9 Stability by solution perturbation: Mathieu's equation

Stability or instability of nonlinear systems can often be tested by an approximate procedure which leads to a *linear* equation describing the growth of the difference between the test solution and its neighbours. By Theorem 8.9 the stability or instability of the original system resolves itself into the question of the boundedness or otherwise of the solutions of the linear equation. This 'variational equation' often turns out to have a periodic coefficient (Mathieu's equation) and the properties of such equations are derived in this chapter. The fact that the solutions to be tested are themselves usually known only approximately can also be assimilated into this theory.

9.1 The stability of forced oscillations by solution perturbation

Consider the general n-dimensional autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t). \tag{9.1}$$

The stability of a solution $\boldsymbol{x}^*(t)$ can be reduced to consideration of the zero solution of a related system. Let $\boldsymbol{x}(t)$ be any other solution, and write

$$\boldsymbol{x}(t) = \boldsymbol{x}^*(t) + \boldsymbol{\xi}(t). \tag{9.2}$$

Then $\boldsymbol{\xi}(t)$ represents a **perturbation**, or disturbance, of the original solution: it seems reasonable to see what happens to $\xi(t)$, since the question of stability is whether such (small) disturbances grow or not. Equation (9.1) can be written in the form

$$\dot{\boldsymbol{x}}^* + \dot{\boldsymbol{\xi}} = \boldsymbol{f}(\boldsymbol{x}^*, t) + \{\boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\xi}, t) - \boldsymbol{f}(\boldsymbol{x}^*, t)\}.$$

Since $\boldsymbol{x}^*$ satisfies (9.1), this becomes

$$\dot{\boldsymbol{\xi}} = \boldsymbol{f}(\boldsymbol{x}^* + \boldsymbol{\xi}, t) - \boldsymbol{f}(\boldsymbol{x}^*, t) = \boldsymbol{h}(\boldsymbol{\xi}, t), \tag{9.3}$$

say, since $\boldsymbol{x}^*(t)$ is assumed known. By (9.2), the stability properties of $\boldsymbol{x}^*(t)$ are the same as those of the zero solution of (9.3), $\boldsymbol{\xi}(t) \equiv \boldsymbol{0}$.

The right-hand side of (9.3) may have a linear approximation for small $\boldsymbol{\xi}$, in which case

$$\dot{\boldsymbol{\xi}} = \boldsymbol{h}(\boldsymbol{\xi}, t) \approx \boldsymbol{A}(t)\dot{\boldsymbol{\xi}}. \tag{9.4}$$

Here, $\boldsymbol{A}(t) = \boldsymbol{J}(0, t)$, where $\boldsymbol{J}(\boldsymbol{\xi}, t)$ is the Jacobian matrix (see also Section 8.9) of first partial derivatives given by

$$\boldsymbol{J}[\boldsymbol{\xi}, t] = \left[\frac{\partial h_i(\boldsymbol{\xi}, t)}{\partial \xi_j}\right] \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n).$$

The properties of this linear system may correctly indicate that of the zero solution of the exact system (9.3). The approximation (9.4) is called the **first variational equation**. This process is not rigorous: it is generally necessary to invoke an approximation not only at the stage (9.4), but also in representing $\boldsymbol{x}^*(t)$, which, of course, will not generally be known exactly.

We shall illustrate the procedure in the case of the two-dimensional forced, undamped pendulum-type equation (a form of Duffing's equation)

$$\ddot{x} + x + \varepsilon x^3 = \Gamma \cos \omega t. \tag{9.5}$$

In order to match the notation of the theory of linear systems of Chapter 8 we will express it in the form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -x - \varepsilon x^3 + \Gamma \cos \omega t \end{bmatrix}. \tag{9.6}$$

To obtain the variational equation define $\boldsymbol{\xi} = (\xi, \eta)^{\mathrm{T}}$ by

$$\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{x}^*, \tag{9.7}$$

where

$$\boldsymbol{x}^* = (x^*, y^*)^{\mathrm{T}}$$

and $\boldsymbol{x}^*$ is the solution to be tested. Substitution for x and y from (9.7) into (9.6) gives

$$\dot{\xi} + \dot{x}^* = \eta + y^*,$$
$$\dot{\eta} + \dot{y}^* = -\xi - x^* - \varepsilon(\xi + x^*)^3 + \Gamma \cos \omega t.$$

By neglecting powers of ξ higher than the first, and using the fact that x^*, y^* satisfy (9.6), the system simplifies to

$$\dot{\xi} = \eta, \quad \dot{\eta} = -\xi - 3\varepsilon x^{*2} \xi, \tag{9.8}$$

corresponding to (9.4).

From Section 7.2 we know that there are periodic solutions of (9.5) which are approximately of the form

$$x = a \cos \omega t,$$

where possible real values of the amplitude a are given by the equation

$$\tfrac{3}{4}\varepsilon a^3 - (\omega^2 - 1)a - \Gamma = 0. \tag{9.9}$$

We shall test the stability of one of these solutions by treating it as being sufficiently close to the corresponding exact form of x^* required by (9.8). By eliminating η between eqns (9.8) we obtain

$$\ddot{\xi} + (1 + 3\varepsilon x^{*2})\xi = 0.$$

When x^* is replaced by its appropriate estimate, $x^* = a\cos\omega t$, with a given by (9.9), this equation becomes

$$\ddot{\xi} + (1 + \tfrac{3}{2}\varepsilon a^2 + \tfrac{3}{2}\varepsilon a^2 \cos 2\omega t)\xi = 0,$$

and we expect that the stability property of x^* and ξ will be the same. The previous equation can be reduced to a standard form

$$\xi'' + (\alpha + \beta\cos\tau)\xi = 0 \qquad (9.10)$$

by the substitutions

$$\tau = 2\omega t, \quad \xi' = \mathrm{d}\xi/\mathrm{d}\tau, \quad \alpha = (2 + 3\varepsilon a^2)/8\omega^2, \quad \beta = 3\varepsilon a^2/8\omega^2. \qquad (9.11)$$

For general values of α and β equation (9.10) is known as **Mathieu's equation.** By Theorem 8.8 its solutions are stable for values of the parameters, α, β for which all its solutions are bounded. We shall return to the special case under discussion at the end of Section 9.4 after studying the stability of solutions of Mathieu's general equation, to which problems of this kind may often be reduced.

A pendulum suspended from a support vibrating vertically is a simple model which leads to an equation with a periodic coefficient. Assuming that friction is negligible, consider a rigid pendulum of length a with a bob of mass m suspended from a point which is constrained to oscillate vertically with prescribed displacement $\zeta(t)$ as shown in Fig. 9.1.

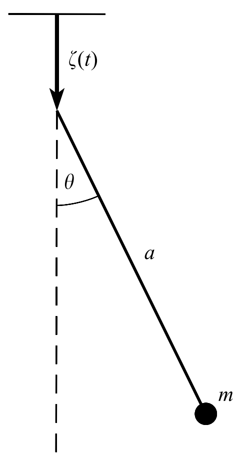

Figure 9.1 Pendulum with vertical forcing: $\zeta(t)$ is the displacement of the support.

The kinetic energy $\mathcal{T}$ and potential energy $\mathcal{V}$ are given by

$$\mathcal{T} = \tfrac{1}{2}m[(\dot{\zeta} - a\sin\theta\dot{\theta})^2 + a^2\cos^2\theta\dot{\theta}^2],$$
$$\mathcal{V} = -mg(\zeta + a\cos\theta).$$

Lagranges's equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{T}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{T}}{\partial\theta} = -\frac{\partial\mathcal{V}}{\partial\theta}$$

becomes

$$a\ddot{\theta} + (g - \ddot{\zeta})\sin\theta = 0,$$

which, for oscillations of small amplitude, reduces to

$$a\ddot{\theta} + (g - \ddot{\zeta})\theta = 0.$$

As a standardized form for this equation we may write

$$\ddot{x} + (\alpha + p(t))x = 0.$$

When $p(t)$ is periodic this equation is known as **Hill's equation**. For the special case $p(t) = \beta\cos t$,

$$\ddot{x} + (\alpha + \beta\cos t)x = 0$$

which is Mathieu's equation (9.10). This type of forced motion, in which $p(t)$ acts as an energy source, is an instance of **parametric excitation**.

Exercise 9.1
Show that the damped equation

$$\ddot{x} + k\dot{x} + (\gamma + \beta\cos t)x = 0$$

can be transformed into a Mathieu equation by the change of variable $x = ze^{\mu t}$ for a suitable choice for μ.

9.2 Equations with periodic coefficients (Floquet theory)

Equation (9.10) is a particular example of an equation associated with the general n-dimensional first-order system

$$\dot{\boldsymbol{x}} = \boldsymbol{P}(t)\boldsymbol{x}, \tag{9.12}$$

where $\boldsymbol{P}(t)$ is periodic with minimal period T; that is, T is the smallest positive number for which

$$\boldsymbol{P}(t+T) = \boldsymbol{P}(t), \quad -\infty < t < \infty. \tag{9.13}$$

($\boldsymbol{P}(t)$, of course, also has periods $2T$, $3T$, ...) The *solutions* are not necessarily periodic, as can be seen from the one-dimensional example

$$\dot{x} = \boldsymbol{P}(t)x = (1 + \sin t)x;$$

the coefficient $P(t)$ has period 2π, but all solutions are given by

$$x = c\mathrm{e}^{t-\cos t},$$

where c is any constant, so only the solution $x = 0$ is periodic. Similarly, the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & \cos t \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

has no periodic solutions apart from the trivial case $x = y = 0$.

In the following discussions remember that the displayed solution vectors may consist of complex solutions.

Theorem 9.1 (Floquet's theorem) *The regular system $\dot{\boldsymbol{x}} = \boldsymbol{P}(t)\boldsymbol{x}$, where $\boldsymbol{P}$ is an $n \times n$ matrix function with minimal period T, has at least one non-trivial solution $\boldsymbol{x} = \boldsymbol{\chi}(t)$ such that*

$$\boldsymbol{\chi}(t + T) = \mu\boldsymbol{\chi}(t), \quad -\infty < t < \infty, \tag{9.14}$$

where μ is a constant.

Proof Let $\boldsymbol{\Phi}(t) = [\phi_{ij}(t)]$ be a fundamental matrix for the system. Then $\dot{\boldsymbol{\Phi}}(t) = \boldsymbol{P}(t)\boldsymbol{\Phi}(t)$. Since $\boldsymbol{P}(t+T) = \boldsymbol{P}(t)$, $\boldsymbol{\Phi}(t+T)$ satisfies the same equation, and by Theorem 8.5, $\det \boldsymbol{\Phi}(t+T) \neq 0$, so $\boldsymbol{\Phi}(t+T)$ is another fundamental matrix. The columns (solutions) in $\boldsymbol{\Phi}(t+T)$ are linear combinations of those in $\boldsymbol{\Phi}(t)$ by Theorem 8.4:

$$\phi_{ij}(t + T) = \sum_{k=1}^{n} \phi_{ik}(t)e_{kj}$$

for some constants e_{kj}, so that

$$\boldsymbol{\Phi}(t + T) = \boldsymbol{\Phi}(t)\boldsymbol{E}, \tag{9.15}$$

where $\boldsymbol{E} = [e_{kj}]$. $\boldsymbol{E}$ is nonsingular, since $\det \boldsymbol{\Phi}(t+T) = \det \boldsymbol{\Phi}(t) \det(\boldsymbol{E})$, and therefore $\det(\boldsymbol{E}) \neq 0$. The matrix $\boldsymbol{E}$ can be found from $\boldsymbol{\Phi}(t_0 + T) = \boldsymbol{\Phi}(t_0)\boldsymbol{E}$ where t_0 is a convenient value of t. Thus

$$\boldsymbol{E} = \boldsymbol{\Phi}^{-1}(t_0)\boldsymbol{\Phi}(t_0 + T).$$

Let μ be an eigenvalue of $\boldsymbol{E}$:

$$\det(\boldsymbol{E} - \mu\boldsymbol{I}) = 0, \tag{9.16}$$

and let $\boldsymbol{s}$ be an eigenvector corresponding to μ:

$$(\boldsymbol{E} - \mu \boldsymbol{I})\boldsymbol{s} = 0. \tag{9.17}$$

Consider the solution $\boldsymbol{x} = \boldsymbol{\Phi}(t)\boldsymbol{s} = \boldsymbol{\chi}(t)$ (being a linear combination of the columns of $\boldsymbol{\Phi}$, $\boldsymbol{\chi}$ is a solution of (9.12)). Then

$$\begin{aligned}
\boldsymbol{\chi}(t+T) &= \boldsymbol{\Phi}(t+T)\boldsymbol{s} \\
&= \boldsymbol{\Phi}(t)\boldsymbol{E}\boldsymbol{s} = \boldsymbol{\Phi}(t)\mu\boldsymbol{s} \quad &&\text{(by (9.17))} \\
&= \mu\boldsymbol{\chi}(t). &&\text{(by (9.14))}
\end{aligned}$$

■

The eigenvalues μ of $\boldsymbol{E}$ are called **characteristic numbers** or **multipliers** of eqn (9.12) (not to be confused with the *eigenvalues* of $\boldsymbol{P}(t)$, which will usually be dependent on t). The importance of this theorem is the possibility of a characteristic number with a special value implying the existence of a periodic solution (though not necessarily of period T).

Example 9.1 *Find a fundamental matrix for the periodic differential equation*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \boldsymbol{P}(t)\boldsymbol{x} = \begin{bmatrix} 1 & 1 \\ 0 & h(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{9.18}$$

where $h(t) = (\cos t + \sin t)/(2 + \sin t - \cos t)$, *and determine the characteristic numbers.*
From (9.18),

$$(2 + \sin t - \cos t)\dot{x}_2 = (\cos t + \sin t)x_2,$$

which has the solution

$$x_2 = b(2 + \sin t - \cos t)$$

where b is any constant. Then x_1 satisfies

$$\dot{x}_1 - x_1 = x_2 = b(2 + \sin t - \cos t)$$

and therefore

$$x_1 = a\mathrm{e}^t - b(2 + \sin t).$$

where a is any constant. A fundamental matrix $\boldsymbol{\Phi}(t)$ can be obtained by putting, say, $a = 0$, $b = 1$, and $a = 1$, $b = 0$:

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} -2 - \sin t & \mathrm{e}^t \\ 2 + \sin t - \cos t & 0 \end{bmatrix}.$$

The matrix $\boldsymbol{P}(t)$ has minimal period $T = 2\pi$, and $\boldsymbol{E}$ in (9.15) must satisfy $\boldsymbol{\Phi}(t + 2\pi) = \boldsymbol{\Phi}(t)\boldsymbol{E}$ for all t. Therefore $\boldsymbol{\Phi}(2\pi) = \boldsymbol{\Phi}(0)\boldsymbol{E}$ and

$$\boldsymbol{E} = \boldsymbol{\Phi}^{-1}(0)\boldsymbol{\Phi}(2\pi) = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2\pi} \end{bmatrix}.$$

The eigenvalues μ of $\boldsymbol{E}$ satisfy

$$\begin{vmatrix} 1-\mu & 0 \\ 0 & \mathrm{e}^{2\pi}-\mu \end{vmatrix} = 0,$$

so $\mu = 1$ or $\mathrm{e}^{2\pi}$. From (9.14), since one eigenvalue is unity there exist solutions such that $\boldsymbol{\chi}(t+2\pi) = \boldsymbol{\chi}(t)$: that is, solutions with period 2π. We have already found these: they correspond to a $a = 0$. ●

Theorem 9.2 *The constants μ in Theorem 9.1 are independent of the choice of $\boldsymbol{\Phi}$.*

Proof Let $\boldsymbol{\Phi}(t)$, $\boldsymbol{\Phi}^*(t)$ be two fundamental matrices; then

$$\boldsymbol{\Phi}^*(t) = \boldsymbol{\Phi}(t)\boldsymbol{C}, \tag{9.19}$$

where $\boldsymbol{C}$ is some constant, nonsingular matrix (nonsingular since $\boldsymbol{\Phi}(t)$ and $\boldsymbol{\Phi}^*(t)$ are nonsingular by Theorem 8.5). Let T be the minimal period of $\boldsymbol{P}(t)$. Then

$$\begin{aligned} \boldsymbol{\Phi}^*(t+T) &= \boldsymbol{\Phi}(t+T)\boldsymbol{C} && \text{(by (9.19))} \\ &= \boldsymbol{\Phi}(t)\boldsymbol{E}\boldsymbol{C} && \text{(by (9.15))} \\ &= \boldsymbol{\Phi}^*(t)\boldsymbol{C}^{-1}\boldsymbol{E}\boldsymbol{C} && \text{(by (9.19))} \\ &= \boldsymbol{\Phi}^*(t)\boldsymbol{D}, \end{aligned}$$

say, where $\boldsymbol{D} = \boldsymbol{C}^{-1}\boldsymbol{E}\boldsymbol{C}$ and $\boldsymbol{C}$ is nonsingular. We may write

$$\begin{aligned} \det(\boldsymbol{D}-\mu\boldsymbol{I}) &= \det(\boldsymbol{C}^{-1}\boldsymbol{E}\boldsymbol{C}-\mu\boldsymbol{I}) = \det[\boldsymbol{C}^{-1}(\boldsymbol{E}-\mu\boldsymbol{I})\boldsymbol{C}] \\ &= \det(\boldsymbol{C}^{-1}\boldsymbol{C})\det(\boldsymbol{E}-\mu\boldsymbol{I}) = \det(\boldsymbol{E}-\mu\boldsymbol{I}) \end{aligned}$$

(using the product rule for determinants). Since $\det(D-\mu I)$ is zero if and only if $\det(E-\mu I)$ is zero, D and E have the same eigenvalues. ■

We can therefore properly refer to '*the* characteristic numbers of the system'. Note that when $\boldsymbol{\Phi}$ is chosen as real $\boldsymbol{E}$ is real, and the characteristic equation for the numbers μ has real coefficients. Therefore if μ (complex) is a characteristic number, then so is its complex conjugate $\bar{\mu}$.

Definition 9.1 *A solution of* (9.12) *satisfying* (9.14) *is called a* ***normal solution****.*

Definition 9.2 (Characteristic exponent) *Let μ be a characteristic number, real or complex, of the system* (9.12), *corresponding to the minimal period T of $\boldsymbol{P}(t)$. Then ρ, defined by*

$$e^{\rho T} = \mu \tag{9.20}$$

is called a ***characteristic exponent*** *of the system. Note that ρ is defined only to an additive multiple of $2\pi\mathrm{i}/T$. It will be fixed by requiring $-\pi < \mathrm{Im}(\rho T) \le \pi$, or by $\rho = (1/T)Ln(\mu)$, where the principal value of the logarithm is taken.*

Theorem 9.3 *Suppose that $\boldsymbol{E}$ of Theorem 9.1 has n distinct eigenvalues, μ_i, $i = 1, 2, \ldots, n$. Then* (9.12) *has n linearly independent* ***normal solutions*** *of the form*

$$\boldsymbol{x}_i = \boldsymbol{p}_i(t)\mathrm{e}^{\rho_i t} \tag{9.21}$$

(ρ_i are the characteristic exponents corresponding to μ_i), where the $\boldsymbol{p}_i(t)$ are vector functions with period T.

Proof To each μ_i corresponds a solution $\boldsymbol{x}_i(t)$ satisfying (9.14): $\boldsymbol{x}_i(t+T) = \mu_i \boldsymbol{x}_i(t) = \mathrm{e}^{\rho_i T}\boldsymbol{x}_i(t)$. Therefore, for every t,

$$\boldsymbol{x}_i(t+T)\mathrm{e}^{-\rho_i(t+T)} = \boldsymbol{x}_i(t)\mathrm{e}^{-\rho_i t}. \tag{9.22}$$

Writing

$$\boldsymbol{p}_i(t) = \mathrm{e}^{-\rho_i t}\boldsymbol{x}_i(t),$$

(9.22) implies that $\boldsymbol{p}_i(t)$ has period T.

The linear independence of the $\boldsymbol{x}_i(t)$ is implied by their method of construction in Theorem 9.1: from (9.17), they are given by $\boldsymbol{x}_i(t) = \boldsymbol{\Phi}(t)\boldsymbol{s}_i$; $\boldsymbol{s}_i$ are the eigenvectors corresponding to the *different* eigenvalues μ_i, and are therefore linearly independent. Since $\boldsymbol{\Phi}(t)$ is non-singular it follows that the $\boldsymbol{x}_i(t)$ are also linearly independent. ■

When the eigenvalues of $\boldsymbol{E}$ are not all distinct, the coefficients corresponding to the $\boldsymbol{p}_i(t)$ are more complicated.

Under the conditions of Theorem 9.3, periodic solutions of period T exist when E has an eigenvalue

$$\mu = 1.$$

The corresponding normal solutions have period T, the minimal period of $\boldsymbol{P}(t)$. This can be seen from (9.14) or from the fact that the corresponding ρ is zero.

There are periodic solutions whenever $\boldsymbol{E}$ has an eigenvalue μ which is one of the mth roots of unity:

$$\mu = 1^{1/m}, \quad m \text{ a positive integer.} \tag{9.23a}$$

In this case, from (9.14),

$$\boldsymbol{\chi}(t+mT) = \mu\boldsymbol{\chi}\{t+(m-1)T\} = \cdots = \mu^m\boldsymbol{\chi}(t) = \boldsymbol{\chi}(t), \tag{9.23b}$$

so that $\boldsymbol{\chi}(t)$ has period mT.

Example 9.2 *Identify the periodic vectors $\boldsymbol{p}_i(t)$ (see eqn (9.21)) in the solution of the periodic differential equation in Example 9.1.*

The characteristic numbers were shown to be $\mu_1 = 1$, $\mu_2 = \mathrm{e}^{2\pi}$. The corresponding characteristic exponents (Definition 9.2) are $\rho_1 = 0$, $\rho_2 = 1$. From Example 9.1, a fundamental matrix is

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} -2-\sin t & \mathrm{e}^t \\ 2+\sin t-\cos t & 0 \end{bmatrix}.$$

From the columns we can identify the 2π-periodic vectors

$$\boldsymbol{p}_1(t) = a\begin{bmatrix} -2-\sin t \\ -2+\sin t-\cos t \end{bmatrix}, \quad \boldsymbol{p}_2(t) = b\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where a and b are any constants. In terms of normal solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a\begin{bmatrix} -2-\sin t \\ 2+\sin t-\cos t \end{bmatrix}\mathrm{e}^0 + b\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathrm{e}^t. \qquad \bullet$$

In the preceding theory, $\det \boldsymbol{\Phi}(t)$ appeared repeatedly, where $\boldsymbol{\Phi}$ is a fundamental matrix of the regular system $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$. This has a simple representation, as follows.

Definition 9.3 *Let $[\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)]$ be a matrix whose columns are any solutions of the n-dimensional system $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$. Then*

$$W(t) = \det[\boldsymbol{\phi}_1(t), \boldsymbol{\phi}_2(t), \ldots, \boldsymbol{\phi}_n(t)] \tag{9.24}$$

*is called the **Wronskian** of this set of solutions, taken in order.*

Theorem 9.4 *For any t_0, the Wronskian of $\dot{\boldsymbol{x}} = A(t)\boldsymbol{x}$ is*

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} \mathrm{tr}\{\boldsymbol{A}(s)\}\mathrm{d}s\right), \tag{9.25}$$

*where $\mathrm{tr}\{\boldsymbol{A}(s)\}$ is the **trace** of $\boldsymbol{A}(s)$ (the sum of the elements of its principal diagonal).*

Proof If the solutions are linearly dependent, $W(t) \equiv 0$ by Theorem 8.5, and the result is true trivially.

If not, let $\boldsymbol{\Phi}(t)$ be any fundamental matrix of solutions, with $\boldsymbol{\Phi}(t) = [\phi_{ij}(t)]$. Then $\mathrm{d}W/\mathrm{d}t$ is equal to the sum of n determinants Δ_k, $k = 1, 2, \ldots, n$, where Δ_k is the same as $\det[\phi_{ij}(t)]$, except for having $\dot{\phi}_{kj}(t)$, $j = 1, 2, \ldots, n$ in place of $\phi_{kj}(t)$ in its kth row. Consider one of the Δ_k, say Δ_1:

$$\Delta_1 = \begin{vmatrix} \dot{\phi}_{11} & \dot{\phi}_{12} & \cdots & \dot{\phi}_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = \begin{vmatrix} \sum\limits_{m=1}^{n} a_{1m}\phi_{m1} & \sum\limits_{m=1}^{n} a_{1m}\phi_{m2} & \cdots & \sum\limits_{m=1}^{n} a_{1m}\phi_{mn} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

(from eqn (8.14))

$$= \sum_{m=1}^{n} a_{1m}\begin{vmatrix} \phi_{m1} & \phi_{m2} & \cdots & \phi_{mn} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} = a_{11}W(t),$$

since all the other determinants have repeated rows, and therefore vanish. In general $\Delta_k = a_{kk}W(t)$. Therefore

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \mathrm{tr}\{A(t)\}W(t),$$

which is a differential equation for W having solution (9.25). ■

For periodic systems we have the following result.

Theorem 9.5 *For the system $\dot{\boldsymbol{x}} = \boldsymbol{P}(t)\boldsymbol{x}$, where $\boldsymbol{P}(t)$ has minimal period T, let the characteristic numbers of the system be $\mu_1, \mu_2, \ldots, \mu_n$. Then*

$$\mu_1\mu_2\ldots\mu_n = \exp\left(\int_0^T \mathrm{tr}\{P(s)\}\mathrm{d}s\right),$$

a repeated characteristic number being counted according to its multiplicity.

Proof Let $\boldsymbol{\Psi}(t)$ be the fundamental matrix of the system for which

$$\boldsymbol{\Psi}(0) = \boldsymbol{I}. \tag{9.26}$$

Then, (eqn (9.15)),

$$\boldsymbol{\Psi}(T) = \boldsymbol{\Psi}(0)\boldsymbol{E} = \boldsymbol{E}, \tag{9.27}$$

in the notation of Theorem 9.1. The characteristic numbers μ_i are the eigenvalues of $\boldsymbol{E}$, given by

$$\det(\boldsymbol{E} - \mu\boldsymbol{I}) = 0.$$

This is an nth-degree polynomial in μ, and the product of the roots is equal to the constant term: that is, equal to the value taken when $\mu = 0$. Thus, by (9.27),

$$\mu_1\mu_2\ldots\mu_n = \det(\boldsymbol{E}) = \det\boldsymbol{\Psi}(T) = W(T),$$

but by Theorem 9.4 with $t_0 = 0$ and $t = T$,

$$W(T) = W(0)\int_0^T \mathrm{tr}\{\boldsymbol{P}(s)\}\mathrm{d}s$$

and $W(0) = 1$ by (9.26). ■

Example 9.3 *Verify the formula in Theorem 9.5 for the product of the characteristic numbers of Example 9.1.*
In Example 9.1, $T = 2\pi$ and

$$\boldsymbol{P}(t) = \begin{bmatrix} 1 & 1 \\ 0 & (\cos t + \sin t)/(2 + \sin t - \cos t) \end{bmatrix}.$$

Then

$$\int_0^{2\pi} \mathrm{tr}\{\boldsymbol{P}(s)\}\mathrm{d}s = \int_0^{2\pi}\left[1+\frac{\cos s+\sin s}{2+\sin s-\cos s}\right]\mathrm{d}s = \int_0^{2\pi}\left[1+\frac{\mathrm{d}(\sin s-\cos s)/\mathrm{d}s}{2+\sin s-\cos s}\right]\mathrm{d}s$$

$$= [s+\log(2+\sin s-\cos s)]_0^{2\pi} = 2\pi.$$

Therefore

$$\exp\left[\int_0^{2\pi}\mathrm{tr}\{\boldsymbol{P}(s)\}\,\mathrm{d}s\right] = \mathrm{e}^{2\pi} = \mu_1\mu_2,$$

by Example 9.1. ●

Exercise 9.2
Find the matrix E for the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & \cos t-1\\ 0 & \cos t\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},$$

and obtain its characteristic numbers. Verify the result in Theorem 9.5.

9.3 Mathieu's equation arising from a Duffing equation

We now return to look in more detail at Mathieu's equation (9.10)

$$\ddot{x}+(\alpha+\beta\cos t)x = 0, \tag{9.28}$$

As a first-order system it can be expressed as

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\alpha-\beta\cos t & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, \tag{9.29}$$

In the notation of the previous section,

$$\boldsymbol{P}(t) = \begin{bmatrix}0 & 1\\ -\alpha-\beta\cos t & 0\end{bmatrix}. \tag{9.30}$$

Clearly $\boldsymbol{P}(t)$ is periodic with minimal period 2π. The general structure of the solution is determined by Theorem 9.3, whilst the question of the stability of a solution can be decided, through Theorem 8.9, by the boundedness or otherwise of the solution for given values of the parameters α and β. We are not particularly interested in periodic solutions as such, though we shall need them to settle the stability question.

From eqn (9.30),

$$\mathrm{tr}\{\boldsymbol{P}(t)\} = 0. \tag{9.31}$$

Therefore, by Theorem 9.5,

$$\mu_1\mu_2 = e^0 = 1, \tag{9.32}$$

where μ_1, μ_2 are the characteristic numbers of $\boldsymbol{P}(t)$. They are solutions of a quadratic characteristic equation (9.16), with real coefficients, which by (9.16) has the form

$$\mu^2 - \phi(\alpha, \beta)\mu + 1 = 0.$$

where the value of ϕ, depending on E (eqn (9.27)) can, in principle, be found in a particular case. The solutions μ are given by

$$\mu_1, \mu_2 = \tfrac{1}{2}[\phi \pm \surd(\phi^2 - 4)]. \tag{9.33}$$

Although $\phi(\alpha, \beta)$ is not specified explicitly, we can make the following deductions.

(i) $\phi > 2$. The characteristic numbers are real, different, and positive, and by (9.32), one of them, say μ_1, exceeds unity. The corresponding characteristic exponents (9.20) are real and have the form $\rho_1 = \sigma > 0$, $\rho_2 = -\sigma < 0$. The general solution is therefore of the form (Theorem 9.3)

$$x(t) = c_1 e^{\sigma t} p_1(t) + c_2 e^{-\sigma t} p_2(t),$$

where c_1, c_2 are constants and p_1, p_2 have minimal period 2π. The parameter region $\phi(\alpha, \beta) > 2$ therefore contains unbounded solutions, and is called an **unstable parameter region.**

(ii) $\phi = 2$. Then $\mu_1 = \mu_2 = 1$, $\rho_1 = \rho_2 = 0$. By (9.21), there is *one solution of period* 2π *on the curves* $\phi(\alpha, \beta) = 2$. (The other solution is unbounded.)

(iii) $-2 < \phi < 2$. The characteristic numbers are complex, and $\mu_2 = \bar{\mu}_1$. Since also $|\mu_1| = |\mu_2| = 1$, we must have $\rho_1 = i\nu$, $\rho_2 = -i\nu$, ν real. The general solution is of the form

$$x(t) = c_1 e^{i\nu t} p_1(t) + c_2 e^{-i\nu t} p_2(t) \quad (p_1, p_2 \text{ period } 2\pi).$$

and *all solutions in the parameter region* $-2 < \phi(\alpha, \beta) < 2$ *are bounded*. This is called the **stable parameter region**. The solutions are oscillatory, but not in general periodic, since the two frequencies ν and 2π are present.

(iv) $\phi = -2$. Then $\mu_1 = \mu_2 = -1(\rho_1 = \rho_2 = \frac{1}{2}i)$, so by Theorem 9.1, eqn (9.14), there is *one solution with period* 4π *at every point on* $\phi(\alpha, \beta) = -2$. (The other solution is in fact unbounded.)

(v) $\phi < -2$. Then μ_1 and μ_2 are real and negative. Since, also, $\mu_1\mu_2 = 1$, the general solution is of the form

$$x(t) = c_1 e^{(\sigma + \frac{1}{2}i)t} p_1(t) + c_2 e^{\left(-\sigma + \frac{1}{2}i\right)t} p_2(t),$$

where $\sigma > 0$ and p_1, p_2 have period 2π. For later purposes it is important to notice that *the solutions have the alternative form*

$$c_1 e^{\sigma t} q_1(t) + c_2 e^{-\sigma t} q_2(t), \tag{9.34}$$

where q_1, q_2 have period 4π.

From (i) to (v) it can be seen that certain curves, of the form

$$\phi(\alpha, \beta) = \pm 2,$$

separate parameter regions where unbounded solutions exist ($|\phi(\alpha, \beta)| > 2$) from regions where all solutions are bounded ($|\phi(\alpha, \beta)| < 2$) (Fig. 9.2). We do not specify the function $\phi(\alpha, \beta)$ explicitly, but we do know that these are also the curves on which periodic solutions, period 2π or 4π, occur. Therefore, if we can establish, by any method, the parameter values for which such periodic solutions can occur, then we have also found the boundaries between the stable and unstable region by Theorem 8.8. These boundaries are called **transition curves**.

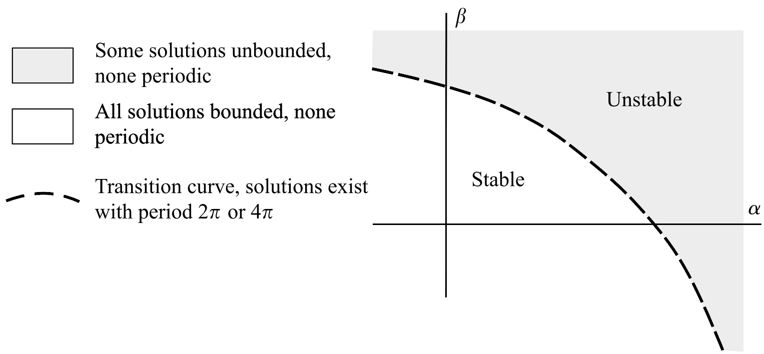

Figure 9.2

First, we find what parameter values α, β give periodic solutions of period 2π. Represent such a solution by the complex Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

We now adopt the following formal procedure which assumes convergence where necessary. Substitute the series into Mathieu's equation

$$\ddot{x} + (\alpha + \beta \cos t)x = 0$$

replacing $\cos t$ by $\frac{1}{2}(e^{it} + e^{-it})$. The result is

$$-\sum_{n=-\infty}^{\infty} c_n n^2 e^{int} + \left[\alpha + \tfrac{1}{2}\beta(e^{it} + e^{-it})\right] \sum_{n=-\infty}^{\infty} c_n e^{int} = 0.$$

which becomes after re-ordering the summation,

$$\sum_{n=-\infty}^{\infty} [\beta c_{n+1} + 2(\alpha - n^2)c_n + \beta c_{n-1}]\mathrm{e}^{\mathrm{i}nt} = 0.$$

This equation can only be satisfied for all t if the coefficients of $\mathrm{e}^{\mathrm{i}nt}$ are all zero, that is if

$$\beta c_{n+1} + 2(\alpha - n^2)c_n + \beta c_{n-1} = 0, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Assume that $\alpha \neq n^2$, and express this equation in the form

$$\gamma_n c_{n+1} + c_n + \gamma_n c_{n-1} = 0, \quad \text{where } \gamma_n = \frac{\beta}{2(\alpha - n^2)}, (n = 0, \pm 1, \pm 2, \ldots), \tag{9.35}$$

but observe that $\gamma_{-n} = \gamma_n$. The infinite set of homogeneous linear equations in (9.35) for the sequence $\{c_n\}$ has nonzero solutions if the **infinite determinant** (Whittaker and Watson, 1962), known as a **Hill determinant**, formed by their coefficients is zero, namely if

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma_1 & 1 & \gamma_1 & 0 & 0 & \cdots \\ \cdots & 0 & \gamma_0 & 1 & \gamma_0 & 0 & \cdots \\ \cdots & 0 & 0 & \gamma_1 & 1 & \gamma_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{9.36}$$

The condition that $\gamma_n = O(n^{-2})$ (from (9.35)) ensures the convergence of the determinant. This equation is equivalent to $\phi(\alpha, \beta) = 2$ (see Section 9.3(ii)).

The determinant in (9.36) is **tridiagonal** (zero elements everywhere except on the leading diagonal and the diagonals immediately above and below it), and a recurrence relation can be established for $n \times n$ approximations. Let

$$D_{m,n} = \begin{vmatrix} 1 & \gamma_m & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{m-1} & 1 & \gamma_{m-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & \gamma_0 & 1 & \gamma_0 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \gamma_{n-1} & 1 & \gamma_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \gamma_n & 1 \end{vmatrix}. \tag{9.37}$$

$D_{m,n}$ is a determinant with $m + n + 1$ rows and columns. Expansion by the first row leads to

$$D_{m,n} = D_{m-1,n} - \gamma_m \gamma_{m-1} D_{m-2,n}.$$

Note that $D_{m,n} = D_{n,m}$. Let $E_n = D_{n,n}$, $P_n = D_{n-1,n}$ and $Q_n = D_{n-2,n}$. Put $m = n, n+1, n+2$ successively in (9.37) resulting in

$$E_n = P_n - \gamma_n \gamma_{n-1} Q_n, \tag{9.38}$$

$$P_{n+1} = E_n - \gamma_{n+1} \gamma_n P_n, \tag{9.39}$$

$$Q_{n+2} = P_{n+1} - \gamma_{n+2} \gamma_{n+1} E_n. \tag{9.40}$$

Eliminate Q_n between (9.38) and (9.40), so that

$$E_{n+2} = P_{n+2} - \gamma_{n+2}\,\gamma_{n+1} P_{n+1} + \gamma_{n+2}^2\,\gamma_{n+1}^2 E_n. \tag{9.41}$$

Now eliminate E_n between (9.39) and (9.41), so that

$$2\gamma_{n+1}\,\gamma_{n+2} P_{n+1} = E_{n+1} - E_{n+2} + \gamma_{n+1}^2\,\gamma_{n+2}^2 E_n.$$

Finally substitute this formula for P_n back into (9.39) to obtain the following third-order difference equation

$$E_{n+2} = (1 - \gamma_{n+1}\,\gamma_{n+2})E_{n+1} - \gamma_{n+1}\,\gamma_{n+2}(1 - \gamma_{n+1}\,\gamma_{n+2})E_n + \gamma_n^2\gamma_{n+1}^3\,\gamma_{n+2} E_{n-1},$$

for $n \geq 1$. In order to solve this difference equation we require E_0, E_1 and E_2, which are given by

$$E_0 = 1, \quad E_1 = \begin{vmatrix} 1 & \gamma_1 & 0 \\ \gamma_0 & 1 & \gamma_0 \\ 0 & \gamma_0 & 1 \end{vmatrix} = 1 - 2\gamma_0\gamma_1,$$

$$E_2 = \begin{vmatrix} 1 & \gamma_2 & 0 & 0 & 0 \\ \gamma_1 & 1 & \gamma_1 & 0 & 0 \\ 0 & \gamma_0 & 1 & \gamma_0 & 0 \\ 0 & 0 & \gamma_1 & 1 & \gamma_1 \\ 0 & 0 & 0 & \gamma_2 & 1 \end{vmatrix} = (\gamma_1\gamma_2 - 1)(\gamma_1\gamma_2 - 1 + 2\gamma_0\gamma_1).$$

The sequence of determinants $\{E_n\}$ is said to converge if there exists a number E such that

$$\lim_{n\to\infty} E_n = E.$$

It can be shown (see Whittaker and Watson (1962), Section 2.8) that E_n converges if the sum of the non-diagonal elements converges absolutely. The sum is

$$2\gamma_0 + 4\sum_{i=1}^{\infty} \gamma_i,$$

which is absolutely convergent since $|\gamma_n| = O(n^{-2})$ as $n \to \infty$.

Given β we solve the equations $E_i = 0$ for α for i increasing from 1 until α is obtained to the required accuracy. However there can be convergence problems if α is close to $1, 2^2, 3^2, \ldots$. To avoid this numerical problem rescale the rows in E to eliminate the denominators $\alpha - n^2$. Hence we consider instead the zeros of (we need not consider E_0)

$$H_1(\alpha,\beta) = \begin{vmatrix} 2(\alpha - 1^2) & \beta & 0 \\ \beta & 2\alpha & \beta \\ 0 & \beta & 2(\alpha - 1^2) \end{vmatrix} = 2^3\alpha(\alpha - 1^2)^2 E_1,$$

$$H_2(\alpha,\beta) = \begin{vmatrix} 2(\alpha-2^2) & \beta & 0 & 0 & 0 \\ \beta & 2(\alpha-1^2) & \beta & 0 & 0 \\ 0 & \beta & 2\alpha & \beta & 0 \\ 0 & 0 & \beta & 2(\alpha-1^2) & \beta \\ 0 & 0 & 0 & \beta & 2(\alpha-2^2) \end{vmatrix}$$
$$= 2^5\alpha(\alpha-1^2)^2(\alpha-2^2)^2E_2,$$

and so on. The evaluations of the first three determinants lead to

$$H_1(\alpha,\beta) = 4(\alpha-1)(-2\alpha+2\alpha^2-\beta^2),$$
$$H_2(\alpha,\beta) = 2(16-20\alpha+4\alpha^2-\beta^2)(16\alpha-20\alpha^2+4\alpha^3+8\beta^2-3\alpha\beta^2),$$
$$H_3(\alpha,\beta) = 8(-72+98\alpha-28\alpha^2+2\alpha^3+5\beta^2-\alpha\beta^2)$$
$$(-288\alpha+392\alpha^2-112\alpha^3+8\alpha^4-144\beta^2+72\alpha\beta^2-8\alpha^2\beta^2+\beta^4),$$

(computer software is needed to expand and factorize these determinants). It can be seen from the determinants $H_i(\alpha,0)$ that $H_i(\alpha,0)=0$ if $\alpha_j=j^2$ for $j\le i$. These are the critical values on the α axis shown in Fig. 9.3.

The table shows the solutions of the equations $H_i(\alpha,\beta)=0$ for $i=1,2,3$ for values of $\beta=0,0.4,0.8,1.2$. As the order of the determinant is increased an increasing number of solutions for α for fixed β appear. The results of a more comprehensive computation are shown

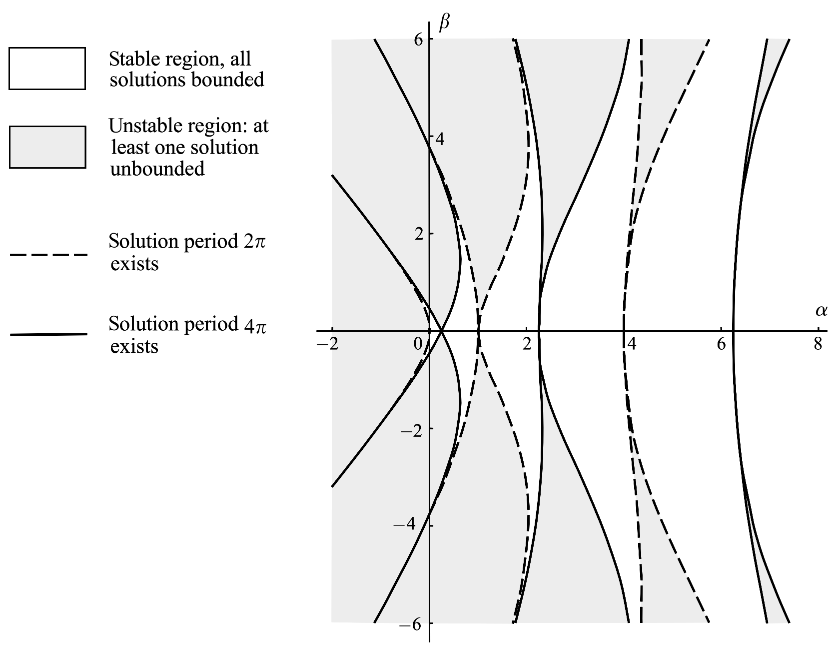

Figure 9.3 Stability diagram for Mathieu's equation $\ddot{x}+(\alpha+\beta\cos t)x=0$.

in Fig. 9.3. The dashed curves show the parameter values of α and β on which 2π-periodic solutions of the Mathieu equation exist. The curves are symmetric about the α axis, one passes through the origin, and the others have cusps at $\alpha = 1, 4, 9, \ldots$.

	$H_1(\alpha,\beta)=0$	$H_2(\alpha,\beta)=0$	$H_3(\alpha,\beta)=0$
$\beta = 0$	$\alpha = 0$	$\alpha = 0$	$\alpha = 0$
	$\alpha = 1$	$\alpha = 1$	$\alpha = 1$
		$\alpha = 4$	$\alpha = 4$
			$\alpha = 9$
$\beta = 0.4$	$\alpha = -0.074$	$\alpha = -0.075$	$\alpha = -0.075$
	$\alpha = 1.000$	$\alpha = 0.987$	$\alpha = 0.987$
	$\alpha = 1.074$	$\alpha = 1.062$	$\alpha = 1.062$
		$\alpha = 4.013$	$\alpha = 4.005$
		$\alpha = 4.013$	$\alpha = 4.005$
$\beta = 0.8$	$\alpha = -0.255$	$\alpha = -0.261$	$\alpha = -0.261$
	$\alpha = 1.000$	$\alpha = 0.948$	$\alpha = 0.947$
	$\alpha = 1.256$	$\alpha = 1.208$	$\alpha = 1.207$
		$\alpha = 4.052$	$\alpha = 4.021$
		$\alpha = 4.053$	$\alpha = 4.022$
$\beta = 1.2$	$\alpha = -0.485$	$\alpha = -0.505$	$\alpha = -0.505$
	$\alpha = 1.000$	$\alpha = 0.884$	$\alpha = 0.883$
	$\alpha = 1.485$	$\alpha = 1.383$	$\alpha = 1.381$
		$\alpha = 4.116$	$\alpha = 4.046$
		$\alpha = 4.122$	$\alpha = 4.052$

For the 4π periodic solutions, let

$$x(t) = \sum_{n=-\infty}^{\infty} d_n e^{\frac{1}{2}int}.$$

As in the previous case substitute $x(t)$ into eqn (9.28) and equate to zero the coefficients of $e^{\frac{1}{2}int}$ so that

$$\tfrac{1}{2}\beta d_{n+2} + (\alpha - \tfrac{1}{4}n^2)d_n + \tfrac{1}{2}d_{n-1} = 0, \quad (n = 0, 1, 2 \ldots).$$

This set of equations split into two independent sets for $\{d_n\}$. If n is even then the equations reproduce those of (9.35) for the 2π period solutions. Therefore we need only consider solutions for n odd, and can put $d_{2m} = 0$ for all m. For n odd, the set of equations have a nontrivial solutions if, and only if,

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \delta_2 & 1 & \delta_2 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \delta_1 & 1 & \delta_1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \delta_1 & 1 & \delta_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \delta_2 & 1 & \delta_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

where

$$\delta_1 = \frac{\beta}{2(\alpha - \frac{1}{4}1^2)}, \quad \delta_2 = \frac{\beta}{2(\alpha - \frac{1}{4}3^2)},$$

$$\delta_m = \frac{\beta}{2[\alpha - \frac{1}{4}(2m-1)^2]}, \quad m = 1, 2, \ldots,$$

provided $\alpha \neq \frac{1}{4}(2m-1)^2$. The numerical relation between α and β can be computed by taking finite approximations to the infinite determinant. The transition curves corresponding to the 4π periodic solutions are shown in Fig. 9.3. The curves pass through the critical points $\beta = 0$, $\alpha = \frac{1}{4}(2m-1)^2$, $(m = 1, 2, 3, \ldots)$.

Exercise 9.3
For the 4π-periodic solutions of Mathieu's equation, let

$$G_1(\alpha, \beta) = \begin{vmatrix} 2(\alpha - \frac{1}{4}) & \beta \\ \beta & 2(\alpha - \frac{1}{4}) \end{vmatrix},$$

$$G_2(\alpha, \beta) = \begin{vmatrix} 2(\alpha - \frac{9}{4}) & \beta & 0 & 0 \\ \beta & 2(\alpha - \frac{1}{4}) & \beta & 0 \\ 0 & \beta & 2(\alpha - \frac{1}{4}) & \beta \\ 0 & 0 & \beta & 2(\alpha - \frac{9}{4}) \end{vmatrix}.$$

Obtain the relations between β and α in the first two approximations to the zeros of $G_1(\alpha, \beta) = 0$ and $G_2(\alpha, \beta) = 0$.

9.4 Transition curves for Mathieu's equation by perturbation

For small values of $|\beta|$ a perturbation method can be used to establish the transition curves. In the equation

$$\ddot{x} + (\alpha + \beta \cos t)x = 0, \tag{9.42}$$

suppose that the transition curves are given by

$$\alpha = \alpha(\beta) = \alpha_0 + \beta\alpha_1 + \beta^2\alpha_2 \cdots, \tag{9.43}$$

and that the corresponding solutions have the form

$$x(t) = x_0(t) + \beta x_1(t) + \beta^2 x_2(t) + \cdots, \tag{9.44}$$

where $x_0, x_1, \ldots$ all have either minimal period 2π or 4π.

When (9.43) and (9.44) are substituted into (9.42) and the coefficients of powers of β are equated to zero in the usual perturbation way, we have

$$\ddot{x}_0 + \alpha_0 x_0 = 0, \tag{9.45a}$$

$$\ddot{x}_1 + \alpha_0 x_1 = -(\alpha_1 + \cos t)x_0, \tag{9.45b}$$

$$\ddot{x}_2 + \alpha_0 x_2 = -\alpha_2 x_0 - (\alpha_1 + \cos t)x_1, \tag{9.45c}$$

$$\ddot{x}_3 + \alpha_0 x_3 = -\alpha_3 x_0 - \alpha_2 x_1 - (\alpha_1 + \cos t)x_2, \tag{9.45d}$$

and so on.

From the analysis in the Section 9.3, we are searching for solutions with minimum period 2π if $\alpha_0 = n^2$, $n - 0, 1, 2, \ldots$, and for solutions of minimum period 4π if $\alpha_0 = (n + \frac{1}{2})^2$, $n = 0, 1, 2, \ldots$. Both cases can be covered by defining $\alpha_0 = \frac{1}{4}n^2$, $n = 0, 1, 2, \ldots$ We consider the cases $n = 0$ and $n = 1$.

(i) $n = 0$. In this case $\alpha_0 = 0$ so that $\ddot{x}_0 = 0$. The periodic solution of (9.45a) is $x_0 = a_0$, where we assume that a_0 is any nonzero constant. Equation (9.45b) becomes

$$\ddot{x}_1 = -(\alpha_1 + \cos t)a_0,$$

which has periodic solutions only if $\alpha_1 = 0$. We need only choose the *particular* solution

$$x_1 = a_0 \cos t.$$

(inclusion of complementary solutions does not add generality since further arbitrary constants can be amalgamated). Equation (9.45c) becomes

$$\ddot{x}_2 = -a_0\alpha_2 - \tfrac{1}{2}a_0 - a_0\cos^2 t = -a_0\alpha_2 - \tfrac{1}{2}a_0 - \tfrac{1}{2}a_0\cos 2t,$$

which generates a periodic solution of 2π (and π) only if $\alpha_2 + \frac{1}{2}a_0 = 0$, that is, if $\alpha_2 = -\frac{1}{2}a_0$. Therefore choose

$$x_2 = \tfrac{1}{8}a_0 \cos 2t.$$

From (9.45d),

$$\begin{aligned}\ddot{x}_3 &= -\alpha_3 x_0 - \alpha_2 a_0 \cos t - \tfrac{1}{8}a_0 \cos t \cos 2t \\ &= a_0[-\alpha_3 - (\alpha_2 + \tfrac{1}{16})\cos t - \tfrac{1}{16}\cos 3t].\end{aligned}$$

Solutions will only be periodic if $\alpha_3 = 0$. Therefore, for small $|\beta|$,

$$\alpha = -\tfrac{1}{2}\beta^2 + O(\beta^4),$$

which is a parabolic approximation to the curve through the origin in the α, β shown in Fig. 9.3. The corresponding 2π periodic solution is

$$x = a_0[1 + \beta \cos t + \tfrac{1}{8}\beta^2 \cos 2t] + O(\beta^3).$$

(ii) $n = 1$. In this case $\alpha_0 = \frac{1}{4}$, and $x_0 = a_0 \cos \frac{1}{2}t + b_0 \sin \frac{1}{2}t$. Equation (9.45b) becomes

$$\begin{aligned} \ddot{x}_1 + \tfrac{1}{4}x_1 &= -(\alpha_1 + \cos t)(a_0 \cos \tfrac{1}{2}t + b_0 \sin \tfrac{1}{2}t) \\ &= -a_0(\alpha_1 + \tfrac{1}{2}) \cos \tfrac{1}{2}t - b_0(\alpha_1 - \tfrac{1}{2}) \sin \tfrac{1}{2}t - \tfrac{1}{2}a_0 \cos \tfrac{3}{2}t - \tfrac{1}{2}b_0 \sin \tfrac{3}{2}t \end{aligned} \tag{9.46}$$

There are periodic solutions of period 4π only if either $b_0 = 0, \alpha_1 = -\frac{1}{2}$, or $a_0 = 0, \alpha_1 = \frac{1}{2}$. Here are two cases to consider.

(a) $b_0 = 0,\ \alpha_1 = -\frac{1}{2}$. It follows that the particular solution of (9.46) is $x_1 = \frac{1}{4}a_0 \cos \frac{3}{2}t$. Equation (9.45c) for x_2 is

$$\ddot{x}_2 + \tfrac{1}{4}x_2 = -(\alpha_2 + \tfrac{1}{8})a_0 \cos \tfrac{1}{2}t + \tfrac{1}{8}a_0 \cos \tfrac{3}{2}t - \tfrac{1}{8}a_0 \cos \tfrac{5}{2}t.$$

Secular terms can be eliminated by putting $\alpha_2 = -\frac{1}{8}$. Hence one transition curve through $\alpha = \frac{1}{4}$, $\beta = 0$ is

$$\alpha = \tfrac{1}{4} - \tfrac{1}{2}\beta - \tfrac{1}{8}\beta^2 + O(\beta^3). \tag{9.47}$$

(b) $a_0 = 0,\ \alpha_1 = \frac{1}{2}$. From (9.46), $x_1 = \frac{1}{4}b_0 \sin \frac{3}{2}t$. Equation (9.445c) becomes

$$\ddot{x}_2 + \tfrac{1}{4}x_2 = -(\alpha_2 + \tfrac{1}{8})b_0 \sin \tfrac{1}{2}t - \tfrac{1}{8}b_0 \sin \tfrac{3}{2}t - \tfrac{1}{8}b_0 \sin \tfrac{5}{2}t.$$

Secular terms can be eliminated by putting $\alpha_2 = -\frac{1}{8}$. Therefore the other transition curve is given by

$$\alpha = \tfrac{1}{4} + \tfrac{1}{2}\beta - \tfrac{1}{8}\beta^2 + O(\beta^3). \tag{9.48}$$

The transition curves given by (9.47) and (9.48) approximate to the computed curves through $\alpha = \frac{1}{4}$, $\beta = 0$ shown in Fig. 9.3.

The same perturbation method can be applied to approximate to the transition curves through $\alpha = 1, \frac{9}{4}, 4, \frac{25}{4} \ldots$. A more extensive investigation of perturbation methods applied to Mathieu's equation is given by Nayfeh and Mook (1979, Chapter 5).

9.5 Mathieu's damped equation arising from a Duffing equation

As we saw in Section 9.1, the variational equation for the undamped forced, pendulum is Mathieu's equation (9.10) or (9.28). With dissipation included, the Duffing equation in standardized form is

$$\ddot{x} + k\dot{x} + x + \varepsilon x^3 = \Gamma \cos \omega t. \tag{9.49}$$

In Chapter 7 we also showed that this equation has periodic solutions which are approximately of the form $a \cos \omega t + b \sin \omega t$ where $r = \surd(a^2 + b^2)$ satisfies

$$\left\{(\omega^2 - 1 - \tfrac{3}{4}\varepsilon r^2)^2 + \omega^2 k^2\right\} r^2 = \Gamma^2, \tag{9.50}$$

which reduces to eqn (7.23) if ε replaces β in the earlier notation. Following the notation and procedure of Section 9.1, write (9.49) as the first-order system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y \\ -ky - x - \varepsilon x^3 + \Gamma \cos \omega t \end{bmatrix} \tag{9.51}$$

and put (approximately)

$$\boldsymbol{x}^* = a \cos \omega t + b \sin \omega t, \quad y^* = -a\omega \sin \omega t + b\omega \cos \omega t.$$

The variations $\xi = x - x^*$ and $\eta = y - y^*$ satisfy

$$\dot{\xi} + \dot{x}^* = \eta + y^*,$$
$$\dot{\eta} + \dot{y}^* = -k(\eta + y^*) - (\xi + x^*) - \varepsilon(\xi + x^*)^3 + \Gamma \cos \omega t.$$

By using (9.51) and retaining only the first powers of ξ and η we obtain corresponding linearized equations

$$\dot{\xi} = \eta, \quad \dot{\eta} = -k\eta - \xi - 3\varepsilon x^{*2}\xi.$$

Elimination of η leads to the second-order equation

$$\ddot{\xi} + k\dot{\xi} + (1 + 3\varepsilon x^{*2})\xi = 0.$$

By substituting for x^* its approximate form $a \cos \omega t + b \sin \omega t$ we obtain

$$\ddot{\xi} + k\dot{\xi} + \left\{1 + \tfrac{3}{2}\varepsilon r^2 + \tfrac{3}{2}\varepsilon r^2 \cos(2\omega t + 2c)\right\} \xi = 0, \tag{9.52}$$

where r, c are defined by

$$a \cos \omega t + b \sin \omega t = r \cos(\omega t + c).$$

We can reduce the eqn (9.52) to 'standard' form by putting

$$\tau = 2\omega t + 2\gamma, \quad \xi' \equiv \frac{d\xi}{d\tau},$$

$$\kappa = \frac{k}{2\omega} = \kappa_1\varepsilon, \quad \nu = \frac{2+3\varepsilon r^2}{8\omega^2} = \nu_0 + \varepsilon\nu_1, \quad \beta = \frac{3\varepsilon r^2}{8\omega^2} = \beta_1\varepsilon,$$

say, so that

$$\xi'' + \kappa\xi' + (\nu + \beta\cos\tau)\xi = 0, \tag{9.53}$$

This is known as **Mathieu's equation with damping**.

We assume that $0 < \varepsilon \ll 1$, $k = O(\varepsilon)$ and $\omega \approx 1 = 1 + O(\varepsilon)$ (near resonance). For near resonance $\nu = \frac{1}{4} + O(\varepsilon)$. Let $\eta = \xi'$ in (9.53). Then the corresponding first-order system is

$$\boldsymbol{\zeta}' = \begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\nu - \beta\cos\tau & -\kappa \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \boldsymbol{P}(\tau)\boldsymbol{\zeta}, \tag{9.54}$$

say. The characteristic numbers of $\boldsymbol{P}(\tau)$ satisfy (see Theorem 9.5)

$$\mu_1\mu_2 = \exp\left[\int_0^{2\pi} \mathrm{tr}\{\boldsymbol{P}(\tau)\}d\tau\right] = \exp\left[-\int_0^{2\pi} \kappa\, d\tau\right] = e^{-2\pi\kappa}.$$

The numbers μ_1 and μ_2 are solutions of a characteristic equation of the form

$$\mu^2 - \phi(\nu, \beta, \kappa)\mu + e^{-2\pi\kappa} = 0. \tag{9.55}$$

The two solutions are

$$\mu_1, \mu_2 = \tfrac{1}{2}[\phi \pm \surd\{\phi^2 - 4e^{-2\pi\kappa}\}]. \tag{9.56}$$

For distinct values of μ_1 and μ_2, (9.54) has 2 linearly independent solutions of the form (see Theorem 9.3)

$$\boldsymbol{\zeta}_i = \mathbf{p}_i(\tau)e^{\rho_i\tau} \quad (i = 1, 2),$$

where $e^{2\rho_i\pi} = \mu_i$, $(i = 1, 2)$ and $\mathbf{p}_i$ are functions of period 2π.

From (9.55), the general solution for ξ, the first component of $\boldsymbol{\zeta}$, is given by

$$\xi = c_1 q_1(\tau)e^{\rho_1\tau} + c_2 q_2(\tau)e^{\rho_2\tau}, \tag{9.57}$$

where c_1, c_2 are constants and $q_1(\tau)$, $q_2(\tau)$ have minimum period 2π: η can then be found from $\eta' = \xi$. The stability or otherwise of the periodic solution of (9.53) will be determined by the behaviour of ξ in (9.57). If the solution for ξ is damped then we can infer its stability. The characteristic exponents may be complex, so that the limit $\xi \to 0$ as $\tau \to \infty$ will occur if both

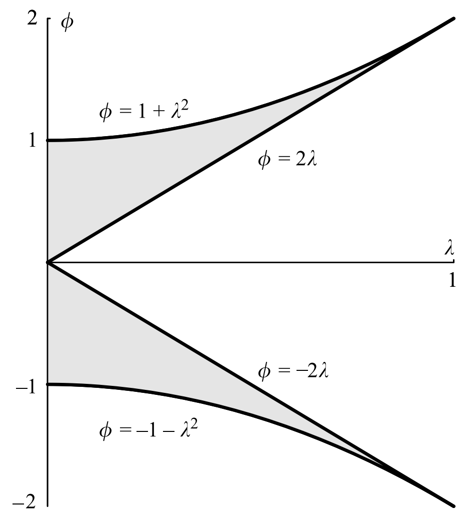

Figure 9.4 The boundaries of the shaded region are $\phi = 2\lambda$, $\phi = 1 + \lambda^2$, and $\phi - 2\lambda$, $\phi = -1 - \lambda^2$, where $\lambda = \mathrm{e}^{-\pi\kappa}$.

$\mathrm{Re}(\rho_1) < 0$ and $\mathrm{Re}(\rho_2) < 0$. This is equivalent to $|\mu_1| < 1$ and $|\mu_2| < 1$. There are three cases to consider.

- $\phi^2 > 4\mathrm{e}^{-2\pi\kappa}$. μ_1 and μ_2 are both real and positive, or both real and negative according to the sign of ϕ: in both cases $\mu_2 < \mu_1$. If they are both positive, then the periodic solution is stable if

$$\mu_1 = \tfrac{1}{2}[\phi + \sqrt{(\phi^2 - 4\mathrm{e}^{-2\pi\kappa})}] < 1, \text{ or, } \phi < 1 + \mathrm{e}^{-2\pi\kappa}. \tag{9.58}$$

  For $\kappa > 0$, this lower bound is always greater than $2\mathrm{e}^{-\pi\kappa}$. The shaded region in $\phi > 0$ in Fig. 9.4 shows the stability domain. Similarly if $\phi < -2\mathrm{e}^{-\pi\kappa}$, then the stability boundaries are $\phi = -2\mathrm{e}^{-\pi\kappa}$ and $\phi = -1 - \mathrm{e}^{-2\pi\kappa}$, which are also shown in Fig. 9.4.
- $\phi^2 = 4\mathrm{e}^{-2\pi\kappa}$. In this case $\mu_1 = \mu_2 = \frac{1}{2}\phi = \pm\mathrm{e}^{-\pi\kappa} = \mu$, say. If $\mu = \mathrm{e}^{-\pi\kappa}$, then $\rho = -\frac{1}{2}\kappa$, and if $\mu = -\mathrm{e}^{-\pi\kappa}$, then $\mathrm{Re}(\rho) = -\frac{1}{2}\kappa$. In both cases the solution is stable, also shown shaded in Fig. 9.4.
- $\phi^2 < 4\mathrm{e}^{-2\pi\kappa}$. μ_1 and μ_2 are complex conjugates given by $\frac{1}{2}(\phi \pm \mathrm{i}\theta)$, where $\theta = \sqrt{[4\mathrm{e}^{-2\pi\kappa} - \phi^2]}$. The system is therefore stable if $|\phi| < 2$.

As in Section 9.5, we can search for periodic solutions of periods 2π and 4π by using Fourier series. Let

$$\xi(\tau) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\tau}.$$

Substitute this series into (9.50) so that

$$\sum_{n=-\infty}^{\infty} [\beta c_{n+1} + 2\{\nu - n^2 + \mathrm{i}\kappa n\}c_n + \beta c_{n-1}]\mathrm{e}^{\mathrm{i}n\tau} = 0.$$

where we have used $\cos\tau = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\tau} + \mathrm{e}^{-\mathrm{i}\tau})$. These equations will only be satisfied for all τ if

$$\beta c_{n+1} + 2\{\nu - n^2 + \mathrm{i}\kappa n\}c_n + \beta c_{n-1} = 0, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{9.59}$$

Let

$$\gamma_n = \frac{\beta}{2(\nu - n^2 + \mathrm{i}\kappa n)}, \tag{9.60}$$

and express eqns (9.59) in the form

$$\gamma_{n+1}c_{n+1} + c_n + \gamma_{n-1}c_{n-1} = 0. \tag{9.61}$$

There are non-zero solutions for the sequence $\{c_n\}$ if, and only if the infinite determinant is zero, that is,

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma_1 & 1 & \gamma_1 & 0 & 0 & \cdots \\ \cdots & 0 & \gamma_0 & 1 & \gamma_0 & 0 & \cdots \\ \cdots & 0 & 0 & \gamma_{-1} & 1 & \gamma_{-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \gamma_1 & 1 & \gamma_1 & 0 & 0 & \cdots \\ \cdots & 0 & \gamma_0 & 1 & \gamma_0 & 0 & \cdots \\ \cdots & 0 & 0 & \overline{\gamma}_1 & 1 & \overline{\gamma}_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

since $\gamma_{-n} = \overline{\gamma}_n$ $(n = 1, 2, \ldots)$, the conjugate of γ_n. We can approximate to the determinant by choosing a finite number of rows. Let

$$E_1 = \begin{vmatrix} 1 & \gamma_1 & 0 \\ \gamma_0 & 1 & \gamma_0 \\ 0 & \overline{\gamma}_1 & 1 \end{vmatrix} = 1 - \frac{\beta^2(\nu - 1)}{2\nu[(\nu - 1)^2 + \kappa^2]}.$$

With $\nu = \frac{1}{4} + O(\varepsilon)$,

$$E_1 = 1 + \tfrac{8}{3}\beta_2\varepsilon^2 + o(\varepsilon^2).$$

Hence E_1 cannot be zero for ε small. The implication is that there are no 2π periodic solutions in the variable τ.

To search for 4π periodic solutions, let

$$\xi = \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\frac{1}{2}in\tau}.$$

Substitution of this series into (9.50) leads to

$$\sum_{n=-\infty}^{\infty} [\beta d_{n+2} + 2\{\nu - \tfrac{1}{4}n^2 + \tfrac{1}{2}\mathrm{i}\kappa n\}d_n + \beta d_{n-2}]\mathrm{e}^{\frac{1}{2}\mathrm{i}n\tau} = 0.$$

These equations will only be satisfied for all τ if

$$\beta d_{n+2} + 2\{\nu - \tfrac{1}{4}n^2 + \tfrac{1}{2}\mathrm{i}\kappa n\}d_n + \beta d_{n-2} = 0. \tag{9.62}$$

As in Section 9.3 there are two independent sets of equations for n even and for n odd. The even case duplicates the previous case for 2π periodic solutions so that we need not consider it. For the case of n odd, let

$$\delta_m = \frac{\beta}{2[\nu - \frac{1}{4}(2m-1)^2 + \frac{1}{2}(2m-1)\mathrm{i}\kappa]} \quad (m \geq 1)$$

Elimination of $\{d_n\}$ in (9.62) results in the infinite determinant equation

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \delta_2 & 1 & \delta_2 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \delta_1 & 1 & \delta_1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \overline{\delta}_1 & 1 & \overline{\delta}_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \overline{\delta}_2 & 1 & \overline{\delta}_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0,$$

where $\overline{\delta}_m$ is the conjugate of δ_m. The value of this determinant can be approximated by

$$G_1 = \begin{vmatrix} 1 & \delta_1 \\ \overline{\delta}_1 & 1 \end{vmatrix} = 1 - \delta_1\overline{\delta}_1 = 1 - \frac{\beta_1^2}{4\nu_1^2 + \kappa_1^2} + O(\varepsilon).$$

To lowest order, $G_1 = 0$ if

$$1 - \frac{\beta_1^2}{4\nu_1^2 + \kappa_1^2} = 0, \quad \text{or} \quad \nu_1 = \pm\tfrac{1}{2}\sqrt{(\beta_1^2 - \kappa_1^2)}.$$

Therefore the stability boundaries are given by

$$\nu = \tfrac{1}{4} \pm \tfrac{1}{2}\varepsilon\sqrt{(\beta_1^2 - \kappa_1^2)} + O(\varepsilon)$$

Note that if $\kappa_1 = 0$ (no damping), then the stability boundaries given by (9.45) can be recovered. Note also that $\kappa_1 < \beta_1$ is required. Using the stability boundaries, the periodic solution of the Duffing equation is stable in the domain defined by

$$\nu < \tfrac{1}{4} - \tfrac{1}{2}\varepsilon\sqrt{(\beta_1^2 - \kappa_1^2)}, \quad \text{or} \quad \nu > \tfrac{1}{4} + \tfrac{1}{2}\varepsilon\sqrt{(\beta_1^2 - \kappa_1^2)},$$

or

$$(\nu - \tfrac{1}{4})^2 > \tfrac{1}{4}\varepsilon^2(\beta_1^2 - \kappa_1^2).$$

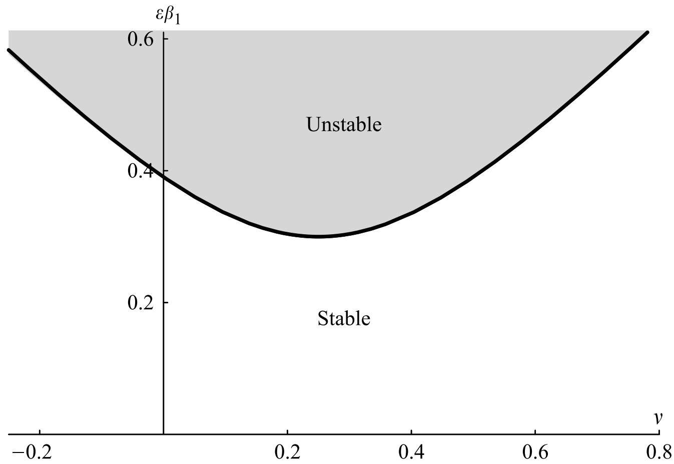

Figure 9.5 The stability for the damped Duffing equation with $\varepsilon\kappa_1 = 0.3$ and $\omega \approx 1$.

A typical stability domain is shown in Fig. 9.5 for $\varepsilon\kappa_1 = 0.3$ in the neighbourhood of resonance near $\omega = 1$. In terms of the original variables, the condition becomes

$$\left(\frac{2 + 3\varepsilon r^2}{8\omega^2} - \frac{1}{4}\right)^2 - \frac{1}{4}\left[\left(\frac{3\varepsilon r^2}{8\omega^2}\right)^2 - \frac{k^2}{4\omega^2}\right] > 0,$$

or

$$(1 - \omega^2)^2 + 3\varepsilon(1 - \omega^2)r^2 + \tfrac{27}{16}\varepsilon^2 r^4 + k^2\omega^2 > 0. \tag{9.63}$$

Since the solutions of the damped Mathieu equation tend to zero in the stable region, asymptotic stability is predicted, confirming the analysis of Chapter 7.

We can also confirm the remark made in Section 5.5 (vi): that stability is to be expected when

$$\frac{\mathrm{d}(\Gamma^2)}{\mathrm{d}(r^2)} > 0, \tag{9.64}$$

that is, when an increase or decrease in magnitude of Γ results in an increase or decrease respectively in the amplitude. From (9.51) it is easy to verify that $\mathrm{d}(\Gamma^2)/\mathrm{d}(r^2)$ is equal to the expression on the left of (9.63), and the speculation is therefore confirmed.

In general, when periodic solutions of the original equation are expected the reduced equation (9.4) is the more complicated Hill type (see Problem 9.11). The stability regions for this equation and examples of the corresponding stability estimates may be found in Hayashi (1964).

Problems

9.1 The system

$$\dot{x}_1 = (-\sin 2t)x_1 + (\cos 2t - 1)x_2, \quad \dot{x}_2 = (\cos 2t + 1)x_1 + (\sin 2t)x_2$$

has a fundamental matrix of normal solutions:

$$\begin{bmatrix} e^t(\cos t - \sin t) & e^{-t}(\cos t + \sin t) \\ e^t(\cos t + \sin t) & e^{-t}(-\cos t + \sin t) \end{bmatrix}.$$

Obtain the corresponding $\boldsymbol{E}$ matrix (Theorem 9.1), the characteristic numbers, and the characteristic exponents.

9.2 Let the system $\dot{\boldsymbol{x}} = \boldsymbol{P}(t)\boldsymbol{x}$ have a matrix of coefficients $\boldsymbol{P}$ with minimal period T (and therefore also with periods $2T, 3T, \ldots$). Follow the argument of Theorem 9.1, using period $mT, m > 1$, to show that $\boldsymbol{\Phi}(t + mT) = \boldsymbol{\Phi}(t)\boldsymbol{E}^m$. Assuming that if the eigenvalues of $\boldsymbol{E}$ are μ_i, then those of $\boldsymbol{E}^m$ are μ_i^m, discuss possible periodic solutions.

9.3 Obtain Wronskians for the following linear systems:

(i) $\dot{x}_1 = x_1 \sin t + x_2 \cos t,\ \dot{x}_2 = -x_1 \cos t + x_2 \sin t$,

(ii) $\dot{x}_1 = f(t)x_2,\ \dot{x}_2 = g(t)x_1$.

9.4 By substituting $x = c + a\cos t + b\sin t$ into Mathieu's equation

$$\ddot{x} + (\alpha + \beta\cos t)x = 0,$$

obtain by harmonic balance an approximation to the transition curve near $\alpha = 0$, $\beta = 0$, (compare with Section 9.4).

By substituting $x = c + a\cos\frac{1}{2}t + b\sin\frac{1}{2}t$, find the transition curves near $\alpha = \frac{1}{4}$, $\beta = 0$.

9.5 Figure 9.6 represents a particle of mass m attached to two identical linear elastic strings of stiffness λ and natural length l. The ends of the strings pass through frictionless guides A and B at a distance $2L$, $l < L$, apart. The particle is set into lateral motion at the mid-point, and symmetrical displacements $a + b\cos\omega t$, $a > b$, are imposed on the ends of the string. Show that, for $x \ll L$,

$$\ddot{x} + \left(\frac{2\lambda(L - l + a)}{mL} + \frac{2\lambda b}{mL}\cos\omega t\right)x = 0.$$

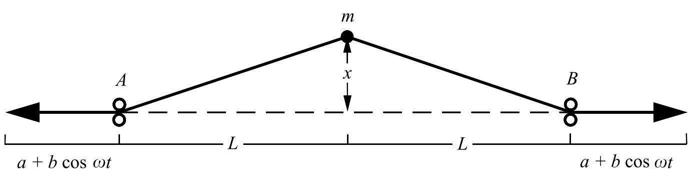

Figure 9.6

Analyse the motion in terms of suitable parameters, using the information of Sections 9.3 and 9.4 on the growth or decay, periodicity and near periodicity of the solutions of Mathieu's equation in the regions of its parameter plane.

9.6 A pendulum with a light, rigid suspension is placed upside-down on end, and the point of suspension is caused to oscillate vertically with displacement y upwards given by $y = \varepsilon\cos\omega t$, $\varepsilon \ll 1$. Show that the equation of motion is

$$\ddot{\theta} + \left(-\frac{g}{a} - \frac{1}{a}\ddot{y}\right)\sin\theta = 0,$$

where a is the length of the pendulum, g is gravitational acceleration, and θ the inclination to the vertical. Linearize the equation for small amplitudes and show that the vertical position is stable (that is, the motion of the pendulum restricts itself to the neighbourhood of the vertical: it does not topple over) provided $\varepsilon^2\omega^2/(2ag) > 1$. For further discussion of the inverted pendulum and its stability see Acheson (1997).

9.7 Let $\mathbf{\Phi}(t) = (\phi_{ij}(t))$, $i, j = 1, 2$, be the fundamental matrix for the system $\dot{x}_1 = x_2$, $\dot{x}_2 = -(\alpha + \beta\cos t)x_1$, satisfying $\mathbf{\Phi}(0) = \boldsymbol{I}$ (Mathieu's equation). Show that the characteristic numbers μ satisfy the equation

$$\mu^2 - \mu\{\phi_{11}(2\pi) + \phi_{22}(2\pi)\} + 1 = 0.$$

9.8 In Section 9.3, for the transition curves of Mathieu's equation for solutions period 2π, let $D_{m,n}$ be the tridiagonal determinant given by

$$D_{m,n} = \begin{vmatrix} 1 & \gamma_m & 0 & & & & & & \\ \gamma_{m-1} & 1 & \gamma_{m-1} & & & & & & \\ & & & \cdots & & & & & \\ & & & & \gamma_0 & 1 & \gamma_0 & & \\ & & & & & & \cdots & & \\ & & & & & & \gamma_{n-1} & 1 & \gamma_{n-1} \\ & & & & & & 0 & \gamma_n & 1 \end{vmatrix}$$

for $m \geq 0, n \geq 0$. Show that

$$D_{m,n} = D_{m-1,n} - \gamma_m\gamma_{m-1}D_{m-2,n}.$$

Let $E_n = D_{n,n}$ and verify that

$$E_0 = 1, \quad E_1 = 1 - 2\gamma_0\gamma_1, \quad E_2 = (1 - \gamma_1\gamma_2)^2 - 2\gamma_0\gamma_1(1 - \gamma_1\gamma_2).$$

Prove that, for $n \geq 1$,

$$\begin{aligned} E_{n+2} = {} & (1 - \gamma_{n+1}\gamma_{n+2})E_{n+1} - \gamma_{n+1}\gamma_{n+2}(1 - \gamma_{n+1}\gamma_{n+2})E_n \\ & + \gamma_n^2\gamma_{n+1}^3\gamma_{n+2}E_{n-1}. \end{aligned}$$

9.9 In eqn (9.38), for the transition curves of Mathieu's equation for solutions of period 4π, let

$$F_{m,n} = \begin{vmatrix} 1 & \delta_m & & & & \\ \delta_{m-1} & 1 & \delta_{m-1} & & & \\ & & & \cdots & & \\ & & & \delta_{n-1} & 1 & \delta_{n-1} \\ & & & & \delta_n & 1 \end{vmatrix}.$$

Show as in the previous problem that $G_n = F_{n,n}$ satisfies the same recurrence relation as E_n for $n \geq 2$ (see Problem 9.8). Verify that

$$\begin{aligned} G_1 &= 1 - \delta_1^2, \\ G_2 &= (1 - \delta_1\delta_2)^2 - \delta_1^2, \\ G_3 &= (1 - \delta_1\delta_2 - \delta_2\delta_3)^2 - \delta_1^2(1 - \delta_2\delta_3)^2. \end{aligned}$$

9.10 Show, by the perturbation method, that the transition curves for Mathieu's equation

$$\ddot{x} + (\alpha + \beta\cos t)x = 0,$$

near $\alpha = 1$, $\beta = 0$, are given approximately by $\alpha = 1 + \frac{1}{12}\beta^2$, $\alpha = 1 - \frac{5}{12}\beta^2$.

9.11 Consider Hill's equation $\ddot{x} + f(t)x = 0$, where f has period 2π, and

$$f(t) = \alpha + \sum_{r=1}^{\infty} \beta_r \cos rt$$

is its Fourier expansion, with $\alpha \approx \frac{1}{4}$ and $|\beta_r| \ll 1$, $r = 1, 2, \ldots$. Assume an approximate solution $e^{\sigma t} q(t)$, where σ is real and q has period 4π as in (9.34). Show that

$$\ddot{q} + 2\sigma\dot{q} + \left(\sigma^2 + \alpha + \sum_{r=1}^{\infty} \beta_r \cos rt\right) q = 0.$$

Take $q \approx \sin(\frac{1}{2}t + \gamma)$ as the approximate form for q and match terms in $\sin \frac{1}{2}t$, $\cos \frac{1}{2}t$, on the assumption that these terms dominate. Deduce that

$$\sigma^2 = -(\alpha + \tfrac{1}{4}) + \tfrac{1}{2}\sqrt{(4\alpha + \beta_1^2)}$$

and that the transition curves near $\alpha = \frac{1}{4}$ are given by $\alpha = \frac{1}{4} \pm \frac{1}{2}\beta_1$. ($\beta_n$ is similarly the dominant coefficient for transition curves near $\alpha = \frac{1}{4}n^2$, $n \geq 1$.)

9.12 Obtain, as in Section 9.4, the boundary of the stable region in the neighbourhood of $\nu = 1$, $\beta = 0$ for Mathieu's equation with damping,

$$\ddot{x} + \kappa\dot{x} + (\nu + \beta \cos t)x = 0,$$

where $\kappa = O(\beta^2)$.

9.13 Solve Meissner's equation

$$\ddot{x} + (\alpha + \beta f(t))x = 0$$

where $f(t) = 1$, $0 \leq t < \pi$; $f(t) = -1$, $\pi \leq t < 2\pi$ and $f(t + 2\pi) = f(t)$ for all t. Find the conditions on α, β, for periodic solutions by putting $x(0) = x(2\pi)$, $\dot{x}(0) = \dot{x}(2\pi)$ and by making x and $\dot{x}$ continuous at $t = \pi$. Find a determinant equation for α and β.

9.14 By using the harmonic balance method of Chapter 4, show that the van der Pol equation with parametric excitation,

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + (1 + \beta \cos t)x = 0$$

has a 2π-periodic solution with approximately the same amplitude as the unforced van der Pol equation.

9.15 The male population M and female population F for a bird community have a constant death rate k and a variable birth rate $\mu(t)$ which has period T, so that

$$\dot{M} = -kM + \mu(t)F, \quad \dot{F} = -kF + \mu(t)F.$$

The births are seasonal, with rate

$$\mu(t) = \begin{cases} \delta, & 0 < t \leq \varepsilon; \\ 0, & \varepsilon < t \leq T. \end{cases}$$

Show that periodic solutions of period T exist for M and F if $kT = \delta\varepsilon$.

9.16 A pendulum bob is suspended by a light rod of length a, and the support is constrained to move vertically with displacement $\zeta(t)$. Show (by using the Lagrange's equation method or otherwise) that the equation of motion is

$$a\ddot{\theta} + (g + \ddot{\zeta}(t)) \sin\theta = 0,$$

where θ is the angle of inclination to the downward vertical. Examine the stablity of the motion for the case when $\zeta(t) = c\sin\omega t$, on the assumption that it is permissible to put $\sin\theta \approx \theta$.

9.17 A pendulum, with bob of mass m and rigid suspension of length a, hangs from a support which is constrained to move with vertical and horizontal displacements $\zeta(t)$ and $\eta(t)$ respectively. Show that the inclination θ of the pendulum satisfies the equation

$$a\ddot{\theta} + (g + \ddot{\zeta})\sin\theta + \ddot{\eta}\cos\theta = 0.$$

Let $\zeta = A\sin\omega t$ and $\eta = B\sin 2\omega t$, where $\omega = \surd(g/a)$. Show that after linearizing this equation for small amplitudes, the resulting equation has a solution

$$\theta = -(8B/A)\cos\omega t.$$

Determine the stability of this solution.

9.18 The equation

$$\ddot{x} + (\tfrac{1}{4} - 2\varepsilon b\cos^2 \tfrac{1}{2}t)x + \varepsilon x^3 = 0$$

has the exact solution $x^*(t) = \surd(2b)\cos\frac{1}{2}t$. Show that the solution is stable by constructing the variational equation.

9.19 Consider the equation $\ddot{x} + (\alpha + \beta\cos t)x = 0$, where $|\beta| \ll 1$ and $\alpha = \frac{1}{4} + \beta c$. In the unstable region near $\alpha = \frac{1}{4}$ (Section 9.4) this equation has solutions of the form $c_1 e^{\sigma t} q_1(t) + c_2 e^{-\sigma t} q_2(t)$, where σ is real, $\sigma > 0$ and q_1, q_2 have period 4π. Construct the equation for q_1, q_2, and show that $\sigma \approx \pm\beta\surd(\frac{1}{4} - c^2)$.

9.20 By using the method of Section 9.5 show that a solution of the equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = \Gamma\cos\omega t$$

where $|\varepsilon| \ll 1$, $\omega = 1 + \varepsilon\omega_1$, of the form $x^* = r_0\cos(\omega t + \alpha)$ (α constant) is asymptotically stable when

$$4\omega_1^2 + \tfrac{3}{16}r_0^4 - r_0^2 + 1 < 0.$$

(Use the result of Problem 9.19.)

9.21 The equation

$$\ddot{x} + \alpha x + \varepsilon x^3 = \varepsilon\gamma\cos\omega t$$

has the exact subharmonic solution

$$x = (4\gamma)^{1/3}\cos\tfrac{1}{3}\omega t,$$

when

$$\omega^2 = 9\left(\alpha + \frac{3}{4^{1/3}}\varepsilon\gamma^{2/3}\right).$$

If $0 < \varepsilon \ll 1$, show that the solution is stable.

9.22 Analyse the stability of the equation

$$\ddot{x} + \varepsilon x\dot{x}^2 + x = \Gamma\cos\omega t$$

for small ε: assume $\Gamma = \varepsilon\gamma$. (First find approximate solutions of the form $a\cos\omega t + b\cos\omega t$ by the harmonic balance method of Chapter 4, then perturb the solution by the method of Section 9.4.)

9.23 The equation $\ddot{x} + x + \varepsilon x^3 = \Gamma\cos\omega t(\varepsilon \ll 1)$ has an approximate solution $x^*(t) = a\cos\omega t$ where (eqn (7.10)) $\frac{3}{4}\varepsilon a^3 - (\omega^2 - 1)a - \Gamma = 0$: Show that the first variational equation (Section 9.4) is $\xi + \{1 + 3\varepsilon x^{*2}(t)\}\xi = 0$. Reduce this to Mathieu's equation and find conditions for stability of $x^*(t)$ if $\Gamma = \varepsilon\gamma$.

9.24 The equation $\ddot{x} + x - \frac{1}{6}x^3 = 0$ has an approximate solution $a \cos \omega t$ where $\omega^2 = 1 - \frac{1}{8}a^2$, $a \ll 1$ (Example 4.10). Use the method of Section 9.4 to show that the solution is unstable.

9.25 Show that a fundamental matrix of the differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x},$$

where

$$\boldsymbol{A}(t) = \begin{bmatrix} \beta \cos^2 t - \sin^2 t & 1 - (1+\beta)\sin t \cos t \\ -1 - (1+\beta)\sin t \cos t & -1 + (1+\beta)\sin^2 t \end{bmatrix}$$

is

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} \mathrm{e}^{\beta t}\cos t & \mathrm{e}^{-t}\sin t \\ -\mathrm{e}^{\beta t}\sin t & \mathrm{e}^{-t}\cos t \end{bmatrix}.$$

Find the characteristic multipliers of the system. For what value of β will periodic solutions exist?

Find the eigenvalues of $\boldsymbol{A}(t)$ and show that they are independent of t. Show that for $0 < \beta < 1$ the eigenvalues have negative real parts. What does this problem indicate about the relationship between the eigenvalues of a linear system with a variable coefficients and the stability of the zero solution?

9.26 Find a fundamental matrix for the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x},$$

where

$$\boldsymbol{A}(t) = \begin{bmatrix} \sin t & 1 \\ -\cos t + \cos^2 t & -\sin t \end{bmatrix}.$$

Show that the characteristic multipliers of the system are $\mu_1 = \mathrm{e}^{2\pi}$ and $\mu_2 = \mathrm{e}^{-2\pi}$. By integration confirm that

$$\exp\left(\int_0^{2\pi} \operatorname{tr}\{\boldsymbol{A}(s)\}\mathrm{d}s\right) = \mu_1\mu_2 = 1.$$