$$l(\Gamma) := \lim_{\gamma \to \beta^-} \int_\alpha^\gamma \|Dg(t)\| dt.$$

Prove that: If Γ is a half-open curve in E with C^1-parametrization $g \in C^1([\alpha, \beta), E)$ and finite length, then Γ is relatively compact.

Application: If φ is a semiflow on E and if $\gamma^+(x)$ has finite length, i.e.,

$$l(\gamma^+(x)) := \lim_{t \to t^+(x)^-} \int_0^t \left\| \frac{d}{d\tau}(\tau \cdot x) \right\| d\tau < \infty, \tag{37}$$

then $\gamma^+(x)$ is relatively compact.

(*Hint*: Since E is complete, it suffices to show that Γ is *totally bounded*, i.e., for every $\epsilon > 0$ there exist finitely many points $x_1, \ldots, x_m \in \Gamma$ such that $\Gamma \subseteq \bigcup_{j=1}^m \mathbb{B}(x_j, \epsilon)$. This follows easily from (36) and (37) by considering sufficiently small parametrization intervals and using the fundamental theorem of calculus.)

7. Let E be a real Banach space, $g \in C^1(E, \mathbb{R})$ and $v \in C^{1-}(E, E)$. Then v is called a *pseudo-gradient vector field* (PGVF) *for* g if

$$\|v(x)\| \le 2\|Dg(x)\| \quad \text{and} \quad \langle Dg(x), v(x)\rangle \ge \|Dg(x)\|^2/2$$

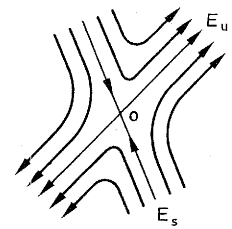
for all $x \in E$. If $E = (E, (\cdot \mid \cdot))$ is a Hilbert space, then clearly $v :=$grad g is a PGVF for g. PGVF's play an important role in the calculus of variations in the large and nonlinear functional analysis, where one tries to obtain statements about the existence and multiplicity of critical points of g.

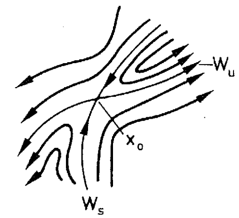
Show that: If $\gamma^+(x)$ is a positive semitrajectory of the flow induced by $-v$ and if $\gamma^+(x)$ has finite length, then g has a critical point at the level $\alpha := \inf g(\gamma^+(x))$, i.e., $g^{-1}(\alpha) \cap \{x \in E \mid Dg(x) = 0\} \neq \emptyset$.

19. Linearizations

In this section we let $(E, |\cdot|)$ denote a finite dimensional Banach space, M an open subset of E and $f \in C^1(M, E)$.

The main goal is to study the flow φ, induced by f, in the neighborhood of a critical point x_0, in particular, in situations in which the principle of linearized stability (theorem (15.6)) is not applicable. For reasons of simplicity, we only consider the case when $Df(x_0)$ induces a hyperbolic linear flow. We will show that locally, i.e., near x_0, the flow φ is flow equivalent to the linear flow $e^{tDf(x_0)}$, that is to say, the structure of a saddle is preserved. In addition, we will derive precise statements about the "stable and unstable manifolds" W_s and W_u.





Let X and Y be either metric spaces or differentiable manifolds and assume that $\varphi : \Omega_\varphi \to X$ and $\psi : \Omega_\psi \to Y$ are flows on X and Y, respectively. We say that φ *at* $x_0 \in X$ *is (locally)* C^k*-flow equivalent to* ψ *at* $y_0 \in Y$, or for short: $\varphi | x_0$ *is* C^k*-flow equivalent to* $\psi | y_0$, $0 \le k \le \infty$, if there exist neighborhoods U and V of x_0 and y_0, respectively, such that the flow φ restricted to U is C^k-flow equivalent to the flow ψ restricted to V, for short: $\varphi | U$ and $\psi | V$ are C^k-flow equivalent.

In what follows, we always let $\varphi : \Omega \to M$ denote the flow induced by f on M and, again, we write $t \cdot x$ for $\varphi(t, x)$.

The first proposition shows that one can "straighten out the trajectories" in a neighborhood of a *regular point*, i.e., a noncritical point. Here we have $\dim_\mathbb{R}(E) = 2\dim_\mathbb{C}(E)$ whenever $\mathbb{K} = \mathbb{C}$ ("decomposition into real and imaginary parts").

(19.1) Proposition. *Let $x_0 \in M$ be a regular point of φ and let ψ denote the flow induced by the constant vector field $y \mapsto e_1 = (1, 0, \ldots, 0)$ on $\mathbb{R}^m$, $m := \dim_\mathbb{R}(E)$, that is to say,*

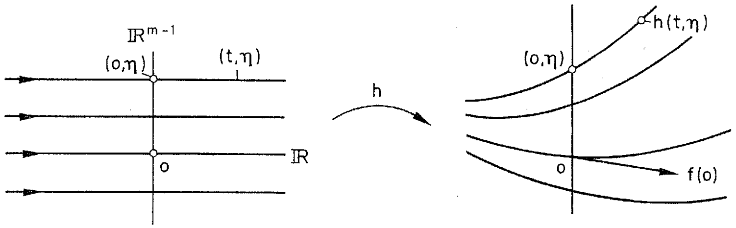
$$\psi(t, y) = y + te_1, \quad \forall (t, y) \in \mathbb{R} \times \mathbb{R}^m.$$

Then $\varphi | x_0$ and $\psi | 0$ are C^1-flow equivalent.

Proof. By decomposing f and x into real and imaginary parts, we may assume that $\mathbb{K} = \mathbb{R}$. By means of a translation we can obtain $x_0 = 0$, and by introducing a basis with first basis vector $f(0)$, we may identify E with $\mathbb{R}^m$ and $f(0)$ with e_1. Then there exists some appropriate neighborhood U of $0 \in \mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$ such that the function $h : U \to \mathbb{R}^m$, defined by

$$h(t, \eta) := \varphi(t, (0, \eta)),$$

is well-defined and of class C^1.



For all $(s, \eta), (t, \eta) \in U$ such that $(s + t, \eta) \in U$ we then have

$$\begin{aligned} h \circ \psi(t, (s, \eta)) &= h((s, \eta) + te_1) = h(s + t, \eta) \\ &= \varphi(s + t, (0, \eta)) = \varphi(t, \varphi(s, (0, \eta))) = \varphi(t, h(s, \eta)), \end{aligned}$$

that is, near $0 \in \mathbb{R}$ we have

$$h \circ \psi = \varphi \circ (id \times h). \tag{1}$$

Since $h|\{0\} \times \mathbb{R}^{m-1} = (0, id_{\mathbb{R}^{m-1}})$ and $D_1 h(0) = D_1 \varphi(0, 0) = f(0) = e_1$, it follows that

$$Dh(0) = \begin{bmatrix} 1 & 0 \\ 0 & id_{\mathbb{R}^{m-1}} \end{bmatrix} = id_{\mathbb{R}^m}.$$

Hence, by the inverse function theorem, h is a local C^1-diffeomorphism near 0 and therefore – because of (1) – a local C^1-flow equivalence. □

(19.2) Remarks. (a) It follows from the differentiability theorem (9.5) and the proof above that $\varphi|x_0$ and $\psi|0$ are C^k-flow equivalent whenever $f \in C^k(M, E), 1 \leq k \leq \infty$, and $\mathbb{K} = \mathbb{R}$.

(b) The proof above shows that $\varphi|x_0$ and $\psi|0$ are *isochron flow equivalent*, i.e., the time variable is unchanged. □

The Linearization Theorem

We now turn to the main task of this section, the study of a flow in a neighborhood of a *hyperbolic critical point*. Here the critical point x_0 of the flow induced by f is called *hyperbolic* if $\sigma_n(Df(x_0)) = \emptyset$, that is, if the linear flow $e^{tDf(x_0)}$ is hyperbolic.

There exists a close connection between flows and homeomorphisms. In fact, according to theorem (10.14), φ^t is a homeomorphism from Ω_t onto Ω_{-t} for all $t \in \mathbb{R}$. In particular, for every linear flow e^{tA} it follows that $e^{tA} \in \mathcal{GL}(E)$ for

every $t \in \mathbb{R}$, that is, e^{tA} is an automorphism on E. It is therefore reasonable (and useful for applications) to consider the case of homeomorphisms first.

To motivate the next definition, we first prove the following special case of the *spectral mapping theorem* (see, for example, Yosida [1]).

(19.3) Lemma. *If $A \in \mathcal{L}(E)$, then*

$$\sigma(e^A) = e^{\sigma(A)} := \{e^\lambda \mid \lambda \in \sigma(A)\}.$$

Proof. Since $\sigma(A) := \sigma(A_{\mathbb{C}})$, we may assume – by passing to the complexification – that E is a complex Banach space. If $\lambda_1, \ldots, \lambda_k$ denote the distinct eigenvalues of A, we know from section 12 that E has the direct sum decomposition $E = E_1 \oplus \cdots \oplus E_k$ which reduces A as well as e^A. Hence it suffices to prove that $\sigma(e^{A_j}) = e^{\sigma(A_j)}$, where $A_j := A|E_j$ for $j = 1, \ldots, k$. We may therefore assume (cf. section 12) that the following holds: $\sigma(A) = \{\lambda\}$ and $A = \lambda + N$ for some nilpotent operator $N \in \mathcal{L}(E)$. It follows that there exists some $x \in E \setminus \{0\}$ such that $Ax = \lambda x$, i.e., $Nx = 0$. From this we deduce that

$$e^A x = e^\lambda e^N x = e^\lambda x,$$

that is, $\sigma(e^A) \supseteq e^{\sigma(A)}$. Conversely, if we have $e^A y = \mu y$ for some $\mu \in \mathbb{C}$ and $y \in E \setminus \{0\}$, then

$$\mu y = e^\lambda e^N y = e^\lambda \sum_{k=0}^{m} \frac{1}{k!} N^k y. \tag{2}$$

Then there exists a smallest index l such that $0 \leq l \leq m$ and $N^{l+1} y = 0$. Applying N^l to (2), we obtain

$$\mu N^l y = e^\lambda N^l y.$$

Since $N^l y \neq 0$, we have $\mu = e^\lambda$, which implies that $\sigma(e^A) \subseteq e^{\sigma(A)}$. □

Assume now that $A \in \mathcal{L}(E)$ and let A induce the hyperbolic linear flow e^{tA}, that is, $\sigma_n(A) = \sigma(A) \cap i\mathbb{R} = \emptyset$. It then follows from lemma (19.3) that $\sigma(e^A) \cap \mathbb{S}_{\mathbb{C}} = \emptyset$, where $\mathbb{S}_{\mathbb{C}} := \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the unit circle in the complex plane. In other words, $e^A \in \mathcal{GL}(E)$ has no eigenvalues of norm 1. In general, if T has no eigenvalues of norm 1, i.e., if $\sigma(T) \cap \mathbb{S}_{\mathbb{C}} = \emptyset$, *the automorphism* $T \in \mathcal{GL}(E)$ is called *hyperbolic.*

If $T \in \mathcal{GL}(E)$ is hyperbolic, then

$$\sigma(T) = \sigma_0(T) \cup \sigma_\infty(T),$$

where

$$\sigma_0(T) := \{\lambda \in \sigma(T) \mid |\lambda| < 1\}$$

and

$$\sigma_\infty(T) := \{\lambda \in \sigma(T) \mid |\lambda| > 1\}.$$

If, again, we let $m(\lambda)$ denote the algebraic multiplicity of the eigenvalue $\lambda \in \sigma(T)$, it follows – for the case $\mathbb{K} = \mathbb{C}$ – that

$$E_0 := \bigoplus_{\lambda \in \sigma_0(T)} \ker[(\lambda - T)^{m(\lambda)}]$$

and

$$E_\infty := \bigoplus_{\lambda \in \sigma_\infty(T)} \ker[(\lambda - T)^{m(\lambda)}]$$

are invariant subspaces of E which reduce T, that is,

$$E = E_0 \oplus E_\infty \quad \text{and} \quad T = T_0 \oplus T_\infty, \tag{3}$$

and it also follows that

$$\sigma(T_0) = \sigma_0(T) \quad \text{and} \quad \sigma(T_\infty) = \sigma_\infty(T). \tag{4}$$

If $\mathbb{K} = \mathbb{R}$, we apply this decomposition to the complexification and subsequently restrict to the real subspaces, i.e.,

$$E_0 := (E_\mathbb{C})_0 \cap E \quad \text{and} \quad E_\infty := (E_\mathbb{C})_\infty \cap E,$$

as well as

$$T_0 := (T_\mathbb{C})_0 | E_0 \quad \text{and} \quad T_\infty := (T_\mathbb{C})_\infty | E_\infty .$$

Then one easily verifies that relations (3) and (4) also hold for the real case (cf. the proof of theorem (13.4)).

The following lemma represents an analogue of lemma (13.1). To simplify the formulation, we make use of the descriptive notation

$$|\sigma(A)| < \alpha \iff |\lambda| < \alpha, \quad \forall \lambda \in \sigma(A).$$

Other inequalities are to be interpreted analogously.

(19.4) Lemma. *Let $T \in \mathcal{GL}(E)$ be hyperbolic and assume that for some $\alpha \in \mathbb{R}_+$ we have*

$$|\sigma(T_0)| < \alpha \quad \text{and} \quad |\sigma((T_\infty)^{-1})| < \alpha.$$

Then there exists a Hilbert norm $\|\cdot\|$ on E such that

$$\max\{\, \|T_0\|,\ \|(T_\infty)^{-1}\| \,\} \le \alpha,$$

and so that E_0 and E_∞ are orthogonal.

Proof. Since $\|A_\mathbb{C}\| = \|A\|$ (cf. the proof of lemma (13.1)), we may, without loss of generality, assume that $\mathbb{K} = \mathbb{C}$. From the proof of lemma (13.1), we know that $T_0 = D + N$ for some nilpotent operator $N \in \mathcal{L}(E_0)$ and some diagonal operator $D = \operatorname{diag}[\mu_1, \ldots, \mu_k]$ (with respect to some appropriate basis), where $\mu_1, \ldots, \mu_k$ are the eigenvalues of T_0, counted according to their multiplicity. In addition, we know that we can choose the basis such that the corresponding Euclidean norm $\|\cdot\|_0$ on E_0 satisfies

$$\|N\|_0 \le \alpha - \max\{|\mu_j| \mid j = 1, \ldots, k\}.$$

From this it immediately follows that

$$\|T_0\|_0 \le \|D\|_0 + \|N\|_0 \le \max\{|\mu_j| \mid 1 \le j \le k\} + \|N\|_0 \le \alpha.$$

Similarly, we find a Hilbert norm $\|\cdot\|_\infty$ for E_∞ such that for the corresponding operator norm we have $\|T_\infty^{-1}\|_\infty \le \alpha$. Then

$$\|x\|^2 := \|x_0\|_0^2 + \|x_\infty\|_\infty^2, \quad \forall x = x_0 + x_\infty \in E_0 \oplus E_\infty = E,$$

defines the desired Hilbert norm on E. □

(19.5) Remark. If $T \in \mathcal{GL}(E)$ is hyperbolic, then

$$|\sigma_0(T)| < 1 < |\sigma_\infty(T)|. \tag{5}$$

Since for each $B \in \mathcal{GL}(E)$ we trivially have

$$\sigma(B^{-1}) = [\sigma(B)]^{-1} := \left\{ \frac{1}{\lambda} \mid \lambda \in \sigma(B) \right\},$$

it follows from (4) and (5) that

$$|\sigma(T_\infty^{-1})| < 1.$$

Then lemma (19.4) implies the existence of some $\alpha < 1$ and a norm $\|\cdot\|$ on E such that

$$\|T_0\| \le \alpha < 1 \quad \text{and} \quad \|T_\infty^{-1}\| \le \alpha < 1.$$

From this it follows that for every $x \in E_0$ we have

$$T^k x = (T_0)^k x \to 0 \quad \text{as} \quad k \to \infty$$

and for every $y \in E_\infty$ we have

$$T^{-k} y = (T_\infty)^{-k} y \to 0 \quad \text{as} \quad k \to \infty.$$

Analogous to the situation for linear flows, E_0 is therefore called the *stable* and E_∞ the *unstable* subspace corresponding to T (or, more precisely: corresponding to the *discrete flow* induced by T). □

For a topological space X we set

$$BC(X, E) := B(X, E) \cap C(X, E),$$

where $B(X, E)$ denotes the Banach space of all bounded maps $u : X \to E$ with the *sup-norm*

$$\|u\|_\infty := \sup_{x \in X} |u(x)|_E.$$

The theorem on the continuity of the limit function of a uniformly convergent sequence of continuous functions implies that $BC(X,E)$ is a closed subspace of $B(X,E)$, and therefore $BC(X,E)$ is itself a Banach space with respect to the sup-norm, the *space of bounded continuous functions* (on X with values in E). If X is compact, then, of course, $BC(X,E) = C(X,E)$ and

$$\|u\|_\infty = \max_{x\in X} |u(x)| =: \|u\|_C.$$

Moreover, it is also clear that if we replace the norm on E by an equivalent norm, we obtain an equivalent norm on $BC(X,E)$.

Let now $E = E_1 \oplus E_2$ be a direct sum decomposition of E and assume that the corresponding projections $P_i : E \to E_i$, $i = 1,2$, satisfy $|P_i| \le 1$, $i = 1,2$. Then every element $u \in B := BC(X,E)$ can be written uniquely in the form

$$u = P_1 u + P_2 u$$

and

$$P_i u \in BC(X, E_i) =: B_i, \quad i = 1,2.$$

Moreover, we trivially have

$$\|P_i u\|_{B_i} = \sup_{x\in X} |P_i u(x)| \le |P_i|\,\|u\|_\infty \le \|u\|_\infty$$

for $i = 1,2$. Consequently

$$(P_i u)(x) := P_i u(x), \quad \forall x \in X,$$

defines continuous projections $P_i : B \to B_i$, $i = 1,2$, satisfying $P_1 + P_2 = \mathrm{id}_B$, i.e., we have:

$$B = B_1 \oplus B_2$$

and $P_i : B \to B_i$ are the corresponding projections. Finally, we set

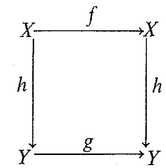
$$\|u\|_B := \max\{\|P_1 u\|_{B_1}, \|P_2 u\|_{B_2}\}.$$

It follows from

$$\begin{aligned}(1/2)\|u\|_\infty &= (1/2)\|P_1 u + P_2 u\|_\infty \le (1/2)[\|P_1 u\|_\infty + \|P_2 u\|_\infty] \\ &= (1/2)[\|P_1 u\|_{B_1} + \|P_2 u\|_{B_2}] \le \|u\|_B \le \|u\|_\infty\end{aligned} \tag{6}$$

that $\|\cdot\|_B$ is an equivalent norm on B.

What is still needed is an analogue of the concept of "flow equivalence" for the case of homeomorphisms. So let X and Y be topological spaces and let $f : X \to X$ and $g : Y \to Y$ be homeomorphisms. Then a homeomorphism $h : X \to Y$ is called a *topological conjugacy from f to g* if $h \circ f = g \circ h$, that is to say, if the diagram



commutes. If X and Y are differentiable manifolds (e.g., open subsets of Banach spaces) and if f, g and h are C^k-diffeomorphisms, $1 \le k \le \infty$, then h is called a C^k*-conjugacy*. Lastly, f and g are said to be *topologically* (resp. C^k-)*conjugate* if there exists a topological (resp. C^k-)conjugacy from f to g. This trivially defines an equivalence relation in the class of all homeomorphisms (resp. C^k-diffeomorphisms).

After these preliminaries we now can prove the *global Hartman linearization theorem.*

(19.6) Proposition. *If $T \in \mathcal{GL}(E)$ is hyperbolic and if $g \in BC(E,E)$ is a uniformly Lipschitz continuous function with a sufficiently small Lipschitz constant, then the maps T and $T+g$ are topologically conjugate.*

Proof. By lemma (19.4) and remark (19.5), there exists a Hilbert norm $\|\cdot\|$ on E such that

$$\max\{\|T_0\|, \|T_\infty^{-1}\|\} \le \alpha < 1. \tag{7}$$

Since the passage to an equivalent norm on E implies that the Lipschitz constant of g will be multiplied by a positive factor, we may assume that the norm $\|\cdot\|$ on E satisfies

$$\|g(x) - g(y)\| \le \lambda \|x - y\|, \quad \forall x, y \in E, \tag{8}$$

where $2\lambda < \min\{1-\alpha, \|T^{-1}\|^{-1}\}$.

(i) First we show that $T + g \in C(E,E)$ is a homeomorphism. Since for every $z \in E$ the equation $Tx + g(x) = z$ is equivalent to the fixed point equation

$$x = T^{-1}(z - g(x)) =: f_z(x),$$

it follows that $T + g$ is bijective whenever $f_z : E \to E$ has a unique fixed point $x(z)$. That f_z has a unique fixed point follows, however, from

$$\begin{aligned}\|f_z(x) - f_z(y)\| &\le \|T^{-1}\|\,\|g(y) - g(x)\| \le \lambda\|T^{-1}\|\,\|x - y\| \\ &\le (1/2)\|x - y\|\end{aligned} \tag{9}$$

for all $x, y \in E$ and the Banach fixed point theorem. From (9) we obtain

$$\begin{aligned}\|x(z)-x(\tilde z)\| &= \|f_z(x(z))-f_{\tilde z}(x(\tilde z))\|\\ &\le \|f_z(x(z))-f_z(x(\tilde z))\|+\|f_z(x(\tilde z))-f_{\tilde z}(x(\tilde z))\|\\ &\le (1/2)\|x(z)-x(\tilde z)\|+\|T^{-1}\|\,\|z-\tilde z\|.\end{aligned}$$

for all $z, \tilde z \in E$ (cf. problem 2 in section 7), hence $\|x(z)-x(\tilde z)\| \le 2\|T^{-1}\|\,\|z-\tilde z\|$. Consequently $x(\cdot) = (T+g)^{-1} : E \to E$ is (uniformly) Lipschitz continuous.

(ii) Let now $h \in BC(E,E) =: B$ be a second function which is uniformly Lipschitz continuous with Lipschitz constant λ. Assume further that corresponding to every pair (g,h) of such functions there exists a unique $H := H(g,h) \in C(E,E)$ such that

$$H - id \in B \tag{10}$$

and

$$(T+g)\circ H = H\circ(T+h). \tag{11}$$

Then for $a := H(g,0)$ we have

$$(T+g)\circ a = a\circ T, \tag{12}$$

and for $b := H(0,g)$ we have

$$T\circ b = b\circ(T+g). \tag{13}$$

It follows from (12) and (13) that

$$(T+g)\circ a\circ b = a\circ T\circ b = a\circ b\circ(T+g). \tag{14}$$

Since $a = id + u$ and $b = id + v$ for some $u, v \in B$, it follows that $a\circ b = id + w$, where $w = v + u\circ b \in B$. Based on the uniqueness of H, we thus obtain from (14) that $a\circ b = H(g,g) = id$. Similarly it follows that $b\circ a = id$. Therefore a is a homeomorphism from E onto E, hence – by (12) – a topological conjugacy from $T+g$ to T.

(iii) With $H = id+u$ it remains to be shown that there exists a unique $u \in B$ such that

$$(T+g)\circ(id+u) = (id+u)\circ(T+h). \tag{15}$$

Since, according to (i), $T+h$ is a homeomorphism, (15) is equivalent to

$$\begin{aligned}id+u &= (T+g)\circ(id+u)\circ(T+h)^{-1}\\ &= g\circ(id+u)\circ(T+h)^{-1}+T(T+h)^{-1}+Tu\circ(T+h)^{-1}.\end{aligned}$$

Because $id = (T+h)\circ(T+h)^{-1}$, the last equation is equivalent to

$$u = Tu\circ(T+h)^{-1} + G(u) =: \tilde F(u), \tag{16}$$

where

$$G(u) := g\circ(id+u)\circ(T+h)^{-1} - h\circ(T+h)^{-1}. \tag{17}$$

Clearly $\tilde F$ maps the Banach space B into itself. It remains to show that $\tilde F$ has a unique fixed point in B.

Because $E = E_0 \oplus E_\infty$ and since E_0 and E_∞ are orthogonal, it follows that for the corresponding projections we have $\|P_0\|, \|P_\infty\| \le 1$ (cf. the proof of theorem (15.5)). The fixed point equation (16) is therefore equivalent to the system of equations

$$P_0u = T_0P_0u\circ(T+h)^{-1} + P_0G(u) =: F_0(u) \tag{18}$$

$$P_\infty u = T_\infty P_\infty u\circ(T+h)^{-1} + P_\infty G(u). \tag{19}$$

Since equation (19) is transformed into the equivalent equation

$$P_\infty u = T_\infty^{-1}P_\infty u\circ(T+h) - T_\infty^{-1}P_\infty G(u)\circ(T+h) =: F_\infty(u) \tag{20}$$

by multiplying from the left by T_∞^{-1} and from the right by $T+h$, (16) is equivalent to the system (18) and (20).

By the considerations preceding this proposition, the decomposition $E = E_0 \oplus E_\infty$ induces the decomposition $B = B_0 \oplus B_\infty$ and for the norm

$$\|u\|_B := \max\{\|P_0u\|_\infty, \|P_\infty u\|_\infty\}$$

we have

$$(1/2)\|u\|_\infty \le \|u\|_B \le \|u\|_\infty, \quad \forall u \in B.$$

Hence

$$F := F_0 + F_\infty$$

defines a map from B into itself such that the fixed point problem $u = F(u)$ is equivalent to the fixed point problem $u = \tilde F(u)$.

For $u, v \in B$ and $x \in E$ we set $y := (T+h)^{-1}(x)$ and $z := (T+h)(x)$, and so by making use of (7) we obtain the estimates

$$\begin{aligned}\|F_0(u)(x)-F_0(v)(x)\| &\le \alpha\|P_0u(y)-P_0v(y)\| + \|g(y+u(y))-g(y+v(y))\|\\ &\le \alpha\|P_0(u-v)\|_\infty + \lambda\|u-v\|_\infty\end{aligned}$$

and

$$\begin{aligned}\|F_\infty(u)(x)-F_\infty(v)(x)\| &\le \alpha\|P_\infty u(z)-P_\infty v(z)\|\\ &\quad + \|g(x+u(x))-g(x+v(x))\|\\ &\le \alpha\|P_\infty(u-v)\|_\infty + \lambda\|u-v\|_\infty.\end{aligned}$$

Therefore

$$\begin{aligned}\|F_0(u)-F_0(v)\|_\infty &\le \alpha\|P_0(u-v)\|_\infty + 2\lambda\|u-v\|_B\\ &\le (\alpha+2\lambda)\|u-v\|_B\end{aligned}$$

and

$$\|F_\infty(u)-F_\infty(v)\|_\infty \le (\alpha+2\lambda)\|u-v\|_B,$$

and so, since $F_0(B) \subseteq B_0$ and $F_\infty(B) \subseteq B_\infty$, we deduce that

$$\|F(u)-F(v)\|_B \le (\alpha+2\lambda)\|u-v\|_B, \qquad \forall u,v \in B.$$

The existence of a unique fixed point of F now follows from the Banach fixed point theorem and the fact that $\alpha + 2\lambda < 1$. □

(19.7) Remarks. (a) The proof above shows that there exists a unique topological conjugacy h from T to $T+g$ which satisfies $h-\mathrm{id} \in BC(E,E)$ (of course, if the Lipschitz constant of g is sufficiently small).

(b) If $g \in C^k(E,E)$, $1 \le k \le \infty$, one would naturally expect the topological conjugacy from $T+g$ to T to have the corresponding differentiability properties, i.e., that $T+g$ and T are C^k-conjugate. This, however, is in general not true. For further investigations along these lines we refer to Hartman [1]. □

We need the following simple lemma for the local version of the above linearization theorem.

(19.8) Lemma. *Let F be an arbitrary NVS and let $r_\alpha : F \to \overline{\mathbb{B}}(0,\alpha)$ denote the radial retraction, that is,*

$$r_\alpha(x) := \begin{cases} x, & \text{if } |x| \le \alpha \\ \alpha x/|x|, & \text{if } |x| \ge \alpha. \end{cases}$$

Then r_α is uniformly Lipschitz continuous with Lipschitz constant 2.

Proof. For $|x| > \alpha \ge |y|$ we have

$$\begin{aligned} |r_\alpha(x) - r_\alpha(y)| &= |\alpha|x|^{-1}x - y| \le \alpha|x|^{-1}|x-y| + \left|\alpha|x|^{-1}y - y\right| \\ &\le |x-y| + |x|^{-1}|y|(|x|-\alpha) \\ &\le |x-y| + |x| - |y| \le 2|x-y|. \end{aligned}$$

If $|x| > \alpha$ and $|y| > \alpha$, we obtain

$$\begin{aligned} |r_\alpha(x) - r_\alpha(y)| &= \left|\alpha|x|^{-1}x - \alpha|y|^{-1}y\right| \\ &\le \alpha|x|^{-1}|x-y| + \alpha|y|\left||x|^{-1} - |y|^{-1}\right| \\ &\le |x-y| + \big||x|-|y|\big| \le 2|x-y|. \end{aligned}$$

This proves the lemma. □

After these preliminaries we can now easily prove the main result of this section. To do this, we recall that E is a finite dimensional Banach space, that M is open in E, and that $f \in C^1(M,E)$.

(19.9) Theorem *(Grobman, Hartman). Let x_0 be a hyperbolic critical point of φ. Then $\varphi|x_0$ and $e^{tDf(x_0)}|0$ are isochronally flow equivalent.*

Proof. (i) Since a translation is evidently an isochron flow equivalence, we may assume, without loss of generality, that $x_0 = 0$. If $\lambda > 0$ is arbitrary, there exists some $\alpha > 0$ so that $|Df(x) - Df(0)| \le \lambda/2$ for all $x \in \overline{\mathbb{B}}(0,\alpha)$. It follows from the mean value theorem that the function $x \mapsto f(x) - Df(0)x$ is uniformly Lipschitz continuous on $\overline{\mathbb{B}}(0,\alpha)$ with Lipschitz constant $\lambda/2$. Employing the radial retraction $r_\alpha : E \to \overline{\mathbb{B}}(0,\alpha)$, we now define $g \in BC(E,E)$ by

$$g := [f - Df(0)] \circ r_\alpha.$$

It follows from lemma (19.8) (and the proof of lemma (8.1 iii) that g is globally Lipschitz continuous with Lipschitz constant λ. Setting $A := Df(0) \in \mathcal{L}(E)$, we have

$$(A+g)|\mathbb{B}(0,\alpha) = f|\mathbb{B}(0,\alpha). \tag{21}$$

If ψ denotes the flow on E, induced by $A+g$, then (21) shows that ψ coincides with φ on $\mathbb{B}(0,\alpha)$ (cf. theorem (10.3) and remark (10.4 b)). It therefore suffices to show that ψ and e^{tA} are isochronally flow equivalent.

(ii) Because g is globally Lipschitz continuous, g – and therefore $A+g$ – is linearly bounded. Hence by proposition (7.8), ψ is a global flow. Since $\dot{x} = Ax + g(x)$, we deduce from the variation of constants formula (cf. formula (5) in section 15) that

$$\psi^t(x) = e^{tA}x + \int_0^t e^{(t-\tau)A} g(\psi^\tau(x))d\tau, \qquad \forall t \in \mathbb{R}. \tag{22}$$

This implies that

$$|\psi^t(x) - \psi^t(y)| \le e^{t|A|}|x-y| + \int_0^t e^{(t-\tau)|A|}\lambda|\psi^\tau(x) - \psi^\tau(y)|d\tau$$

for all $t \ge 0$ and all $x,y \in E$. After multiplying this inequality by $e^{-t|A|}$, we may apply Gronwall's inequality (corollary (6.2)) and obtain

$$|\psi^t(x) - \psi^t(y)| \le |x-y|e^{(\lambda+|A|)t}, \qquad \forall x,y \in E, \quad t \ge 0. \tag{23}$$

Since $g \in BC(E,E)$, it follows from (22) that

$$|\psi^t(x) - e^{tA}x| \le \|g\|_\infty \left|\int_0^t e^{|t-\tau||A|}d\tau\right|$$

for all $t \in \mathbb{R}$ and and $x \in E$, hence

$$\psi^t - e^{tA} \in BC(E,E), \qquad \forall t \in \mathbb{R}. \tag{24}$$

Finally, it follows from (22) and (23) that

$$
\begin{aligned}
|(\psi^t - e^{tA})(x) - (\psi^t - e^{tA})(y)| &\le \int_0^t e^{(t-\tau)|A|}\lambda|\psi^\tau(x) - \psi^\tau(y)|d\tau \\
&\le \lambda|x-y|e^{t|A|}\int_0^t e^{\lambda\tau}\,d\tau \qquad (25)\\
&= |x-y|e^{t|A|}(e^{\lambda t} - 1)
\end{aligned}
$$

for all $t \ge 0$ and $x,\ y \in E$.

(iii) By assumption, 0 is a hyperbolic critical point and thus $\sigma(A) \cap i\mathbb{R} = \emptyset$. It therefore follows from lemma (19.3) that $T := e^A$ is a hyperbolic automorphism on E. Since λ can be chosen arbitrarily small, it follows from (25) that the Lipschitz constant of $\tilde{g} := \psi^1 - T$ can be made arbitrarily small. Because $\tilde{g} \in BC(E, E)$ (cf. (24)), we may assume, by proposition (19.6), that T and $\psi^1 = T + \tilde{g}$ are topologically conjugate. From remark (19.7 a) we also know that there exists a unique topological conjugacy h from T to ψ^1 satisfying $h - id \in BC(E, E)$.

From $h \circ T = \psi^1 \circ h$ we deduce that for each $t \in \mathbb{R}$ we have

$$
\begin{aligned}
\psi^1 \circ (\psi^t \circ h \circ T^{-t}) &= \psi^t \circ \psi^1 \circ h \circ T^{-t} = \psi^t \circ h \circ T \circ T^{-t}\\
&= (\psi^t \circ h \circ T^{-t}) \circ T,
\end{aligned}
$$

where $T^t := e^{tA}$. Therefore $\psi^t \circ h \circ T^{-t}$ is also a topological conjugacy from T to ψ^1. Because we have

$$\psi^t \circ h \circ T^{-t} - id = (\psi^t - T^t) \circ h \circ T^{-t} + T^t \circ (h - id) \circ T^{-t},$$

it follows from (24) that $\psi^t \circ h \circ T^{-t} - id \in BC(E, E)$. Therefore $\psi^t \circ h \circ T^{-t} = h$ and hence $\psi^t \circ h = h \circ e^{tA}$ for all $t \in \mathbb{R}$, that is to say, ψ and e^{tA} are isochronally flow equivalent. □

Stable Manifolds

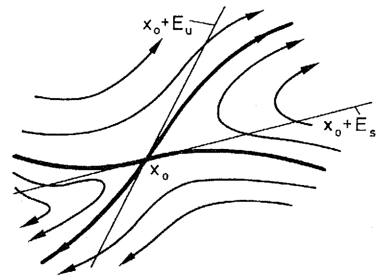
The preceding theorem states that in a neighborhood of a hyperbolic critical point x_0, the phase portrait of φ has the same topological structure as the phase portrait of the linearization in a neighborhood of 0. Analogous to the stable subspace E_s and unstable subspace E_u of a linear flow, one defines the *stable manifold* $W_s(x_0)$ *of* φ *at* x_0 and the *unstable manifold* $W_u(x_0)$ *of* φ *at* x_0 by

$$W_s(x_0) := \{x \in E \mid t^+(x) = \infty \ \text{ and } \ t \cdot x \to x_0 \ \text{ as } \ t \to \infty\}$$

and

$$W_u(x_0) := \{x \in E \mid t^-(x) = -\infty \ \text{ and } \ t \cdot x \to x_0 \ \text{ as } \ t \to -\infty\},$$

respectively. Clearly $x_0 \in W_s(x_0) \cap W_u(x_0)$. We will now show that in a neighborhood of x_0 the sets $W_s(x_0)$ and $W_u(x_0)$ are indeed differentiable submanifolds of E which intersect transversely at x_0, and that the tangent spaces of $W_s(x_0)$ and $W_u(x_0)$ are translates of E_s and E_u, respectively, that is to say, $T_{x_0}W_s(x_0) = x_0 + E_s$ and $T_{x_0}W_u(x_0) = x_0 + E_u$.



In order to prove these facts, we again consider first homeomorphisms from E onto itself. If $h : E \to E$ is a homeomorphism such that $h(0) = 0$, then the set

$$W_0 := \{x \in E \mid h^n(x) \to 0 \ \text{ as } \ n \to \infty\}$$

is called the *stable set* of h at 0, where h^n denotes the nth iterate of h. Similarly, we define the *unstable set* of h at 0 by

$$W_\infty := \{x \in E \mid h^{-n}(x) \to 0 \ \text{ as } \ n \to \infty\},$$

where we have set $h^{-n} := (h^{-1})^n$. Evidently W_0 and W_∞ change into each other if we replace h by h^{-1}. Hence it suffices to consider W_0.

Now let $T \in \mathcal{GL}(E)$ be hyperbolic and let

$$E = E_0 \oplus E_\infty, \qquad T = T_0 \oplus T_\infty$$

be the decomposition into the stable and unstable subspace, introduced a few pages back.

(19.10) Proposition. *Let $g : E \to E$ be uniformly Lipschitz continuous and assume that $g(0) = 0$. If the Lipschitz constant of g is sufficiently small, there exists a unique uniformly Lipschitz continuous function $h : E_0 \to E_\infty$ such that the graph of h is the stable set W_0 of $T + h$ at 0. If in some neighborhood of 0, g is of class C^k, $1 \le k \le \infty$, then so is the function h. In this case there exists a neighborhood V of 0 in E_0 such that*

$$W_0^V := \{(x, h(x)) \mid x \in V\}$$

is a C^k-manifold. If, in addition, $Dg(0) = 0$, then $T_0 W_0^V = E_0$.

Proof. Set

$$B_0 := \{u : \mathbb{N} \to E \mid u(k) \to 0 \text{ as } k \to \infty\}.$$

Then it is easily seen that B_0 is a closed subspace of the Banach space of all bounded sequences in $E, (B(\mathbb{N}, E), \|\cdot\|_\infty)$. Hence B_0 itself is a Banach space with respect to the sup-norm.

Let

$$\mathbb{W}_0 := \{u \in B_0 \mid u(k) = (T+g)^k(x), k \in \mathbb{N}, x \in E\}.$$

Then clearly

$$W_0 = \{u(0) \mid u \in \mathbb{W}_0\}.$$

Moreover, we have

$$u \in \mathbb{W}_0 \quad \Leftrightarrow \quad u(k+1) = (T+g)(u(k)), \quad \forall k \in \mathbb{N}.$$

Using the canonical projections $P_0 : E \to E_0$ and $P_\infty : E \to E_\infty$, the last equation is equivalent to the system

$$\begin{aligned} P_0 u(k+1) &= T_0 P_0 u(k) + P_0 g(u(k)) \\ P_\infty u(k+1) &= T_\infty P_\infty u(k) + P_\infty g(u(k)), \end{aligned}$$

and hence to the system

$$\begin{aligned} P_0 u(k+1) &= T_0 P_0 u(k) + P_0 g(u(k)) \\ P_\infty u(k) &= T_\infty^{-1} P_\infty u(k+1) - T_\infty^{-1} P_\infty g(u(k)) \end{aligned}$$

for all $k \in \mathbb{N}$. If for $x \in E$ and $u \in B_0$ we now set

$$F_x(u)(k) := \begin{cases} T_0 P_0 u(k-1) + T_\infty^{-1} P_\infty u(k+1) + P_0 g(u(k-1)) \\ \qquad\qquad -T_\infty^{-1} P_\infty g(u(k)) \\ P_0 x + T_\infty^{-1}(P_\infty u(1) - P_\infty g(u(0))), \quad \text{if } k = 0, \end{cases} \tag{26}$$

we see that $x \in E_0$ belongs to W_0 if and only if u is a fixed point of F_x in B_0.

As in the proof of proposition (19.6), we may assume that

$$|T_0|, \ |T_\infty^{-1}| \le \alpha < 1, \quad |P_0|, \ |P_\infty| \le 1$$

and that

$$|g(x) - g(y)| \le \lambda |x - y|, \quad \forall x, y \in E, \tag{27}$$

where $2\lambda < 1 - \alpha$. In addition, we may employ the equivalent norm $\|u\|_B := \max\{\|P_0 u\|_\infty, \|P_\infty u\|_\infty\}$ in B_0, for which we have

$$(1/2)\|u\|_\infty \le \|u\|_B \le \|u\|_\infty, \quad \forall u \in B_0.$$

For $x \in E$ and $u, v \in B_0$ we then easily obtain the estimates

$$\|F_x(u) - F_x(v)\|_B \le (\alpha + 2\lambda)\|u - v\|_B \tag{28}$$

and

$$|F_x(u)(k)| \le (\alpha + \lambda)|u(k-1)| + \alpha\lambda|u(k)| + \alpha|u(k+1)| \tag{29}$$

for all $k \ge 1$. It follows from (29) that F_x maps the Banach space B_0 into itself, and (28) shows that $F_x : B_0 \to B_0$ is a contraction with contraction constant $\alpha + 2\lambda < 1$, which is independent of $x \in E$. Therefore the Banach fixed point theorem implies the existence of a unique fixed point u_x of F_x in B_0. From the estimate

$$\begin{aligned} \|u_x - u_y\|_B &= \|F_x(u_x) - F_y(u_y)\|_B \\ &\le \|F_x(u_x) - F_x(u_y)\|_B + \|F_x(u_y) - F_y(u_y)\|_B \\ &\le (\alpha + 2\lambda)\|u_x - u_y\|_B + |P_0 x - P_0 y| \end{aligned}$$

we deduce that

$$\|u_x - u_y\|_B \le |P_0 x - P_0 y|/(1 - \alpha - 2\lambda), \quad \forall x, y \in E. \tag{30}$$

We now set

$$h(x) := P_\infty u_x(0), \quad \forall x \in E_0.$$

Because of (20), h is locally uniformly Lipschitz continuous and maps the space E_0 into E_∞, and from (26) we read off the relation

$$W_0 = \{(x, h(x)) \mid x \in E_0\} = \text{graph}(h).$$

We let $S_1, S_{-1} : B_0 \to B_0$ denote the "shift operators"

$$(S_{-1}u)(k) := u(k+1), \quad \forall k \in \mathbb{N},$$

and

$$(S_1 u)(k) := \begin{cases} u(k-1), & \text{if } k \in \mathbb{N}^* \\ 0, & \text{if } k = 0. \end{cases}$$

Clearly S_{-1} and S_1 are continuous and

$$\|S_{-1}\|_B, \ \|S_1\|_B \quad \le 1. \tag{31}$$

Finally, we define $G : B_0 \to B_0$ by

$$G(u)(k) := g(u(k)), \quad \forall k \in \mathbb{N}.$$

If for some $\beta > 0$ and some $k \in \mathbb{N}^* \cup \{\infty\}$ we have

$$g \in C^k(\mathbb{B}_E(0, \beta), E),$$

then one easily verifies that

$$G \in C^k(\mathbb{B}_{B_0}(0, \beta), B_0) \tag{32}$$

and (since $Dg(0) = 0$)

$$DG(0) = 0.$$

Using the "unit vector" $e_0 := (1, 0, \ldots) \in B_0, F_x$ can be written in the form

$$F_x = (P_0 x)e_0 + T_0 P_0(S_1 + G \circ S_1) + T_\infty^{-1} P_\infty(S_{-1} - G).$$

Setting

$$H(x,u) := u - F_x(u)$$

it then follows from (32) that

$$H \in C^k(E \times \mathbb{B}_{B_0}(0,\beta), B_0)$$

and

$$D_2H(0,0) = id_{B_0} - T_0P_0S_1 - T_\infty^{-1}P_\infty S_{-1} =: id_{B_0} - K. \tag{33}$$

With the aid of (31) we obtain the estimate

$$\|K\|_{\mathcal{L}(B_0)} \leq \alpha < 1,$$

from which it easily follows (by means of the Neumann series (e.g., Yosida [1])) that $D_2H(0,0) \in \mathcal{GL}(B_0)$ (cf. remark (25.6 a)). Because $H(0,0) = 0$ and since for each $x \in E_0$, the element $u_x \in B_0$ is the unique solution of the equation

$$H(x,u) = 0,$$

it follows from the implicit function theorem (cf. Dieudonné [1]) that in some neighborhood of $x = 0$, the function

$$E_0 \to B_0, \quad x \mapsto u_x$$

is of class C^k. Therefore the function $h : E_0 \to E_\infty$ is – because the "evaluation map" $B_0 \to E$, $u \mapsto u(0)$, is clearly linear and continuous – a C^k-function in some neighborhood V of 0. Therefore W_0^V is, as a graph of a C^k-function, a C^k-manifold of dimension $\dim_{\mathbb{R}}(E_0)$. Since

$$V \ni x \mapsto (x, h(x)) \in E$$

defines a parametrization on W_0^V, we know that

$$T_0W_0^V = im(id_{E_0}, Dh(0))$$

(where we may assume, without loss of generality, that $\mathbb{K} = \mathbb{R}$). By differentiating the identity $H(x,u_x) = 0$ at the point $x = 0$, it follows that

$$D_1H(0,0) + D_2H(0,0)Du_0 = 0, \tag{34}$$

where $Du_0 \in \mathcal{L}(E_0, B_0)$ and

$$D_1H(0,0)\xi = -(P_0\xi)e_0, \quad \forall \xi \in E.$$

We thus obtain $P_\infty D_1H(0,0)\xi = 0$, and from (33) and (34) we deduce (by applying P_∞ to (34)) that

$$P_\infty Du_0 - T_\infty^{-1}P_\infty S_{-1}Du_0 = 0,$$

that is,

$$P_\infty[(Du_0)x](k) = T_\infty^{-1}P_\infty[(Du_0)x](k+1)$$

for all $k \in \mathbb{N}$ and all $x \in E_0$. From this it follows that

$$|P_\infty[(Du_0)x](k)| \leq \alpha\|P_\infty Du_0\|_{\mathcal{L}(E_0,B_0)}\|x\|_E, \quad \forall k \in \mathbb{N},$$

and therefore

$$\|P_\infty Du_0\|_{\mathcal{L}(E_0,B_0)} \leq \alpha\|P_\infty Du_0\|_{\mathcal{L}(E_0,B_0)},$$

i.e., $P_\infty Du_0 = 0$, since $\alpha < 1$. Because

$$Dh(0)\xi = P_\infty[(Du_0)\xi](0)$$

for all $\xi \in E_0$, we obtain $Dh(0) = 0$ and hence the assertion. □

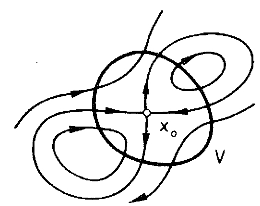
Let x_0 be a critical point of the flow φ. If V is a neighborhood of x_0, we define the *local stable* and *unstable manifolds of* φ *at* x_0 *with respect to* V by, respectively,

$$W_s^V(x_0) := \{x \in W_s(x_0) \mid t \cdot x \in V \text{ for all } t \geq 0\}$$

and

$$W_u^V(x_0) := \{x \in W_u(x_0) \mid t \cdot x \in V \text{ for all } t \leq 0\}.$$

As the adjoining figure shows, it need not be true, in general, that $W_s^V(x_0) = W_s(x_0) \cap V$ or $W_u^V(x_0) = W_u(x_0) \cap V$.



After these preliminaries we can prove the announced *theorem about the local stable and unstable manifolds*, which essentially goes back to Hadamard and Perron.

(19.11) Theorem. *Let M be open in the finite dimensional real Banach space E and let $f \in C^k(M,E)$ for some $k \in \mathbb{N}^* \cup \{\infty\}$. Moreover, let x_0 be a hyperbolic critical point of the flow φ induced by f. Then there exists a neighborhood V of x_0 such that $W_s^V(x_0)$ and $W_u^V(x_0)$ are C^k-manifolds. In addition, we have*

$$T_{x_0}W_s^V(x_0) = x_0 + E_s \quad \textit{and} \quad T_{x_0}W_u^V(x_0) = x_0 + E_u,$$

where E_s and E_u denote the stable and unstable subspaces of the linear flow $e^{tDf(x_0)}$.

Proof. Without loss of generality we may assume that $x_0 = 0$. As in the proof of theorem (19.9), we set

$$g := [f - Df(0)] \circ r_\alpha,$$

where $r_\alpha : E \to \overline{\mathbb{B}}(0, \alpha)$ denotes the radical retraction. Then g is uniformly Lipschitz continuous, $g(0) = 0$, $Dg(0) = 0$ and $g \in C^k(\mathbb{B}(0, \alpha), E)$. Moreover, by a judicious choice of $\alpha > 0$, the Lipschitz constant of g can be made arbitrarily small. With $A := Df(0) \in \mathcal{L}(E)$ we have

$$(A + g)|\mathbb{B}(0, \alpha) = f|\mathbb{B}(0, \alpha).$$

Hence the global flow ψ induced by $A + g$ agrees with the flow φ induced by f on $\mathbb{B}(0, \alpha)$.

We now set $T := e^A$. Then T is a hyperbolic automorphism and, as in the proof of theorem (19.9), it follows that $\tilde{g} := \psi^1 - T$ is globally Lipschitz continuous, where, by an appropriate choice of α, the Lipschitz constant of $\tilde{g}$ can be made arbitrarily small (cf. (25)). It also follows from theorem (10.3) that in some neighborhood of 0, $\tilde{g}$ is a C^k-function. Evidently $\tilde{g}(0) = 0$, and since theorem (9.2) implies that $D_2\psi(\cdot, 0)$ is the solution of the linearized IVP

$$\dot{z} = [A + Dg(\psi(t, 0))]z, \quad z(0) = id_E,$$

it follows from $\psi(t, 0) = 0$ and $Dg(0) = 0$ that $D_2\psi(t, 0) = e^{tA}$, therefore $D\tilde{g}(0) = 0$. It follows that T and $\tilde{g}$ satisfy the assumptions of proposition (19.10). Therefore the stable set W_0 of $\psi^1 = T + \tilde{g}$ can be represented as the graph of a globally Lipschitz continuous function $h : E_0 \to E_\infty$. Moreover, there exists a neighborhood V_0 of 0 in E_0 such that $h \in C^k(V_0, E_\infty)$ and so that

$$W_0^{V_0} := \{(x, h(x)) \in E \mid x \in V_0\}$$

is a C^k-manifold and $T_0 W_0^{V_0} = E_0 = E_s$.

We now assert that $W_0 = \tilde{W}_s(0)$, where $\tilde{W}_s(0)$ is the stable manifold of Ψ at the point 0. Since $\lim_{t\to\infty} \psi^t(x) = 0$ implies $\psi^k(x) \to 0$ as $k \to \infty$, it follows that $\tilde{W}_s(0) \subseteq W_0$. To prove the reverse inclusion, we first note that for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|\psi^t(x)| \le \epsilon \quad \text{for all} \quad |t| \le 1 \text{ and } \quad |x| \le \delta. \tag{35}$$

For otherwise there would exist some $\epsilon > 0$ and a sequence (t_k, x_k) in $[-1, 1] \times E$ such that $x_k \to 0$ and $|\psi^{t_k}(x_k)| \ge \epsilon$. By taking an appropriate subsequence, we may assume that $t_k \to \bar{t} \in [-1, 1]$, from which we deduce $|\psi^{\bar{t}}(0)| \ge \epsilon$, which contradicts $\psi(\cdot, 0) = 0$.

Let $\epsilon > 0$ now be arbitrary and assume that

$$\psi^k(x) \to 0 \quad \text{as} \quad k \to \infty.$$

Then there exists some $k(\epsilon) \in \mathbb{N}$ so that $|\psi^k(x)| \le \delta$ for all $k \ge k(\epsilon)$, where $\delta > 0$ is chosen as in (35). For $t \ge k(\epsilon)$ and $k := [t]$ it follows from (35) that

$$|\psi^t(x)| = |\psi^{t-k}(\psi^k(x))| \le \epsilon,$$

i.e., $\psi^t(x) \to 0$ as $t \to \infty$ and therefore $W_0 \subseteq \tilde{W}_s(0)$.

It follows from the proof of proposition (19.10) that $h(x) = P_\infty u_x(0)$ for all $x \in E_0$, where the function

$$E \to B_0, \quad y \mapsto u_y,$$

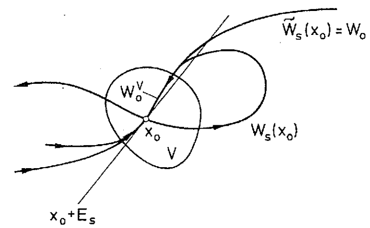
is continuous, vanishes at $y = 0$, and satisfies $u_y(k) = (T + \tilde{g})^k(y) = \psi^k(y)$ for all $k \in \mathbb{N}$. Hence for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\{(x, h(x)) \mid |x| \le \delta\} \subseteq \{y \in W_0 \mid |\psi^k(y)| \le \epsilon, \ \ \forall k \in \mathbb{N}\}.$$

As in the proof of the inclusion $W_0 \subseteq \tilde{W}_s(0)$, it follows from (35) that – for a given $\epsilon > 0$ – the number $\delta > 0$ can be chosen such that

$$\{(x, h(x)) \mid |x| \le \delta\} \subseteq \{y \in W_0 \mid |\psi^t(y)| \le \epsilon, \ \ \forall t \ge 0\}.$$

Because $W_0 = \tilde{W}_s(0)$, there exist neighborhoods of 0, V in E and $\hat{V} \subseteq V_0$ in E_0, such that $W_0^{\hat{V}} \subseteq \tilde{W}_s^V(0)$. Since the flows agree in a neighborhood of 0, we can choose V so small that $\tilde{W}_s^V(0) = W_s^V(0)$. In particular, we have $W_s^V(0) \subseteq W_0$. And since W_0 is the graph of a function defined on E_0, we have $W_s^V(0) \subseteq W_0^{V_0}$ for some sufficiently small neighborhood V of 0. Therefore $W_s^V(0)$ is a C^k-



manifold and $T_{x_0} W_s^V(0) = E_s$. The assertion about $W_u^V(0)$ now follows by "reversing the time." □

(19.12) Remarks. (a) The fact that E is finite dimensional was only used in the proofs of lemmas (19.3) and (19.4). By Dunford's operational calculus (e.g., Dunford-Schwartz [1], Yosida [1]) lemma (19.3) also holds for an arbitrary Banach space E. And in this case $T \in \mathcal{GL}(E)$ is also called hyperbolic if $\sigma(T) \cap \mathbb{S}^1_{\mathbb{C}} = \emptyset$ (that is to say, if the whole spectrum of T is a positive distance away from the unit circle in the complex plane). With the aid of the Dunford calculus, one then, again, shows that we have the decomposition $E = E_0 \oplus E_\infty$ and $T = T_0 \oplus T_\infty$, and that the following holds: $\sigma(T_0) = \sigma_0(T) := \sigma(T) \cap \mathbb{B}_{\mathbb{C}}(0, 1)$ and

$\sigma(T_\infty) = \sigma_\infty(T) := \sigma(T) \setminus \sigma(T_0)$. Using spectral theory, one can also show that there exists an equivalent norm $\|\cdot\|$ on E (but in general no Hilbert norm) satisfying

$$\max\{\|T_0\|, \|T_\infty^{-1}\|\} \le \alpha < 1.$$

Then proposition (19.6), theorem (19.9), proposition (19.10) and theorem (19.11) remain correct with these modifications also when $\dim(E) = \infty$ (we merely have to replace the inequality $\|u\|_B \le \|u\|_\infty$ by $\|u\|_B \le \beta\|u\|_\infty$, where $\beta := \max\{\|P_0\|, \|P_\infty\|\}$).

(b) The theorem of Grobman and Hartman can also be expressed in the following manner: If x_0 is a hyperbolic critical point of the vector field $f \in C^1(M, E)$, then there exists a homeomorphism $h : U \to V$, from some neighborhood U of x_0 onto some neighborhood V of 0, satisfying $h(x_0) = 0$ and such that the solutions of the differential equation

$$\dot{x} = f(x)$$

in U are mapped homeomorphically by $y = h(x)$ onto the solutions of the linear differential equation

$$\dot{y} = Ay$$

in V, where $A := Df(x_0)$. Naturally, the inevitable question about what happens if x_0 is a nonhyperbolic critical point now arises. In this case there exists a decomposition

$$E = E_n \oplus E_h, \qquad A = A_n \oplus A_h$$

such that

$$\sigma(A_n) = \sigma_n(A) = \sigma(A) \cap i\mathbb{R}$$

and

$$\sigma(A_h) = \sigma(A) \setminus \sigma_n(A).$$

In other words: A_h induces a hyperbolic linear flow on E_h. In this case one can "partially linearize," i.e., there exists a homeomorphism h, from a neighborhood U of x_0 onto a neighborhood V of 0 satisfying $h(x_0) = 0$, and a function $g \in C^1(V, E^n)$ such that the solutions of the differential equation

$$\dot{x} = f(x)$$

in U are mapped homeomorphically by $y = h(x)$ onto the solutions of the differential equation

$$\begin{aligned} \dot{y}_n &= A_n y_n + g(y_n, y_h) \\ \dot{y}_h &= A_h y_h, \end{aligned} \tag{36}$$

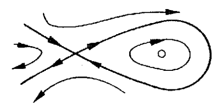
where $y = (y_n, y_h) \in V$. One can show, furthermore, that there exists a neighborhood V_n of 0 in E_n and a function $G \in C^1(V_n, E_h)$ satisfying $G(0) = 0$ and $DG(0) = 0$, and such that for every solution v in V_0 of the equation

$$\dot{v} = A_n v + g(v, G(v)),$$

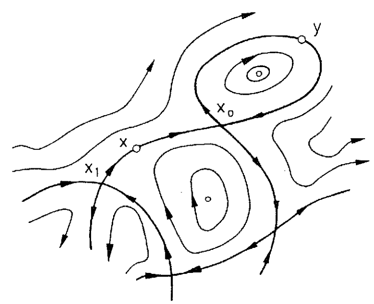
the function $t \mapsto (v(t), G(v(t)))$ is a solution of the "full" system (36). Therefore the graph of G is a C^1-manifold which has the space E_n as tangent space at the point 0. This manifold Z – the *center manifold* – is characterized by requiring that it contain all

solutions of (36) with bounded projections y_h in E_h. For proofs and further investigations in this direction we refer to the literature (e.g., Palmer [1 – 3], Abraham-Robbin [1], Marsden-McCracken [1], Knobloch-Kappel [1]).

(c) If x_0 is a hyperbolic critical point of the flow induced by $f \in C^k(M, E)$, one can show that $W_s(x_0)$ and $W_u(x_0)$ are *immersed C^k-manifolds*, that is to say, there exist C^k-atlases for $W_s(x_0)$ and $W_u(x_0)$ such that the inclusion maps $W_s(x_0) \hookrightarrow E$ and $W_u(x_0) \hookrightarrow E$ are immersions (i.e., at every point the tangent maps are injective) (see, for example, Irwin [1]). In general, however, $W_s(x_0)$ and $W_u(x_0)$ are *not* embedded submanifolds of E, as the following figure shows.



(d) A point $x \in M$ is called *heteroclinic* if $x \in W_s(x_0) \cap W_u(x_1)$, where x_0 and x_1 are distinct hyperbolic critical points. A point $y \in M$ is called *homoclinic* if $y \in W_s(x_0) \cap W_u(x_0)$. In these cases we also say that $\gamma(x)$ is a *heteroclinic* and that $\gamma(y)$ is a *homoclinic orbit*. The phase portrait of a flow is in general extremely complicated (see, for example, the figures in Abraham-Marsden [1]) and even in relatively easy cases, one is very remote from a complete description. □



Problems

In what follows we always assume that $f \in C^1(\mathbb{R},\mathbb{R})$ and

$$f(0) = f(1) = 0,$$

and we consider the simple reaction-diffusion equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = f(v). \tag{37}$$

In many cases (e.g., chemistry or biology), so-called *traveling waves* are of interest, that is, solutions of (37) which have the form

$$v(t,x) = u(x+ct), \qquad (t,x) \in \mathbb{R}\times\mathbb{R},$$

so that $u \in C^2(\mathbb{R},\mathbb{R})$ and $c \neq 0$, and such that the limits

$$u(\pm\infty) := \lim_{\xi\to\pm\infty} u(\xi)$$

exist. Here a traveling wave is said to be of *wave front type* if

$$0 \le u \le 1 \quad \text{and} \quad u(-\infty) = 0, \quad u(\infty) = 1,$$

and it is called a *soliton* if

$$u \neq 0, \quad u \ge 0 \quad \text{and} \quad u(-\infty) = u(\infty) = 0.$$

1. Clarify these concepts geometrically.

2. Show that traveling waves of wave front type can be characterized as the heteroclinic orbits of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= cy - f(x), \end{aligned} \tag{38}$$

which lie in $[0,1]\times\mathbb{R}$ and connect the critical points $(0,0)$ and $(1,0)$. The solitons correspond to the homoclinic orbits in $\mathbb{R}_+\times\mathbb{R}$ of the critical point $(0,0)$.

3. Show that the heteroclinic orbits, which correspond to solutions of the wave front type, lie completely in $(0,1)\times(0,\infty)$. In the special case when

$$f(u) = ku(1-u),$$

where $k > 0$ is some constant, (37) is called *Fisher's equation*, named after R.A. Fisher who suggested and analyzed it in 1937 as a model in genetics.

4. Show that for every $c > 2\sqrt{k}$, Fisher's equation has a unique traveling wave of wave front type.

5. Show (in the general case) that if $f'(0) > 0$, then (37) has no solitons.

6. Let M be open in $\mathbb{R}^2$ and let $g \in C^1(M,\mathbb{R}^2)$. Furthermore, let $x_0 \in M$ be a critical point of g such that 0 is a spiral point of the linearized equation $\dot{y} = Dg(x_0)y$. By introducing polar coordinates, show that x_0 is also a "spiral" for the nonlinear equation $\dot{x} = g(x)$, i.e., show that, in a neighborhood of x_0, the orbits have the same structure as the orbits of $e^{tDg(x_0)}$: They either spiral toward the point x_0 or away from it.

7. Make use of problem 6 to show that if $c^2 < 4f'(0)$, then (37) has no solutions of wave front type.