c^*(E) = 0$ then, for each n in $\mathbb{N}$, there is a bounded open set ω_n such that $E \subseteq \omega_n$ and $c(\omega_n) < n^{-1}$. Let $F = \bigcap_n \omega_n$ and let R be such that $F \subseteq B(0, R)$. Then $E \subseteq F$ and $c^*(F) = 0$. If K is any compact subset of F, then $c(K) = 0$ and so K is polar. Hence ${}^R C(K) = 0$ for all such K, so ${}^R C_*(F) = 0$. Since also F is a Borel set, F is polar and thus E is polar. □

Theorem 5.8.8. *If E is a bounded set in $\mathbb{R}^2$ and $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a contraction, then $c^*(f(E)) \leq c^*(E)$.*

Proof. We may assume that $f(E)$ is non-polar, for otherwise there is nothing to prove, in view of Corollary 5.8.7(ii). We first consider the case where E is compact. Let $\varepsilon > 0$. There is an open set ω such that $E \subseteq \omega$ and $c(\omega) < c(E) + \varepsilon$ and hence there is a compact set K such that $E \subseteq K^\circ$ and $c(K) < c(E) + \varepsilon$. If h_K denotes the function of Lemma 5.8.1, then $h_K = r_K - U\mu$ for some unit measure μ on ∂K, and $U\mu = r_K$ on K°. By the uniform continuity of $(x, y) \mapsto \log\|x - y\|$ on $E \times \partial K$ there are points $y_1, y_2, \ldots, y_n$ in ∂K and non-negative constants $a_1, a_2, \ldots, a_n$ such that

$$\sum_{k=1}^{n} a_k \log\|x - y_k\| \leq -r_K + \varepsilon \qquad (x \in E)$$

and $\sum_k a_k = 1$. If we define

$$u(x) = \sum_{k=1}^{n} a_k \log\|x - f(y_k)\| \qquad (x \in \mathbb{R}^2),$$

then $u(f(x)) \leq -r_K + \varepsilon$ when $x \in E$ since f is a contraction. Hence $u \leq -r_K + \varepsilon$ on the compact set $f(E)$, and it follows from Theorem 5.2.6 that

$$u(x) + r_K - \varepsilon - h_{f(E)}(x) \leq 0 \qquad (x \in \mathbb{R}^2 \backslash f(E)).$$

Hence

$$\begin{aligned} r_{f(E)} &= \lim_{x\to\infty} \left(h_{f(E)}(x) - \log\|x\|\right) \\ &\geq \lim_{x\to\infty} (u(x) - \log\|x\|) + r_K - \varepsilon = r_K - \varepsilon, \end{aligned}$$

and so

$$c(f(E)) \leq e^\varepsilon c(K) < e^\varepsilon (c(E) + \varepsilon).$$

Since ε is arbitrary, we obtain $c(f(E)) \leq c(E)$.

Now let E be an arbitrary bounded set and let ω be a bounded open set such that $E \subseteq \omega$ and $c(\omega) < c^*(E) + \varepsilon$. Let (K_n) be an increasing sequence of compact sets such that $\bigcup_n K_n = \omega$. Then, by Theorems 5.5.6(i), 5.8.4 and 5.8.6, $c(K_n) \to c(\omega)$ and $c(f(K_n)) \to c(f(\omega))$, so

$$c^*(f(E)) \leq c(f(\omega)) = \lim_{n\to\infty} c(f(K_n)) \leq \lim_{n\to\infty} c(K_n) = c(\omega) < c^*(E) + \varepsilon.$$

Since ε is arbitrary, the result follows. □

Corollary 5.8.9. *If E is a polar subset of $\mathbb{R}^2$, then E is totally disconnected; that is, every component of E is a singleton.*

Proof. It is enough to show that, if $x \in E$, then there are rectangles K_x of arbitrarily small diameter such that $x \in K_x^\circ$ and $E \cap \partial K_x = \emptyset$. Further, it is enough to prove this for bounded polar sets E. So we suppose that E is bounded and note from Corollary 5.8.7 that $c^*(E) = 0$. From the above theorem the projections of E onto each of the coordinate axes have outer logarithmic capacity 0. These projections cannot contain any line segment, in view of Example 5.1.6. Hence there are rectangles K_x as described above. □

We remark that, although logarithmic capacity has many properties in common with Green capacity, it is not subadditive (see Exercise 5.17).

5.9. Hausdorff measure and capacity

In this section we give some metric estimates of the size of polar sets in terms of Hausdorff measures.

Definition 5.9.1. Any increasing function $\phi:(0,+\infty)\to(0,+\infty]$ such that $\phi(t)\to 0$ as $t\to 0$ is called a *measure function*. If E is a bounded set in $\mathbb{R}^N$ and $\rho\in(0,+\infty]$, we define

$$M_\phi^{(\rho)}(E)=\inf\left\{\sum_k \phi(r_k): E\subseteq\bigcup_k B(x_k,r_k) \text{ and } r_k<\rho \text{ for each } k\right\},$$

where the infimum is over all possible coverings of E by a countable collection of balls $\{B(x_k,r_k):k\geq 1\}$ such that $r_k<\rho$ for each k. Since $M_\phi^{(\rho)}(E)$ is decreasing as a function of ρ, we can define

$$m_\phi(E)=\lim_{\rho\to 0} M_\phi^{(\rho)}(E),$$

which is called the *Hausdorff ϕ-measure* of E. Clearly $0\leq m_\phi(E)\leq+\infty$. In the special case where $\phi(t)=t^\alpha$ $(\alpha>0)$, we also write $M_{(\alpha)}^{(\rho)}(E)$ for $M_\phi^{(\rho)}(E)$ and $m_{(\alpha)}(E)$ for $m_\phi(E)$.

Lemma 5.9.2. *If E is a bounded set in $\mathbb{R}^N$, then there exists a number α_E in the interval $[0,N]$ such that*

$$m_{(\alpha)}(E)=\begin{cases}+\infty & (\alpha<\alpha_E)\\ 0 & (\alpha>\alpha_E).\end{cases}$$

Proof. First we note that $m_{(\alpha)}(E)=0$ when $\alpha>N$. To see this, let K be a cube which contains E, let $2a$ be its diameter and let $n\in\mathbb{N}$. If we divide K into n^N identical cubes, we see that E can be covered by n^N balls of radius a/n, and so

$$m_{(\alpha)}(E)\leq\lim_{n\to\infty} n^N(a/n)^\alpha=0 \qquad (\alpha>N).$$

We now define $\alpha_E=\inf\{\alpha>0: m_{(\alpha)}(E)=0\}$, and observe that $0\leq\alpha_E\leq N$.

Let $\alpha>\alpha_E$ and $\rho>0$. Then there exists β in (α_E,α) such that $m_{(\beta)}(E)=0$, and so there is a countable covering $\{B(x_k,r_k):k\geq 1\}$ of E such that $r_k<\rho$ for each k and $\sum_k r_k^\beta<1$. Hence

$$M_{(\alpha)}^{(\rho)}(E)\leq\sum_k r_k^\alpha=\sum_k r_k^{\alpha-\beta}r_k^\beta<\rho^{\alpha-\beta}\to 0 \qquad (\rho\to 0).$$

Thus $m_{(\alpha)}(E)=0$ when $\alpha>\alpha_E$.

If $0<\alpha<\alpha_E$, then there exists β in (α,α_E) such that $m_{(\beta)}(E)>0$. If $\rho>0$ and $\{B(x_k,r_k):k\geq 1\}$ is any countable covering of E such that $r_k<\rho$ for each k, then

$$\sum_k r_k^\alpha=\sum_k r_k^{\alpha-\beta}r_k^\beta>\rho^{\alpha-\beta}M_{(\beta)}^{(\rho)}(E)$$

so

$$M_{(\alpha)}^{(\rho)}(E)\geq\rho^{\alpha-\beta}M_{(\beta)}^{(\rho)}(E)\to+\infty \qquad (\rho\to 0)$$

and hence $m_{(\alpha)}(E)=+\infty$. This completes the proof of the lemma. □

Definition 5.9.3. The number α_E of Lemma 5.9.2 is called the *Hausdorff dimension* of E.

We recall that $V_N(t)=t^{2-N}$ $(N\geq 3)$ and $V_2(t)=\log(1/t)$ when $t>0$. Below we interpret $1/V_2^+(t)$ as $+\infty$ when $t\geq 1$.

Theorem 5.9.4. *If E is a bounded analytic set such that $m_\phi(E)<+\infty$, where $\phi(t)=1/V_N^+(t)$, then E is polar.*

Proof. Suppose that E is not polar. Then there is a non-polar compact subset K of E, in view of the analyticity of E and Corollary 5.5.7. We can choose K to be contained in a ball B_0 of diameter 1. It follows from Theorem 5.3.7 that there is a non-zero measure μ with support in K such that the superharmonic function $u(x)=\int V_N(\|x-y\|)\,d\mu(y)$ is finite-valued and continuous on $\mathbb{R}^N$. Let

$$u_\rho(x)=2^{-1}\int V_N(\|x-y\|)(2-\|x-y\|/\rho)^+\,d\mu(y) \qquad (x\in\mathbb{R}^N;0<\rho<1).$$

By Fatou's lemma,

$$\begin{aligned}u_\rho(z)\leq\liminf_{x\to z}u_\rho(x)&\leq\limsup_{x\to z}u_\rho(x)\\&=u(z)-\liminf_{x\to z}(u-u_\rho)(x)\leq u_\rho(z) \qquad (z\in\mathbb{R}^N),\end{aligned}$$

so u_ρ is continuous. Further, $\mu(\{x\})=0$ for every x by the finiteness of u, so $u_\rho\downarrow 0$ on $\overline{B_0}$ as $\rho\to 0$, and this convergence is uniform by Dini's theorem. Hence there is a decreasing sequence (ρ_n) in $(0,1)$ such that $u_{\rho_n}<2^{-n}$ on $\overline{B_0}$. We define

$$f(t)=2^{-1}V_N(t)\sum_{n=1}^\infty\chi_{[0,\rho_n]}(t) \qquad (t\geq 0), \tag{5.9.1}$$

where χ_A denotes the characteristic function valued 1 on $A \subseteq [0,+\infty)$ and 0 elsewhere on $[0,+\infty)$, and observe that

$$\int_K f(\|x-y\|)\,d\mu(y) \le 1 \qquad (x \in \overline{B_0}). \tag{5.9.2}$$

Now let n be large enough so that $\rho_n < \operatorname{dist}(K, \mathbb{R}^N \backslash B_0)$ and let $\{B(x_k, r_k): k \ge 1\}$ be a (finite) covering of the compact set K such that $r_k < \rho_n$ for each k. If we discard any ball which does not intersect K, then $x_k \in B_0$ for each k. Since f is a decreasing function, (5.9.2) yields $f(r_k)\mu(B(x_k, r_k)) \le 1$ and it is clear from (5.9.1) that $f(r_k) \ge nV_N(r_k)/2$. Hence

$$\mu(K) \le \sum_k \mu(B(x_k, r_k)) \le \sum_k \{f(r_k)\}^{-1} \le 2n^{-1} \sum_k \{V_N(r_k)\}^{-1},$$

and so

$$M_\phi^{(\rho_n)}(E) \ge M_\phi^{(\rho_n)}(K) \ge \frac{n}{2}\mu(K) \to +\infty \qquad (n \to \infty).$$

This leads to the contradictory conclusion that $m_\phi(E) = +\infty$, so E must be polar. □

If E is a bounded analytic set in $\mathbb{R}^N$ $(N \ge 3)$ and the Hausdorff dimension α_E of E satisfies $\alpha_E < N-2$, then it is clear from the above result that E is polar. In the opposite direction we will see below that, if E is a polar set, then $\alpha_E \le N-2$. First we give a preparatory lemma.

Lemma 5.9.5. *Let $u = \int U_y d\mu(y)$, where $\operatorname{supp}\mu$ is compact, and suppose that ϕ is a measure function such that*

$$\int_0^1 t^{1-N}\phi(t)dt < +\infty. \tag{5.9.3}$$

Then there is a constant C, depending only on N and the value of the integral in (5.9.3), such that

$$u(z) \le C \sup\left\{\frac{\mu(B(z,r))}{\phi(r)} : r > 0\right\} + \mu(\mathbb{R}^N) \qquad (z \in \mathbb{R}^N).$$

Proof. Let $m_z(r) = \mu(B(z,r))$. If $N \ge 3$, then integration by parts yields

$$\begin{aligned} u(z) &\le \int_0^1 t^{2-N} dm_z(t) + \mu(\mathbb{R}^N \backslash B(z,1)) \\ &\le (N-2)\int_0^1 t^{1-N} m_z(t)dt + \mu(\mathbb{R}^N). \end{aligned}$$

If $N = 2$, then similarly $u(z) \le \int_0^1 t^{-1} m_z(t)dt$. In either case, the result follows. □

Theorem 5.9.6. *If E is a bounded polar set, then $m_\phi(E) = 0$ for any measure function ϕ which satisfies (5.9.3). In particular, $m_\alpha(E) = 0$ when $\alpha > N-2$ and so the Hausdorff dimension of E is at most $N-2$.*

Proof. By Theorem 5.1.3 there is a measure μ such that the function $\int U_y d\mu(y)$ is superharmonic on $\mathbb{R}^N$ and valued $+\infty$ on E. Further, since E is bounded, we can arrange (by choosing a suitable restriction of μ) that μ has compact support. If we define $\phi_1(t) = \min\{\phi(5t), \phi(1/2)\}$, then (5.9.3) holds when ϕ is replaced by ϕ_1. Let $a > 0$. If $x \in E$ then $u(x) = +\infty$, so we see from Lemma 5.9.5 that there exists $r_x > 0$ such that $\mu(B(x, r_x)) > a\phi_1(r_x)$ and r_x is at most the diameter of E. By Lemma 4.6.1 there is a countable disjoint subcollection $\{B(x_k, r_{x_k}): k \ge 1\}$ such that $E \subseteq \bigcup_k B(x_k, 5r_{x_k})$. Thus

$$\sum_k \phi_1(r_{x_k}) \le a^{-1} \sum_k \mu(B(x_k, r_{x_k})) \le a^{-1}\mu(\mathbb{R}^N).$$

Hence, for large a, we have $\phi_1(r_{x_k}) = \phi(5r_{x_k})$ for all k and so

$$M_\phi^{(+\infty)}(E) \le \sum_k \phi(5r_{x_k}) = \sum_k \phi_1(r_{x_k}) \le a^{-1}\mu(\mathbb{R}^N) \to 0 \qquad (a \to +\infty).$$

Since ϕ is increasing and positive, it follows easily that $M_\phi^{(\rho)}(E) = 0$ for any $\rho > 0$ and hence $m_\phi(E) = 0$. □

5.10. Exercises

Exercise 5.1. (i) Let $u \in \mathcal{U}_+(\mathbb{R}^{N-1} \times (0,+\infty))$, where $N \ge 3$, and let

$$u'(x') = \int_{\mathbb{R}} u(t, x')dt \qquad (x' \in \mathbb{R}^{N-2} \times (0,+\infty)).$$

Show that either u' is superharmonic or $u' \equiv +\infty$.
(ii) Deduce that, if $E' \subseteq \mathbb{R}^{N-1}$ and $E' \times \mathbb{R}$ is polar in $\mathbb{R}^N$, then E' is polar in $\mathbb{R}^{N-1}$. Is the converse true?

Exercise 5.2. Let $\Omega \subseteq \mathbb{C}$ and let E be a relatively closed polar subset of Ω. Show that, if f is holomorphic on $\Omega \backslash E$ and bounded near points of E, then f has a unique holomorphic extension to Ω.

Exercise 5.3. Let $h \in \mathcal{H}(B \backslash E)$, where E is a relatively closed polar subset of B. Show that, if $\int_B \|\nabla h\|^2 d\lambda < +\infty$, then h has a unique harmonic extension to B. (Hint: begin by using Corollary 4.4.6 to see that the subharmonic function $s = h^2$ has a harmonic majorant v on $B\backslash E$. Show that, if $m \in \mathbb{N}$ and $r \in (0,1)$, then $m|h| - v$ is bounded above on $B\backslash E$, and hence that $|h| - I_{|h|,0,r} \le m^{-1}\{v - I_{v,0,r}\}$ on $B(0,r)\backslash E$. Deduce that h is bounded near points of E.)

Exercise 5.4. Given a point $x = (x_1, \dots, x_{N+2}) \in \mathbb{R}^{N+2}$, we write $t_x = (x_N^2 + x_{N+1}^2 + x_{N+2}^2)^{1/2}$, and we define $E' = \{x \in \mathbb{R}^{N+2} : t_x = 0\}$. Let h be harmonic on $D = \mathbb{R}^{N-1} \times (0, +\infty)$. Show that the function H defined by

$$H(x_1, \dots, x_{N+2}) = \frac{h(x_1, \dots, x_{N-1}, t_x)}{t_x}$$

is harmonic on $\mathbb{R}^{N+2}\backslash E'$. Show further that if $h \ge 0$ on D, then H has a superharmonic extension $\widetilde{H}$ to $\mathbb{R}^{N+2}$ and apply the Riesz decomposition theorem to $\widetilde{H}$ to deduce that h has the Poisson integral representation given in Theorem 1.7.3.

Exercise 5.5. Let $\Omega \subseteq \mathbb{C}$ be a domain and $E \subseteq \mathbb{C}$ be a polar set.
(i) Show that, if $f : \Omega \to \mathbb{C}$ is holomorphic, then either $f^{-1}(E) = \Omega$ or $f^{-1}(E)$ is polar. (Hint: use Corollary 3.3.4.)
(ii) Suppose further that E is closed. Show that, if $f : \Omega \to \mathbb{C}$ is continuous and f is holomorphic on $\Omega\backslash f^{-1}(E)$, then f is holomorphic on Ω.

Exercise 5.6. Show that, if K is a compact polar set in $\mathbb{R}^N$ $(N \ge 3)$, then there is a Newtonian potential u such that $u = +\infty$ on K and u is harmonic on $\mathbb{R}^N\backslash K$. (Hint: choose a decreasing sequence (K_n) of non-polar compact sets such that $\bigcap_n K_n = K$, and consider the Newtonian potentials $\{\mathcal{C}(K_n)\}^{-1}\widehat{R}_1^{K_n}$.)

Exercise 5.7. In $\mathbb{R}^2$ let $\Omega = B(0, \sqrt{12})$ and $E = \overline{B((0,-1), \sqrt{6})}$. Show that the reduced function R_1^E relative to E in Ω is given on $\Omega\backslash E$ by

$$R_1^E(x_1, x_2) = \log\Big(\frac{3}{4} \cdot \frac{x_1^2 + (x_2+4)^2}{x_1^2 + (x_2+3)^2}\Big) \Big/ \log\frac{9}{8}$$

and on E by $R_1^E(x) = 1$.

Exercise 5.8. Let $\Omega = \mathbb{R}^{N-1} \times (0, +\infty)$ and $r > 0$.
(i) If $u(x) = x_N \|x\|^{-N}$, find $R_u^{\Omega\cap B(0,r)}$ and also show that

$$R_u^{\Omega\backslash B(0,r)}(x) = x_N \min\{\|x\|^{-N}, r^{-N}\}.$$

(ii) If $v(x) = x_N$ and $E = \mathbb{R}^{N-1} \times (0, r]$, find R_v^E.

Exercise 5.9. Let $E = \{x' \in \mathbb{R}^{N-1} : \|x'\| \le 1\} \times \mathbb{R}$, where $N \ge 3$. Show that

$$R_1^E(x', x_N) = \begin{cases} 1 & (N = 3) \\ \min\{\|x'\|^{3-N}, 1\} & (N \ge 4) \end{cases} \qquad ((x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}),$$

the reduced function being relative to E in $\mathbb{R}^N$.

Exercise 5.10. Show that, if $\overline{E}$ is a compact subset of a Greenian open set Ω, then

$$\mathcal{C}_*(E) = \sup\Big\{\int G_\Omega \mu \, d\mu : \ \operatorname{supp} \mu \subseteq E \text{ and } G_\Omega\mu \le 1 \text{ on } \Omega\Big\}$$

and

$$\mathcal{C}^*(E) = \inf\Big\{\int G_\Omega \mu \, d\mu : \ G_\Omega \mu \ge 1 \text{ quasi-everywhere on } E\Big\}.$$

Exercise 5.11. Let E be a compact subset of $\mathbb{R}^N$, where $N \ge 3$, such that $\{rx : r \ge 1\} \cap E$ is non-empty for each $x \in S$. Show that $\mathcal{C}(E) \ge 1$.

Exercise 5.12. Show that, if $u \in \mathcal{U}(\mathbb{R}^N)$ where $N \ge 3$, then there is a polar set E contained in the unit sphere S such that the function $r \mapsto u(ry)$ is continuous on $(0, +\infty)$ whenever $y \in S\backslash E$. (Hint: use Theorem 5.5.8.)

Exercise 5.13. Let Ω be Greenian and let $\mu, \mu_1, \mu_2, \dots$ be measures with support contained in a compact set $K \subset \Omega$ such that (μ_n) is w^*-convergent to μ; that is,

$$\int f \, d\mu_n \to \int f \, d\mu \qquad (f \in C(K)).$$

Show that $\liminf_{n\to\infty} G_\Omega \mu_n = G_\Omega \mu$ quasi-everywhere on Ω. (Hint: first show that $\liminf G_\Omega \mu_n \ge G_\Omega \mu$ on Ω. Next assume that the (Borel) set where $\liminf G_\Omega \mu_n > G_\Omega \mu$ has positive capacity and use Theorem 5.3.7.)

Exercise 5.14. Let $u \in \mathcal{U}_+(\mathbb{R} \times \Omega')$, where Ω' is a non-empty open set in $\mathbb{R}^{N-1}$ $(N \ge 3)$, and define $u'(x') = \inf_t u(t, x')$ for each $x' \in \Omega'$. Show that there exists $v' \in \mathcal{U}(\Omega')$ such that $u' = v'$ quasi-everywhere on Ω'. (Hint: use Exercise 5.1.)

Exercise 5.15. Show that, if K is a closed ellipse with semi-axes of length a and b, then $c(K) = (a+b)/2$. (Hint: dismissing the case of the disc, we may assume that $a > b$. The function $\psi(z) = (a^2 - b^2)^{1/2}(z + z^{-1})/2$ maps $\{z : |z|^2 > (a+b)/(a-b)\}$ bijectively to $\mathbb{C}\backslash K$.)

Exercise 5.16. Let E be a compact subset of $\mathbb{R}$ of Lebesgue measure l. Show that $c(E \times \{0\}) \ge l/4$. (Hint: consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x_1, x_2) = (\lambda(E \cap (-\infty, x_1]), x_2)$.)

Exercise 5.17. For each $n \in \mathbb{Z}$ let $K_n = [n-2, n+2] \times \{0\}$ and let h_n denote the Green function for $\mathbb{R}^2 \backslash K_n$ with pole at ∞.
(i) Show that, if we fix $m > 0$ sufficiently large, then $h_m > \log m$ on K_{-m}.
(ii) Now let $\Omega = \mathbb{R}^2 \backslash (K_m \cup K_{-m})$ and let h denote the Green function for Ω with pole at ∞. By applying Theorem 5.2.6 to the function $h + \log \sqrt{m} - (h_m + h_{-m})/2$ on Ω, show that

$$c(K_m \cup K_{-m}) \geq \sqrt{m} = \sqrt{m/4}\{c(K_m) + c(K_{-m})\}.$$

(Thus, in particular, logarithmic capacity is not subadditive.)

Exercise 5.18. Let E' be a non-polar compact subset of $\mathbb{R}^{N-1}$, where $N \geq 3$. Show that $\mathcal{C}(E' \times \mathbb{R}) = +\infty$, where $\mathcal{C}(\cdot)$ denotes Newtonian capacity.

Chapter 6. The Dirichlet Problem

6.1. Introduction

In its simplest form the Dirichlet problem may be stated as follows: for a given function $f \in C(\partial^\infty \Omega)$, determine, if possible, a function $h \in \mathcal{H}(\Omega)$ such that $h(x) \to f(y)$ as $x \to y$ for each $y \in \partial^\infty \Omega$. Such a function h is called the *(classical) solution of the Dirichlet problem* on Ω with boundary function f, and the maximum principle guarantees the uniqueness of the solution if it exists. For example, if Ω is either a ball or a half-space and $f \in C(\partial^\infty \Omega)$, then the solution of the Dirichlet problem certainly exists and is given by the Poisson integral of f. This follows immediately from Theorems 1.3.3 and 1.7.5. On the other hand, there are quite simple examples in which there is no such solution.

Example 6.1.1. If $\Omega = B \backslash \{0\}$ and $f : \partial \Omega \to \mathbb{R}$ is defined by $f(0) = 1$ and $f(x) = 0$ when $x \in S$, then the Dirichlet problem on Ω with boundary function f has no classical solution.

To verify this, suppose that a solution h exists. Then h is bounded on Ω and has a harmonic continuation $\overline{h}$ to B, by Theorem 1.3.7. Since $\overline{h}$ has limit 0 at each point of S, it follows that $\overline{h} \equiv 0$, contrary to the requirement that $\overline{h}(0) = \lim_{x \to 0} h(x) = f(0) = 1$.

In this chapter we will discuss the Perron–Wiener–Brelot (PWB) approach to the Dirichlet problem. For a wide class of functions f on $\partial^\infty \Omega$, the PWB approach associates with f a corresponding function $H_f \in \mathcal{H}(\Omega)$ in such a way that H_f is equal to the classical solution of the Dirichlet problem on Ω with boundary function f whenever this classical solution exists. Even in the absence of a classical solution, H_f provides a slightly weaker solution in the sense that $H_f(x) \to f(y)$ as $x \to y$ at most points of continuity of f.

We shall assume throughout this chapter that Ω is Greenian, and except where the contrary is stated, this will be the only restriction on Ω. For non-Greenian sets most of the major results would become either false or trivial.

Two important ideas which are discussed in this chapter are harmonic measure and regularity. Harmonic measure, which is introduced in Section 6.4, is a measure μ_x on $\partial^\infty \Omega$, depending on the point $x \in \Omega$, which allows the representation

$$H_f(x) = \int_{\partial^\infty \Omega} f \, d\mu_x$$

whenever the PWB solution H_f exists. In a ball, for example, the above integral is just the Poisson integral of f (see Example 6.4.9 below). A point y of $\partial^\infty \Omega$ will be called regular if $H_f(x) \to f(y)$ whenever $f \in C(\partial^\infty \Omega)$. It turns out that regularity is intimately connected with the idea of thinness, a major topic in the next chapter, in which an important criterion for regularity in terms of thinness is established. In Section 6.6 another criterion is given which leads to some simple and useful geometric conditions that are sufficient for the regularity of a boundary point.

6.2. Upper and lower PWB solutions

It is convenient to start by enlarging the classes $\mathcal{U}(\Omega)$ and $\mathcal{S}(\Omega)$.

Definition 6.2.1. A function u is called *hyperharmonic* on Ω if on each component of Ω either $u \equiv +\infty$ or u is superharmonic. A function s is called *hypoharmonic* on Ω if $-s$ is hyperharmonic on Ω.

Definition 6.2.2. Let f be an extended real-valued function defined at least on $\partial^\infty \Omega$. Families of functions on Ω are defined by

$$\Phi_f^\Omega = \{u : u \text{ is hyperharmonic and bounded below on } \Omega \text{ and } \liminf_{x\to y} u(x) \ge f(y) \text{ for each } y \in \partial^\infty \Omega\},$$

$$\Psi_f^\Omega = \{s : s \text{ is hypoharmonic and bounded above on } \Omega \text{ and } \limsup_{x\to y} s(x) \le f(y) \text{ for each } y \in \partial^\infty \Omega\}.$$

When there is no risk of ambiguity we simply write Φ_f and Ψ_f. We note that Φ_f and Ψ_f are never empty, since they contain the constant functions $+\infty$ and $-\infty$ respectively.

Definition 6.2.3. Let f be an extended real-valued function defined at least on $\partial^\infty \Omega$. The *upper* and *lower* PWB *solutions* $\overline{H}_f^\Omega$ and $\underline{H}_f^\Omega$ are defined on Ω by

$$\overline{H}_f^\Omega(x) = \inf\{u(x) : u \in \Phi_f^\Omega\}, \qquad \underline{H}_f^\Omega(x) = \sup\{s(x) : s \in \Psi_f^\Omega\}.$$

We sometimes write simply $\overline{H}_f$ and $\underline{H}_f$.

Lemma 6.2.4. *Let ω be a component of Ω. For every function f on $\partial^\infty \Omega$, we have $\overline{H}_f^\Omega = \overline{H}_f^\omega$ on ω.*

Proof. Note that $\partial^\infty \omega \subseteq \partial^\infty \Omega$, so f is defined on $\partial^\infty \omega$. If $u \in \Phi_f^\Omega$, then $u|_\omega \in \Phi_f^\omega$, so $\overline{H}_f^\omega \le \overline{H}_f^\Omega$ on ω. Conversely, if $v \in \Phi_f^\omega$, then the function equal to v on ω and $+\infty$ on $\Omega \backslash \omega$ belongs to Φ_f^Ω, so $\overline{H}_f^\omega \ge \overline{H}_f^\Omega$ on ω. □

Theorem 6.2.5. *For every function $f : \partial^\infty \Omega \to [-\infty, +\infty]$ the following statements hold.*
(i) $\overline{H}_f = -\underline{H}_{-f}$.
(ii) On each component of Ω each of the functions $\overline{H}_f, \underline{H}_f$ is identically $+\infty$, identically $-\infty$, or harmonic.
(iii) $\underline{H}_f \le \overline{H}_f$ on Ω.

Proof. (i) It is easy to check that $\Phi_f = \{-u : u \in \Psi_{-f}\}$, and this gives the result.

(ii) In view of (i) it is enough to consider $\overline{H}_f$, and by Lemma 6.2.4, it suffices to treat the case where Ω is connected. If $\Phi_f = \{+\infty\}$, then $\overline{H}_f \equiv +\infty$. Otherwise $\overline{H}_f = \inf \phi_f$, where $\phi_f = \Phi_f \cap \mathcal{U}(\Omega)$. It is easy to verify that ϕ_f is saturated (Definition 3.6.1). Hence, by Theorem 3.6.2, either $\overline{H}_f \equiv -\infty$ or $\overline{H}_f \in \mathcal{H}(\Omega)$.

(iii) Again we may suppose that Ω is connected. It is enough to show that if $u \in \Phi_f \cap \mathcal{U}(\Omega)$ and $s \in \Psi_f \cap \mathcal{S}(\Omega)$, then $u \ge s$ on Ω. This inequality will follow from the maximum principle if we show that

$$\limsup_{x\to y}(s-u)(x) \le 0 \tag{6.2.1}$$

for each $y \in \partial^\infty \Omega$. If $f(y)$ is finite, then

$$\begin{aligned}\limsup_{x\to y}(s-u)(x) &\le \limsup_{x\to y} s(x) - \liminf_{x\to y} u(x)\\ &\le f(y) - f(y)\\ &= 0.\end{aligned}$$

If $f(y) = +\infty$, then $u(x) \to +\infty$ as $x \to y$, while s is bounded above on Ω, and hence $(s-u)(x) \to -\infty$ as $x \to y$. A similar argument yields the same conclusion if $f(y) = -\infty$. Hence (6.2.1) holds at each point $y \in \partial^\infty \Omega$, as required. □

Definition 6.2.6. A function $f : \partial^\infty \Omega \to [-\infty, +\infty]$ is called *resolutive* if $\overline{H}_f$ and $\underline{H}_f$ are equal and finite-valued (and hence harmonic) on Ω. If f is resolutive, then we define $H_f = \overline{H}_f$ $(= \underline{H}_f)$ and call H_f the *PWB* (or *generalized*) *solution of the Dirichlet problem* on Ω with boundary function f. The class of all resolutive functions on $\partial^\infty \Omega$ is denoted by $\mathcal{R}(\Omega)$.

Remark 6.2.7. Suppose that $f \in C(\partial^\infty \Omega)$ and that the classical solution h of the Dirichlet problem on Ω with boundary function f exists. Note that

f is bounded, since $\partial^\infty\Omega$ is compact in the topology of $\mathbb{R}^N \cup \{\infty\}$. Since $h(x) \to f(y)$ as $x \to y$ for each $y \in \partial^\infty\Omega$, it follows that h is bounded on Ω, and hence $h \in \Phi_f \cap \Psi_f$. This implies that $\overline{H}_f \le h \le \underline{H}_f$ on Ω. Combining these inequalities with the inequality $\underline{H}_f \le \overline{H}_f$ of Theorem 6.2.5(iii), we obtain $\overline{H}_f = h = \underline{H}_f$. This shows that if the classical solution exists, then so also does the PWB solution and the two are equal. In particular, it shows that constant functions are resolutive and that, if $f \equiv a$, then $H_f \equiv a$. However there are many instances in which the PWB solution exists but the classical solution does not. Indeed a key result (Theorem 6.3.8 below) is that every finite-valued continuous function on $\partial^\infty\Omega$ is resolutive. Thus, for example, if Ω and f are as in Example 6.1.1, then the PWB solution exists but the classical one does not. (In fact, $H_f^\Omega \equiv 0$ in this case since $0 \in \Psi_f$ and $\varepsilon U_0 \in \Phi_f$ for every $\varepsilon > 0$.) Also highly discontinuous functions may be resolutive, as the following example shows.

Example 6.2.8. Let $\Omega = B$, let E be a countable dense subset of $\partial\Omega$, and let f be equal to 1 on E and 0 on $\partial\Omega\backslash E$. Then f is nowhere continuous on $\partial\Omega$, but $f \in \mathcal{R}(\Omega)$ and $H_f = 0$ on Ω.

To check the resolutivity of f, note that E is polar, so there exists a positive superharmonic function u on an open neighbourhood of $\overline{\Omega}$ such that $u = +\infty$ on E. For every positive number ε, we have $\varepsilon u \in \Phi_f$. Thus $\overline{H}_f \le 0$ quasi-everywhere and hence, by continuity, everywhere on Ω. Obviously $0 \in \Psi_f$, so $\underline{H}_f \ge 0$. Since $\underline{H}_f \le \overline{H}_f$, we have $\underline{H}_f = 0 = \overline{H}_f$. Thus $f \in \mathcal{R}(\Omega)$ and $H_f = 0$.

Remark 6.2.9. A function f on $\partial^\infty\Omega$ is resolutive if each component of Ω contains a point at which $\overline{H}_f$ and $\underline{H}_f$ are finite and equal. To verify this, let ω be such a component and suppose that $\overline{H}_f(x) = \underline{H}_f(x) \in \mathbb{R}$ for some $x \in \omega$. Then $\overline{H}_f, \underline{H}_f \in \mathcal{H}(\Omega)$ and $\underline{H}_f - \overline{H}_f \le 0$ on ω with equality at x. Hence, by the maximum principle, $\underline{H}_f = \overline{H}_f$ on ω.

Remark 6.2.10. The results in this section do not depend on our assumption that Ω is Greenian and remain valid without this assumption. However, the results are of little interest in the non-Greenian case, as we now explain. Suppose, for the moment, that Ω is a non-Greenian open subset of $\mathbb{R}^2$. By Theorem 5.3.8 and Corollary 5.1.5, Ω is connected and the only lower-bounded superharmonic functions on Ω are constant. Hence, for any function f on $\partial^\infty\Omega$, the class Φ_f contains only constant functions (including $+\infty$). Therefore $\overline{H}_f$ is constant, and similarly so is $\underline{H}_f$. Further, $\overline{H}_f = \underline{H}_f$ if and only if f is constant, so the only resolutive functions are the finite constant functions.

6.3. Further properties of PWB solutions

Theorem 6.3.1. *Let $f, g : \partial^\infty\Omega \to [-\infty, +\infty]$ and let $\alpha \in \mathbb{R}$.*
(i) If $(f+g)(y)$ is defined arbitrarily at points y of $\partial^\infty\Omega$ where $f(y)+g(y)$ has the indeterminate form $(\pm\infty)+(\mp\infty)$, then

$$\overline{H}_{f+g} \le \overline{H}_f + \overline{H}_g, \tag{6.3.1}$$

provided the right-hand side is well-defined on Ω, and with the same proviso

$$\underline{H}_{f+g} \ge \underline{H}_f + \underline{H}_g. \tag{6.3.2}$$

(ii) If $\alpha \ge 0$, then

$$\overline{H}_{\alpha f} = \alpha\overline{H}_f, \qquad \underline{H}_{\alpha f} = \alpha\underline{H}_f. \tag{6.3.3}$$

If $\alpha \le 0$, then

$$\overline{H}_{\alpha f} = \alpha\underline{H}_f, \qquad \underline{H}_{\alpha f} = \alpha\overline{H}_f. \tag{6.3.4}$$

Proof. (i) We may assume that Ω is connected. If $\overline{H}_f \equiv +\infty$ or $\overline{H}_g \equiv +\infty$, then (6.3.1) is trivial (or vacuous if $\overline{H}_f + \overline{H}_g$ is indeterminate). Hence we may suppose that $\Phi_f \cap \mathcal{U}(\Omega)$ and $\Phi_g \cap \mathcal{U}(\Omega)$ are non-empty. Let u, v respectively belong to these classes. Then $u+v$ is superharmonic and bounded below on Ω and

$$\liminf_{x\to y}(u+v)(x) \ge \liminf_{x\to y} u(x) + \liminf_{x\to y} v(x) \ge (f+g)(y)$$

at each point $y \in \partial^\infty\Omega$. (In the case where $f(y)+g(y)$ is indeterminate, it is easy to see that $u(x)+v(x) \to +\infty$ as $x \to y$.) Hence $u+v \in \Phi_{f+g}$. Since u, v are arbitrary elements of their respective classes, it follows in the case where $\overline{H}_f > -\infty$ that $\overline{H}_f + v \ge \overline{H}_{f+g}$ for all $v \in \Phi_f \cap \mathcal{U}(\Omega)$ and hence that (6.3.2) holds. In the case where $\overline{H}_f \equiv -\infty$, the same argument shows that for such v we have $-\infty + v \ge \overline{H}_{f+g}$ at all points where $v < +\infty$, so that $\overline{H}_{f+g} = -\infty$ at some, and hence all, points of Ω. The inequality (6.3.2) follows easily using Theorem 6.2.5(i).

(ii) If $\alpha = 0$, then the equations are trivial. If $\alpha > 0$, then $\alpha u \in \Phi_{\alpha f}$ is equivalent to $u \in \Phi_f$, and hence $\overline{H}_{\alpha f} = \alpha\overline{H}_f$. Similarly, $\underline{H}_{\alpha f} = \alpha\underline{H}_f$ when $\alpha > 0$. The equations (6.3.4) follow from (6.3.3) by Theorem 6.2.5(i). □

Corollary 6.3.2. *If $f, g \in \mathcal{R}(\Omega)$ and $\alpha \in \mathbb{R}$, then (with the convention of Theorem 6.3.1 regarding $f+g$) we have $f+g, \alpha f \in \mathcal{R}(\Omega)$ and*

$$H_{f+g} = H_f + H_g, \quad H_{\alpha f} = \alpha H_f.$$

Proof. Theorems 6.3.1 and 6.2.5(iii) give

$$H_f + H_g \le \underline{H}_{f+g} \le \overline{H}_{f+g} \le H_f + H_g$$

and $\overline{H}_{\alpha f} = \alpha H_f = \underline{H}_{\alpha f}$, and the corollary follows. □

Theorem 6.3.3. *Let (f_n) be a sequence of finite-valued functions in $\mathcal{R}(\Omega)$. If (f_n) converges uniformly on $\partial^\infty \Omega$ to a function f, then $f \in \mathcal{R}(\Omega)$ and $H_f = \lim H_{f_n}$.*

Proof. Fix a positive number ε and let n be so large that $|f_n - f| < \varepsilon$ on $\partial^\infty \Omega$. If $u \in \Phi_{f_n}$, then $u + \varepsilon \in \Phi_f$. Hence $\overline{H}_f \le H_{f_n} + \varepsilon$. Similarly $H_{f_n} - \varepsilon \le \underline{H}_f$. Since also $\underline{H}_f \le \overline{H}_f$, it follows that $\underline{H}_f$ and $\overline{H}_f$ are finite-valued, and that $\lim H_{f_n}$ exists and is equal to both $\overline{H}_f$ and $\underline{H}_f$ on Ω. □

Corollary 6.3.4. *The set of all bounded resolutive functions on $\partial^\infty \Omega$ is a vector space which, equipped with the norm $\|f\| = \sup_{\partial^\infty \Omega} |f|$, is a Banach space.*

Proof. It follows immediately from Corollary 6.3.2 that the bounded resolutive functions form a vector space. If (f_n) is a Cauchy sequence of such functions, then (f_n) converges uniformly on $\partial^\infty \Omega$ to some bounded function f, and by Theorem 6.3.3, $f \in \mathcal{R}(\Omega)$. Hence the bounded functions in $\mathcal{R}(\Omega)$ form a Banach space. □

Theorem 6.3.5. *Let (f_n) be an increasing sequence of extended real-valued functions on $\partial^\infty \Omega$, and let $f = \lim f_n$. If $\overline{H}_{f_m} > -\infty$ for some m, then $\overline{H}_f = \lim \overline{H}_{f_n}$ on Ω.*

Proof. Again we may suppose that Ω is connected. Since $(\overline{H}_{f_n})$ is increasing on Ω, we may also suppose that $\overline{H}_{f_n} > -\infty$ for all n. If $\overline{H}_{f_m} \equiv +\infty$ for some m, then $\overline{H}_{f_n} \equiv +\infty$ for all $n \ge m$ and $\overline{H}_f \equiv +\infty$. Now suppose that each $\overline{H}_{f_n}$ is finite-valued (and hence harmonic) on Ω. Fix a point $x_0 \in \Omega$ and a positive number ε. For each n, there exists $u_n \in \Phi_{f_n}$ such that

$$u_n(x_0) - \overline{H}_{f_n}(x_0) < \varepsilon 2^{-n}. \tag{6.3.5}$$

By Theorem 3.1.4, the function

$$u = \lim_{n\to\infty} \overline{H}_{f_n} + \sum_{n=1}^{\infty} (u_n - \overline{H}_{f_n}) \tag{6.3.6}$$

is hyperharmonic on Ω. Also $u \ge \overline{H}_{f_n} + (u_n - \overline{H}_{f_n}) = u_n$ for each n, so u is bounded below on Ω and

$$\liminf_{x\to y} u(x) \ge f_n(y) \qquad (y \in \partial^\infty \Omega; n \in \mathbb{N}).$$

Hence

$$\liminf_{x\to y} u(x) \ge f(y) \qquad (y \in \partial\Omega),$$

so that $u \in \Phi_f$ and $u \ge \overline{H}_f$. In particular,

$$\overline{H}_f(x_0) \le u(x_0) \le \lim_{n\to\infty} \overline{H}_{f_n}(x_0) + \varepsilon,$$

by (6.3.5) and (6.3.6). Since ε is arbitrary,

$$\overline{H}_f(x_0) \le \lim_{n\to\infty} \overline{H}_{f_n}(x_0).$$

Also, it is clear that $\lim \overline{H}_{f_n} \le \overline{H}_f$ on Ω. Since $(\overline{H}_{f_n})$ is an increasing sequence in $\mathcal{H}(\Omega)$, its limit is either identically $+\infty$ or harmonic on Ω. In the former case, $\overline{H}_f \equiv +\infty$. In the latter case, $\overline{H}_f - \lim \overline{H}_{f_n}$ belongs to $\mathcal{H}_+(\Omega)$ and attains the value 0 at x_0, so this function is identically 0, by the minimum principle. □

Theorem 6.3.6. *Let $f : \partial^\infty \Omega \to [-\infty, +\infty]$ and let F be defined on $\Omega \cup \partial^\infty \Omega$ by*

$$F(x) = f(x) \quad (x \in \partial^\infty \Omega), \qquad F(x) = \overline{H}_f^\Omega(x) \qquad (x \in \Omega).$$

If ω is an open subset of Ω, then $\overline{H}_F^\omega = \overline{H}_f^\Omega$ on ω. If $f \in \mathcal{R}(\Omega)$, then $F \in \mathcal{R}(\omega)$ and $H_F^\omega = H_f^\Omega$ on ω.

Proof. We may suppose that ω and Ω are connected. If $u \in \Phi_f^\Omega$, then clearly $u|_\omega \in \Phi_F^\omega$. Hence $\overline{H}_F^\omega \le \overline{H}_f^\Omega$ on ω. Clearly equality holds if $\overline{H}_f^\Omega \equiv -\infty$. If $\overline{H}_f^\Omega \equiv +\infty$, then it is easy to see that any element u of Φ_F^ω can be extended to an element of Φ_f^Ω by defining $u = +\infty$ on $\Omega \backslash \omega$, and since $\Phi_f^\Omega = \{+\infty\}$, we have $\Phi_F^\omega = \{+\infty\}$ and $\overline{H}_F^\omega \equiv +\infty$. Now suppose that $\overline{H}_f^\Omega$ is finite-valued (and hence harmonic) on Ω. If $u \in \Phi_F^\omega \cap \mathcal{U}(\omega)$, then by Corollary 3.2.4, the function $\tilde{u}$, defined by

$$\tilde{u} = \begin{cases} \min\{u, \overline{H}_f^\Omega\} & \text{on } \omega \\ \overline{H}_f^\Omega & \text{on } \Omega\backslash\omega, \end{cases}$$

belongs to $\mathcal{U}(\Omega)$. Let $v \in \Phi_f^\Omega \cap \mathcal{U}(\Omega)$ and define $w = \tilde{u} + v - \overline{H}_f^\Omega$. Then $w \in \mathcal{U}(\Omega)$. Also $w = v$ on $\Omega\backslash\omega$ and at points of ω where $\overline{H}_f^\Omega \le u$, and $w \ge u$ at other points of ω. Hence w is bounded below on Ω and

$$\liminf_{x\to y, x\in\Omega} w(x) \ge f(y) \qquad (y \in \partial^\infty \Omega \backslash \partial^\infty \omega),$$

$$\liminf_{x\to y, x\in\Omega} w(x) \ge \min\{\liminf_{x\to y, x\in\Omega} v(x), \liminf_{x\to y, x\in\omega} u(x)\} \ge f(y) \quad (y \in \partial^\infty \Omega \cap \partial^\infty \omega).$$

It follows that $w \in \Phi_f^\Omega$, so $\tilde{u} + v = \overline{H}_f^\Omega + w \ge 2\overline{H}_f^\Omega$ on Ω. Since this holds for all $v \in \Phi_f^\Omega \cap \mathcal{U}(\Omega)$, we see that $\tilde{u} \ge \overline{H}_f^\Omega$ on Ω, and in particular $u \ge \overline{H}_f^\Omega$

on ω. Since u is an arbitrary element of $\Phi_F^\omega \cap \mathcal{U}(\omega)$, we have $\overline{H}_F^\omega \geq \overline{H}_f^\Omega$ on ω, as required.

By Theorem 6.2.5(i) the same result holds with lower solutions in place of upper solutions, so the stated results for resolutive f follow. □

The next lemma is in preparation for the proof of the fact that $C(\partial^\infty\Omega) \subseteq \mathcal{R}(\Omega)$, which is important in itself and is used in Section 6.4 to show the existence of harmonic measure.

Lemma 6.3.7. *Let u_1, u_2 be finite-valued superharmonic functions on Ω, each possessing a subharmonic minorant there. If $u_1 - u_2$ has a finite limit $f(y)$ at each point $y \in \partial^\infty\Omega$, then $f \in \mathcal{R}(\Omega)$.*

Proof. Clearly $f \in C(\partial^\infty\Omega)$, so f is bounded on $\partial^\infty\Omega$. Let h_j $(j = 1, 2)$ be the greatest harmonic minorant of u_j on Ω. (The existence of h_j is guaranteed by Theorem 3.6.3.) Then $u_1 - h_2 \in \mathcal{U}(\Omega)$ and

$$\liminf_{x\to y}(u_1 - h_2)(x) \geq f(y) \qquad (y \in \partial^\infty\Omega).$$

From this it also follows, by the minimum principle, that $u_1 - h_2$ is bounded below on Ω. Hence $u_1 - h_2 \in \Phi_f$ and therefore $h_2 + \overline{H}_f$ is a harmonic minorant of u_1 on Ω, so $h_2 + \overline{H}_f \leq h_1$. Similarly, $h_1 - \underline{H}_f \leq h_2$. Since $\underline{H}_f \leq \overline{H}_f$, it follows that $\underline{H}_f = h_1 - h_2 = \overline{H}_f$. Thus $f \in \mathcal{R}(\Omega)$. □

Theorem 6.3.8. $C(\partial^\infty\Omega) \subseteq \mathcal{R}(\Omega)$.

Proof. By Theorem 6.3.3 the uniform limit of a sequence in $\mathcal{R}(\Omega)$ also belongs to $\mathcal{R}(\Omega)$, so it is enough to find a family $\mathcal{F}$ in $C(\partial^\infty\Omega) \cap \mathcal{R}(\Omega)$ that is dense in $C(\partial^\infty\Omega)$.

We suppose first that $N \geq 3$. Let $\mathcal{G} = \mathcal{U}_+(\mathbb{R}^N) \cap C(\mathbb{R}^N \cup \{\infty\})$, let $\mathcal{G}_d = \{u_1 - u_2 : u_1, u_2 \in \mathcal{G}\}$, and let $\mathcal{F} = \{u|_{\partial^\infty\Omega} : u \in \mathcal{G}_d\}$. Clearly $\mathcal{F}$ is a vector subspace of $C(\partial^\infty\Omega)$ and by Lemma 6.3.7, $\mathcal{F} \subseteq \mathcal{R}(\Omega)$. The result will be established if we prove that $\mathcal{F}$ satisfies the hypotheses of the Stone–Weierstrass theorem (see Appendix, Theorem A.12), for it will then follow that $\mathcal{F}$ is dense in $C(\partial^\infty\Omega)$, as required. Clearly $1 \in \mathcal{F}$. Also $\mathcal{F}$ separates points of $\partial^\infty\Omega$, since $\mathcal{F}$ contains every function of the form $\min\{U_y, c\}$ (defined to be 0 at ∞), where $y \in \mathbb{R}^N$ and $c \in (0, +\infty)$. Finally, if $u, v \in \mathcal{G}$, then $\min\{u, v\} \in \mathcal{G}$, so that if $u_1, u_2, v_1, v_2 \in \mathcal{G}$, then

$$\max\{u_1 - v_1, u_2 - v_2\} = u_1 + u_2 - \min\{u_2 + v_1, u_1 + v_2\} \in \mathcal{G}_d,$$

and hence $\max\{f, g\} \in \mathcal{F}$ whenever $f, g \in \mathcal{F}$.

In the case $N = 2$, the classes $\mathcal{G}, \mathcal{G}_d, \mathcal{F}$ need to be modified. Let $\mathcal{G}$ be the class of functions u in $\mathcal{U}(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ satisfying

$$u(x) = -\alpha_u \log\|x\| + \beta_u \qquad (\|x\| > R_u)$$

where $\alpha_u, \beta_u, R_u \in \mathbb{R}$ and $\alpha_u \geq 0$. Let

$$\mathcal{G}_d = \{u \in C(\mathbb{R}^2 \cup \{\infty\}) : u|_{\mathbb{R}^2} = u_1 - u_2, \text{ where } u_1, u_2 \in \mathcal{G} \text{ and } \alpha_{u_1} = \alpha_{u_2}\},$$

and let $\mathcal{F} = \{u|_{\partial^\infty\Omega} : u \in \mathcal{G}_d\}$. Then $\mathcal{F}$ is a vector subspace of $C(\partial^\infty\Omega)$. Since Ω is Greenian the elements of $\mathcal{G}$ have harmonic minorants on Ω by Theorem 5.3.8. Hence Lemma 6.3.7 is applicable and shows that $\mathcal{F} \subseteq \mathcal{R}(\Omega)$. Also $1 \in \mathcal{F}$, and an argument similar to that for the case where $N \geq 3$ shows that if $f, g \in \mathcal{F}$, then $\max\{f, g\} \in \mathcal{F}$. To show that $\mathcal{F}$ separates points of $\partial^\infty\Omega$, let y_1, y_2 be distinct points of $\partial^\infty\Omega$ and choose a point $y_0 \in \mathbb{R}^2$ such that there is a disc $B(y_0, r)$ containing 0 and exactly one of y_1, y_2, say y_1. Define v to be the superharmonic function on $\mathbb{R}^2$ obtained by replacing U_0 on $B(y_0, r)$ by its Poisson integral there, and define $w = v - \min\{U_0, -\log(r - \|y_0\|)\}$ on $\mathbb{R}^2$ and $w(\infty) = 0$. Then $w \in \mathcal{G}_d$. Also $w < 0$ on $B(y_0, r)$ and $w = 0$ on $\mathbb{R}^2 \backslash B(y_0, r)$. Define $f = w$ on $\partial^\infty\Omega$. Then $f \in \mathcal{F}$ and $f(y_1) < 0 = f(y_2)$. It now follows from the Stone–Weierstrass theorem that $\mathcal{F}$ is dense in $C(\partial^\infty\Omega)$. □

Remark 6.3.9. In the above proof we used the hypotheses that Ω is Greenian, and Theorem 6.3.8 is actually false for any non-Greenian domain Ω with more than one point in $\partial^\infty\Omega$ (that is, for any non-Greenian $\Omega \neq \mathbb{R}^2$); see Remark 6.2.10.

Theorem 6.3.10. *Let (Ω_n) be an increasing sequence of open sets such that $\bigcup_{n=1}^\infty \Omega_n = \Omega$. If $f \in C(\Omega \cup \partial^\infty\Omega)$, then $H_f^{\Omega_n} \to H_f^\Omega$ on Ω as $n \to \infty$.*

Proof. Suppose that $u \in \Phi_f^\Omega$ and $\varepsilon > 0$. Since

$$\liminf_{x\to y, x\in\Omega}(u - f)(x) \geq 0 \qquad (y \in \partial^\infty\Omega)$$

and $\partial^\infty\Omega$ is compact, there is a compact subset E of Ω such that $u - f > -\varepsilon$ on $\Omega\backslash E$. If n is sufficiently large, then $\partial^\infty\Omega_n \subset \partial^\infty\Omega \cup (\Omega\backslash E)$ and hence

$$\liminf_{x\to y, x\in\Omega_n}(u - f)(x) \geq -\varepsilon \qquad (y \in \partial^\infty\Omega_n),$$

so $u + \varepsilon \in \Phi_f^{\Omega_n}$ for all such n. Thus $u + \varepsilon \geq H_f^{\Omega_n}$ on Ω_n for all large n, so

$$u + \varepsilon \geq \limsup_{n\to\infty} H_f^{\Omega_n}$$

on Ω. Since u is an arbitrary element of Φ_f^Ω and ε is an arbitrary positive number, it follows that

$$H_f^\Omega \geq \limsup_{n\to\infty} H_f^{\Omega_n}$$

on Ω. A similar argument shows that

$$H_f^\Omega \leq \liminf_{n\to\infty} H_f^{\Omega_n},$$

and the result now follows. □

6.4. Harmonic measure

Our aim is to show that given a point $z \in \Omega$, there exists a measure μ_z, depending on z and Ω, such that

$$H_f(z) = \int_{\partial^\infty \Omega} f \, d\mu_z \tag{6.4.1}$$

for each $f \in \mathcal{R}(\Omega)$, and further that, if Ω is connected, then every μ_z-integrable function f is resolutive and satisfies (6.4.1). We lead up to these facts through a sequence of preliminary results.

Theorem 6.4.1. *If $z \in \Omega$, then there exists a unique Borel measure μ_z on $\partial^\infty \Omega$ such that (6.4.1) holds for every $f \in C(\partial^\infty \Omega)$. Further, $\mu_z(\partial^\infty \Omega) = 1$.*

Proof. By Theorem 6.3.8, $C(\partial^\infty \Omega) \subseteq \mathcal{R}(\Omega)$, and by Corollary 6.3.2, the mapping $f \mapsto H_f(z)$ is a linear functional on $C(\partial^\infty \Omega)$. By the minimum principle this functional is positive: that is, $H_f(z) \geq 0$ when $f \geq 0$ on $\partial^\infty \Omega$. The existence and uniqueness of the Borel measure μ_z on $\partial^\infty \Omega$ satisfying (6.4.1) for each $f \in C(\partial^\infty \Omega)$ follows from the Riesz representation theorem (see Appendix). Finally, $\mu_z(\partial^\infty \Omega) = H_1(z) = 1$. □

Remark 6.4.2. If μ is a Borel measure, then the class of all sets of the form $E \cup Y$, where E is a Borel set and Y is contained in a Borel set of μ-measure 0, is a σ-algebra, which we denote by $\mathcal{B}\mu$. Also μ can be extended to a measure on $\mathcal{B}\mu$ by defining $\mu(E \cup Y) = \mu(E)$, where E, Y are as above. This extended measure is called the *completion* of μ.

Definition 6.4.3. If $z \in \Omega$, then the completion of the measure μ_z is called *harmonic measure* relative to Ω and z. This harmonic measure is also denoted by μ_z or sometimes by μ_z^Ω. A function f on $\partial^\infty \Omega$ is called *μ_z-measurable* if f is $\mathcal{B}\mu_z$-measurable.

Lemma 6.4.4. *If $z \in \Omega$ and $f : \partial^\infty \Omega \to (-\infty, +\infty]$ is a lower semicontinuous function on $\partial^\infty \Omega$, then*

$$\overline{H}_f(z) = \int_{\partial^\infty \Omega} f \, d\mu_z; \tag{6.4.2}$$

also $f \in \mathcal{R}(\Omega)$ provided that $\overline{H}_f < +\infty$ on Ω.

Proof. By Lemma 3.2.1 there is an increasing sequence (f_n) in $C(\partial^\infty \Omega)$ such that $f = \lim f_n$ on $\partial^\infty \Omega$, and by Theorem 6.3.5, $\overline{H}_f = \lim H_{f_n}$. Also $H_{f_n} \leq \underline{H}_f$ for each n. Hence $-\infty < \underline{H}_f = \overline{H}_f$ on Ω, so that $f \in \mathcal{R}(\Omega)$ if $\overline{H}_f < +\infty$. By Theorem 6.4.1 and monotone convergence,

$$\overline{H}_f(z) = \lim H_{f_n}(z) = \lim \int_{\partial^\infty \Omega} f_n \, d\mu_z = \int_{\partial^\infty \Omega} f \, d\mu_z. \qquad \square$$

Lemma 6.4.5. *If $z \in \Omega$ and $f : \partial^\infty \Omega \to [-\infty, +\infty]$ and A is a number such that $\overline{H}_f(z) < A$, then there exists a lower semicontinuous function $g : \partial^\infty \Omega \to (-\infty, +\infty]$ such that $f \leq g$ on $\partial^\infty \Omega$ and $\overline{H}_g(z) < A$.*

Proof. Let $u \in \Phi_f$ be such that $u(z) < A$ and define g on $\partial^\infty \Omega$ by $g(y) = \liminf_{x \to y} u(x)$. Then g is lower semicontinuous and $f \leq g$ on $\partial^\infty \Omega$. Also $u \in \Phi_g$, so $\overline{H}_g(z) \leq u(z) < A$. □

Theorem 6.4.6. *Suppose that $z \in \Omega$ and $f : \partial^\infty \Omega \to [-\infty, +\infty]$.*
(i) If f is μ_z-measurable and the integral below exists, then

$$\underline{H}_f(z) = \overline{H}_f(z) = \int_{\partial^\infty \Omega} f \, d\mu_z. \tag{6.4.3}$$

(ii) If $\underline{H}_f(z) = \overline{H}_f(z) \in \mathbb{R}$, then f is μ_z-integrable and (6.4.3) holds.

Proof. (i) We prove (6.4.3) for increasingly general classes of functions.

(A) First let f be the characteristic function of a relatively open subset E of $\partial^\infty \Omega$. Then f is lower semicontinuous on $\partial^\infty \Omega$ and (6.4.3) follows from Lemma 6.4.4.

(B) Next we prove (6.4.3) in the case where f is the characteristic function χ_E of a Borel subset E of $\partial^\infty \Omega$. Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\partial^\infty \Omega$, and let $\mathcal{F}$ be the class of sets $E \in \mathcal{B}$ such that (6.4.3) holds with $f = \chi_E$. The result in (A) says that $\mathcal{F}$ contains all relatively open subsets of $\partial^\infty \Omega$. Hence, to prove that $\mathcal{F} = \mathcal{B}$, it is enough to show that $\mathcal{F}$ is a σ-algebra. Clearly, $\partial^\infty \Omega \in \mathcal{F}$. Suppose that $E \in \mathcal{F}$ and define $E' = (\partial^\infty \Omega) \backslash E$. Then

$$\begin{aligned} 1 - H_{\chi_E}(z) = H_1(z) + H_{-\chi_E}(z) &\leq \underline{H}_{\chi_{E'}}(z) \leq \overline{H}_{\chi_{E'}}(z) \\ &\leq H_1(z) + H_{-\chi_E}(z) = 1 - H_{\chi_E}(z), \end{aligned}$$

by Theorem 6.3.1, so that

$$\underline{H}_{\chi_{E'}}(z) = \overline{H}_{\chi_{E'}}(z) = 1 - H_{\chi_E}(z) = 1 - \mu_z(E) = \mu_z(E'),$$

and hence $E' \in \mathcal{F}$. Now let (F_n) be an increasing sequence in $\mathcal{F}$ and define $F = \bigcup_{n=1}^\infty F_n$. Then

$$\underline{H}_{\chi_F}(z) \geq \lim H_{\chi_{F_n}}(z) = \overline{H}_{\chi_F}(z) \geq \underline{H}_{\chi_F}(z),$$

the last-written equation following from Theorem 6.3.5. Hence

$$\underline{H}_{\chi_F}(z) = \overline{H}_{\chi_F}(z) = \lim H_{\chi_{F_n}}(z) = \lim \mu_z(F_n) = \mu_z(F),$$

so $F \in \mathcal{F}$. Thus $\mathcal{F}$ is a σ-algebra, so $\mathcal{F} = \mathcal{B}$.

(C) Now suppose that $E \in \mathcal{B}\mu_z$. Again we want to prove (6.4.3) with $f = \chi_E$. We can write $E = F \cup Y$, where F is a Borel set and $Y \subseteq Z$ for some Borel set Z with $\mu_z(Z) = 0$. Then $\mu_z(E) = \mu_z(F)$ and

$$H_{\chi_F}(z) \leq \underline{H}_{\chi_E}(z) \leq \overline{H}_{\chi_E}(z) \leq H_{\chi_{F\cup Z}}(z) \leq H_{\chi_F}(z) + H_{\chi_Z}(z).$$

The result in (B) shows that (6.4.3) holds with $f = \chi_F$ and that $H_{\chi_Z}(z) = 0$. Hence

$$\underline{H}_{\chi_E}(z) = \overline{H}_{\chi_E}(z) = \mu_z(F) = \mu_z(E).$$

(D) Next let f be a non-negative μ_z-measurable simple function. Thus there exist sets $E_1, \ldots, E_n \in \mathcal{B}\mu_z$ and positive numbers $a_1, \ldots, a_n$ such that $f = \sum_{k=1}^n a_k \chi_{E_k}$. By Theorem 6.3.1 and the result of (C),

$$\begin{aligned}\sum_{k=1}^n a_k\mu_z(E_k) = \sum_{k=1}^n a_k H_{\chi_{E_k}}(z) \leq \underline{H}_f(z) \leq \overline{H}_f(z) \\ \leq \sum_{k=1}^n a_k H_{\chi_{E_k}}(z) = \sum_{k=1}^n a_k \mu_z(E_k),\end{aligned}$$

and hence

$$\underline{H}_f(z) = \overline{H}_f(z) = \sum_{k=1}^n a_k\mu_z(E_k) = \int_{\partial^\infty \Omega} f \, d\mu_z.$$

(E) If f is a non-negative μ_z-measurable function, then f is the limit of some increasing sequence (f_j) of non-negative μ_z-measurable simple functions. By the result of (D) and monotone convergence,

$$H_{f_j}(z) = \int_{\partial^\infty \Omega} f_j \, d\mu_z \to \int_{\partial^\infty \Omega} f \, d\mu_z,$$

and by Theorem 6.3.5,

$$\underline{H}_f(z) \geq \lim_{j\to\infty} H_{f_j}(z) = \overline{H}_f(z) \geq \underline{H}_f(z).$$

It follows that (6.4.3) holds.

(F) Finally, if f is any μ_z-measurable function for which the integral in (6.4.3) exists, then

$$\int_{\partial^\infty \Omega} f \, d\mu_z = \int_{\partial^\infty \Omega} f^+ d\mu_z - \int_{\partial^\infty \Omega} f^- d\mu_z = \overline{H}_{f^+}(z) - \underline{H}_{f^-}(z) \geq \overline{H}_f(z)$$

by Theorem 6.3.1, and similarly

$$\int_{\partial^\infty \Omega} f d\mu_z = \underline{H}_{f^+}(z) - \overline{H}_{f^-}(z) \leq \underline{H}_f(z).$$

Hence (6.4.3) holds.

(ii) By Lemma 6.4.5 there is a sequence (f_n) of lower semicontinuous functions on $\partial^\infty \Omega$ such that $f_n \geq f$ and $\overline{H}_{f_n}(z) < \overline{H}_f(z) + n^{-1}$. Similarly, there is a sequence (g_n) of upper semicontinuous functions on $\partial^\infty \Omega$ such that $g_n \leq f$ and $\underline{H}_{g_n}(z) > \underline{H}_f(z) - n^{-1}$. By (6.4.2),

$$\overline{H}_f(z) = \inf_n \overline{H}_{f_n}(z) = \inf_n \int_{\partial^\infty \Omega} f_n d\mu_z \geq \int_{\partial^\infty \Omega} f^* d\mu_z,$$

where $f^* = \inf_n f_n$. Similarly,

$$\underline{H}_f(z) \leq \int_{\partial^\infty \Omega} g_* d\mu_z,$$

where $g_* = \sup_n g_n$. Since $g_* \leq f \leq f^*$ on $\partial^\infty \Omega$ and f^*, g_* are Borel measurable, it follows that there is a Borel set Z with $\mu_z(Z) = 0$ such that $g_* = f = f^*$ on $\partial^\infty \Omega \backslash Z$. All subsets of Z belong to $\mathcal{B}\mu_z$, so f is μ_z-measurable and

$$\overline{H}_f(z) = \underline{H}_f(z) \leq \int_{\partial^\infty \Omega} g_* d\mu_z \leq \int_{\partial^\infty \Omega} f d\mu_z \leq \int_{\partial^\infty \Omega} f^* d\mu_z \leq \overline{H}_f(z),$$

so (6.4.3) holds. □

Corollary 6.4.7. *(i) If f is Borel measurable and $-\infty < \underline{H}_f \leq \overline{H}_f < +\infty$ on Ω, then $f \in \mathcal{R}(\Omega)$ and*

$$H_f(x) = \int_{\partial^\infty \Omega} f \, d\mu_x \tag{6.4.4}$$

for each $x \in \Omega$.
(ii) If Ω is connected, then the following are equivalent:
(a) $f \in \mathcal{R}(\Omega)$;
(b) f is μ_x-integrable for some $x \in \Omega$;
(c) f is μ_x-integrable for all $x \in \Omega$.
If any of these conditions holds, then (6.4.4) holds for all $x \in \Omega$.

Proof. (i) If the hypotheses of (i) hold, then f^+ is μ_x-measurable for each $x \in \Omega$ and by Theorem 6.4.6(i),

$$\underline{H}_{f^+}(x) = \overline{H}_{f^+}(x) = \int_{\partial^\infty \Omega} f^+ d\mu_x.$$

Since $\overline{H}_f(x) < +\infty$, there is an element $u \in \Phi_f$ such that $u(x) < +\infty$, and since u is bounded below on Ω, we have $u + A \in \Phi_{f^+}$ for some real number A. Hence $\overline{H}_{f^+}(x) < +\infty$. Since x is an arbitrary point of Ω, it follows that $f^+ \in \mathcal{R}(\Omega)$ and (6.4.4) holds with f^+ in place of f. Similarly $f^- \in \mathcal{R}(\Omega)$ and (6.4.4) holds for f^-. The required conclusion now follows from Corollary 6.3.2.

(ii) If (a) holds, then by Theorem 6.4.6(ii), condition (c) holds. Clearly (c) implies (b). If (b) holds, then by Theorem 6.4.6(i), $\underline{H}_f$ and $\overline{H}_f$ are finite and equal at some point, and hence by connectedness and the maximum principle, at every point of Ω, so $f \in \mathcal{R}(\Omega)$. Thus (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a). Finally, if (c) holds, then (6.4.4) holds for all $x \in \Omega$ by Theorem 6.4.6(i). □

Theorem 6.4.8. *Suppose that ω is an open subset of Ω, that $z \in \omega$, and that $E \subseteq \partial^\infty \omega \cap \partial^\infty \Omega$. If E is μ_z^Ω-measurable, then E is μ_z^ω-measurable and $\mu_z^\omega(E) \le \mu_z^\Omega(E)$ with equality in the case where ω is a component of Ω.*

Proof. Suppose that E is μ_z^Ω-measurable and let χ_E denote the characteristic function of E. We define a function F on $\Omega \cup \partial^\infty \Omega$ to be equal to χ_E on $\partial^\infty \Omega$ and $\overline{H}^\Omega_{\chi_E}$ on Ω. By Theorem 6.4.6(i), the equations (6.4.3) hold with $f = \chi_E$. Hence, by Theorem 6.3.6 (and its counterpart for lower solutions) $\underline{H}^\omega_F(z) = F(z) = \overline{H}^\omega_F(z)$ so that, by Theorem 6.4.6(ii), F is μ_z^ω-measurable. Since $\chi_E = F\chi_{\partial^\infty \Omega}$, it follows that χ_E is μ_z^ω-measurable and

$$\mu_z^\omega(E) = \int_{\partial^\infty \omega} \chi_E d\mu_z^\omega \le \int_{\partial^\infty \omega} F d\mu_z^\omega = \overline{H}^\omega_F(z) = H^\Omega_{\chi_E}(z) = \mu_z^\Omega(E).$$

If ω is a component of Ω, then $E \subseteq \partial^\infty \omega \subseteq \partial^\infty \Omega$, so $F = \chi_E$ on $\partial^\infty \omega$ and the above inequality is an equality. □

Example 6.4.9. (i) Harmonic measure relative to B and $z \in B$ is given by

$$d\mu_z = K_{0,1}(z, \cdot) d\sigma, \tag{6.4.5}$$

where σ is surface measure on S and $K_{0,1}$ is the Poisson kernel of B given by (1.3.1).

To see this, we note that if μ_z is given by (6.4.5), then by Theorem 1.3.3, for any $f \in C(S)$, the function $z \mapsto \int_S f \, d\mu_z$ is the classical, and hence also the PWB, solution of the Dirichlet problem on B with boundary function f. The assertion follows by the uniqueness of harmonic measure (Theorem 6.4.1).

(ii) Harmonic measure relative to $D = \mathbb{R}^{N-1} \times (0, +\infty)$ and $z \in D$ is given on $\partial^\infty D$ by

$$d\mu_z = \mathcal{K}(z, \cdot) d\lambda' \text{ on } \partial D, \quad \mu_z(\{\infty\}) = 0,$$

where λ' is $(N-1)$-dimensional Lebesgue measure on ∂D and $\mathcal{K}$ is the Poisson kernel of D, given in Definition 1.7.1. This is proved by arguing as in (i), and using Theorem 1.7.5 in place of Theorem 1.3.3.

Theorem 6.4.10. *Let (Ω_n) be an increasing sequence of bounded open sets such that $\overline{\Omega}_n \subset \Omega$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$, and let $u \in \mathcal{U}(\Omega)$. Then:*
(i) for each $m \in \mathbb{N}$ the sequence $(H_u^{\Omega_n})_{n \ge m}$ is decreasing on Ω_m;
(ii) u has a harmonic minorant on Ω if and only if $\lim H_u^{\Omega_n} > -\infty$ on Ω;

(iii) if u has a harmonic minorant on Ω, then its greatest harmonic minorant is $\lim H_u^{\Omega_n}$.

Proof. For each n the restriction of u to $\partial \Omega_n$ is lower semicontinuous and bounded below, and $u \in \Phi_u^{\Omega_n}$. Hence $H_u^{\Omega_n}$ exists and is a harmonic minorant of u on Ω_n. If $s \in \Psi_u^{\Omega_{n+1}}$, then $s \le u$ on Ω_{n+1} by the maximum principle, so $s \in \Psi_u^{\Omega_n}$. It follows that $H_u^{\Omega_{n+1}} \le H_u^{\Omega_n}$ on Ω_n and so (i) holds.

Let $h = \lim H_u^{\Omega_n}$. If $h > -\infty$ on Ω, then h is a harmonic minorant of u on each Ω_n, and hence on Ω. Conversely, if h_1 is a harmonic minorant of u on Ω, then $h_1 \in \Psi_u^{\Omega_n}$ so $h_1 \le H_u^{\Omega_n}$ for each n, and hence $h \ge h_1$ on Ω. This proves (ii) and (iii). □

6.5. Negligible sets

Definition 6.5.1. A subset E of $\partial^\infty \Omega$ is called *negligible (for Ω)* if $\mu_x(E) = 0$ for each $x \in \Omega$ or, equivalently, if $\overline{H}_{\chi_E} \equiv 0$, where χ_E is the characteristic function of E.

Theorem 6.5.2. *A subset E of $\partial^\infty \Omega$ is negligible if and only if there exists $u \in \mathcal{U}_+(\Omega)$ such that $u(x) \to +\infty$ as $x \to y$ for each $y \in E$.*

Proof. If there is such a function u, then $\varepsilon u \in \Phi_{\chi_E}$ for each $\varepsilon > 0$. Hence $\overline{H}_{\chi_E} = 0$ on the set where $u < +\infty$ and therefore, by continuity, everywhere on Ω, so E is negligible.

To prove the converse, suppose first that Ω is connected. Suppose also that $\overline{H}_{\chi_E} \equiv 0$ and fix $z \in \Omega$. For each $n \in \mathbb{N}$, there exists $u_n \in \Phi_{\chi_E}$ such that $u_n(z) < 2^{-n}$. We note that $u_n \in \mathcal{U}_+(\Omega)$. Define $u = \sum_{n=1}^\infty u_n$ on Ω. Then $u(z) < +\infty$, so $u \in \mathcal{U}_+(\Omega)$. Also,

$$\liminf_{x \to y} u(x) \ge \sum_{n=1}^\infty \liminf u_n(x) \ge \sum_{n=1}^\infty 1 = +\infty \qquad (y \in E).$$

Now consider the general case. Let $\{\omega_j : j \in J\}$, where $J \subseteq \mathbb{N}$, be the set of components of Ω. The result in the preceding paragraph yields, for each $j \in J$, a function $v_j \in \mathcal{U}_+(\omega_j)$ such that $v_j(x) \to +\infty$ as $x \to y$ for each $y \in E \cap \partial^\infty \omega_j$. Define v on Ω by putting $v = j + v_j$ on ω_j for each $j \in J$. Then $v \in \mathcal{U}_+(\Omega)$. Fix $y \in E$ and $A \in (1, +\infty)$. To show that $v(x) \to +\infty$ as $x \to y$, we must prove that there is a neighbourhood ω of y such that $j + v_j > A$ on $\omega \cap \omega_j$ whenever $\omega \cap \omega_j \ne \emptyset$. If $j \in J$ and $j \le A$, then there is a neighbourhood ω_j' of y such that either $\omega_j \cap \omega_j' = \emptyset$ or $v_j > A$ on $\omega_j \cap \omega_j'$. If $j \in J$ and $j > A$, then $j + v_j > A$ on ω_j. Hence the neighbourhood $\omega = \bigcap_{j \le A} \omega_j'$ has the required property. □

Lemma 6.5.3. *A subset E of $\partial^\infty\Omega$ is negligible for Ω if and only if every component ω of Ω contains a point x_ω such that $\mu^\omega_{x_\omega}(E\cap\partial^\infty\omega)=0$. In particular, if $E\cap\partial^\infty\omega=\emptyset$ for each component ω, then E is negligible for Ω.*

Proof. If E is negligible for Ω and x is an arbitrary point of some component ω of Ω, then by Theorem 6.4.8,

$$\mu^\omega_x(E\cap\partial^\infty\omega)=\mu^\Omega_x(E\cap\partial^\infty\omega)\le\mu^\Omega_x(E)=0.$$

To prove the converse, let ω be any component of Ω. If $\mu^\omega_{x_\omega}(E\cap\partial^\infty\omega)=0$ for some $x_\omega\in\omega$, then by the minimum principle, $\overline{H}^\omega_\chi\equiv 0$, where χ is the characteristic function of $E\cap\partial^\infty\omega$. By Theorem 6.3.6, $\overline{H}^\Omega_{\chi_E}=0$ on ω and therefore E is negligible for Ω. □

We now use this lemma to give an example in which $\partial\Omega$ contains a negligible set of positive λ-measure.

Example 6.5.4. Let (q_n) be a dense sequence in $(0,1)$ and let

$$\Omega=\left\{(x_1,\ldots,x_N)\in(0,1)^N : x_N\in\bigcup_{n=1}^\infty(q_n,q_n+2^{-n-1})\right\}.$$

The density of (q_n) implies that $\overline{\Omega}$ is the cube $[0,1]^N$. Also

$$\lambda(\partial\Omega)=\lambda(\overline{\Omega})-\lambda(\Omega)\ge 1-\sum_{n=1}^\infty 2^{-n-1}=\frac{1}{2}.$$

Let E be the set of points in $\partial\Omega$ that are not in the boundary of any component of Ω. By Lemma 6.5.3, E is negligible. If ω is a component of Ω, then clearly ω is an N-dimensional rectangle, so $\lambda(\partial\omega)=0$. Hence $\lambda(E)=\lambda(\partial\Omega)\ge\frac{1}{2}$.

Theorem 6.5.5. *If $E\subseteq\partial\Omega$ and E is polar, then E is negligible.*

Proof. Suppose first that $N\ge 3$. By Theorem 5.1.3(i) there exists $u\in\mathcal{U}_+(\mathbb{R}^N)$ such that $u=+\infty$ on E. Hence, for each $y\in E$, we have $u(x)\to+\infty$ as $x\to y$ with $x\in\Omega$. It follows from Theorem 6.5.2 that E is negligible.

In the case where $N=2$, for each $z\in\partial\Omega$ let r_z be a positive number such that $\Omega\cup B(z,r_z)$ is Greenian; such r_z exist by Theorem 5.3.8. Since E may be covered by a countable union of balls $B(z,r_z)$, it is enough to show that $E\cap B(z,r_z)$ is negligible for each $z\in\partial\Omega$. Since $\Omega\cup B(z,r_z)$ is Greenian, there exists a potential u on $\Omega\cup B(z,r_z)$ such that $u=+\infty$ on $E\cap B(z,r_z)$, by Theorem 5.1.3(i). If $y\in E\cap B(z,r_z)$, then $u(x)\to+\infty$ as $x\to y$, and therefore $E\cap B(z,r_z)$ is negligible, as required. □

It follows, in particular, that any one-point subset of $\partial\Omega$ is negligible. The question whether the one-point set $\{\infty\}$ is negligible is more complicated, as we illustrate below. A complete characterization of unbounded sets Ω in $\mathbb{R}^N$, where $N\ge 3$, for which $\{\infty\}$ is negligible will be given in Theorem 7.6.5(ii).

Example 6.5.6. (i) If Ω is an unbounded Greenian open subset of $\mathbb{R}^2$, then $\{\infty\}$ is negligible for Ω.

To prove this, we note first that $\mathbb{R}^2\backslash\Omega$ is a closed non-polar subset of $\mathbb{R}^2$ and therefore contains a compact non-polar set K. By Lemma 5.8.1 there is a positive harmonic function h on $\mathbb{R}^2\backslash K$ such that $h(x)\to+\infty$ as $x\to\infty$. The result follows as an application of Theorem 6.5.2 with $u=h|_\Omega$.

(ii) If $\Omega=\mathbb{R}^N\backslash K$, where $N\ge 3$ and K is compact, then $\{\infty\}$ is non-negligible for Ω.

To prove this, we take a ball $B(0,r)$ containing K and define

$$s(x)=1-(r/\|x\|)^{N-2}\qquad(x\in\Omega).$$

Then $s\in\Psi_{\chi_{\{\infty\}}}$. Hence $H_{\chi_{\{\infty\}}}\ge s$ on Ω.

(iii) If Ω is a half-space, then $\{\infty\}$ is negligible for Ω by Example 6.4.9(ii).

6.6. Boundary behaviour

We have already seen that the PWB approach always yields the classical solution to the Dirichlet problem when the classical solution exists. We will now see that much more is true: if $f\in C(\partial^\infty\Omega)$, then $\lim_{x\to y}H_f(x)=f(y)$ for most points y in $\partial^\infty\Omega$ even if the classical solution does not exist. The points y at which this equation fails for some $f\in C(\partial^\infty\Omega)$ are called "irregular". We shall show that the irregular points in $\partial\Omega$ always form a polar set and shall characterize regular points of $\partial\Omega$ by the existence of so-called "barrier functions". Several sufficient geometric conditions for regularity will be given. These all suggest that a point $y\in\partial\Omega$ is regular if $\mathbb{R}^N\backslash\Omega$ is not too "thin" at y. A precise characterization of (ir)regularity in terms of thinness is proved in the next chapter. For bounded, not necessarily continuous, resolutive functions f, we shall show that the behaviour of H_f near a regular boundary point y is determined by the local behaviour of f near y; this is not generally true for unbounded resolutive functions.

Definition 6.6.1. A point y of $\partial^\infty\Omega$ is called *regular (for Ω)* if

$$\lim_{x\to y}H_f(x)=f(y)\qquad\text{for each } f\in C(\partial^\infty\Omega).$$

Otherwise y is called *irregular*. We say that the set Ω is *regular* if every point of $\partial^\infty\Omega$ is regular.

Definition 6.6.2. A function u is called a *barrier (for Ω) at $y \in \partial\Omega$* if u is positive and superharmonic on $\Omega \cap \omega$ for some open neighbourhood ω of y and $\lim_{x\to y} u(x) = 0$.

We first aim to show that $y \in \partial\Omega$ is regular if and only if there is a barrier at y. Conditions for ∞ to be a regular boundary point will be discussed separately in Section 6.7.

Lemma 6.6.3. *If there is a barrier at $y \in \partial\Omega$, then there exists a barrier v at y such that $v \in \mathcal{U}_+(\Omega)$ and $\inf_{\Omega\setminus\omega} v > 0$ for every open neighbourhood ω of y.*

Proof. Let u be a barrier at y and choose $r > 0$ such that u is positive and superharmonic on $\Omega \cap B(y,r)$ and $\Omega \cap S(y,r) \neq \emptyset$. Define $\Omega' = \Omega \cap B(y,r)$ and suppose for the moment that there exists $w \in \mathcal{U}_+(\Omega')$ such that $\lim_{x\to y} w(x) = 0$ and $\inf_{\Omega'\setminus\omega} w > 0$ for every open set ω such that $y \in \omega \subseteq B(y, r/2)$. Let $\alpha = \inf\{w(x) : x \in \Omega'\setminus B(y,r/2)\}$. Then the function v, defined to be equal to $\min\{w,\alpha\}$ on $\Omega\cap B(y,r/2)$ and equal to α elsewhere on Ω, has the properties we require. Hence it is enough to show that such a function w exists.

Define f on $\partial\Omega'$ by $f(x) = \|y - x\|$. Then $f \in \mathcal{R}(\Omega')$, and we aim to show that $H_f^{\Omega'}$ has the properties required of w. Note that the function $x \mapsto \|y-x\|$ has positive Laplacian on $\mathbb{R}^N\setminus\{y\}$ and so belongs to the lower class $\Psi_f^{\Omega'}$. Therefore $\inf_{\Omega'\setminus\omega} H_f^{\Omega'} > 0$ for every open set ω such that $y \in \omega \subseteq B(y,r/2)$. It remains to show that $H_f^{\Omega'}(x) \to 0$ as $x \to y$. To do this, we take a number ρ such that $0 < \rho < r$ and $\Omega \cap S(y,\rho) \neq \emptyset$ and show that

$$\limsup_{x\to y} H_f^{\Omega'}(x) < 2\rho. \tag{6.6.1}$$

Since arbitrarily small values of ρ can be chosen, the required conclusion will then follow.

Fix such a number ρ, write $B_\rho = B(y,\rho)$ and let E be a non-empty compact subset of $\Omega\cap\partial B_\rho$ such that $\sigma((\Omega\cap\partial B_\rho)\setminus E) < (\rho/r)\sigma(\partial B_\rho)$. Also let $k = \inf_E u$ and note that $k > 0$. Define g to be equal to r on $(\Omega\cap\partial B_\rho)\setminus E$ and equal to 0 elsewhere on ∂B_ρ. Let s be an arbitrary element of $\Psi_f^{\Omega'}$ and define s_0 on $\Omega\cap B_\rho$ by

$$s_0 = s - \rho - k^{-1}ru - I_{g,y,\rho}.$$

Then $s_0 \in \mathcal{S}(\Omega\cap B_\rho)$. We wish to show that

$$\limsup_{x\to z} s_0(x) \le 0 \qquad (z \in \partial(\Omega\cap B_\rho)). \tag{6.6.2}$$

We have

$$\lim_{x\to z} I_{g,y,\rho}(x) = r \qquad (z \in (\Omega\cap\partial B_\rho)\setminus E),$$

$$\liminf_{x\to z, x\in\Omega\cap B_\rho} u(x) \ge u(z) \ge k \qquad (z\in E),$$

$$\limsup_{x\to z, x\in\Omega\cap B_\rho} s(x) \le f(z) \le \rho \quad (z \in \overline{B_\rho}\cap\partial\Omega);$$

also

$$I_{g,y,\rho}(x) \ge 0, \quad u(x) > 0, \quad s(x) \le \sup_{\partial\Omega'} f \le r \quad (x \in \Omega\cap B_\rho).$$

By using these inequalities and considering separately the cases where $z \in E$, $z \in (\Omega\cap\partial B_\rho)\setminus E$, and $z \in \overline{B_\rho}\cap\partial\Omega$, we find that (6.6.2) holds. Hence, by the maximum principle, $s_0 \le 0$ on $\Omega\cap B_\rho$, so

$$s \le \rho + k^{-1}ru + I_{g,y,\rho}$$

on $\Omega\cap B_\rho$. Since s is an arbitrary element of $\Psi_f^{\Omega'}$, it follows that

$$H_f^{\Omega'} \le \rho + k^{-1}ru + I_{g,y,\rho}$$

on $\Omega\cap B_\rho$. Now

$$\lim_{x\to y} I_{g,y,\rho}(x) = I_{g,y,\rho}(y) = \mathcal{M}(g;y,\rho) < r\rho/r = \rho,$$

and by hypothesis, $\lim_{x\to y} u(x) = 0$. Hence (6.6.1) holds, as required. □

Theorem 6.6.4. *If there is a barrier at $y\in\partial\Omega$, then for any function f on $\partial^\infty\Omega$ which is bounded above,*

$$\limsup_{x\to y, x\in\Omega} \overline{H}_f(x) \le \limsup_{x\to y, x\in\partial\Omega} f(x). \tag{6.6.3}$$

If, further, f is bounded on $\partial^\infty\Omega$ and continuous at y, then

$$\lim_{x\to y} \overline{H}_f(x) = \lim_{x\to y} \underline{H}_f(x) = f(y). \tag{6.6.4}$$

Proof. By Lemma 6.6.3, there is a barrier v at y such that $v \in \mathcal{U}_+(\Omega)$ and $\inf_{\Omega\setminus\omega} v > 0$ for every open neighbourhood ω of y. We choose a number A such that $\limsup_{x\to y} f(x) < A < +\infty$. Let ω be an open neighbourhood of y such that $f < A$ on $(\overline{\omega}\cap\partial\Omega)\setminus\{y\}$ and $\Omega\setminus\omega \neq \emptyset$. Also, let c be a positive number such that $A + c\inf_{\Omega\setminus\omega} v > \sup_{\partial^\infty\Omega} f$ and define $u = A + cv$. Then $u \in \mathcal{U}(\Omega)$ and u is bounded below on Ω. Further,

$$\liminf_{x\to z} u(x) \ge A + c\inf_{\Omega\setminus\omega} v > \sup_{\partial^\infty\Omega} f \ge f(z) \quad (z \in (\partial^\infty\Omega)\setminus\overline{\omega})$$

and

$$\liminf_{x\to z} u(x) \ge A > f(z) \qquad (z \in (\partial\Omega\cap\overline{\omega})\setminus\{y\}).$$

The singleton $\{y\}$ is negligible, so by Theorem 6.5.2 there exists $w \in \mathcal{U}_+(\Omega)$ such that $w(x) \to +\infty$ as $x \to y$. If $\varepsilon > 0$, then $u + \varepsilon w \in \Phi_f$. Hence $\overline{H}_f \le u$ on $\{x \in \Omega : w(x) < +\infty\}$ and therefore on Ω. It follows that

$$\limsup_{x\to y} \overline{H}_f(x) \le \limsup_{x\to y} u(x) = A + c \lim_{x\to y} v(x) = A,$$

and hence (6.6.3) holds.

If f is now supposed to be bounded below on $\partial^\infty\Omega$, then by applying the result just established to $-f$, we obtain

$$\liminf_{x\to y} \underline{H}_f(x) \ge \liminf_{x\to y} f(x). \tag{6.6.5}$$

If f is bounded on $\partial^\infty\Omega$ and continuous at y, then (6.6.3) and (6.6.5) yield (6.6.4). □

Theorem 6.6.5. *A point $y \in \partial\Omega$ is regular if and only if there is a barrier at y.*

Proof. If there is a barrier at y and $f \in C(\partial^\infty\Omega)$, then by Theorem 6.6.4, $H_f(x) \to f(y)$ as $x \to y$, so that y is regular.

Conversely, suppose that y is regular. We define a function g on $\partial\Omega$ by $g(x) = \min\{1, ||x-y||\}$ and let $g(\infty) = 1$ if Ω is unbounded. Then $g \in C(\partial^\infty\Omega)$ and since y is regular $H_g(x) \to g(y) = 0$ as $x \to y$. Also, since $g > 0$ on $\partial^\infty\Omega$, except on the negligible set $\{y\}$, we have $H_g > 0$ on Ω. Hence H_g is a barrier at y. □

Corollary 6.6.6. *Let $y \in \partial\Omega$ be regular and $f : \partial^\infty\Omega \to [-\infty, +\infty]$.*
(i) If f is bounded above on $\partial^\infty\Omega$, then (6.6.3) holds.
(ii) If f is resolutive and bounded on $\partial^\infty\Omega$ and f is continuous at y, then $H_f(x) \to f(y)$ as $x \to y$.

Proof. This follows immediately from Theorems 6.6.4 and 6.6.5. □

The boundedness of f in the above result is essential, as we will see in Example 6.6.18 below.

Theorem 6.6.7. *A point $y \in \partial\Omega$ is irregular for Ω if and only if there is some component ω of Ω such that $y \in \partial\omega$ and y is irregular for ω. In particular, y is regular for Ω if y is not in the boundary of any component of Ω.*

Proof. Let the set of components of Ω be $\{\Omega_j : j \in J\}$, where $J \subseteq \mathbb{N}$. If y is irregular for some Ω_j, then by Theorem 6.6.5 there is no barrier for Ω_j at y. Hence there is no barrier for Ω at y, and therefore y is irregular for Ω.

To prove the converse, suppose that there is no Ω_j for which y is an irregular boundary point. Then for each $j \in J$, either y is a regular boundary point of Ω_j or $y \notin \partial\Omega_j$. In the former case, let v_j be a barrier for Ω_j at y such that $v_j \in \mathcal{U}_+(\Omega_j)$; such a function v_j exists by Lemma 6.6.3. Now define u on Ω by putting $u = \min\{v_j, 1/j\}$ on Ω_j if $y \in \partial\Omega_j$ and $u = 1/j$ on Ω_j if $y \notin \partial\Omega_j$. Then u is a positive superharmonic function on Ω, and we claim that u is a barrier for Ω at y. Fix a positive number ε and let J' denote the finite set $\{j \in J : 1/j \ge \varepsilon\}$. If $J' = \emptyset$, then $u < \varepsilon$ on Ω. Otherwise, for each $j \in J'$ there exists an open neighbourhood ω_j of y such that either $\omega_j \cap \Omega_j = \emptyset$ or $u < \varepsilon$ on $\omega_j \cap \Omega_j$. Let $\omega = \bigcap_{j\in J'} \omega_j$. Then ω is an open neighbourhood of y and $u < \varepsilon$ on $\omega \cap \Omega$. Hence $\lim_{x\to y} u(x) = 0$ and u is a barrier for Ω at y, and therefore y is regular for Ω. □

Theorem 6.6.8. *The set of irregular boundary points in $\partial\Omega$ is polar.*

Proof. Suppose first that Ω is connected and fix an arbitrary point $z \in \Omega$. By Theorem 5.7.4(i), the Green function $G_\Omega(\cdot, z)$ has the property that $G_\Omega(x, z) \to 0$ as $x \to y$ for all $y \in \partial\Omega \backslash P$, where P is a polar set. Thus $G_\Omega(\cdot, z)$ is a barrier at y for each $y \in \partial\Omega\backslash P$ and therefore each such y is regular. Thus the irregular points of $\partial\Omega$ belong to the polar set P.

In the general case, Theorem 6.6.7 allows us to conclude that the irregular boundary points in $\partial\Omega$ are contained in a countable union of polar sets and therefore form a polar set. □

Corollary 6.6.9. *(i) If E is a relatively open subset of $\partial\Omega$ which is negligible for Ω, then each point of E is irregular and the set E is polar.*
(ii) If E is a relatively open subset of $\partial\Omega$ which is polar, then each point of E is irregular.

Proof. (i) If $z \in E$ then we choose $f \in C(\partial^\infty\Omega)$ such that $f = 0$ on $\partial^\infty\Omega\backslash E$ and $f(z) \neq 0$. Since $H_f \equiv 0$, it follows that z is irregular. Hence E is polar, by Theorem 6.6.8.

(ii) This follows from (i) and Theorem 6.5.5. □

We now provide a supplement to Theorem 6.4.8.

Theorem 6.6.10. *Let ω be an open subset of Ω and let E be a subset of $\partial\Omega\backslash\overline{\Omega\backslash\omega}$. Then E is negligible for ω if and only if E is negligible for Ω.*

Proof. The "if" part is immediate from Theorem 6.4.8. To prove the converse we suppose that E is negligible for ω. Since a countable union of negligible sets is negligible, it is enough to treat the case where E is bounded and $\overline{E} \subseteq \partial\Omega\backslash\overline{\Omega\backslash\omega}$. Since E is contained in a Borel set that is negligible for ω, we may also suppose that E is a Borel set and hence that the characteristic function χ_E is resolutive for Ω. We define a function F to be equal to χ_E on

$\partial^\infty \Omega$ and equal to $H^\Omega_{\chi_E}$ on Ω. Then $0 \le F \le 1$ on $\Omega \cup \partial^\infty \Omega$ and $F|_\Omega \in \mathcal{H}(\Omega)$. By Corollary 6.6.6,

$$\lim_{x \to y, x \in \Omega} F(x) = 0 \tag{6.6.6}$$

for every $y \in \partial\Omega \backslash \overline{E}$ that is regular for Ω. By Theorem 6.3.6, Corollary 6.3.2 and the assumption that E is negligible for ω,

$$F(x) = H^\omega_F(x) = H^\omega_{F\chi_{\Omega \cap \partial\omega}}(x) + H^\omega_{\chi_E}(x) = H^\omega_{F\chi_{\Omega\cap\partial\omega}}(x) \qquad (x \in \omega),$$

and hence (6.6.6) holds also for every $y \in \overline{E}$ that is regular for ω. Thus (6.6.6) holds for quasi-every $y \in \partial\Omega$ by Theorem 6.6.8. Also, in the case $N \ge 3$, some multiple of U_0 belongs to Φ^Ω_F, and so $F(x) \to 0$ as $x \to \infty$ if Ω is unbounded. It now follows from Theorem 5.2.6 that $F = 0$ on Ω, so E is negligible for Ω. □

Example 6.6.11. Let $D = \mathbb{R} \times (0, +\infty)$ and $E_t = (0,1) \times \{t\}$, and define $\Omega = D \backslash (\bigcup_{n=1}^\infty \overline{E_{1/n}})$. Then E_0 is negligible for Ω. To see this, let $\omega = \Omega \cap (0,1)^2$. Since E_0 is negligible for ω, by Lemma 6.5.3, E_0 is negligible for Ω, by Theorem 6.6.10.

We now give some simple geometric sufficient conditions for regularity.

Theorem 6.6.12. *If $y \in \partial\Omega$ and there is a ball B_0 such that $\overline{B_0} \cap \overline{\Omega} = \{y\}$, then y is regular.*

Proof. Let $B_0 = B(z,r)$. Then the function $U_z(y) - U_z$ is a barrier at y, so the result follows from Theorem 6.6.5. □

The above theorem implies, in particular, that any bounded domain with a (one-sided) C^2 boundary is regular.

Corollary 6.6.13. *Any open set Ω is the union of a sequence (Ω_n) of bounded regular open sets such that $\overline{\Omega}_n \subset \Omega_{n+1}$ for each n.*

Proof. For the purposes of this proof, we say that a non-empty bounded open set ω is *admissible* if $\omega = \omega_0 \backslash (\overline{B}_1 \cup \ldots \cup \overline{B}_m)$, where ω_0 is a bounded open set and $B_1, \ldots, B_m$ are open balls whose union covers $\partial\omega_0$. It follows from Theorem 6.6.12 that any admissible set is regular, and it is easy to show that any open set Ω is the union of a sequence (Ω_n) of admissible sets such that $\overline{\Omega}_n \subset \Omega_{n+1}$ for each n. □

Even with elementary methods, we can give a much stronger result than Theorem 6.6.12; for example, in $\mathbb{R}^3$ one such result says that y is a regular boundary point for Ω if y is the vertex of a plane triangle lying outside Ω. For this, we need the following lemma.

Lemma 6.6.14. *Let*

$$E = \{x \in \mathbb{R}^N : \|x\| < \delta, \quad x_1 = 0, \quad x_2 \ge 0\},$$

where $\delta > 0$. If $y \in \partial\Omega$ and there is an isometry T such that $T(0) = y$ and $T(E) \subseteq \mathbb{R}^N \backslash \Omega$, then y is regular for Ω.

Proof. It is enough to deal with the case where $y = 0$ and T is the identity mapping on $\mathbb{R}^N$. Let (r, θ) be polar coordinates such that $x_1 = r\sin\theta$, $x_2 = r\cos\theta$. The function $(x_1, \ldots, x_N) \mapsto r^{1/2}\sin(\theta/2)$ is harmonic and positive on $\mathbb{R}^N \backslash \{x : x_1 = 0, x_2 \ge 0\}$ and vanishes at 0. Hence 0 is regular. □

The strengthening of Theorem 6.6.12 mentioned above is as follows.

Theorem 6.6.15. *(i) If $y \in \partial\Omega$, where $\Omega \subset \mathbb{R}^2$, and y is an endpoint of some line-segment lying in $\mathbb{R}^2 \backslash \Omega$, then y is regular.*
(ii) Suppose that $y \in \partial\Omega$ and $\Omega \subset \mathbb{R}^N$, where $N \ge 3$. Suppose also that there exists a cone Γ of vertex y and an $(N-1)$-dimensional hyperplane P containing the axis of Γ such that $\Gamma \cap P \cap B(y,\delta) \subset \mathbb{R}^N \backslash \Omega$ for some $\delta > 0$. Then y is regular.

Proof. (i) This is simply a reformulation of Lemma 6.6.14 for the case $N = 2$.
(ii) We may suppose that $y = 0$. Define $\Omega_0 = B \backslash \overline{\Gamma \cap P}$. Then any barrier for Ω_0 at y, suitably restricted, will be a barrier for Ω. Define $f(x) = \|x\|$ for $x \in \partial\Omega_0$. We will show that $H^{\Omega_0}_f$ is a barrier for Ω_0 at 0 and thus complete the proof. Since the function $x \mapsto \|x\|$ belongs to $\Psi^{\Omega_0}_f$, we have $H^{\Omega_0}_f(x) \ge \|x\|$ on Ω_0, so it is enough to show that $H^{\Omega_0}_f(x) \to 0$ as $x \to 0$. By Lemma 6.6.14, every point of $\partial\Omega_0 \backslash \{0\}$ is regular and therefore $H^{\Omega_0}_f(x) \to f(z)$ as $x \to z$ for each $z \in \partial\Omega_0 \backslash \{0\}$. Since $f \le 1$ on $\partial\Omega_0$ and Ω_0 is connected, either $H^{\Omega_0}_f < 1$ or $H^{\Omega_0}_f = 1$ on Ω_0, but the latter is clearly impossible. Let $\Omega_1 = \Omega_0 \cap B(0, 1/2)$ and define g to be equal to f on $\partial\Omega_1 \cap \partial\Omega_0$ and equal to $H^{\Omega_0}_f$ on $\Omega_0 \cap \partial\Omega_1$. By Theorem 6.3.6, $H^{\Omega_0}_f = H^{\Omega_1}_g$ on Ω_1 and hence $\sup_{\Omega_1} H^{\Omega_0}_f \le \sup_{\partial\Omega_1} g < 1$. Let $\alpha = \max\{1/2, \sup_{\Omega_1} H^{\Omega_0}_f\}$. The function

$$x \mapsto H^{\Omega_0}_f(x) - \alpha H^{\Omega_0}_f(2x)$$

is bounded and harmonic on Ω_1 and has a non-positive limit at each point of $\partial\Omega_1 \backslash \{0\}$. Hence by the maximum principle (Theorem 5.2.6), this function is non-positive on Ω_1 and therefore

$$0 \le \limsup_{x \to 0} H^{\Omega_0}_f(x) \le \alpha \limsup_{x \to 0} H^{\Omega_0}_f(x) < +\infty.$$

Since $0 < \alpha < 1$, it follows that $H^{\Omega_0}_f(x) \to 0$ as $x \to 0$. □

Theorem 6.6.15 implies that if $0 \in \partial\Omega$, then a sufficient condition (called the *Poincaré exterior cone condition*) for 0 to be regular is that $\mathbb{R}^N \backslash \Omega$ contains a cone of revolution with vertex 0. In the opposite direction, the next result describes some sets of revolution E for which 0 is an irregular boundary point of $\mathbb{R}^N \backslash E$.

Theorem 6.6.16. *Let*

$$E = \{(x_1, \ldots, x_N) : \|(x_1, \ldots, x_{N-1})\| < f(x_N^+)\},$$

where $N \geq 3$ and $f : [0, +\infty) \to [0, +\infty)$ is increasing. If

$$\int_0^1 t^{2-N}\{f(t)\}^{N-3} dt < +\infty \qquad (N \geq 4), \tag{6.6.7}$$

$$\int_0^1 \frac{dt}{t\{1 + \log^+(t/f(t))\}} < +\infty \qquad (N = 3),$$

then 0 is irregular for $\mathbb{R}^N \backslash \overline{E}$.

Proof. We prove the case where $N \geq 4$ and leave the case where $N = 3$, which is similar, as an exercise. Points of $\mathbb{R}^N$ will be written as (x', x_N), where $x' \in \mathbb{R}^{N-1}$. Let

$$u_a(x) = \int_{-a}^{a} \{\|x'\|^2 + (x_N - t)^2\}^{(2-N)/2} dt \qquad (0 < a \leq +\infty).$$

Then $u_{+\infty}$ is harmonic on $(\mathbb{R}^{N-1} \backslash \{0'\}) \times \mathbb{R}$ and depends only on $\|x'\|$, so the function $x' \mapsto u_{+\infty}(x', 0)$ is harmonic on $\mathbb{R}^{N-1} \backslash \{0'\}$. Further, by monotone convergence, this function has limit $+\infty$ at $0'$ and limit 0 as $\|x'\| \to +\infty$. Hence, by Theorem 1.1.2, $u_{+\infty}(x) = c_1 \|x'\|^{3-N}$ for some positive constant c_1. Since $u_{+\infty} - u_1$ has a harmonic continuation to B, there is a positive constant c_2 such that $u_1(x', 0) \geq 2c_2 \|x'\|^{3-N}$ when $\|x'\| \leq 1$. If $|x_N| \leq 1$ and $\|x'\| \leq 1$, then

$$u_1(x', x_N) \geq \frac{1}{2} \int_{x_N - 1}^{x_N + 1} \{\|x'\|^2 + (x_N - t)^2\}^{(2-N)/2} dt \geq c_2 \|x'\|^{3-N}.$$

Since $u_1(a^{-1}x', a^{-1}x_N) = a^{N-3} u_a(x', x_N)$, we obtain

$$u_a(x', x_N) \geq c_2 \|x'\|^{3-N} \qquad (\|x'\| \leq a; |x_N| \leq a; 0 < a < +\infty). \tag{6.6.8}$$

Now let E and f be as stated, and let $\Omega = \mathbb{R}^N \backslash \overline{E}$. It follows from (6.6.7) that

$$\sum_{n=1}^{\infty} \{2^n f(2^{-n})\}^{N-3} < +\infty.$$

Clearly $f(2^{-n}) < 2^{-n-2}$ for all sufficiently large n. Also, we can choose a sequence (b_n) of positive numbers such that $b_n \to +\infty$ and

$$\sum_{n=1}^{\infty} b_n \{2^n f(2^{-n})\}^{N-3} < +\infty. \tag{6.6.9}$$

Let

$$v(x) = \sum_{n=1}^{\infty} b_n \{f(2^{-n})\}^{N-3} \int_{2^{-n-1}}^{2^{-n}} \{\|x'\|^2 + (x_N - t)^2\}^{(2-N)/2} dt.$$

Then v is a potential on $\mathbb{R}^N$ and v is harmonic on $\mathbb{R}^N \backslash (\{0'\} \times [0, 1])$. It is clear from (6.6.9) that $v(0) < +\infty$. However, if $2^{-n-1} \leq x_N \leq 2^{-n}$ and $\|x'\| \leq f(2^{-n})$, then it follows from (6.6.8) with $a = 2^{-n-2}$ and a translation that $v(x) \geq b_n c_2$, provided n is large enough to ensure that $f(2^{-n}) < 2^{-n-2}$. Hence $v(x) \to +\infty$ as $x \to 0$ along $\overline{E}$. Now define g on $\partial^\infty \Omega$ by writing $g(0) = 2v(0), g(\infty) = 0$ and $g = \min\{v, 2v(0)\}$ elsewhere. Then $g \in C(\partial^\infty \Omega)$ and $v + \varepsilon U_0 \in \Phi_g$ for every positive number ε, so $H_g \leq v$ on Ω. However, it is clear from the definition of v that $v(0', x_N) < v(0) = g(0)/2$ when $x_N < 0$, so we cannot have $H_g(x) \to g(0)$ as $x \to 0$, and therefore 0 is irregular for Ω. □

Remark 6.6.17. It follows from the above theorem that if $N \geq 4$, then 0 is an irregular boundary point for sets of the form

$$\mathbb{R}^N \backslash \{(x', x_N) : x_N \geq 0, \quad \|x'\| \leq x_N^\alpha\}$$

when $\alpha > 1$. However, as we have already observed, Theorem 6.6.15 shows that when $\alpha = 1$, the point 0 is regular for this set. In the case $N = 3$, Theorem 6.6.16 shows that 0 is an irregular boundary point of the set $\mathbb{R}^3 \backslash F$, where

$$F = \{0\} \cup \{(x', x_3) : x_3 > 0, \|x'\| \leq \exp(-x_3^{-\varepsilon})\}$$

and $\varepsilon > 0$. In the case where $\varepsilon = 1$ the set F is referred to as the *Lebesgue spine*.

We conclude this section with an example showing that the boundedness of f in Corollary 6.6.6(ii) cannot be dispensed with.

Example 6.6.18. For each $n \in \mathbb{N}$ let $\Omega_n = (-1, 1) \times ((n+1)^{-1}, n^{-1})$, and let $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Then 0 is a regular boundary point of $\partial\Omega$, but there is a resolutive function f, continuous at 0, for which $\lim_{x \to 0} H_f(x)$ does not exist.

To see this, let x_n be the midpoint of the rectangle Ω_n. Then there exists $f_n \in C(\partial\Omega_n)$ such that $f_n = 0$ on $[-1, 1] \times \{(n+1)^{-1}, n^{-1}\}$ and $H_{f_n}^{\Omega_n}(x_n) = 2$. Also, since the point $(0, n^{-1})$ is regular for Ω_n, there exists $y_n \in \Omega_n \cap (\{0\} \times \mathbb{R})$ such that $H_{f_n}^{\Omega_n}(y_n) < 1$. Define f on $\partial\Omega$ by putting $f = f_n$ on $\{-1, 1\} \times ((n+1)^{-1}, n^{-1})$ for each n and $f = 0$ elsewhere. Then f is continuous at 0 and $H_f^\Omega = H_{f_n}^{\Omega_n}$ on Ω_n for each n. In particular,

$$H_f^\Omega(x_n) = 2, \qquad H_f^\Omega(y_n) < 1 \qquad (n \in \mathbb{N}).$$

Since $x_n \to 0$ and $y_n \to 0$, it follows that $\lim_{x\to 0} H_f^\Omega(x)$ does not exist. We note that, by Theorem 6.6.15(i), all points of $\partial\Omega$ are regular.

6.7. Behaviour near infinity

Theorem 6.7.1. *If Ω is an unbounded open subset of $\mathbb{R}^N$, where $N \geq 3$, then ∞ is regular for Ω.*

Proof. Let $f \in C(\partial^\infty\Omega)$. We have to show that $H_f(x) \to f(\infty)$ as $x \to \infty$. By adding a constant to f, we may suppose that $f(\infty) = 0$, and by multiplying f by a suitable positive constant, we may also suppose that $|f| \leq 1$ on $\partial^\infty\Omega$. Given $\varepsilon > 0$, let $R > 0$ be such that $|f(y)| < \varepsilon$ when $y \in \partial\Omega$ and $||y|| > R$. Then the function $u_\varepsilon(x) = \varepsilon + (R/||x||)^{N-2}$ belongs to Φ_f and $-u_\varepsilon \in \Psi_f$. Hence $|H_f| \leq u_\varepsilon$ on Ω. Since $u_\varepsilon(x) \to \varepsilon$ as $x \to \infty$ and ε is an arbitrary positive number, the required conclusion follows. □

In $\mathbb{R}^2$ the question of whether ∞ is a regular boundary point of Ω is more complicated. An answer will appear as a corollary of the following theorem. We use the notation of Definition 1.6.2 with the convention that the inverse ∞^* of ∞ with respect to $S(y,a)$ is y and $y^* = \infty$, and we then modify the definition of the inverse of a set E by putting $E^* = \{x^* : x \in E\}$ for any $E \subseteq \mathbb{R}^N \cup \{\infty\}$.

Theorem 6.7.2. *Let Ω be a Greenian open subset of $\mathbb{R}^2$, and let Ω^* be its inverse with respect to some sphere $S(y,a)$, where $y \in \mathbb{R}^2\backslash\Omega$. If $f : \partial^\infty\Omega \to [-\infty,+\infty]$, then $\overline{H}_{f^*}^{\Omega^*}(x) = \overline{H}_f^\Omega(x^*)$ for each $x \in \Omega^*$.*

Proof. Suppose that $u \in \Phi_f^\Omega$. Then u^* is hyperharmonic and bounded below on Ω^*. Also, by the continuity of the mapping $x \mapsto x^*$,

$$\liminf_{x\to z, x\in\Omega^*} u^*(x) = \liminf_{x\to z, x\in\Omega^*} u(x^*) = \liminf_{x^*\to z^*, x^*\in\Omega} u(x^*) \geq f(z^*) = f^*(z)$$

for each $z \in \partial^\infty\Omega^* = (\partial^\infty\Omega)^*$. Hence $u^* \in \Phi_{f^*}^{\Omega^*}$. We now have

$$\overline{H}_{f^*}^{\Omega^*}(x) \leq \inf\{u^*(x) : u \in \Phi_f^\Omega\} = \inf\{u(x^*) : u \in \Phi_f^\Omega\} = \overline{H}_f^\Omega(x^*)$$

for each $x \in \Omega^*$. Since the mapping $x \mapsto x^*$ is its own inverse, the same argument shows that $\overline{H}_{f^*}^{\Omega^*}(x) \geq \overline{H}_f^\Omega(x^*)$ for each $x \in \Omega^*$. □

Corollary 6.7.3. *Let $N = 2$. With the notation of Theorem 6.7.2, a point z of $\partial^\infty\Omega$ is regular for Ω if and only if z^* is regular for Ω^*.*

Proof. Since $(x^*)^* = x$, it is enough to prove the "if" statement. Suppose that z^* is regular for Ω^*, and let $f \in C(\partial^\infty\Omega)$. Then $f^* \in C(\partial^\infty\Omega^*)$ and by Theorem 6.7.2 (and its counterpart for lower solutions), $H_f^\Omega(x) = H_{f^*}^{\Omega^*}(x^*)$ for each $x \in \Omega$. By the regularity of z^* for Ω^* and the continuity of the mapping $x \mapsto x^*$, we have

$$\lim_{x\to z} H_f^\Omega(x) = \lim_{x^*\to z^*} H_{f^*}^{\Omega^*}(x^*) = f^*(z^*) = f(z),$$

so that z is regular for Ω. □

Example 6.7.4. (i) If Ω is a Greenian open subset of $\mathbb{R}^2$ and $\mathbb{R}^2\backslash\Omega$ is compact, then ∞ is irregular for Ω. To see this, let Ω^* be the inverse of Ω with respect to $S(y,1)$, where $y \in \mathbb{R}^2\backslash\Omega$. Then $y = \infty^*$ under this inversion and y is an isolated, and hence irregular (see Corollary 6.6.9(ii)), point of $\partial\Omega^*$. Thus by Corollary 6.7.3, ∞ is an irregular point of $\partial^\infty\Omega$.

(ii) If Ω is an unbounded open subset of $\mathbb{R}^2$ and $\mathbb{R}^2\backslash\Omega$ contains a half-line, then ∞ is regular for Ω. To see this, let Ω^* be the inverse of Ω with respect to $S(y,1)$, where y is a point of some half-line in $\mathbb{R}^2\backslash\Omega$. Then $y = \infty^*$ and there is a line-segment in $\mathbb{R}^2\backslash\Omega^*$ having y as an endpoint. Hence y is regular for Ω^* by Theorem 6.6.15(i), and ∞ is regular for Ω by Corollary 6.7.3.

Theorem 6.7.5. *A Greenian set Ω in $\mathbb{R}^N$ is regular if and only if each component of Ω is regular.*

Proof. This follows from Theorems 6.6.7 and 6.7.1 and Corollary 6.7.3. □

6.8. Regularity and the Green function

Theorem 6.8.1. *If $\Omega \subseteq \mathbb{R}^N$, where $N \geq 3$, or Ω is a bounded open subset of $\mathbb{R}^2$, then $G_\Omega(\cdot,y) = U_y - H_{U_y}$ on Ω for each $y \in \Omega$. (Here we define $U_y(\infty) = 0$ if Ω is unbounded.)*

Proof. Fix $y \in \Omega$ and note that under the stated hypotheses $U_y \in C(\partial^\infty\Omega)$ and $U_y \in \Phi_{U_y}$, so that $U_y \geq H_{U_y}$ on Ω. Since $G_\Omega(\cdot,y) = U_y - h_y$, where h_y is the greatest harmonic minorant of U_y on Ω, we see that $h_y \geq H_{U_y}$ on Ω. On the other hand, $h_y \leq U_y$ on Ω, so $h_y \in \Psi_{U_y}$ and therefore $h_y \leq H_{U_y}$ on Ω. Thus $h_y = H_{U_y}$ and the proof is complete. □

Example 6.8.2. Let $\Omega = \mathbb{R}^2\backslash K$, where K is compact and non-polar and let $y \in \Omega$. Then $G_\Omega(\cdot,y) \neq U_y - H_{U_y}$ (no matter what value is assigned to $U_y(\infty)$). Thus Theorem 6.8.1 does not extend to all Greenian sets in $\mathbb{R}^2$.

To see this, we note that $\{\infty\}$ is a negligible subset of $\partial^\infty\Omega$ (see Example 6.5.6(i)) and U_y is bounded on $\partial\Omega$, so H_{U_y} is bounded on Ω. Hence $U_y - H_{U_y}$ takes negative values at some points of Ω, but $G_\Omega(\cdot,y)$ does not.

Theorem 6.8.3. *Suppose that $z \in \partial^\infty \Omega$. The following are equivalent:*
(a) z is regular for Ω;
(b) for every $y \in \Omega$,

$$\lim_{x \to z} G_\Omega(x,y) = 0; \tag{6.8.1}$$

(c) every component of Ω contains a point y for which (6.8.1) holds.

Proof. We first treat the case where $z \in \partial\Omega$. Suppose that (c) holds. Then, for each component ω of Ω which satisfies $z \in \partial\omega$, there is a point $y \in \omega$ such that $G_\Omega(\cdot, y)$ is a barrier for ω at z, and thus z is regular for ω. It follows from Theorem 6.6.7 that (a) holds.

In the case where Ω satisfies the hypotheses of Theorem 6.8.1, it is clear that (a) implies (b). In order to include the case where Ω is an unbounded subset of $\mathbb{R}^2$, we give a different argument. Fix $y \in \Omega$ and choose $r > 0$ such that $\overline{B(y,r)} \subset \Omega$. If z is regular, then there is a barrier v at z and, by Lemma 6.6.3, we may assume that $v \in \mathcal{U}_+(\Omega)$. We can arrange that $v > G_\Omega(\cdot, y)$ on $S(y,r)$ by working with a suitable multiple of v. Define $u = G_\Omega(\cdot, y)$ on $\overline{B(y,r)}$ and $u = \min\{v, G_\Omega(\cdot, y)\}$ on $\Omega \backslash \overline{B(y,r)}$. Then $u \in \mathcal{U}_+(\Omega)$ and u has the form $U_y + w$, where $w \in \mathcal{U}(\Omega)$. Since $G_\Omega(\cdot, y)$ is the minimal function of this form in $\mathcal{U}_+(\Omega)$, we have $0 \le G_\Omega(\cdot, y) \le u \le v$ on $\Omega \backslash \overline{B(y,r)}$ and hence (6.8.1) holds. Since (b) clearly implies (c), this completes the proof of the theorem, apart from the case where $z = \infty$.

Finally, suppose that Ω is unbounded and $z = \infty$. In the case where $N \ge 3$, the point ∞ is necessarily regular, by Theorem 6.7.1, and for each fixed $y \in \Omega$,

$$0 \le G_\Omega(x,y) \le U_y(x) \to 0 \qquad (x \to \infty, x \in \Omega).$$

In the case where $N = 2$, Theorem 4.1.11 and Corollary 6.7.3 enable us to deduce the theorem for $z = \infty$ from the already established case where $z \in \partial\Omega$. □

Corollary 6.8.4. *The irregular boundary points of Ω form an F_σ set.*

Proof. It is sufficient to show that the set E of finite irregular boundary points is F_σ, since the possible addition of ∞ to E would preserve this property. In view of Theorem 6.6.7, and the fact that a countable union of F_σ sets is F_σ, we may assume that Ω is connected. Let $y \in \Omega$. Theorem 6.8.3 shows that (6.8.1) holds for a point $z \in \partial\Omega$ if and only if $z \notin E$. Thus $E = \{z \in \partial\Omega : f(z) > 0\}$, where

$$f(z) = \limsup_{x \to z} G_\Omega(x,y) \qquad (z \in \partial\Omega).$$

Since f is upper semicontinuous on $\partial\Omega$, the set $\{z \in \partial\Omega : f(z) \ge n^{-1}\}$ is closed for each $n \in \mathbb{N}$. Thus E, being the union of these sets, is an F_σ set. □

6.9. PWB solutions and reduced functions

The first result in this section is a generalization of the fact that if we take a superharmonic function and replace it in some ball by its Poisson integral then superharmonicity is preserved (see Corollary 3.2.5).

Theorem 6.9.1. *Suppose that $u \in \mathcal{U}_+(\Omega)$, let E be a relatively closed subset of Ω, and define $u_1 = u$ on Ω and $u_1 = 0$ on $\partial^\infty\Omega$. Then $u_1 \in \mathcal{R}(\Omega\backslash E)$ and*

$$R_u^E(x) = \begin{cases} u(x) & (x \in E) \\ H_{u_1}^{\Omega\backslash E}(x) & (x \in \Omega\backslash E). \end{cases}$$

In particular, $H_{u_1}^{\Omega\backslash E} \le u$ on $\Omega\backslash E$.

Proof. Since u_1 is lower semicontinuous on $\partial^\infty(\Omega\backslash E)$ and $u \in \Phi_{u_1}^{\Omega\backslash E}$, it follows that $u_1 \in \mathcal{R}(\Omega\backslash E)$. If $v \in \mathcal{U}_+(\Omega)$ and $v \ge u$ on E, then $v \in \Phi_{u_1}^{\Omega\backslash E}$, so $R_u^E \ge H_{u_1}^{\Omega\backslash E}$ on $\Omega\backslash E$. To prove the reverse inequality, let $w \in \Phi_{u_1}^{\Omega\backslash E} \cap \mathcal{U}(\Omega\backslash E)$ and define $\tilde{w}$ to be equal to u on E and $\min\{w, u\}$ on $\Omega\backslash E$. By Corollary 3.2.4, $\tilde{w} \in \mathcal{U}_+(\Omega)$ and hence $\tilde{w} \ge R_u^E$ on Ω. Thus $w \ge R_u^E$ on $\Omega\backslash E$, and since w is an arbitrary superharmonic element of $\Phi_{u_1}^{\Omega\backslash E}$, it follows that $H_{u_1}^{\Omega\backslash E} \ge R_u^E$ on $\Omega\backslash E$. □

Corollary 6.9.2. *Let ω be a bounded open set such that $\overline{\omega} \subset \Omega$, let $u \in \mathcal{U}(\Omega)$ and define*

$$v(x) = \begin{cases} u(x) & (x \in \Omega\backslash\omega) \\ H_u^\omega(x) & (x \in \omega). \end{cases}$$

Then $\hat{v} \in \mathcal{U}(\Omega)$ and $\hat{v} \le u$ on Ω.

Proof. This follows from Theorem 6.9.1 by defining $E = \Omega\backslash\omega$. □

Corollary 6.9.3. *If E is a relatively closed subset of Ω and $u, v \in \mathcal{U}_+(\Omega)$, then*

$$R_{u+v}^E = R_u^E + R_v^E$$

and

$$\widehat{R}_{u+v}^E = \widehat{R}_u^E + \widehat{R}_v^E$$

on Ω.

Proof. By Theorem 6.9.1 the first equation holds on Ω, and hence the second equation holds quasi-everywhere on Ω. The second equation therefore holds everywhere on Ω, by Corollary 3.2.7. □

6.10. Superharmonic extension

Newtonian potentials on $\mathbb{R}^N$ (or logarithmic potentials in the case $N = 2$) are convenient functions to work with, especially when their Riesz measures have compact support. It is therefore of some interest to know when a superharmonic function on an open set can be represented, at least locally, as such a potential plus a constant. The proof of the following theorem makes use of properties of the PWB solution of a Dirichlet problem.

Theorem 6.10.1. *Let K be a compact subset of $\mathbb{R}^N$ such that $\mathbb{R}^N\backslash K$ is connected. If u is superharmonic on some open set containing K, then there exists $\overline{u} \in \mathcal{U}(\mathbb{R}^N)$ such that $\overline{u} = u$ on K, and there exist $\alpha, \beta, \rho \in \mathbb{R}$ with $\beta, \rho > 0$ such that $\overline{u} = \alpha + \beta U_0$ on $\mathbb{R}^N\backslash B(0,\rho)$.*

Proof. Let Ω be a bounded open set containing K such that u is superharmonic and bounded below on Ω. By adding a suitable constant to u, we may suppose that $u > 0$ on Ω. Let L be a compact set such that $K \subset L^\circ$ and $L \subset \Omega$. By initially taking L to be a finite union of cubes of equal size, we can arrange that $\mathbb{R}^N\backslash L$ has only finitely many bounded components. If ω is such a component, then there exists a tract T from a point of ω to ∞ (see Section 2.6) such that $\overline{T} \subset \mathbb{R}^N\backslash K$. By removing such tracts from L we arrange that $\mathbb{R}^N\backslash L$ is connected. Let R be such that $\overline{\Omega} \subset B(0, R)$. Our first aim is to show that there is a superharmonic function w on $B(0, R+1)$ such that $w = u$ on L°. Let $v = \widehat{R}_u^L$, the balayage of u relative to L in Ω. Then $v \in \mathcal{U}(\Omega) \cap \mathcal{H}(\Omega\backslash L)$ and $0 \le v \le u$ on Ω with $v = u$ on L°. Let Ω_0 be a regular domain of the form $B(0, R+1)\backslash E$ where E is a compact subset of Ω such that $L \subset E^\circ$. Let $g_1 = v$ on $\partial E, g_1 = 0$ on $S(0, R+1), g_2 = 0$ on $\partial E, g_2 = 1$ on $S(0, R+1)$, and define $h_k = H_{g_1}^{\Omega_0} - kH_{g_2}^{\Omega_0}$ for each $k \in \mathbb{N}$. Since $H_{g_2}^{\Omega_0} > 0$ on Ω_0 by Corollary 6.6.9(i), we see that h_k decreases to $-\infty$ on Ω_0 as $k \to \infty$. By Dini's theorem there exists m such that $h_m \le 0$ on $\partial\Omega$. Since Ω_0 is regular, $h_m(x) \to v(y)$ as $x \to y$ for each $y \in \partial E$. Since, also, $h_m \le 0$ on $\partial\Omega$ and $v \ge 0$ on Ω, the minimum principle yields $h_m \le v$ on Ω. Now define $w = h_m$ on Ω_0 and $w = v$ on E. By Corollary 3.2.4, $w \in \mathcal{U}(\Omega)$. Since $w \in \mathcal{H}(\Omega_0)$, we conclude that $w \in \mathcal{U}(B(0, R+1)) \cap \mathcal{H}(B(0, R+1)\backslash E)$.

We note that $w = v = u$ on L° and w tends to $-m$ on $S(0, R+1)$. Let $M = \sup_{S(0,R)} w$ and choose numbers α, β with $\beta > 0$ such that

$$\alpha + \beta U_0(x) \ge M \quad (x \in S(0, R)), \quad \alpha + \beta U_0(x) \le -m \quad (x \in S(0, R+1)).$$

We define

$$\overline{u}(x) = \begin{cases} w(x) & (x \in \overline{B(0,R)}) \\ \min\{w(x), \alpha + \beta U_0(x)\} & (x \in B(0, R+1)\backslash\overline{B(0,R)}) \\ \alpha + \beta U_0(x) & (x \in \mathbb{R}^N\backslash B(0, R+1)). \end{cases}$$

Then $\overline{u} = u$ on L° and it follows from two applications of Corollary 3.2.4 that $\overline{u} \in \mathcal{U}(\mathbb{R}^N)$. □

Remark 6.10.2. The function $\overline{u}$ in Theorem 6.10.1 is harmonic outside some ball and agrees with u, not just on K, but on a neighbourhood of K. It follows that the Riesz measure associated with $\overline{u}$ has compact support and the Riesz measures associated with u and $\overline{u}$ agree on K.

Example 6.10.3. There is no function $\overline{u} \in \mathcal{U}(\mathbb{R}^N)$ such that $\overline{u} = -U_0$ on $\{x : 1 \le \|x\| \le 2\}$. For if there were such a function $\overline{u}$, then $\mathcal{M}(\overline{u}; 0, \cdot)$ would be strictly increasing on $[1, 2]$, which is impossible. This shows that the hypothesis in Theorem 6.10.1 that $\mathbb{R}^N\backslash K$ is connected cannot be dispensed with.

Also, if u is superharmonic on a bounded open set Ω, there will not in general exist a function $\overline{u} \in \mathcal{U}(\mathbb{R}^N)$ such that $\overline{u} = u$ on Ω even if $\mathbb{R}^N\backslash\Omega$ and $\mathbb{R}^N\backslash\overline{\Omega}$ are connected, as we now show.

Example 6.10.4. Let $\Omega = \{x \in \mathbb{R}^N : \|x\| < 1, x_N > 0\}$. If $u(x) = \sqrt{x_N}$ on Ω, then $u \in \mathcal{U}(\Omega)$ but there is no $\overline{u} \in \mathcal{U}(\mathbb{R}^N)$ such that $\overline{u} = u$ on Ω. In fact, $\Delta u(x) = -x_N^{-3/2}/4$, so Δu is negative and not integrable on Ω. Thus, if there were such a function $\overline{u}$, its associated Riesz measure ν would be such that $\nu(\Omega) = +\infty$ (see Theorem 4.3.2(i) and Corollary 4.3.3), which is impossible.

6.11. Exercises

Exercise 6.1. Let $\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : 0 < \|x'\| < 1\}$, where $N \ge 3$. Find a function in $C(\partial^\infty\Omega)$ for which the classical solution to the Dirichlet problem fails to exist.

Exercise 6.2. Let $N = 2$, let $\Omega = B(0, e)\backslash\overline{B}$ and let $f = 1$ on $S(0, e)$ and $f = 0$ on S. Show directly from Definition 6.2.3 that $\overline{H}_f^\Omega = \underline{H}_f^\Omega = -U_0$.

Exercise 6.3. Show that Theorem 6.3.5 may fail if $\overline{H}_{f_n} \equiv -\infty$ for all n.

Exercise 6.4. Let (Ω_n) be an increasing sequence of open sets such that $\bigcup_n \Omega_n = \Omega$, where Ω is connected, let $h \in \mathcal{H}(\Omega)$ and $x_1 \in \Omega_1$. Show that h can be written as $h_1 - h_2$ where $h_1, h_2 \in \mathcal{H}_+(\Omega)$ if and only if $(H_{|h|}^{\Omega_n}(x_1))$ is bounded.

Exercise 6.5. Let $\Omega = \mathbb{R}^2\backslash\overline{B}$. Use the Kelvin transform to show that harmonic measure for Ω and $z \in \Omega$ is given by

$$d\mu_z(y) = \frac{1}{2\pi}\frac{\|z\|^2 - 1}{\|z - y\|^2} d\sigma(y),$$

where σ denotes arc length measure on S.

Exercise 6.6. Let $\Omega = \mathbb{R}^N \backslash \overline{B}$, where $N \geq 3$. Use the Kelvin transform and the maximum principle to show that, if $f \in C(\partial^\infty \Omega)$, then

$$H_f^\Omega(z) = \frac{1}{\sigma_N} \int_S \frac{\|z\|^2 - 1}{\|z-y\|^N} f(y) d\sigma(y) + f(\infty)\{1 - \|z\|^{2-N}\}.$$

Exercise 6.7. Let $\Omega = \mathbb{R} \times (-\pi/2, \pi/2)$, $D = (0, +\infty) \times \mathbb{R}$ and $f \in C(\partial^\infty \Omega)$, and define $g \in C(\partial^\infty D)$ by $g(\infty) = g(0) = f(\infty)$ and

$$g(0,t) = \begin{cases} f(\log t, \pi/2) & (t > 0) \\ f(\log |t|, -\pi/2) & (t < 0). \end{cases}$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, show that $H_f^\Omega(z) = H_g^D(e^z)$ when $z \in \Omega$, and deduce that

$$H_f^\Omega(\xi + i\eta) = \frac{1}{2\pi} \sum_{k=0}^{1} \int_{-\infty}^{+\infty} \frac{\cos \eta}{\cosh(\xi - t) - (-1)^k \sin \eta} f(t, (-1)^k \pi/2) dt.$$

Exercise 6.8. Let f be a bounded resolutive function on $\partial^\infty \Omega$ and let f_0 be defined on $\partial^\infty \Omega$ by

$$f_0(y) = \min\{f(y), \limsup_{x \to y} H_f^\Omega(x)\}.$$

Show that $H_f = H_{f_0}$ and deduce that

$$\limsup_{x \to y} H_f^\Omega(x) \geq f(y)$$

for all but a negligible set of points y in $\partial^\infty \Omega$.

Exercise 6.9. Let $D = \mathbb{R}^{N-1} \times (0, +\infty)$ and let A be a relatively open subset of ∂D. Show that there exists $h \in \mathcal{H}_+(D)$ such that h has limit 1 at each point of A and $h(0', 1) \leq \lambda'(A)$. Deduce that, if $E \subseteq \partial D$ and $\lambda'(E) = 0$, then there exists $u \in \mathcal{H}_+(D)$ such that u has limit $+\infty$ at each point of E.

Exercise 6.10. Let $S_t = \{x \in S(0,t) : x_N \geq 0\}$ and define $\Omega = B \backslash \left(\bigcup_{k=2}^\infty S_{1-k^{-1}} \right)$. Show that S_1 is negligible for Ω. (Hint: first show that $\{x \in S : x_N > 0\}$ is negligible.)

Exercise 6.11. Let $\Omega = B \backslash \left(\{0\} \cup \bigcup_{n=2}^\infty S(0, 1/n) \right)$. By constructing a barrier for Ω at 0, show that 0 is a regular boundary point of $\partial\Omega$.

Exercise 6.12. Let $\Omega = \bigcup_{k=1}^\infty \left[\left(\frac{1}{k+1}, \frac{1}{k} \right) \times (-2, k) \right]$. Find $f : \partial^\infty \Omega \to [0, +\infty)$ such that f is continuous on $\partial\Omega$ and H_f^Ω exists, yet

$$\limsup_{x \to 0} H_f^\Omega(x) = +\infty.$$

Why does this not contradict Theorem 6.6.7?

Exercise 6.13. Let $N = 2$ and $0 \in \partial\Omega$, and suppose that there is a one-to-one continuous function $g : [0,1] \to \mathbb{R}^2 \backslash \Omega$ such that $g(0) = 0$. Show that 0 is regular for Ω. (Hint: let $t \in (0,1]$ be the smallest number such that $\|g(t)\| = \|g(1)\|$ and let $\omega = B(0,t) \backslash g([0,t])$. Then ω is simply connected, so there is a holomorphic function f on ω such that $e^{f(z)} = z$ there. Now consider $\mathrm{Re}(1/f)$.)

Exercise 6.14. Show that a subset E of $\partial^\infty \Omega$ is negligible if and only if there exists $h \in \mathcal{H}_+(\Omega)$ such that $h(x) \to +\infty$ as $x \to y$ for each $y \in E$. (Hint: to prove the "only if" part, adapt the proof of Theorem 6.5.2 to show that there exists $h_1 \in \mathcal{H}(\Omega)$ such that $h_1(x) \to +\infty$ as $x \to y$ for each regular point of E. Then use the approach of Exercise 5.6 to complete the argument.)

Exercise 6.15. Justify the case $N = 3$ of Theorem 6.6.16.

Exercise 6.16. Let E be a relatively closed subset of Ω.
(i) Show that, if $u_n \in \mathcal{U}_+(\Omega)$ for each n and $\sum u_n$ converges somewhere in each component of Ω, then $R^E_{\sum u_n} = \sum R^E_{u_n}$ on Ω.
(ii) Show that, if $G_\Omega \mu$ is a potential on Ω, then $R^E_{G_\Omega \mu} = \int R^E_{G_\Omega(\cdot, y)} d\mu(y)$ on Ω.

Exercise 6.17. Let K be a compact subset of an open set Ω in $\mathbb{R}^N$ such that every bounded component of $\mathbb{R}^N \backslash K$ contains a point of $\mathbb{R}^N \backslash \Omega$. Show that, if u is superharmonic on some open set containing K, then there exists $\overline{u} \in \mathcal{U}(\Omega)$ such that $\overline{u} = u$ on K.

Exercise 6.18. Let K be a compact subset of $\mathbb{R}^N$, let $\widetilde{K}$ denote the union of K with the bounded components of $\mathbb{R}^N \backslash K$, and suppose that $\partial K \neq \partial \widetilde{K}$. Find a superharmonic function u on some neighbourhood of K with the following property: there is no superharmonic function $\overline{u}$ on $\mathbb{R}^N$ such that $\overline{u} = u$ on K.