is diagonal. Hint: Use the density of the diagonalizable matrices and the continuity of the eigenvalues of a matrix with respect to its components (see Exercises 2.66 and 8.1). (c) Prove that the origin is asymptotically stable for the system $\dot{x} = Ax + g(x)$ where

$$A := \begin{pmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \qquad g(u, v, w) := \begin{pmatrix} u^2 + uv + v^2 + wv^2 \\ w^2 + uvw \\ w^3 \end{pmatrix}$$

and construct the corresponding matrix B that solves Lyapunov's equation.

Exercise 2.81. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is *conservative;* that is, there is some function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = \operatorname{grad} g(x)$. Also, suppose that M and Λ are symmetric positive definite $n \times n$ matrices. Consider the differential equation

$$M\ddot{x} + \Lambda\dot{x} + f(x) = 0, \qquad x \in \mathbb{R}^n$$

and note that, in case M and Λ are diagonal, the differential equation can be viewed as a model of n particles each moving according to Newton's second law in a conservative force field with viscous damping. (a) Prove that the function $V : \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(x, y) := \frac{1}{2}\langle My, y\rangle + \int_0^1 \langle f(sx), x\rangle \, ds$$

decreases along orbits of the associated first-order system

$$\dot{x} = y, \qquad M\dot{y} = -\Lambda y - f(x);$$

in fact, $\dot{V} = -\langle \Lambda y, y\rangle$. Conclude that the system has no periodic orbits. (b) Prove that if $f(0) = 0$ and $Df(0)$ is positive definite, then the system has an asymptotically stable rest point at the origin. Prove this fact in two ways: using the function V and by the method of linearization. Hint: To use the function V see Exercise 1.171. To use the method of linearization, note that M is invertible, compute the system matrix for the linearization in block form, suppose there is an eigenvalue λ, and look for a corresponding eigenvector in block form, that is the transpose of a vector (x, y). This leads to two equations corresponding to the block components corresponding to x and y. Reduce to one equation for x and then take the inner product with respect to x.

2.4 Floquet Theory

We will study linear systems of the form

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n \tag{2.27}$$

where $t \to A(t)$ is a T-periodic continuous matrix-valued function. The main theorem in this section, Floquet's theorem, gives a canonical form for fundamental matrix solutions. This result will be used to show that there is a periodic time-dependent change of coordinates that transforms system (2.27) into a homogeneous linear system with constant coefficients.

Floquet's theorem is a corollary of the following result about the range of the exponential map.

Theorem 2.82. *If C is a nonsingular $n \times n$ matrix, then there is an $n \times n$ matrix B (which may be complex) such that $e^B = C$. If C is a nonsingular real $n \times n$ matrix, then there is a real $n \times n$ matrix B such that $e^B = C^2$.*

Proof. If S is a nonsingular $n \times n$ matrix such that $S^{-1}CS = J$ is in Jordan canonical form, and if $e^K = J$, then $Se^K S^{-1} = C$. As a result, $e^{SKS^{-1}} = C$ and $B = SKS^{-1}$ is the desired matrix. Thus, it suffices to consider the nonsingular matrix C or C^2 to be a Jordan block.

For the first statement of the theorem, assume that $C = \lambda I + N$ where N is nilpotent; that is, $N^m = 0$ for some integer m with $0 \leq m < n$. Because C is nonsingular, $\lambda \neq 0$ and we can write $C = \lambda(I + (1/\lambda)N)$. A computation using the series representation of the function $t \mapsto \ln(1 + t)$ at $t = 0$ shows that, formally (that is, without regard to the convergence of the series), if $B = (\ln \lambda)I + M$ where

$$M = \sum_{j=1}^{m-1} \frac{(-1)^{j+1}}{j\lambda^j} N^j,$$

then $e^B = C$. But because N is nilpotent, the series are finite. Thus, the formal series identity is an identity. This proves the first statement of the theorem.

The Jordan blocks of C^2 correspond to the Jordan blocks of C. The blocks of C^2 corresponding to real eigenvalues of C are all of the type $rI + N$ where $r > 0$ and N is real nilpotent. For a real matrix C all the complex eigenvalues with nonzero imaginary parts occur in complex conjugate pairs; therefore, the corresponding real Jordan blocks of C^2 are block diagonal or "block diagonal plus block nilpotent" with 2×2 diagonal subblocks of the form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

as in equation (2.10). Some of the corresponding real Jordan blocks for the matrix C^2 might have real eigenvalues, but these blocks are again all block diagonal or "block diagonal plus block nilpotent" with 2×2 subblocks.

For the case where a block of C^2 is $rI + N$ where $r > 0$ and N is real nilpotent a real "logarithm" is obtained by the matrix formula given above. For block diagonal real Jordan block, write

$$R = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $r > 0$, and note that a real logarithm is given by

$$\ln rI + \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

Finally, for a "block diagonal plus block nilpotent" Jordan block, factor the Jordan block as follows:

$$\mathcal{R}(I + \mathcal{N})$$

where $\mathcal{R}$ is block diagonal with R along the diagonal and $\mathcal{N}$ has 2×2 blocks on its super diagonal all given by R^{-1}. Note that we have already obtained logarithms for each of these factors. Moreover, it is not difficult to check that the two logarithms commute. Thus, a real logarithm of the Jordan block is obtained as the sum of real logarithms of the factors. □

Theorem 2.82 can be proved without reference to the Jordan canonical form (see [5]).

Theorem 2.83 (Floquet's Theorem). *If $\Phi(t)$ is a fundamental matrix solution of the T-periodic system (2.27), then, for all $t \in \mathbb{R}$,*

$$\Phi(t + T) = \Phi(t)\Phi^{-1}(0)\Phi(T).$$

In addition, there is a matrix B (which may be complex) such that

$$e^{TB} = \Phi^{-1}(0)\Phi(T)$$

and a T-periodic matrix function $t \mapsto P(t)$ (which may be complex valued) such that $\Phi(t) = P(t)e^{tB}$ for all $t \in \mathbb{R}$. Also, there is a real matrix R and a real $2T$-periodic matrix function $t \to Q(t)$ such that $\Phi(t) = Q(t)e^{tR}$ for all $t \in \mathbb{R}$.

Proof. Since the function $t \mapsto A(t)$ is periodic, it is defined for all $t \in \mathbb{R}$. Thus, by Theorem 2.4, all solutions of the system are defined for $t \in \mathbb{R}$.

If $\Psi(t) := \Phi(t + T)$, then $\Psi(t)$ is a matrix solution. Indeed, we have that

$$\dot{\Psi}(t) = \dot{\Phi}(t + T) = A(t + T)\Phi(t + T) = A(t)\Psi(t),$$

as required.

Define

$$C := \Phi^{-1}(0)\Phi(T) = \Phi^{-1}(0)\Psi(0),$$

and note that C is nonsingular. The matrix function $t \mapsto \Phi(t)C$ is clearly a matrix solution of the linear system with initial value $\Phi(0)C = \Psi(0)$. By the uniqueness of solutions, $\Psi(t) = \Phi(t)C$ for all $t \in \mathbb{R}$. In particular, we have that

$$\begin{aligned} \Phi(t + T) &= \Phi(t)C = \Phi(t)\Phi^{-1}(0)\Phi(T), \\ \Phi(t + 2T) &= \Phi((t + T) + T) = \Phi(t + T)C = \Phi(t)C^2. \end{aligned}$$

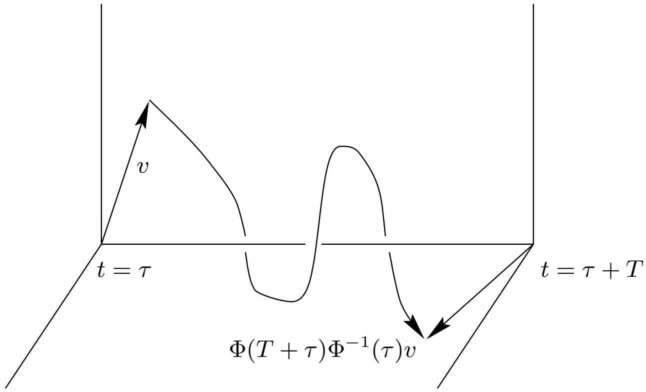

Figure 2.2: The figure depicts the geometry of the monodromy operator for the system $\dot{x} = A(t)x$ in the extended phase space. The vector v in $\mathbb{R}^n$ at $t = \tau$ is advanced to the vector $\Phi(T+\tau)\Phi^{-1}(\tau)v$ at $t = \tau + T$.

By Theorem 2.82, there is a matrix B, possibly complex, such that

$$e^{TB} = C.$$

Also, there is a real matrix R such that

$$e^{2TR} = C^2.$$

If $P(t) := \Phi(t)e^{-tB}$ and $Q(t) := \Phi(t)e^{-tR}$, then

$$P(t+T) = \Phi(t+T)e^{-TB}e^{-tB} = \Phi(t)Ce^{-TB}e^{-tB} = \Phi(t)e^{-tB} = P(t),$$
$$Q(t+2T) = \Phi(t+2T)e^{-2TR}e^{-tR} = \Phi(t)e^{-tR} = Q(t).$$

Thus, we have $P(t+T) = P(t)$, $Q(t+2T) = Q(t)$, and

$$\Phi(t) = P(t)e^{tB} = Q(t)e^{tR},$$

as required. □

The representation $\Phi(t) = P(t)e^{tB}$ in Floquet's theorem is called a *Floquet normal form* for the fundamental matrix $\Phi(t)$. We will use this normal form to study the stability of the zero solution of periodic homogeneous linear systems.

Let us consider a fundamental matrix solution $\Phi(t)$ for the periodic system (2.27) and a vector $v \in \mathbb{R}^n$. The vector solution of the system starting at time $t = \tau$ with initial condition $x(\tau) = v$ is given by

$$t \mapsto \Phi(t)\Phi^{-1}(\tau)v.$$

If the initial vector is moved forward over one period of the system, then we again obtain a vector in $\mathbb{R}^n$ given by $\Phi(T+\tau)\Phi^{-1}(\tau)v$. The operator

$$v \mapsto \Phi(T+\tau)\Phi^{-1}(\tau)v$$

is called a *monodromy operator* (see Figure 2.2). Moreover, if we view the periodic differential equation (2.27) as the autonomous system

$$\dot{x} = A(\psi)x, \qquad \dot{\psi} = 1$$

on the phase cylinder $\mathbb{R}^n \times \mathbb{T}$ where ψ is an angular variable modulo T, then each monodromy operator is a (stroboscopic) Poincaré map for our periodic system. For example, if $\tau = 0$, then the Poincaré section is the fiber $\mathbb{R}^n$ on the cylinder at $\psi = 0$. Of course, each fiber $\mathbb{R}^n$ at $\psi = mT$ where m is an integer is identified with the fiber at $\psi = 0$, and the corresponding Poincaré map is given by

$$v \mapsto \Phi(T)\Phi^{-1}(0)v.$$

The eigenvalues of a monodromy operator are called *characteristic multipliers* of the corresponding time-periodic homogeneous system (2.27). The next proposition states that characteristic multipliers are nonzero complex numbers that are intrinsic to the periodic system—they do not depend on the choice of the fundamental matrix or the initial time.

Proposition 2.84. *The following statements are valid for the periodic linear homogeneous system (2.27).*

(1) Every monodromy operator is invertible. Equivalently, every characteristic multiplier is nonzero.

(2) All monodromy operators have the same eigenvalues. In particular, there are exactly n characteristic multipliers, counting multiplicities.

Proof. The first statement of the proposition is obvious from the definitions.

To prove statement (2), let us consider the principal fundamental matrix $\Phi(t)$ at $t = 0$. If $\Psi(t)$ is a fundamental matrix, then $\Psi(t) = \Phi(t)\Psi(0)$. Also, by Floquet's theorem,

$$\Phi(t+T) = \Phi(t)\Phi^{-1}(0)\Phi(T) = \Phi(t)\Phi(T).$$

Consider the monodromy operator $\mathcal{M}$ given by

$$v \mapsto \Psi(T+\tau)\Psi^{-1}(\tau)v$$

and note that

$$\begin{aligned} \Psi(T+\tau)\Psi^{-1}(\tau) &= \Phi(T+\tau)\Psi(0)\Psi^{-1}(0)\Phi^{-1}(\tau) \\ &= \Phi(T+\tau)\Phi^{-1}(\tau) \\ &= \Phi(\tau)\Phi(T)\Phi^{-1}(\tau). \end{aligned}$$

In particular, the eigenvalues of the operator $\Phi(T)$ are the same as the eigenvalues of the monodromy operator $\mathcal{M}$. Thus, all monodromy operators have the same eigenvalues. $\square$

Because

$$\Phi(t+T) = \Phi(t)\Phi^{-1}(0)\Phi(T),$$

some authors define characteristic multipliers to be the eigenvalues of the matrices defined by $\Phi^{-1}(0)\Phi(T)$ where $\Phi(t)$ is a fundamental matrix. Of course, both definitions gives the same characteristic multipliers. To prove this fact, let us consider the Floquet normal form $\Phi(t) = P(t)e^{tB}$ and note that $\Phi(0) = P(0) = P(T)$. Thus, we have that

$$\Phi^{-1}(0)\Phi(T) = e^{TB}.$$

Also, by using the Floquet normal form,

$$\begin{aligned}\Phi(T)\Phi^{-1}(0) &= P(T)e^{TB}\Phi^{-1}(0)\\ &= \Phi(0)e^{TB}\Phi^{-1}(0)\\ &= \Phi(0)(\Phi^{-1}(0)\Phi(T))\Phi^{-1}(0),\end{aligned}$$

and therefore $\Phi^{-1}(0)\Phi(T)$ has the same eigenvalues as the monodromy operator given by

$$v \mapsto \Phi(T)\Phi^{-1}(0)v.$$

In particular, the traditional definition agrees with our geometrically motivated definition.

Returning to consideration of the Floquet normal form $P(t)e^{tB}$ for the fundamental matrix $\Phi(t)$ and the monodromy operator

$$v \mapsto \Phi(T+\tau)\Phi^{-1}(\tau)v,$$

note that $P(t)$ is invertible and

$$\Phi(T+\tau)\Phi^{-1}(\tau) = P(\tau)e^{TB}P^{-1}(\tau).$$

Thus, the characteristic multipliers of the system are the eigenvalues of e^{TB}. A complex number μ is called a *characteristic exponent* (or a *Floquet exponent*) of the system, if ρ is a characteristic multiplier and $e^{\mu T} = \rho$. Note that if $e^{\mu T} = \rho$, then $\mu + 2\pi ik/T$ is also a Floquet exponent for each integer k. Thus, while there are exactly n characteristic multipliers for the periodic linear system (2.27), there are infinitely many Floquet exponents.

Exercise 2.85. Suppose that $a : \mathbb{R} \to \mathbb{R}$ is a T-periodic function. Find the characteristic multiplier and a Floquet exponent of the T-periodic system $\dot{x} = a(t)x$. Also, find the Floquet normal form for the principal fundamental matrix solution of this system at $t = t_0$.

Exercise 2.86. For the autonomous linear system $\dot{x} = Ax$ a fundamental matrix solution $t \mapsto \Phi(t)$ satisfies the identity $\Phi(T - t) = \Phi(T)\Phi^{-1}(t)$. Show that, in general, this identity does not hold for nonautonomous homogeneous linear systems. Hint: Write down a Floquet normal form matrix $\Phi(t) = P(t)e^{tB}$ that does not satisfy the identity and then show that it is the solution of a (periodic) nonautonomous homogeneous linear system.

Exercise 2.87. Suppose as usual that $A(t)$ is T-periodic and the Floquet normal form of a fundamental matrix solution of the system $\dot{x} = A(t)x$ has the form $P(t)e^{tB}$. (a) Prove that

$$\operatorname{tr} B = \frac{1}{T}\int_0^T \operatorname{tr} A(t)\, dt.$$

Hint: Use Liouville's formula 2.18. (b) By (a), the sum of the characteristic exponents is given by the right-hand side of the formula for the trace of B. Prove that the product of the characteristic multipliers is given by $\exp(\int_0^T \operatorname{tr} A(t)\, dt)$.

Let us suppose that a fundamental matrix for the system (2.27) is represented in Floquet normal form by $P(t)e^{tB}$. We have seen that the characteristic multipliers of the system are the eigenvalues of e^{TB}, but the definition of the Floquet exponents does not mention the eigenvalues of B. Are the eigenvalues of B Floquet exponents? This question is answered affirmatively by the following general theorem about the exponential map.

Theorem 2.88. *If A is an $n \times n$ matrix and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A repeated according to their algebraic multiplicity, then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k and $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are the eigenvalues of e^A.*

Proof. We will prove the theorem by induction on the dimension n.

The theorem is clearly valid for 1×1 matrices. Suppose that it is true for all $(n-1) \times (n-1)$ matrices. Define $\lambda := \lambda_1$, and let $v \neq 0$ denote a corresponding eigenvector so that $Av = \lambda v$. Also, let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the usual basis of $\mathbb{C}^n$. There is a nonsingular $n \times n$ matrix S such that $Sv = \mathbf{e}_1$. (Why?) Thus,

$$SAS^{-1}\mathbf{e}_1 = \lambda \mathbf{e}_1,$$

and it follows that the matrix SAS^{-1} has the block form

$$SAS^{-1} = \begin{pmatrix} \lambda & * \\ 0 & \widetilde{A} \end{pmatrix}.$$

The matrix SA^kS^{-1} has the same block form, only with the block diagonal elements λ^k and $\widetilde{A}^k$. Clearly the eigenvalues of this block matrix are λ^k together with the eigenvalues of $\widetilde{A}^k$. By induction, the eigenvalues of $\widetilde{A}^k$ are the kth powers of the eigenvalues of $\widetilde{A}$. This proves the second statement of the theorem.

Using the power series definition of exp, we see that $e^{SAS^{-1}}$ has block form, with block diagonal elements e^{λ} and $e^{\widetilde{A}}$. Clearly, the eigenvalues of this block matrix are e^{λ} together with the eigenvalues of $e^{\widetilde{A}}$. Again using induction, it follows that the eigenvalues of $e^{\widetilde{A}}$ are $e^{\lambda_2}, \ldots, e^{\lambda_n}$. Thus, the eigenvalues of $e^{SAS^{-1}} = Se^{A}S^{-1}$ are $e^{\lambda_1}, \ldots, e^{\lambda_n}$. □

Theorem 2.88 is an example of a spectral mapping theorem. If we let $\sigma(A)$ denote the spectrum of the matrix A, that is, the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not invertible, then, for our finite dimensional matrix, $\sigma(A)$ coincides with the set of eigenvalues of A. Theorem 2.88 can be restated as follows: $e^{\sigma(A)} = \sigma(e^A)$.

The next result uses Floquet theory to show that the differential equation (2.27) is equivalent to a homogeneous linear system with constant coefficients. This result demonstrates that the stability of the zero solution can often be determined by the Floquet multipliers.

Theorem 2.89. *If the principal fundamental matrix solution of the T-periodic differential equation $\dot{x} = A(t)x$ (system (2.27)) at $t = 0$ is given by $Q(t)e^{tR}$ where Q and R are real, then the time-dependent change of coordinates $x = Q(t)y$ transforms this system to the (real) constant coefficient linear system $\dot{y} = Ry$. In particular, there is a time-dependent ($2T$-periodic) change of coordinates that transforms the T-periodic system to a (real) constant coefficient linear system.*

(1) If the characteristic multipliers of the periodic system (2.27) all have modulus less than one; equivalently, if all characteristic exponents have negative real part, then the zero solution is asymptotically stable.

(2) If the characteristic multipliers of the periodic system (2.27) all have modulus less than or equal to one; equivalently, if all characteristic exponents have nonpositive real part, and if the algebraic multiplicity equals the geometric multiplicity of each characteristic multiplier with modulus one; equivalently, if the algebraic multiplicity equals the geometric multiplicity of each characteristic exponent with real part zero, then the zero solution is Lyapunov stable.

(3) If at least one characteristic multiplier of the periodic system (2.27) has modulus greater than one; equivalently, if a characteristic exponent has positive real part, then the zero solution is unstable.

Proof. We will prove the first statement of the theorem and part (1). The proof of the remaining two parts is left as an exercise. For part (2), note that since the differential equation is linear, the Lyapunov stability may reasonably be determined from the eigenvalues of a linearization.

By Floquet's theorem, there is a real matrix R and a real $2T$-periodic matrix $Q(t)$ such that the *principal* fundamental matrix solution $\Phi(t)$ of the system at $t = 0$ is represented by

$$\Phi(t) = Q(t)e^{tR}.$$

Also, there is a matrix B and a T-periodic matrix P such that

$$\Phi(t) = P(t)e^{tB}.$$

The characteristic multipliers are the eigenvalues of e^{TB}. Because $\Phi(0)$ is the identity matrix, we have that

$$\Phi(2T) = e^{2TR} = e^{2TB},$$

and in particular

$$(e^{TB})^2 = e^{2TR}.$$

By Theorem 2.88, the eigenvalues of e^{2TR} are the squares of the characteristic multipliers. These all have modulus less than one. Thus, by another application of Theorem 2.88, all eigenvalues of the real matrix R have negative real parts.

Consider the change of variables $x = Q(t)y$. Because

$$x(t) = Q(t)e^{tR}x(0)$$

and $Q(t)$ is invertible, we have that $y(t) = e^{tR}x(0)$; and therefore,

$$\dot{y} = Ry.$$

By our previous result about linearization (Lyapunov's indirect method), the zero solution of $\dot{y} = Ry$ is asymptotically stable. In fact, by Theorem 2.61, there are numbers $\lambda > 0$ and $C > 0$ such that

$$|y(t)| \leq Ce^{-\lambda t}|y(0)|$$

for all $t \geq 0$ and all $y(0) \in \mathbb{R}^n$. Because Q is periodic, it is bounded. Thus, by the relation $x = Q(t)y$, the zero solution of $\dot{x} = A(t)x$ is also asymptotically stable. □

While the stability theorem just presented is very elegant, in applied problems it is usually impossible to compute the eigenvalues of e^{TB} explicitly. In fact, because $e^{TB} = \Phi(T)$, it is not at all clear that the eigenvalues can be found without solving the system, that is, without an explicit representation of a fundamental matrix. Note, however, that we only have to approximate *finitely* many numbers (the Floquet multipliers) to determine the stability of the system. This fact is important! For example, the stability can often be determined by applying a numerical method to approximate the Floquet multipliers.

Exercise 2.90. If the planar system $\dot{u} = f(u)$ has a limit cycle, then it is possible to find coordinates in a neighborhood of the limit cycle so that the differential equation has the form

$$\dot{\rho} = h(\rho, \varphi)\rho, \qquad \dot{\varphi} = \omega$$

where ω is a constant and for each ρ the function $\varphi \mapsto h(\rho, \varphi)$ is $2\pi/\omega$-periodic. Prove: If the partial derivative of h with respect to ρ is identically zero, then there is a coordinate system such that the differential equation in the new coordinates has the form

$$\dot{r} = cr, \qquad \dot{\phi} = \omega.$$

Hint: Use Exercise 2.85 and Theorem 2.89.

Exercise 2.91. View the damped periodically-forced Duffing equation $\ddot{x} + \dot{x} - x + x^3 = \epsilon \sin \omega t$ on the phase cylinder. The unperturbed system ($\epsilon = 0$) has a periodic orbit on the phase cylinder with period $2\pi/\omega$ corresponding to its rest point at the origin of the phase plane. Determine the Floquet multipliers associated with this periodic orbit of the unperturbed system; that is, the Floquet multipliers of the linearized system along the periodic orbit.

Exercise 2.92. Consider the system of two coupled oscillators with periodic parametric excitation

$$\ddot{x} + (1 + a \cos \omega t)x = y - x, \quad \ddot{y} + (1 + a \cos \omega t)y = x - y$$

where a and ω are nonnegative parameters. (See Section 3.3 for a derivation of the coupled oscillator model.) (a) Prove that if $a = 0$, then the zero solution is Lyapunov stable. (b) Using a numerical method (or otherwise), determine the Lyapunov stability of the zero solution for fixed but arbitrary values of the parameters. (c) What happens if viscous damping is introduced into the system? Hint: A possible numerical experiment might be designed as follows. For each point in a region of (ω, a)-space, mark the point green if the corresponding system has a Lyapunov stable zero solution; otherwise, mark it red. To decide which regions of parameter space might contain interesting phenomena, recall from your experience with second-order scalar differential equations with constant coefficients (mathematical models of springs) that resonance is expected when the frequency of the periodic excitation is rationally related to the natural frequency of the system. Consider resonances between the frequency ω of the excitation and the frequency of periodic motions of the system with $a = 0$, and explore the region of parameter space near these parameter values. Although interesting behavior does occur at resonances, this is not the whole story. Because the monodromy matrix is symplectic (see [11, Sec. 42]), the characteristic multipliers have two symmetries: If λ is a characteristic multiplier, then so is its complex conjugate and its reciprocal. It follows that on the boundary between the stable and unstable regions a pair of characteristic exponents coalesce on the unit circle. Thus, it is instructive to determine the values of ω, with $a = 0$, for those characteristic multipliers that coalesce. These values of ω determine the points where unstable regions have boundary points on the ω-axis.

Is there a method to determine the characteristic exponents without finding the solutions of the differential equation (2.27) explicitly? An example of Lawrence Marcus and Hidehiko Yamabe shows no such method can be constructed in any obvious way from the eigenvalues of $A(t)$. Consider the π-periodic system $\dot{x} = A(t)x$ where

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}. \tag{2.28}$$

It turns out that $A(t)$ has the (time independent) eigenvalues $\frac{1}{4}(-1 \pm \sqrt{7}\, i)$. In particular, the real part of each eigenvalue is negative. On the other hand,

$$x(t) = e^{t/2} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

is a solution, and therefore the zero solution is unstable!

The situation is not hopeless. An important example (Hill's equation) where the stability of the zero solution of the differential equation (2.27) can be determined in some cases is discussed in the next section.

Exercise 2.93. (a) Find the principal fundamental matrix solution $\Phi(t)$ at $t = 0$ for the Marcus–Yamabe system; its system matrix $A(t)$ is given in display (2.28). (b) Find the Floquet normal form for $\Phi(t)$ and its "real" Floquet normal form. (c) Determine the characteristic multipliers for the system. (d) The matrix function $t \mapsto A(t)$ is isospectral. Find a matrix function $t \mapsto M(t)$ such that $(A(t), M(t))$ is a Lax pair (see Exercise 2.55). Is every isospectral matrix function the first component of a Lax pair?

The Floquet normal form can be used to obtain detailed information about the solutions of the differential equation (2.27). For example, if we use the fact that the Floquet normal form decomposes a fundamental matrix into a periodic part and an exponential part, then it should be clear that for some systems there are periodic solutions and for others there are no nontrivial periodic solutions. It is also possible to have "quasi-periodic" solutions. The next lemma will be used to prove these facts.

Lemma 2.94. *If μ is a characteristic exponent for the homogeneous linear T-periodic differential equation (2.27) and $\Phi(t)$ is the principal fundamental matrix solution at $t = 0$, then $\Phi(t)$ has a Floquet normal form $P(t)e^{tB}$ such that μ is an eigenvalue of B.*

Proof. Let $\mathcal{P}(t)e^{t\mathcal{B}}$ be a Floquet normal form for $\Phi(t)$. By the definition of characteristic exponents, there is a characteristic multiplier λ such that $\lambda = e^{\mu T}$, and, by Theorem 2.88, there is an eigenvalue ν of $\mathcal{B}$ such that $e^{\nu T} = \lambda$. Also, there is some integer $k \neq 0$ such that $\nu = \mu + 2\pi i k/T$.

Define $B := \mathcal{B} - (2\pi ik/T)I$ and $P(t) = \mathcal{P}(t)e^{(2\pi ikt/T)I}$. Note that μ is an eigenvalue of B, the function P is T-periodic, and

$$P(t)e^{tB} = \mathcal{P}(t)e^{t\mathcal{B}}.$$

It follows that $\Phi(t) = P(t)e^{tB}$ is a representation in Floquet normal form where μ is an eigenvalue of B. □

A basic result that is used to classify the possible types of solutions that can arise is the content of the following theorem.

Theorem 2.95. *If λ is a characteristic multiplier of the homogeneous linear T-periodic differential equation (2.27) and $e^{T\mu} = \lambda$, then there is a (possibly complex) nontrivial solution of the form*

$$x(t) = e^{\mu t}p(t)$$

where p is a T-periodic function. Moreover, for this solution $x(t+T) = \lambda x(t)$.

Proof. Consider the principal fundamental matrix solution $\Phi(t)$ at $t = 0$. By Lemma 2.94, there is a Floquet normal form representation $\Phi(t) = P(t)e^{tB}$ such that μ is an eigenvalue of B. Hence, there is a vector $v \neq 0$ such that $Bv = \mu v$. Clearly, it follows that $e^{tB}v = e^{\mu t}v$, and therefore the solution $x(t) := \Phi(t)v$ is also represented in the form

$$x(t) = P(t)e^{tB}v = e^{\mu t}P(t)v.$$

The solution required by the first statement of the theorem is obtained by defining $p(t) := P(t)v$. The second statement of the theorem is proved as follows:

$$x(t+T) = e^{\mu(t+T)}p(t+T) = e^{\mu T}e^{\mu t}p(t) = \lambda x(t).$$ □

Theorem 2.96. *Suppose that λ_1 and λ_2 are characteristic multipliers of the homogeneous linear T-periodic differential equation (2.27) and μ_1 and μ_2 are characteristic exponents such that $e^{T\mu_1} = \lambda_1$ and $e^{T\mu_2} = \lambda_2$. If $\lambda_1 \neq \lambda_2$, then there are T-periodic functions p_1 and p_2 such that*

$$x_1(t) = e^{\mu_1 t}p_1(t) \quad \textit{and} \quad x_2(t) = e^{\mu_2 t}p_2(t)$$

are linearly independent solutions.

Proof. Let $\Phi(t) = P(t)e^{tB}$ (as in Lemma 2.94) be such that μ_1 is an eigenvalue of B. Also, let v_1 be a nonzero eigenvector corresponding to the eigenvalue μ_1. Since λ_2 is an eigenvalue of the monodromy matrix $\Phi(T)$, by Theorem 2.88 there is an eigenvalue μ of B such that $e^{T\mu} = \lambda_2 = e^{T\mu_2}$. It follows that there is an integer k such that $\mu_2 = \mu + 2\pi ik/T$. Also, because $\lambda_1 \neq \lambda_2$, we have that $\mu \neq \mu_1$. Hence, if v_2 is a nonzero eigenvector of

B corresponding to the eigenvalue μ, then the eigenvectors v_1 and v_2 are linearly independent.

As in the proof of Theorem 2.95, there are solutions of the form

$$x_1(t) = e^{\mu_1 t} P(t) v_1, \qquad x_2(t) = e^{\mu t} P(t) v_2.$$

Moreover, because $x_1(0) = v_1$ and $x_2(0) = v_2$, these solutions are linearly independent. Finally, let us note that x_2 can be written in the required form

$$x_2(t) = \big(e^{\mu t} e^{2\pi k i/T}\big)\big(e^{-2\pi k i/T} P(t) v_2\big). \qquad \square$$

The T-periodic system (2.27) has the Floquet normal form

$$t \mapsto Q(t) e^{tR}$$

where Q is a real $2T$-periodic function and R is real matrix. By Theorem 2.37 and 2.89, all solutions of the system are represented as finite sums of real solutions of the two types

$$q(t) r(t) e^{\alpha t} \sin \beta t \quad \text{and} \quad q(t) r(t) e^{\alpha t} \cos \beta t,$$

where q is $2T$-periodic, r is a polynomial of degree at most $n-1$, and $\alpha + i\beta$ is an eigenvalue of R. We will use Theorem 2.95 to give a more detailed description of the nature of these real solutions.

If the characteristic multiplier λ is a positive real number, then there is a corresponding real characteristic exponent μ. In this case, if the periodic function p in Theorem 2.95 is complex, then it can be represented as $p = r + is$ where both r and s are real T-periodic functions. Because our T-periodic system is real, both the real and the imaginary parts of a solution are themselves solutions. Hence, there is a real nontrivial solution of the form $x(t) = e^{\mu t} r(t)$ or $x(t) = e^{\mu t} s(t)$. Such a solution is periodic if and only if $\lambda = 1$ or equivalently if $\mu = 0$. On the other hand, if $\lambda \neq 1$ or $\mu \neq 0$, then the solution is unbounded either as $t \to \infty$ or as $t \to -\infty$. If $\lambda < 1$ (equivalently, $\mu < 0$), then the solution is asymptotic to the zero solution as $t \to \infty$. On the other hand, if $\lambda > 1$ (equivalently, $\mu > 0$), then the solution is unbounded as $t \to \infty$.

If the characteristic multiplier λ is a negative real number, then μ can be chosen to have the form $\nu + \pi i/T$ where ν is real and $e^{T\mu} = \lambda$. Hence, if we again take $p = r + is$, then we have the solution

$$e^{\mu t} p(t) = e^{\nu t} e^{\pi i t/T} (r(t) + i s(t))$$

from which real nontrivial solutions are easily constructed. For example, if the real part of the complex solution is nonzero, then the real solution has the form

$$x(t) = e^{\nu t} (r(t) \cos(\pi t/T) - s(t) \sin(\pi t/T)).$$

Such a solution is periodic if and only if $\lambda = -1$ or equivalently if $\nu = 0$. In this case the solution is $2T$-periodic. If $\nu \neq 0$, then the solution is unbounded. If $\nu < 0$, then the solution is asymptotic to zero as $t \to \infty$. On the other hand, if $\nu > 0$, then the solution is unbounded as $t \to \infty$.

If λ is complex, then we have $\mu = \alpha + i\beta$ and there is a solution given by

$$x(t) = e^{\alpha t}(\cos \beta t + i \sin \beta t)(r(t) + is(t)).$$

Thus, there are real solutions

$$\begin{aligned} x_1(t) &= e^{\alpha t}(r(t) \cos \beta t - s(t) \sin \beta t), \\ x_2(t) &= e^{\alpha t}(r(t) \sin \beta t + s(t) \cos \beta t). \end{aligned}$$

If $\alpha \neq 0$, then both solutions are unbounded. But, if $\alpha < 0$, then these solutions are asymptotic to zero as $t \to \infty$. On the other hand, if $\alpha > 0$, then these solutions are unbounded as $t \to \infty$. If $\alpha = 0$ and there are relatively prime positive integers m and n such that $2\pi m/\beta = nT$, then the solution is nT-periodic. If no such integers exist, then the solution is called *quasi-periodic*.

We will prove in Section 2.4.4 that the stability of a periodic orbit is determined by the stability of the corresponding fixed point of a Poincaré map defined on a Poincaré section that meets the periodic orbit. Generically, the stability of the fixed point of the Poincaré map is determined by the eigenvalues of its derivative at the fixed point. For example, if the eigenvalues of the derivative of the Poincaré map at the fixed point corresponding to the periodic orbit are all inside the unit circle, then the periodic orbit is asymptotically stable. It turns out that the eigenvalues of the derivative of the Poincaré map are closely related to the characteristic multipliers of a time-periodic system, namely, the variational equation along the periodic orbit. We will have much more to say about the general case later. Here we will illustrate the idea for an example where the Poincaré map is easy to compute.

Suppose that

$$\dot{u} = f(u, t), \qquad u \in \mathbb{R}^n \tag{2.29}$$

is a smooth nonautonomous differential equation. If there is some $T > 0$ such that $f(u, t + T) = f(u, t)$ for all $u \in \mathbb{R}^n$ and all $t \in \mathbb{R}$, then the system (2.29) is called T-periodic.

The nonautonomous system (2.29) is made "artificially" autonomous by the addition of a new equation as follows:

$$\dot{u} = f(u, \psi), \qquad \dot{\psi} = 1 \tag{2.30}$$

where ψ may be viewed as an angular variable modulo T. In other words, we can consider $\psi + nT = \psi$ whenever n is an integer. The phase cylinder for system (2.30) is $\mathbb{R}^n \times \mathbb{T}$, where $\mathbb{T}$ (topologically the unit circle) is defined to be $\mathbb{R}$ modulo T. This autonomous system provides the correct geometry with which to define a Poincaré map.

For each $\xi \in \mathbb{R}^n$, let $t \mapsto u(t, \xi)$ denote the solution of the differential equation (2.29) such that $u(0, \xi) = \xi$, and note that $t \mapsto (u(t, \xi), t)$ is the corresponding solution of the system (2.30). The set $\Sigma := \{(\xi, \psi) : \psi = 0\}$ is a Poincaré section, and the corresponding Poincaré map is given by $\xi \mapsto u(T, \xi)$.

If there is a point $p \in \mathbb{R}^n$ such that $f(p, t) = 0$ for all $t \in \mathbb{R}$, then the function $t \mapsto (p, t)$, or equivalently $t \mapsto (u(t, p), t)$, is a periodic solution of the system (2.30) with period T. Moreover, let us note that $u(T, p) = p$. Thus, the periodic solution corresponds to a fixed point of the Poincaré map as it should.

The derivative of the Poincaré map at p is the linear transformation of $\mathbb{R}^n$ given by the partial derivative $u_\xi(T, p)$. Moreover, by differentiating both the differential equation (2.29) and the initial condition $u(0, \xi) = \xi$ with respect to ξ, it is easy to see that the matrix function $t \mapsto u_\xi(t, p)$ is the principal fundamental matrix solution at $t = 0$ of the (T-periodic linear) variational initial value problem

$$\dot{W} = f_u(u(t, p), t)W, \qquad W(0) = I. \tag{2.31}$$

If the solution of system (2.31) is represented in the Floquet normal form $u_\xi(t, p) = P(t)e^{tB}$, then the derivative of the Poincaré map is given by $u_\xi(T, p) = e^{TB}$. In particular, the characteristic multipliers of the variational equation (2.31) coincide with the eigenvalues of the derivative of the Poincaré map. Thus, whenever the principle of linearized stability is valid, the stability of the periodic orbit is determined by the characteristic multipliers of the periodic variational equation (2.31).

As an example, consider the pendulum with oscillating support

$$\ddot{\theta} + (1 + a \cos \omega t) \sin \theta = 0.$$

The zero solution, given by $\theta(t) \equiv 0$, corresponds to a $2\pi/\omega$-periodic solution of the associated autonomous system. A calculation shows that the variational equation along this periodic solution is equivalent to the second order differential equation

$$\ddot{x} + (1 + a \cos \omega t)x = 0,$$

called a *Mathieu equation.* The normal form for the Mathieu equation is

$$\ddot{x} + (a - 2q \cos 2t)x = 0,$$

where a and q are parameters.

Since, as we have just seen (see also Exercise 2.92), equations of Mathieu type arise frequently in applications, the stability analysis of such equations is important (see, for example, [12], [18], [101], [127], [149], and [237]). In Section 2.4.2 we will show how the stability of the zero solution of the Mathieu equation, and, in turn, the stability of the zero solution of the pendulum with oscillating support, is related in a delicate manner to the amplitude a and the frequency ω of the periodic displacement.

Exercise 2.97. This is a continuation of Exercise 2.57. Suppose that $v : \mathbb{R} \to \mathbb{R}^3$ is a periodic function. Consider the differential equation

$$\dot{x} = v(t) \times x$$

and discuss the stability of its periodic solutions.

Exercise 2.98. Determine the stability type of the periodic orbit discussed in Exercise 2.91.

Exercise 2.99. (a) Prove that the system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2), \\ \dot{y} &= x + y - y(x^2 + y^2), \\ \dot{z} &= z + xz - z^3 \end{aligned}$$

has periodic orbits. Hint: Change to cylindrical coordinates, show that the cylinder (with radius one whose axis of symmetry is the z-axis) is invariant, and recall the analysis of equation (1.43). (b) Prove that there is a stable periodic orbit. (c) The stable periodic orbit has three Floquet multipliers. Of course, one of them is unity. Find (exactly) a vector v such that $\Phi(T)v = v$, where T is the period of the periodic orbit and $\Phi(t)$ is the principal fundamental matrix solution at $t = 0$ of the variational equation along the stable periodic solution. (d) Approximate the remaining two multipliers. Note: It is possible to represent these multipliers with integrals, but they are easier to approximate using a numerical method.

2.4.1 Lyapunov Exponents

An important generalization of Floquet exponents, called Lyapunov exponents, are introduced in this section. This concept is used extensively in the theory of dynamical systems (see, for example, [103], [144], [176], and [233]).

Consider a (nonlinear) differential equation

$$\dot{u} = f(u), \qquad u \in \mathbb{R}^n \tag{2.32}$$

with flow φ_t. If $\epsilon \in \mathbb{R}$, ξ, $v \in \mathbb{R}^n$, and $\eta := \xi + \epsilon v$, then the two solutions

$$t \mapsto \varphi_t(\xi), \qquad t \mapsto \varphi_t(\xi + \epsilon v)$$

start at points that are $O(\epsilon)$ close; that is, the absolute value of the difference of the two points in $\mathbb{R}^n$ is bounded by the usual norm of v times ϵ. Moreover, by Taylor expansion at $\epsilon = 0$, we have that

$$\varphi_t(\xi + \epsilon v) - \varphi_t(\xi) = \epsilon D\varphi_t(\xi)v + O(\epsilon^2)$$

where $D\varphi_t(\xi)$ denotes the derivative of the function $u \mapsto \varphi_t(u)$ evaluated at $u = \xi$. Thus, the first order approximation of the difference of the solutions at time t is $\epsilon D\varphi_t(\xi)v$ where $t \mapsto D\varphi_t(\xi)$ is the principal fundamental matrix solution at $t = 0$ of the linearized equation

$$\dot{W} = Df(\varphi_t(\xi))W$$

along the solution of the original system (2.32) starting at ξ. To see this fact, just note that

$$\dot{\varphi}_t(u) = f(\varphi_t(u))$$

and differentiate both sides of this identity with respect to u at $u = \xi$.

If we view v as a vector in the tangent space to $\mathbb{R}^n$ at ξ, denoted $T_\xi\mathbb{R}^n$, then $D\varphi_t(\xi)v$ is a vector in the tangent space $T_{\varphi_t(\xi)}\mathbb{R}^n$. For each such v, if $v \neq 0$, then it is natural to define a corresponding linear operator L, from the linear subspace of $T_\xi\mathbb{R}^n$ generated by v to the linear subspace of $T_{\varphi_t(\xi)}\mathbb{R}^n$ generated by $D\varphi_t(\xi)v$, defined by $L(av) = D\varphi_t(\xi)av$ where $a \in \mathbb{R}$. Let us note that the norm of this operator measures the relative "expansion" or "contraction" of the vector v; that is,

$$\|L\| = \sup_{a \neq 0} \frac{|D\phi_t(\xi)av|}{|av|} = \frac{|D\phi_t(\xi)v|}{|v|}.$$

Our two solutions can be expressed in integral form; that is,

$$\varphi_t(\xi) = \xi + \int_0^t f(\varphi_s(\xi))\,ds,$$
$$\varphi_t(\xi + \epsilon v) = \xi + \epsilon v + \int_0^t f(\varphi_s(\xi + \epsilon v))\,ds.$$

Hence, as long as we consider a finite time interval or a solution that is contained in a compact subset of $\mathbb{R}^n$, there is a Lipschitz constant $\mathrm{Lip}(f) > 0$ for the function f, and we have the inequality

$$|\varphi_t(\xi + \epsilon v) - \varphi_t(\xi)| \le \epsilon|v| + \mathrm{Lip}(f) \int_0^t |\varphi_s(\xi + \epsilon v) - \varphi_s(\xi)|\,ds.$$

By Gronwall's inequality, the separation distance between the solutions is bounded by an exponential function of time. In fact, we have the estimate

$$|\varphi_t(\xi + \epsilon v) - \varphi_t(\xi)| \le \epsilon|v|e^{t\,\mathrm{Lip}(f)}.$$

The above computation for the norm of L and the exponential bound for the separation rate between two solutions motivates the following definition (see [144]).

Definition 2.100. Suppose that $\xi \in \mathbb{R}^n$ and the solution $t \mapsto \varphi_t(\xi)$ of the differential equation (2.32) is defined for all $t \geq 0$. Also, let $v \in \mathbb{R}^n$ be a nonzero vector. The *Lyapunov exponent* at ξ in the direction v for the flow φ_t is defined to be

$$\chi(p, v) := \limsup_{t\to\infty} \frac{1}{t} \ln\Big(\frac{|D\phi_t(\xi)v|}{|v|}\Big).$$

As a simple example, let us consider the planar system

$$\dot{x} = -ax, \qquad \dot{y} = by$$

where a and b are positive parameters, and let us note that its flow is given by

$$\varphi_t(x, y) = (e^{-at}x, e^{bt}y).$$

By an easy computation using the definition of the Lyapunov exponents, it follows that if v is given by $v = (w, z)$ and $z \neq 0$, then $\chi(\xi, v) = b$. If $z = 0$ and $w \neq 0$, then $\chi(\xi, v) = -a$. In particular, there are exactly two Lyapunov exponents for this system. Of course, the Lyapunov exponents in this case correspond to the eigenvalues of the system matrix.

Although our definition of Lyapunov exponents is for autonomous systems, it should be clear that since the definition only depends on the fundamental matrix solutions of the associated variational equations along orbits of the system, we can define the same notion for solutions of abstract time-dependent linear systems. Indeed, for a T-periodic linear system

$$\dot{u} = A(t)u, \qquad u \in \mathbb{R}^n \tag{2.33}$$

with principal fundamental matrix $\Phi(t)$ at $t = 0$, the Lyapunov exponent defined with respect to the nonzero vector $v \in \mathbb{R}^n$ is

$$\chi(v) := \limsup_{t\to\infty} \frac{1}{t} \ln\Big(\frac{|\Phi(t)v|}{|v|}\Big).$$

Proposition 2.101. *If μ is a Floquet exponent of the system (2.33), then the real part of μ is a Lyapunov exponent.*

Proof. Let us suppose that the principal fundamental matrix $\Phi(t)$ is given in Floquet normal form by

$$\Phi(t) = P(t)e^{tB}.$$

If $\mu = a + bi$ is a Floquet exponent, then there is a corresponding vector v such that $e^{TB}v = e^{\mu T}v$. Hence, using the Floquet normal form, we have that

$$\Phi(T)v = e^{\mu T}v.$$

If $t \geq 0$, then there is a nonnegative integer n and a number r such that $0 \leq r < T$ and

$$\begin{aligned}
\frac{1}{t}\ln\Big(\frac{|\Phi(t)v|}{|v|}\Big) &= \frac{1}{T}\Big(\frac{nT}{nT+r}\Big)\Big(\frac{1}{n}\ln\Big(\frac{|P(nT+r)e^{rB}e^{n\mu T}v|}{|v|}\Big)\Big) \\
&= \frac{1}{T}\Big(\frac{nT}{nT+r}\Big)\Big(\frac{1}{n}\ln|e^{nTa}| + \frac{1}{n}\ln\Big(\frac{|P(r)e^{rB}v|}{|v|}\Big)\Big).
\end{aligned}$$

Clearly, $n \to \infty$ as $t \to \infty$. Thus, it is easy to see that

$$\lim_{t\to\infty}\frac{1}{T}\Big(\frac{nT}{nT+r}\Big)\Big(\frac{1}{n}\ln|e^{nTa}| + \frac{1}{n}\ln\Big(\frac{|P(r)e^{rB}v|}{|v|}\Big)\Big) = a. \qquad \square$$

Let us suppose that a differential equation has a compact invariant set that contains an orbit whose closure is dense in the invariant set. Then, the existence of a positive Lyapunov exponent for this orbit ensures that nearby orbits tend to separate exponentially fast from the dense orbit. But, since these orbits are confined to a compact invariant set, they must also be bounded. This suggests that each small neighborhood in the invariant set undergoes both stretching and folding as it evolves with the flow. The subsequent kneading of the invariant set due to this stretching and folding would tend to mix the evolving neighborhoods so that they eventually intertwine in a complicated manner. For this reason, the existence of a positive Lyapunov exponent is often taken as a signature of "chaos." While this criterion is not always valid, the underlying idea that the stretching implied by a positive Lyapunov exponent is associated with complex motions is important in the modern theory of dynamical systems.

Exercise 2.102. Show that if two points are on the same orbit, then the corresponding Lyapunov exponents are the same.

Exercise 2.103. Prove the "converse" of Proposition 2.101; that is, every Lyapunov exponent for a time-periodic system is a Floquet exponent.

Exercise 2.104. If $\dot{x} = f(x)$, determine the Lyapunov exponent $\chi(\xi, f(\xi))$.

Exercise 2.105. How many Lyapunov exponents are associated with an orbit of a differential equation in an n-dimensional phase space.

Exercise 2.106. Suppose that x is in the omega limit set of an orbit. Are the Lyapunov exponents associated with x the same as those associated with the original orbit?

Exercise 2.107. In all the examples in this section, the lim sup can be replaced by lim. Are there examples where the superior limit is a finite number, but the limit does not exist? This is (probably) a challenging exercise! For an answer see [144] and [176].

2.4.2 Hill's Equation

A famous example where Floquet theory applies to give good stability results is Hill's equation,

$$\ddot{u} + a(t)u = 0, \qquad a(t+T) = a(t).$$

It was introduced by George W. Hill in his study of the motions of the moon. Roughly speaking, the motion of the moon can be viewed as a harmonic oscillator in a periodic gravitational field. But this model equation arises in many areas of applied mathematics where the stability of periodic motions is an issue. A prime example, mentioned in the previous section, is the stability analysis of small oscillations of a pendulum whose length varies with time.

If we define

$$x := \begin{pmatrix} u \\ \dot{u} \end{pmatrix},$$

then Hill's equation is equivalent to the first order system $\dot{x} = A(t)x$ where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}.$$

We will apply linear systems theory, especially Floquet theory, to analyze the stability of the zero solution of this linear T-periodic system.

The first step in the stability analysis is an application of Liouville's formula (2.18). In this regard, you may recall from your study of scalar second order linear differential equations that if $\ddot{u} + p(t)\dot{u} + q(t)u = 0$ and the Wronskian of the two solutions u_1 and u_2 is defined by

$$W(t) := \det \begin{pmatrix} u_1(t) & u_2(t) \\ \dot{u}_1(t) & \dot{u}_2(t) \end{pmatrix},$$

then

$$W(t) = W(0)e^{-\int_0^t p(s)\,ds}. \tag{2.34}$$

Note that for the equivalent first order system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} x = B(t)x$$

with fundamental matrix $\Psi(t)$, formula (2.34) is a special case of Liouville's formula

$$\det \Psi(t) = \det \Psi(0) e^{\int_0^t \operatorname{tr} B(s) ds}.$$

At any rate, let us apply Liouville's formula to the principal fundamental matrix $\Phi(t)$ at $t = 0$ for Hill's system to obtain the identity $\det \Phi(t) \equiv 1$. Since the determinant of a matrix is the product of the eigenvalues of the matrix, we have an important fact: The product of the characteristic multipliers of the monodromy matrix, $\Phi(T)$, is 1.

Let the characteristic multipliers for Hill's equation be denoted by λ_1 and λ_2 and note that they are roots of the characteristic equation

$$\lambda^2 - (\operatorname{tr}\Phi(T))\lambda + \det\Phi(T) = 0.$$

For notational convenience let us set $2\phi = \operatorname{tr}\Phi(T)$ to obtain the equivalent characteristic equation

$$\lambda^2 - 2\phi\lambda + 1 = 0$$

whose solutions are given by

$$\lambda = \phi \pm \sqrt{\phi^2 - 1}.$$

There are several cases to consider depending on the value of ϕ.

Case 1: If $\phi > 1$, then λ_1 and λ_2 are distinct positive real numbers such that $\lambda_1\lambda_2 = 1$. Thus, we may assume that $0 < \lambda_1 < 1 < \lambda_2$ with $\lambda_1 = 1/\lambda_2$ and there is a real number $\mu > 0$ (a characteristic exponent) such that $e^{T\mu} = \lambda_2$ and $e^{-T\mu} = \lambda_1$. By Theorem 2.95 and Theorem 2.96, there is a fundamental set of solutions of the form

$$e^{-\mu t}p_1(t), \qquad e^{\mu t}p_2(t)$$

where the real functions p_1 and p_2 are T-periodic. In this case, the zero solution is unstable.

Case 2: If $\phi < -1$, then λ_1 and λ_2 are both real and both negative. Also, since $\lambda_1\lambda_2 = 1$, we may assume that $\lambda_1 < -1 < \lambda_2 < 0$ with $\lambda_1 = 1/\lambda_2$. Thus, there is a real number $\mu > 0$ (a characteristic exponent) such that $e^{2T\mu} = \lambda_1^2$ and $e^{-2T\mu} = \lambda_2^2$. As in Case 1, there is a fundamental set of solutions of the form

$$e^{\mu t}q_1(t), \qquad e^{-\mu t}q_2(t)$$

where the real functions q_1 and q_2 are $2T$-periodic. Again, the zero solution is unstable.

Case 3: If $-1 < \phi < 1$, then λ_1 and λ_2 are complex conjugates each with nonzero imaginary part. Since $\lambda_1\bar{\lambda}_1 = 1$, we have that $|\lambda_1| = 1$, and therefore both characteristic multipliers lie on the unit circle in the complex plane. Because both λ_1 and λ_2 have nonzero imaginary parts, one of these characteristic multipliers, say λ_1, lies in the upper half plane. Thus, there is a real number θ with $0 < \theta T < \pi$ and $e^{i\theta T} = \lambda_1$. In fact, there is a solution of the form $e^{i\theta t}(r(t) + is(t))$ with r and s both T-periodic functions. Hence, there is a fundamental set of solutions of the form

$$r(t)\cos\theta t - s(t)\sin\theta t, \quad r(t)\sin\theta t + s(t)\cos\theta t.$$

In particular, the zero solution is stable (see Exercise 2.113) but not asymptotically stable. Also, the solutions are periodic if and only if there are relatively prime positive integers m and n such that $2\pi m/\theta = nT$. If such integers exist, all solutions have period nT. If not, then these solutions are quasi-periodic.

We have just proved the following facts for Hill's equation: *Suppose that $\Phi(t)$ is the principal fundamental matrix solution of Hill's equation at $t = 0$. If $|\operatorname{tr}\Phi(T)| < 2$, then the zero solution is stable. If $|\operatorname{tr}\Phi(T)| > 2$, then the zero solution is unstable.*

Case 4: If $\phi = 1$, then $\lambda_1 = \lambda_2 = 1$. The nature of the solutions depends on the canonical form of $\Phi(T)$. If $\Phi(T)$ is the identity, then $e^0 = \Phi(T)$ and there is a Floquet normal form $\Phi(t) = P(t)$ where $P(t)$ is T-periodic and invertible. Thus, there is a fundamental set of periodic solutions and the zero solution is stable. If $\Phi(T)$ is not the identity, then there is a nonsingular matrix C such that

$$C\Phi(T)C^{-1} = I + N = e^N$$

where $N \neq 0$ is nilpotent. Thus, $\Phi(t)$ has a Floquet normal form $\Phi(t) = P(t)e^{tB}$ where $B := C^{-1}(\frac{1}{T}N)C$. Because

$$e^{tB} = C^{-1}(I + \frac{t}{T}N)C,$$

the matrix function $t \mapsto e^{tB}$ is unbounded, and therefore the zero solution is unstable.

Case 5: If $\phi = -1$, then the situation is similar to Case 4, except the fundamental matrix is represented by $Q(t)e^{tB}$ where $Q(t)$ is a $2T$-periodic matrix function.

By the results just presented, the stability of Hill's equation is reduced, in most cases, to a determination of the absolute value of the trace of its principal fundamental matrix evaluated after one period. While this is a useful fact, it leaves open an important question: Can the stability be determined without imposing a condition on the solutions of the equation? It turns out that in some special cases this is possible (see [149] and [237]). A theorem of Lyapunov [144] in this direction follows.

Theorem 2.108. *If $a : \mathbb{R} \to \mathbb{R}$ is a positive T-periodic function such that*

$$T\int_0^T a(t)\,dt \leq 4,$$

then all solutions of the Hill's equation $\ddot{x} + a(t)x = 0$ are bounded. In particular, the trivial solution is stable.

The proof of Theorem 2.108 is outlined in Exercises 2.113 and 2.116.

Exercise 2.109. Consider the second order system

$$\ddot{u} + \dot{u} + \cos(t)\,u = 0.$$

Prove: (a) If ρ_1 and ρ_2 are the characteristic multipliers of the corresponding first order system, then $\rho_1\rho_2 = \exp(-2\pi)$. (b) The Poincaré map for the system is dissipative; that is, it contracts area.

Exercise 2.110. Prove: The equation

$$\ddot{u} - (2\sin^2 t)\dot{u} + (1 + \sin 2t)u = 0.$$

does not have a fundamental set of periodic solutions. Does it have a nonzero periodic solution? Is the zero solution stable?

Exercise 2.111. Discuss the stability of the trivial solution of the scalar time-periodic system $\dot{x} = (\cos^2 t)x$.

Exercise 2.112. Prove: The zero solution is unstable for the system $\dot{x} = A(t)x$ where

$$A(t) := \begin{pmatrix} 1/2 - \cos t & 12 \\ 147 & 3/2 + \sin t \end{pmatrix}.$$

Exercise 2.113. Prove: If all solutions of the T-periodic system $\dot{x} = A(t)x$ are bounded, then the trivial solution is Lyapunov stable.

Exercise 2.114. For Hill's equation with period T, if the absolute value of the trace of $\Phi(T)$, where $\Phi(t)$ is the principal fundamental matrix at $t = 0$, is strictly less than two, show that there are no solutions of period T or $2T$. On the other hand, if the absolute value of the trace of $\Phi(T)$ is two, show that there is such a solution. Note that this property characterizes the boundary between the stable and unstable solutions.

Exercise 2.115. Prove: If $a(t)$ is an even T-periodic function, then Hill's equation has a fundamental set of solutions such that one solution is even and one is odd.

Exercise 2.116. Prove Theorem 2.108. Hint: If Hill's equation has an unbounded solution, then there is a real solution $t \mapsto x(t)$ and a real Floquet multiplier such that $x(t + T) = \lambda x(t)$. Define a new function $t \mapsto u(t)$ by

$$u(t) := \frac{\dot{x}(t)}{x(t)},$$

and show that u is a solution of the Riccati equation

$$\dot{u} = -a(t) - u^2.$$

Use the Riccati equation to prove that the solution x has at least one zero in the interval $[0, T]$. Also, show that x has two distinct zeros on some interval whose length does not exceed T. Finally, use the following proposition to finish the proof. If f is a smooth function on the finite interval $[\alpha, \beta]$ such that $f(\alpha) = 0$, $f(\beta) = 0$, and such that f is positive on the open interval (α, β), then

$$(\beta - \alpha) \int_\alpha^\beta \frac{|f''(t)|}{f(t)}\, dt > 4.$$

To prove this proposition, first suppose that f attains its maximum at γ and show that

$$\frac{4}{\beta-\alpha} \le \frac{1}{\gamma-\alpha} + \frac{1}{\beta-\gamma} = \frac{1}{f(\gamma)}\Big(\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha} - \frac{f(\beta)-f(\gamma)}{\beta-\gamma}\Big).$$

Then, use the mean value theorem and the fundamental theorem of calculus to complete the proof.

Exercise 2.117. Prove: If $t \mapsto a(t)$ is negative, then the Hill's equation $\ddot{x} + a(t)x = 0$ has an unbounded solution. Hint: Multiply by x and integrate by parts.

2.4.3 Periodic Orbits of Linear Systems

In this section we will consider the existence and stability of periodic solutions of the time-periodic system

$$\dot{x} = A(t)x + b(t), \qquad x \in \mathbb{R}^n \tag{2.35}$$

where $t \mapsto A(t)$ is a T-periodic matrix function and $t \mapsto b(t)$ is a T-periodic vector function.

Theorem 2.118. *If the number one is not a characteristic multiplier of the T-periodic homogeneous system $\dot{x} = A(t)x$, then (2.35) has at least one T-periodic solution.*

Proof. Let us show first that if $t \mapsto x(t)$ is a solution of system (2.35) and $x(0) = x(T)$, then this solution is T-periodic. Define $y(t) := x(t+T)$. Note that $t \mapsto y(t)$ is a solution of (2.35) and $y(0) = x(0)$. Thus, by the uniqueness theorem $x(t+T) = x(t)$ for all $t \in \mathbb{R}$.

If $\Phi(t)$ is the principal fundamental matrix solution of the homogeneous system at $t = 0$, then, by the variation of parameters formula,

$$x(T) = \Phi(T)x(0) + \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds.$$

Therefore, $x(T) = x(0)$ if and only if

$$(I - \Phi(T))x(0) = \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds.$$

This equation for $x(0)$ has a solution whenever the number one is not an eigenvalue of $\Phi(T)$. (Note that the map $x(0) \mapsto x(T)$ is the Poincaré map. Thus, our periodic solution corresponds to a fixed point of the Poincaré map).

By Floquet's theorem, there is a matrix B such that the monodromy matrix is given by

$$\Phi(T) = e^{TB}.$$

In other words, by the hypothesis, the number one is not an eigenvalue of $\Phi(T)$. □

Corollary 2.119. *If $A(t) = A$, a constant matrix such that A is infinitesimally hyperbolic (no eigenvalues on the imaginary axis), then the differential equation (2.35) has at least one T-periodic solution.*

Proof. The monodromy matrix e^{TA} does not have 1 as an eigenvalue. □

Exercise 2.120. Discuss the uniqueness of the T-periodic solutions of the system (2.35). Also, using Theorem 2.89, discuss the stability of the T-periodic solutions.

In system (2.35) if $b = 0$, then the trivial solution is a T-periodic solution. The next theorem states a general sufficient condition for the existence of a T-periodic solution.

Theorem 2.121. *If the T-periodic system (2.35) has a bounded solution, then it has a T-periodic solution.*

Proof. Consider the principal fundamental matrix solution $\Phi(t)$ at $t = 0$ of the homogeneous system corresponding to the differential equation (2.35). By the variation of parameters formula, we have the equation

$$x(T) = \Phi(T)x(0) + \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds.$$

Also, by Theorem 2.82, there is a constant matrix B such that $\Phi(T) = e^{TB}$. Thus, the stroboscopic Poincaré map P is given by

$$\begin{aligned} P(\xi) &:= \Phi(T)\xi + \Phi(T)\int_0^T \Phi^{-1}(s)b(s)\,ds \\ &= e^{TB}\Big(\xi + \int_0^T \Phi^{-1}(s)b(s)\,ds\Big). \end{aligned}$$

If the solution with initial condition $x(0) = \xi_0$ is bounded, then the sequence $\{P^j(\xi_0)\}_{j=0}^{\infty}$ is bounded. Also, P is an affine map; that is, $P(\xi) = L\xi + y$ where $L = e^{TB} = \Phi(T)$ is a real invertible linear map and y is an element of $\mathbb{R}^n$.

Note that if there is a point $x \in \mathbb{R}^n$ such that $P(x) = x$, then the system (2.35) has a periodic orbit. Thus, if we assume that there are no periodic orbits, then the equation

$$(I - L)\xi = y$$

has no solution ξ. In other words, y is not in the range $\mathcal{R}$ of the operator $I - L$.

There is some vector $v \in \mathbb{R}^n$ such that v is orthogonal to $\mathcal{R}$ and the inner product $\langle v, y\rangle$ does not vanish. Moreover, because v is orthogonal to the range, we have

$$\langle (I-L)\xi, v\rangle = 0$$

for each $\xi \in \mathbb{R}^n$, and therefore

$$\langle \xi, v\rangle = \langle L\xi, v\rangle. \tag{2.36}$$

Using the representation $P(\xi) = L\xi + y$ and an induction argument, it is easy to prove that if j is a nonnegative integer, then $P^j(\xi_0) = L^j\xi_0 + \sum_{k=0}^{j-1} L^k y$. By taking the inner product with v and repeatedly applying the reduction formula (2.36), we have

$$\langle P^j(\xi_0), v\rangle = \langle \xi_0, v\rangle + (j-1)\langle y, v\rangle.$$

Moreover, because $\langle v, y\rangle \neq 0$, it follows immediately that

$$\lim_{j\to\infty} \langle P^j(\xi_0), v\rangle = \infty,$$

and therefore the sequence $\{P^j(\xi_0)\}_{j=0}^{\infty}$ is unbounded, in contradiction. $\square$

2.4.4 Stability of Periodic Orbits

Consider a (nonlinear) autonomous system of differential equations on $\mathbb{R}^n$ given by $\dot{u} = f(u)$ with a periodic orbit Γ. Also, for each $\xi \in \mathbb{R}^n$, define the vector function $t \mapsto u(t, \xi)$ to be the solution of this system with the initial condition $u(0, \xi) = \xi$.

If $p \in \Gamma$ and $\Sigma' \subset \mathbb{R}^n$ is a section transverse to $f(p)$ at p, then, as a corollary of the implicit function theorem, there is an open set $\Sigma \subseteq \Sigma'$ and a function $T : \Sigma \to \mathbb{R}$, the time of first return to Σ', such that for each $\sigma \in \Sigma$, we have $u(T(\sigma), \sigma) \in \Sigma'$. The map $\mathcal{P}$, given by $\sigma \mapsto u(T(\sigma), \sigma)$, is the Poincaré map corresponding to the Poincaré section Σ.

The Poincaré map is defined only on Σ, a manifold contained in $\mathbb{R}^n$. It is convenient to avoid choosing local coordinates on Σ. Thus, we will view the elements in Σ also as points in the ambient space $\mathbb{R}^n$. In particular, if $v \in \mathbb{R}^n$ is tangent to Σ at p, then the derivative of $\mathcal{P}$ in the direction v is given by

$$D\mathcal{P}(p)v = (dT(p)v)f(p) + u_\xi(T(p), p)v. \tag{2.37}$$

The next proposition relates the spectrum of $D\mathcal{P}(p)$ to the Floquet multipliers of the first variational equation

$$\dot{W} = Df(u(t,p))W.$$

Proposition 2.122. *If Γ is a periodic orbit and $p \in \Gamma$, then the union of the set of eigenvalues of the derivative of a Poincaré map at $p \in \Gamma$ and the singleton set $\{1\}$ is the same as the set of characteristic multipliers of the first variational equation along Γ. In particular, zero is not an eigenvalue.*

Proof. Recall that $t \mapsto u_\xi(t, \xi)$ is the principal fundamental matrix solution at $t = 0$ of the first variational equation and, since

$$\frac{d}{dt} f(u(t, \xi)) = Df(u(t, \xi) u_t(t, \xi) = Df(u(t, \xi) f(u(t, \xi)),$$

the vector function $t \mapsto f(u(t, \xi))$ is the solution of the variational equation with the initial condition $W(0) = f(\xi)$. In particular,

$$u_\xi(T(p), p) f(p) = f(u(T(p), p)) = f(p),$$

and therefore $f(p)$ is an eigenvector of the linear transformation $u_\xi(T(p), p)$ with eigenvalue the number one.

Since Σ is transverse to $f(p)$, there is a basis of $\mathbb{R}^n$ of the form

$$f(p), s_1, \dots, s_{n-1}$$

with s_i tangent to Σ at p for each $i = 1, \dots, n-1$. It follows that the matrix $u_\xi(T(p), p)$ has block form, relative to this basis, given by

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

where a is $1 \times (n-1)$ and b is $(n-1) \times (n-1)$. Moreover, each $v \in \mathbb{R}^n$ that is tangent to Σ at p has block form (the transpose of) $(0, v_\Sigma)$. As a result, we have the equality

$$u_\xi(T(p), p) v = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ v_\Sigma \end{pmatrix}.$$

The range of $D\mathcal{P}(p)$ is tangent to Σ at p. Thus, using equation (2.37) and the block form of $u_\xi(T(p), p)$, it follows that

$$D\mathcal{P}(p) v = \begin{pmatrix} dT(p)v + a v_\Sigma \\ b v_\Sigma \end{pmatrix} = \begin{pmatrix} 0 \\ b v_\Sigma \end{pmatrix}.$$

In other words, the derivative of the Poincaré map may be identified with b and the differential of the return time map with $-a$. In particular, the eigenvalues of the derivative of the Poincaré map coincide with the eigenvalues of b. □

Exercise 2.123. Prove that the characteristic multipliers of the first variational equation along a periodic orbit do not depend on the choice of $p \in \Gamma$.

Most of the rest of this section is devoted to a proof of the following fundamental theorem.

Theorem 2.124. *Suppose that Γ is a periodic orbit for the autonomous differential equation $\dot{u} = f(u)$ and $\mathcal{P}$ is a corresponding Poincaré map defined on a Poincaré section Σ such that $p \in \Gamma \cap \Sigma$. If the eigenvalues of the derivative $D\mathcal{P}(p)$ are inside the unit circle in the complex plane, then Γ is asymptotically stable.*

There are several possible proofs of this theorem. The approach used here is adapted from [123].

To give a complete proof of Theorem 2.124, we will require several preliminary results. Our first objective is to show that the point p is an asymptotically stable fixed point of the dynamical system defined by the Poincaré map on Σ.

Let us begin with a useful simple replacement of the Jordan normal form theorem that is adequate for our purposes here (see [129]).

Proposition 2.125. *An $n \times n$ (possibly complex) matrix A is similar to an upper triangular matrix whose diagonal elements are the eigenvalues of A.*

Proof. Let v be a nonzero eigenvector of A corresponding to the eigenvalue λ. The vector v can be completed to a basis of $\mathbb{C}^n$ that defines a matrix Q partitioned by the corresponding column vectors $Q := [v, y_1, \ldots, y_{n-1}]$. Moreover, Q is invertible and

$$[Q^{-1}v, Q^{-1}y_1, \ldots, Q^{-1}y_{n-1}] = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the usual basis elements.

Note that

$$\begin{aligned} Q^{-1}AQ &= Q^{-1}[\lambda v, Ay_1, \ldots, Ay_{n-1}] \\ &= [\lambda \mathbf{e}_1, Q^{-1}Ay_1, \ldots, Q^{-1}Ay_{n-1}]. \end{aligned}$$

In other words, the matrix $Q^{-1}AQ$ is given in block form by

$$Q^{-1}AQ = \begin{pmatrix} \lambda & * \\ 0 & \tilde{A} \end{pmatrix}$$

where $\tilde{A}$ is an $(n-1) \times (n-1)$ matrix. In particular, this proves the theorem for all 2×2 matrices.

By induction, there is an $(n-1) \times (n-1)$ matrix $\tilde{R}$ such that $\tilde{R}^{-1}\tilde{A}\tilde{R}$ is upper triangular. The matrix $(QR)^{-1}AQR$ where

$$R = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{R} \end{pmatrix}$$

is an upper triangular matrix with the eigenvalues of A as its diagonal elements, as required. $\square$

Let $\rho(A)$ denote the *spectral radius* of A, that is, the maximum modulus of the eigenvalues of A.

Proposition 2.126. *Suppose that A is an $n \times n$ matrix. If $\epsilon > 0$, then there is a norm on $\mathbb{C}^n$ such that $\|A\|_\epsilon < \rho(A) + \epsilon$. If A is a real matrix, then the restriction of the "ϵ-norm" to $\mathbb{R}^n$ is a norm on $\mathbb{R}^n$ with the same property.*

Proof. The following proof is adapted from [129]. By Proposition 2.125, there is a matrix Q such that

$$QAQ^{-1} = D + N$$

where D is diagonal with the eigenvalues of A as its diagonal elements, and N is upper triangular with each of its diagonal elements equal to zero.

Let $\mu > 0$, and define a new diagonal matrix S with diagonal elements

$$1, \mu^{-1}, \mu^{-2}, \ldots, \mu^{1-n}.$$

A computation shows that

$$S(D + N)S^{-1} = D + SNS^{-1}.$$

Also, it is easy to show—by writing out the formulas for the components—that every element of the matrix SNS^{-1} is $O(\mu)$.

Define a norm on $\mathbb{C}^n$, by the formula

$$|v|_\mu := |SQv| = \langle SQv, SQv \rangle$$

where the angle brackets on the right hand side denote the usual Euclidean inner product on $\mathbb{C}^n$. It is easy to verify that this procedure indeed defines a norm on $\mathbb{C}^n$ that depends on the parameter μ.

Post multiplication by SQ of both sides of the equation

$$SQAQ^{-1}S^{-1} = D + SNS^{-1}$$

yields the formula

$$SQA = (D + SNS^{-1})SQ.$$

Using this last identity we have that

$$|Av|_\mu^2 = |SQAv|^2 = |(D + SNS^{-1})SQv|^2.$$

Let us define $w := SQv$ and then expand the last norm into inner products to obtain

$$\begin{aligned} |Av|_\mu^2 = {} & \langle Dw, Dw \rangle + \langle SNS^{-1}w, Dw \rangle \\ & + \langle Dw, SNS^{-1}w \rangle + \langle SNS^{-1}w, SNS^{-1}w \rangle. \end{aligned}$$

A direct estimate of the first inner product together with an application of the Schwarz inequality to each of the other inner products yields the following estimate:

$$|Av|_\mu^2 \leq (\rho^2(A) + O(\mu))|w|^2.$$

Moreover, we have that $|v|_\mu = |w|$. In particular, if $|v|_\mu = 1$ then $|w| = 1$, and it follows that

$$\|A\|_\mu^2 \leq \rho^2(A) + O(\mu).$$

Thus, if $\mu > 0$ is sufficiently small, then $\|A\|_\mu < \rho(A) + \epsilon$, as required. □

Corollary 2.127. *If all the eigenvalues of the $n \times n$ matrix A are inside the unit circle in the complex plane, then there is an "adapted norm" and a number λ, with $0 < \lambda < 1$, such that $|Av|_a < \lambda |v|_a$ for all vectors v, real or complex. In particular A is a contraction with respect to the adapted norm. Moreover, for each norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, there is a positive number C such that $|A^n v| \leq C\lambda^n |v|$ for all nonnegative integers n.*

Proof. Under the hypothesis, we have $\rho(A) < 1$; thus, there is a number λ such that $\rho(A) < \lambda < 1$. Using Proposition 2.126, there is an adapted norm so that $\|A\|_a < \lambda$. This proves the first part of the corollary. To prove the second part, recall that all norms on a finite dimensional space are equivalent. In particular, there are positive numbers C_1 and C_2 such that

$$C_1|v| \leq |v|_a \leq C_2|v|$$

for all vectors v. Thus, we have

$$C_1|A^n v| \leq |A^n v|_a \leq |A|_a^n |v|_a \leq C_2 \lambda^n |v|.$$

After dividing both sides of the last inequality by $C_1 > 0$, we obtain the desired estimate. □

We are now ready to return to the dynamics of the Poincaré map $\mathcal{P}$ defined above. Recall that Γ is a periodic orbit for the differential equation $\dot{u} = f(u)$ and $\mathcal{P} : \Sigma \to \Sigma'$ is defined by $\mathcal{P}(\sigma) = u(T(\sigma), \sigma)$ where T is the return time function. Also, we have that $p \in \Gamma \cap \Sigma$.

Lemma 2.128. *Suppose that $V \subseteq \mathbb{R}^n$ is an open set with compact closure $\bar{V}$ such that $\Gamma \subset V$ and $\bar{V}$ is contained in the domain of the function f. If $t_* \geq 0$, then there is an open set $W \subseteq V$ that contains Γ and is such that, for each point $\xi \in W$, the solution $t \mapsto u(t, \xi)$ is defined and stays in V on the interval $0 \leq t \leq t_*$. Moreover, if ξ and ν are both in W and $0 \leq t \leq t_*$, then there is a number $L > 0$ such that*

$$|u(t, \xi) - u(t, \nu)| < |\xi - \nu| e^{Lt_*}.$$

Proof. Note that $\bar{V}$ is a compact subset of the domain of the function f. By Lemma 2.75, f is globally Lipschitz on V with a Lipschitz constant $L > 0$. Also, there is a minimum *positive* distance m from the boundary of V to Γ.

An easy application of Gronwall's inequality can be used to show that if $\xi, \nu \in V$, then

$$|u(t, \xi) - u(t, \nu)| \leq |\xi - \nu| e^{Lt} \tag{2.38}$$

for all t such that both solutions are defined on the interval $[0, t]$.

Define the set

$$W_q := \{\xi \in \mathbb{R}^n : |\xi - q| e^{Lt_*} < m\}$$

and note that W_q is open. If $\xi \in W_q$, then

$$|\xi - q| < m e^{-Lt_*} < m.$$

Thus, it follows that $W_q \subseteq V$.

Using the extension theorem (Theorem 1.263), it follows that if $\xi \in W_q$, then the interval of existence of the solution $t \mapsto u(t, \xi)$ can be extended as long as the orbit stays in the compact set $\bar{V}$. The point q is on the periodic orbit Γ. Thus, the solution $t \to u(t, q)$ is defined for all $t \geq 0$. Using the definition of W_q and an application of the inequality (2.38) to the solutions starting at ξ and q, it follows that the solution $t \mapsto u(t, \xi)$ is defined and stays in V on the interval $0 \leq t \leq t_*$.

The union $W := \bigcup_{q \in \Gamma} W_q$ is an open set in V containing Γ with the property that all solutions starting in W remain in V at least on the time interval $0 \leq t \leq t_*$. $\square$

Define the distance of a point $q \in \mathbb{R}^n$ to a set $S \subseteq \mathbb{R}^n$ by

$$\operatorname{dist}(q, S) = \inf_{x \in S} |q - x|$$

where the norm on the right hand side is the usual Euclidean norm. Similarly, the (minimum) distance between two sets is defined as

$$\operatorname{dist}(A, B) = \inf\{|a - b| : a \in A, b \in B\}.$$

(Warning: dist is not a metric.)

Proposition 2.129. *If* $\sigma \in \Sigma$ *and if* $\lim_{n \to \infty} \mathcal{P}^n(\sigma) = p$, *then*

$$\lim_{t \to \infty} \operatorname{dist}(u(t, \sigma), \Gamma) = 0.$$

Proof. Let $\epsilon > 0$ be given and let Σ_0 be an open subset of Σ such that $p \in \Sigma_0$ and such that $\bar{\Sigma}_0$, the closure of Σ_0, is a compact subset of Σ. The return time map T is continuous; hence, it is uniformly bounded on the set $\bar{\Sigma}_0$, that is,

$$\sup\{T(\eta) : \eta \in \bar{\Sigma}_0\} = T^* < \infty.$$

Let V be an open subset of $\mathbb{R}^n$ with compact closure $\bar{V}$ such that $\Gamma \subset V$ and $\bar{V}$ is contained in the domain of f. By Lemma 2.128, there is an open set $W \subseteq V$ such that $\Gamma \subset W$ and such that, for each $\xi \in W$, the solution starting at ξ remains in V on the interval $0 \le s \le T^*$.

Choose $\delta > 0$ so small that the set

$$\Sigma_\delta := \{\eta \in \Sigma : |\eta - p| < \delta\}$$

is contained in $W \cap \Sigma_0$, and such that

$$|\eta - p|e^{LT^*} < \min\{m, \epsilon\}$$

for all $\eta \in \Sigma_\delta$. By Lemma 2.128, if $\eta \in \Sigma_\delta$, then, for $0 \le s \le T^*$, we have that

$$|u(s, \eta) - u(s, p)| < \epsilon.$$

By the hypothesis, there is some integer $N > 0$ such that $\mathcal{P}^n(\sigma) \in \Sigma_\delta$ whenever $n \ge N$.

Using the group property of the flow, let us note that

$$\mathcal{P}^n(\sigma) = u(\sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)), \sigma).$$

Moreover, if $t \ge \sum_{j=0}^{N-1} T(\mathcal{P}^j(\sigma))$, then there is some integer $n \ge N$ and some number s such that $0 \le s \le T^*$ and

$$t = \sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)) + s.$$

For this t, we have $\mathcal{P}^n(\sigma) \in \Sigma_\delta$ and

$$\begin{aligned}
\operatorname{dist}(u(t, \sigma), \Gamma) &= \min_{q \in \Gamma} |u(t, \sigma) - q| \\
&\le |u(t, \sigma) - u(s, p)| \\
&= |u(s, u(\sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)), \sigma)) - u(s, p)| \\
&= |u(s, P^n(\sigma)) - u(s, p)|.
\end{aligned}$$

It follows that $\operatorname{dist}(u(t, \sigma), \Gamma) < \epsilon$ whenever $t \ge \sum_{j=0}^{N-1} T(\mathcal{P}^j(\sigma))$. In other words,

$$\lim_{t \to \infty} \operatorname{dist}(u(t, \sigma), \Gamma) = 0,$$

as required. □

We are now ready for the proof of Theorem 2.124.

Proof. Suppose that V is a neighborhood of Γ. We must prove that there is a neighborhood U of Γ such that $U \subseteq V$ with the additional property that every solution of $\dot{u} = f(u)$ that starts in U stays in V and is asymptotic to Γ.

We may as well assume that V has compact closure $\bar{V}$ and $\bar{V}$ is contained in the domain of f. Then, by Lemma 2.128, there is an open set W that contains Γ and is contained in the closure of V with the additional property that every solution starting in W exists and stay in V on the time interval $0 \leq t \leq 2\tau$ where $\tau := T(p)$ is the period of Γ.

Also, let us assume without loss of generality that our Poincaré section Σ is a subset of a hyperplane Σ' and that the coordinates on Σ' are chosen so that p lies at the origin. By our hypothesis, the linear transformation $D\mathcal{P}(0) : \Sigma' \to \Sigma'$ has its spectrum inside the unit circle in the complex plane. Thus, by Corollary 2.127, there is an adapted norm on Σ' and a number λ with $0 < \lambda < 1$ such that $\|D\mathcal{P}(0)\| < \lambda$.

Using the continuity of the map $\sigma \to D\mathcal{P}(\sigma)$, the return time map, and the adapted norm, there is an open ball $\Sigma_0 \subseteq \Sigma$ centered at the origin such that $\Sigma_0 \subset W$, the return time map T restricted to Σ_0 is bounded by 2τ, and $\|D\mathcal{P}(\sigma)\| < \lambda$ whenever $\sigma \in \Sigma_0$. Moreover, using the mean value theorem, it follows that

$$|\mathcal{P}(\sigma)| = |\mathcal{P}(\sigma) - \mathcal{P}(0)| < \lambda|\sigma|,$$

whenever $\sigma \in \Sigma_0$. In particular, if $\sigma \in \Sigma_0$, then $\mathcal{P}(\sigma) \in \Sigma_0$.

Let us show that all solutions starting in Σ_0 are defined for all positive time. To see this, consider $\sigma \in \Sigma_0$ and note that, by our construction, the solution $t \mapsto u(t, \sigma)$ is defined for $0 \leq t \leq T(\sigma)$ because $T(\sigma) < 2\tau$. We also have that $u(T(\sigma), \sigma) = \mathcal{P}(\sigma) \in \Sigma_0$. Thus, the solution $t \mapsto u(t, \sigma)$ can be extended beyond the time $T(\sigma)$ by applying the same reasoning to the solution $t \to u(t, \mathcal{P}(\sigma)) = u(t + u(T\sigma), \sigma))$. This procedure can be extended indefinitely, and thus the solution $t \to u(t, \sigma)$ can be extended for all positive time.

Define $U := \{u(t, \sigma) : \sigma \in \Sigma_0 \text{ and } t > 0\}$. Clearly, $\Gamma \subset U$ and also every solution that starts in U stays in U for all $t \geq 0$. We will show that U is open. To prove this fact, let $\xi := u(t, \sigma) \in U$ with $\sigma \in \Sigma_0$. If we consider the restriction of the flow given by $u : (0, \infty) \times \Sigma_0 \to U$, then, using the same idea as in the proof of the rectification lemma (Lemma 1.120), it is easy to see that the derivative $Du(t, \sigma)$ is invertible. Thus, by the inverse function theorem (Theorem 1.121), there is an open set in U at ξ diffeomorphic to a product neighborhood of (t, σ) in $(0, \infty) \times \Sigma_0$. Thus, U is open.

To show that $U \subseteq V$, let $\xi := u(t, \sigma) \in U$ with $\sigma \in \Sigma_0$. There is some integer $n \geq 0$ and some number s such that

$$t = \sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)) + s$$

where $0 \leq s < T(\mathcal{P}^n(\sigma)) < 2\tau$. In particular, we have that $\xi = u(s, \mathcal{P}^n(\sigma))$. But since $0 \leq s < 2\tau$ and $\mathcal{P}^n(\sigma) \in W$ it follows that $\xi \in V$.

Finally, for this same $\xi \in U$, we have as an immediate consequence of Proposition 2.129 that $\lim_{t\to\infty} \operatorname{dist}(u(t, \mathcal{P}^n(\xi)), \Gamma) = 0$. Moreover, for each $t \geq 0$, we also have that

$$\operatorname{dist}(u(t,\xi),\Gamma) = \operatorname{dist}(u(t, u(s, \mathcal{P}^n(\xi))), \Gamma) = \operatorname{dist}(u(s+t, \mathcal{P}^n(\xi)), \Gamma).$$

It follows that $\lim_{t\to\infty} \operatorname{dist}(u(t,\xi),\Gamma) = 0$, as required. □

A useful application of our results can be made for a periodic orbit Γ of a differential equation defined on the plane. In fact, there are exactly two characteristic multipliers of the first variational equation along Γ. Since one of these characteristic multipliers must be the number one, the product of the characteristic multipliers is the eigenvalue of the derivative of every Poincaré map defined on a section transverse to Γ. Because the determinant of a matrix is the product of its eigenvalues, an application of Liouville's formula proves the following proposition.

Proposition 2.130. *If Γ is a periodic orbit of period ν of the autonomous differential equation $\dot{u} = f(u)$ on the plane, and if $\mathcal{P}$ is a Poincaré map defined at $p \in \Gamma$, then, using the notation of this section, the eigenvalue λ_Γ of the derivative of $\mathcal{P}$ at p is given by*

$$\lambda_\Gamma = \det u_\xi(T(p), p) = e^{\int_0^\nu \operatorname{div} f(u(t,p))\, dt}.$$

In particular, if $\lambda_\Gamma < 1$ then Γ is asymptotically stable, whereas if $\lambda_\Gamma > 1$ then Γ is unstable.

The flow near an attracting limit cycle is very well understood. A next proposition states that the orbits of points in the basin of attraction of the limit cycle are "asymptotically periodic."

Proposition 2.131. *Suppose that Γ is an asymptotically stable periodic orbit with period T. There is a neighborhood V of Γ such that if $\xi \in V$, then $\lim_{t\to\infty} |u(t+T,\xi) - u(t,\xi)| = 0$ where $|\ |$ is an arbitrary norm on $\mathbb{R}^n$. (In this case, the point ξ is said to have asymptotic period T.)*

Proof. By Lemma 2.128, there is an open set W such that $\Gamma \subset W$ and the function $\xi \mapsto u(T,\xi)$ is defined for each $\xi \in W$. Using the continuity of this function, there is a number $\delta > 0$ such that $\delta < \epsilon/2$ and

$$|u(T,\xi) - u(T,\eta)| < \frac{\epsilon}{2}$$

whenever $\xi, \eta \in W$ and $|\xi - \eta| < \delta$.

By the hypothesis, there is a number T^* so large that $\operatorname{dist}(u(t,\xi), \Gamma) < \delta$ whenever $t \geq T^*$. In particular, for each $t \geq T^*$, there is some $q \in \Gamma$ such that $|u(t,\xi) - q| < \delta$. Using this fact and the group property of the flow, we have that

$$\begin{aligned} |u(t+T,\xi) - u(t,\xi)| &\leq |u(T, u(t,\xi)) - u(T,q)| + |q - u(t,\xi)| \\ &\leq \frac{\epsilon}{2} + \delta < \epsilon \end{aligned}$$

whenever $t \geq T^*$. Thus, $\lim_{t\to\infty} |u(t+T,\xi) - u(t,\xi)| = 0$, as required. □

A periodic orbit can be asymptotically stable without being hyperbolic. In fact, it is easy to construct a limit cycle in the plane that is asymptotically stable whose Floquet multiplier is the number one. By the last proposition, points in the basin of attraction of such an attracting limit cycle have asymptotic periods equal to the period of the limit cycle. But, if the periodic orbit is hyperbolic, then a stronger result is true: Not only does each point in the basin of attraction have an asymptotic period, each such point has an asymptotic phase. This is the content of the next result.

Theorem 2.132. *If Γ is an attracting hyperbolic periodic orbit, then there is a neighborhood V of Γ such that for each $\xi \in V$ there is some $q \in \Gamma$ such that $\lim_{t\to\infty} |u(t,\xi) - u(t,q)| = 0$. (In this case, ξ is said to have asymptotic phase q.)*

Proof. Let Σ be a Poincaré section at $p \in \Gamma$ with compact closure, return map $\mathcal{P}$, and return-time map T. Without loss of generality, we will assume that for each $\sigma \in \Sigma$ we have (1) $\lim_{n\to\infty} \mathcal{P}^n(\sigma) = p$, (2) $T(\sigma) < 2T(p)$, and (3) $\|DT(\sigma)\| < 2\|DT(p)\|$.

By the hyperbolicity hypothesis, the spectrum of $D\mathcal{P}(p)$ is inside the unit circle; therefore, there are numbers C and λ such that $C > 0$, $0 < \lambda < 1$ and

$$|p - \mathcal{P}^n(\sigma)| < C\lambda^n \|p - \sigma\|.$$

Let

$$K := \frac{2C\|DT(p)\|}{1-\lambda} \sup_{\sigma\in\bar{\Sigma}} \|p - \sigma\| + 3T(p).$$

Using the implicit function theorem, it is easy to construct a neighborhood V of Γ such that for each $\xi \in V$, there is a number $t_\xi \geq 0$ with $\sigma_\xi := u(t_\xi, \xi) \in \Sigma$. Moreover, using Lemma 2.128, we can choose V such that every solution with initial point in V is defined at least on the time interval $-K \leq t \leq K$. Indeed, by the asymptotic stability of Γ, there is a neighborhood V of Γ such that every solution starting in V is defined for all positive time. If we redefine V to be the image of V under the flow for time K, then every solution starting in V is defined at least on the time interval $-K \leq t \leq K$.

We will show that if $\sigma_\xi \in \Sigma$, then there is a point $q_\xi \in \Gamma$ such that

$$\lim_{t\to\infty} |u(t,\sigma_\xi) - u(t,q_\xi)| = 0.$$

Using this fact, it follows that if $r := u(-t_\xi, q_\xi)$, then

$$\begin{aligned}\lim_{t\to\infty} |u(t,\xi) - u(t,r)| &= \lim_{t\to\infty} |u(t-t_\xi, u(t_\xi,\xi)) - u(t-t_\xi, q_\xi)| \\ &= \lim_{t\to\infty} |u(t-t_\xi,\sigma_\xi) - u(t-t_\xi,q_\xi)| = 0.\end{aligned}$$

Thus, it suffices to prove the theorem for a point $\sigma \in \Sigma$.

Given $\sigma \in \Sigma$, define

$$s_n := nT(p) - \sum_{j=0}^{n-1} T(\mathcal{P}^j(\sigma)).$$

Note that

$$(n+1)T(p) - nT(p) = T(\mathcal{P}^n(\sigma)) + s_{n+1} - s_n,$$

and, as a result,

$$|s_{n+1} - s_n| = |T(p) - T(\mathcal{P}^n(\sigma))| \leq 2\|DT(p)\|\|p - \mathcal{P}^n(\sigma)\|.$$

Hence,

$$|s_{n+1} - s_n| < 2\|DT(p)\|C\|p-\sigma\|\lambda^n$$

whenever $n \geq 0$.

Because $s_n = s_1 + \sum_{j=1}^{n-1}(s_{j+1} - s_j)$ and

$$\sum_{j=1}^{n-1} |s_{j+1} - s_j| < 2C\|DT(p)\|\|p-\sigma\| \sum_{j=1}^{n-1} \lambda^j < 2C\|DT(p)\| \frac{\|p-\sigma\|}{1-\lambda},$$

the series $\sum_{j=1}^{\infty}(s_{j+1} - s_j)$ is absolutely convergent—its absolute partial sums form an increasing sequence that is bounded above. Thus, in fact, there is a number s such that $\lim_{n\to\infty} s_n = s$. Also, the sequence $\{s_n\}_{n=1}^{\infty}$ is uniformly bounded; that is,

$$|s_n| \leq |s_1| + 2C\|DT(p)\| \frac{\|p-\sigma\|}{1-\lambda} \leq K.$$

Hence, the absolute value of its limit $|s|$ is bounded by the same quantity.

Let $\epsilon > 0$ be given. By the compactness of its domain, the function

$$u : [-K, K] \times \bar{\Sigma} \to \mathbb{R}^n$$

is uniformly continuous. In particular, there is a number $\delta > 0$ such that if (t_1, σ_1) and (t_2, σ_2) are both in the domain and if $|t_1 - t_2| + |\sigma_1 - \sigma_2| < \delta$, then

$$|u(t_1, \sigma_1) - u(t_2, \sigma_2)| < \epsilon.$$

In view of the equality,

$$u(nT(p), \sigma) = u(s_n, \mathcal{P}^n(\sigma)),$$

which follows from the definition of s_n, we have

$$|u(nT(p), \sigma) - u(s, p)| = |u(s_n, \mathcal{P}^n(\sigma)) - u(s, p)|.$$

Since for sufficiently large n,

$$|s_n - s| + |\mathcal{P}^n(\sigma) - p| < \epsilon,$$

it follows that

$$\lim_{n \to \infty} |u(nT(p), \sigma) - u(s, p)| = 0.$$

Also, for each $t \geq 0$, there is an integer $n \geq 0$ and a number $s(t)$ such that $0 \leq s(t) < T(p)$ and $t = nT(p) + s(t)$. Using this fact, we have the equation

$$|u(t, \sigma) - u(t, u(s, p))| = |u(s(t), u(nT(p), \sigma)) - u(s(t), u(nT(p), u(s, p))|.$$

Also, because $q := u(s, p) \in \Gamma$ and Lemma 2.128, there is a constant $L > 0$ such that

$$\begin{aligned}|u(t, \sigma) - u(t, q)| &= |u(s(t), u(nT(p), \sigma)) - u(s(t), q))| \\ &\leq |u(nT(p), \sigma) - q|e^{LT(p)}.\end{aligned}$$

By passing to the limit as $n \to \infty$, we obtain the desired result. □

Necessary and sufficient conditions for the existence of asymptotic phase are known (see [47, 77]). An alternate proof of Theorem 2.132 is given in [47].

Exercise 2.133. Find a periodic solution of the system

$$\begin{aligned}\dot{x} &= x - y - x(x^2 + y^2), \\ \dot{y} &= x + y - y(x^2 + y^2), \\ \dot{z} &= -z,\end{aligned}$$

and determine its stability type. In particular, compute the Floquet multipliers for the monodromy matrix associated with the periodic orbit [128, p. 120].

Exercise 2.134. (a) Find an example of a planar system with a limit cycle such that some nearby solutions do not have an asymptotic phase. (b) Contrast and compare the asymptotic phase concept for the following planar systems that are defined in the punctured plane in polar coordinates:

1. $\dot{r} = r(1-r), \quad \dot{\theta} = r,$
2. $\dot{r} = r(1-r)^2, \quad \dot{\theta} = r,$
3. $\dot{r} = r(1-r)^n, \quad \dot{\theta} = r.$

Exercise 2.135. Suppose that $v \neq 0$ is an eigenvector for the monodromy operator with associated eigenvalue λ_Γ as in Proposition 2.130. If $\lambda_\Gamma \neq 1$, then v and $f(p)$ are independent vectors that form a basis for $\mathbb{R}^2$. The monodromy operator expressed in this basis is diagonal. (a) Express the operators a and b defined in the proof of Proposition 2.122 in this basis. (b) What can you say about the derivative of the transit time map along a section that is tangent to v at p?

Exercise 2.136. This exercise is adapted from [235]. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and $A := \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ is a regular level set of f. (a) Prove that each bounded component of A is an attracting hyperbolic limit cycle for the differential equation

$$\dot{x} = -f_y - f f_x, \qquad \dot{y} = f_x - f f_y.$$

(b) Prove that the bounded components of A are the only periodic orbits of the system. (c) Draw and explain the phase portrait of the system for the case where

$$f(x, y) = ((x-\epsilon)^2 + y^2 - 1)(x^2 + y^2 - 9).$$

Exercise 2.137. Consider an attracting hyperbolic periodic orbit Γ for an autonomous system $\dot{u} = f(u)$ with flow φ_t, and for each point $p \in \Gamma$, let Γ_p denote the set of all points in the phase space with asymptotic phase p. (a) Construct Γ_p for each p on the limit cycle in the planar system

$$\dot{x} = -y + x(1 - x^2 - y^2), \quad \dot{y} = x + y(1 - x^2 - y^2).$$

(b) Repeat the construction for the planar systems of Exercise 2.134. (c) Prove that $\mathcal{F} := \bigcup_{p\in\Gamma} \Gamma_p$ is an invariant foliation of the phase space in a neighborhood U of Γ. Let us take this to mean that every point in U is in one of the sets in the union $\mathcal{F}$ and the following invariance property is satisfied: If $\xi \in \Gamma_p$ and $s \in \mathbb{R}$, then $\varphi_s(\xi) \in \Gamma_{\varphi_s(p)}$. The second condition states that the flow moves fibers of the foliation (Γ_p is the fiber over p) to fibers of the foliation. (d) Are the fibers of the foliation smooth manifolds?