CHAPTER XIX

MATHIEU FUNCTIONS

19·1. *The differential equation of Mathieu.*

The preceding five chapters have been occupied with the discussion of functions which belong to what may be generally described as the hypergeometric type, and many simple properties of these functions are now well known.

In the present chapter we enter upon a region of Analysis which lies beyond this, and which is, as yet, only very imperfectly explored.

The functions which occur in Mathematical Physics and which come next in order of complication to functions of hypergeometric type are called *Mathieu functions*; these functions are also known as *the functions associated with the elliptic cylinder*. They arise from the equation of two-dimensional wave motion, namely

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}.$$

This partial differential equation occurs in the theory of the propagation of electromagnetic waves; if the electric vector in the wave-front is parallel to OZ and if E denotes the electric force, while $(H_x, H_y, 0)$ are the components of magnetic force, Maxwell's fundamental equations are

$$\frac{1}{c^2}\frac{\partial E}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \quad \frac{\partial H_x}{\partial t} = -\frac{\partial E}{\partial y}, \quad \frac{\partial H_y}{\partial t} = \frac{\partial E}{\partial x},$$

c denoting the velocity of light; and these equations give at once

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}.$$

In the case of the scattering of waves, propagated parallel to OX, incident on an elliptic cylinder for which OX and OY are axes of a principal section, the boundary condition is that E should vanish at the surface of the cylinder.

The same partial differential equation occurs in connexion with the vibrations of a uniform plane membrane, the dependent variable being the displacement perpendicular to the membrane; if the membrane be in the shape of an ellipse with a rigid boundary, the boundary condition is the same as in the electromagnetic problem just discussed.

The differential equation was discussed by Mathieu* in 1868 in connexion with the problem of vibrations of an elliptic membrane in the following manner:

* *Journal de Math.* (2), XIII. (1868), p. 137.

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Suppose that the membrane, which is in the plane XOY when it is in equilibrium, is vibrating with frequency p. Then, if we write

$$V = u(x, y) \cos(pt + \epsilon),$$

the equation becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{p^2}{c^2} u = 0.$$

Let the foci of the elliptic membrane be $(\pm h, 0, 0)$, and introduce new real variables* ξ, η defined by the complex equation

$$x + iy = h \cosh(\xi + i\eta),$$

so that $$x = h \cosh \xi \cos \eta, \quad y = h \sinh \xi \sin \eta.$$

The curves, on which ξ or η is constant, are evidently ellipses or hyperbolas confocal with the boundary; if we take $\xi \geqslant 0$ and $-\pi < \eta \leqslant \pi$, to each point $(x, y, 0)$ of the plane corresponds one and only one† value of (ξ, η).

The differential equation for u transforms into‡

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{h^2 p^2}{c^2} (\cosh^2 \xi - \cos^2 \eta)\, u = 0.$$

If we assume a solution of this equation of the form

$$u = F(\xi)\, G(\eta),$$

where the factors are functions of ξ only and of η only respectively, we see that

$$\left\{\frac{1}{F(\xi)} \frac{d^2 F(\xi)}{d\xi^2} + \frac{h^2 p^2}{c^2} \cosh^2 \xi\right\} = -\left\{\frac{1}{G(\eta)} \frac{d^2 G(\eta)}{d\eta^2} - \frac{h^2 p^2}{c^2} \cos^2 \eta\right\}.$$

Since the left-hand side contains ξ but not η, while the right-hand side contains η but not ξ, $F(\xi)$ and $G(\eta)$ must be such that each side is a constant, A, say, since ξ and η are independent variables.

We thus arrive at the equations

$$\frac{d^2 F(\xi)}{d\xi^2} + \left(\frac{h^2 p^2}{c^2} \cosh^2 \xi - A\right) F(\xi) = 0,$$

$$\frac{d^2 G(\eta)}{d\eta^2} - \left(\frac{h^2 p^2}{c^2} \cos^2 \eta - A\right) G(\eta) = 0.$$

By a slight change of independent variable in the former equation, we see that *both of these equations are linear differential equations, of the second order, of the form*

$$\frac{d^2 u}{dz^2} + (a + 16q \cos 2z)\, u = 0,$$

* The introduction of these variables is due to Lamé, who called ξ the *thermometric parameter*. They are more usually known as *confocal coordinates*. See Lamé, *Sur les fonctions inverses des transcendantes*, 1ère Leçon.

† This may be seen most easily by considering the ellipses obtained by giving ξ various positive values. If the ellipse be drawn through a definite point (ξ, η) of the plane, η is the eccentric angle of that point on the ellipse.

‡ A proof of this result, due to Lamé, is given in numerous text-books; see p. 401, footnote.

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where a and q are constants*. It is obvious that every point (infinity excepted) is a regular point of this equation.

This is the equation which is known as *Mathieu's equation* and, in certain circumstances (§ 19·2), particular solutions of it are called *Mathieu functions.*

19·11. *The form of the solution of Mathieu's equation.*

In the physical problems which suggested Mathieu's equation, the constant a is not given *a priori*, and we have to consider how it is to be determined. It is obvious from physical considerations in the problem of the membrane that $u(x, y)$ is a *one-valued* function of position, and is consequently unaltered by increasing η by 2π; and the condition† $G(\eta + 2\pi) = G(\eta)$ is sufficient to determine a set of values of a in terms of q. And it will appear later (§§ 19·4, 19·41) that, when a has not one of these values, the equation

$$G(\eta + 2\pi) = G(\eta)$$

is no longer true.

When a is thus determined, q (and thence p) is determined by the fact that $F(\xi) = 0$ on the boundary; and so the periods of the free vibrations of the membrane are obtained.

Other problems of Mathematical Physics which involve Mathieu functions in their solution are (i) Tidal waves in a cylindrical vessel with an elliptic boundary, (ii) Certain forms of steady vortex motion in an elliptic cylinder, (iii) The decay of magnetic force in a metal cylinder‡. The equation also occurs in a problem of Rigid Dynamics which is of general interest§.

19·12. *Hill's equation.*

A differential equation, similar to Mathieu's but of a more general nature, arises in G. W. Hill's|| method of determining the motion of the Lunar Perigee, and in Adams'¶ determination of the motion of the Lunar Node. Hill's equation is

$$\frac{d^2u}{dz^2} + \left(\theta_0 + 2\sum_{n=1}^{\infty} \theta_n \cos 2nz\right) u = 0.$$

The theory of Hill's equation is very similar to that of Mathieu's (in spite of the increase in generality due to the presence of the infinite series), so the two equations will, to some extent, be considered together.

* Their actual values are $a = A - h^2p^2/(2c^2)$, $q = h^2p^2/(32c^2)$; the factor 16 is inserted to avoid powers of 2 in the solution.

† An elementary analogue of this result is that a solution of $\frac{d^2u}{dz^2} + au = 0$ has period 2π if, and only if, a is the square of an integer.

‡ R. C. Maclaurin, *Trans. Camb. Phil. Soc.* XVII. p. 41.

§ A. W. Young, *Proc. Edinburgh Math. Soc.* XXXII. p. 81.

|| *Acta Math.* VIII. (1886). Hill's memoir was originally published in 1877 at Cambridge, U.S.A.

¶ *Monthly Notices R.A.S.* XXXVIII. p. 43.

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In the astronomical applications $\theta_0, \theta_1, \ldots$ are *known* constants, so the problem of choosing them in such a way that the solution may be periodic does not arise. The solution of Hill's equation in the Lunar Theory is, in fact, not periodic.

19·2. *Periodic solutions of Mathieu's equation.*

We have seen that in physical (as distinguished from astronomical) problems the constant a in Mathieu's equation has to be chosen to be such a function of q that the equation possesses a periodic solution.

Let this solution be $G(z)$; then $G(z)$, in addition to being periodic, is an integral function of z. Three possibilities arise as to the nature of $G(z)$: (i) $G(z)$ may be an *even* function of z, (ii) $G(z)$ may be an *odd* function of z, (iii) $G(z)$ may be neither even nor odd.

In case (iii), $\frac{1}{2}\{G(z)+G(-z)\}$

is an *even* periodic solution and

$$\frac{1}{2}\{G(z)-G(-z)\}$$

is an *odd* periodic solution of Mathieu's equation, these two solutions forming a fundamental system. It is therefore sufficient to confine our attention to periodic solutions of Mathieu's equation which are either even or odd. These solutions, *and these only*, will be called *Mathieu functions.*

It will be observed that, since the roots of the indicial equation at $z=0$ are 0 and 1, two even (or two odd) periodic solutions of Mathieu's equation cannot form a fundamental system. But, so far, there seems to be no reason why Mathieu's equation, for special values of a and q, should not have one even and one odd periodic solution; for comparatively small values of $|q|$ it can be seen [§ 19·3 example 2, (ii) and (iii)] that Mathieu's equation has two periodic solutions only in the trivial case in which $q=0$; the result that there are never pairs of periodic solutions for larger values of $|q|$ is a special case of a theorem due to Hille, *Proc. London Math. Soc.* (2) XXIII. (1924), p. 224. See also Ince, *Proc. Camb. Phil. Soc.* XXI. (1922), p. 117.

19·21. *An integral equation satisfied by even Mathieu functions*.*

It will now be shewn that, if $G(\eta)$ is any even Mathieu function, *then* $G(\eta)$ *satisfies the homogeneous integral equation*

$$G(\eta)=\lambda\int_{-\pi}^{\pi} e^{k\cos\eta\cos\theta}\,G(\theta)\,d\theta,$$

where $k=\sqrt{(32q)}$. This result is suggested by the solution of Laplace's equation given in § 18·3.

* This integral equation and the expansions of § 19·3 were published by Whittaker, *Proc. Int. Congress of Math.* 1912. The integral equation was known to him as early as 1904; see *Trans. Camb. Phil. Soc.* XXI. (1912), p. 193.

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For, if $x+iy=h\cosh(\xi+i\eta)$ and if $F(\xi)$ and $G(\eta)$ are solutions of the differential equations

$$\frac{d^2F(\xi)}{d\xi^2}-(A+m^2h^2\cosh^2\xi)F(\xi)=0,$$

$$\frac{d^2G(\eta)}{d\eta^2}+(A+m^2h^2\cos^2\eta)G(\eta)=0,$$

then, by § 19·1, $F(\xi)G(\eta)e^{miz}$ is a particular solution of Laplace's equation. If this solution is a special case of the general solution

$$\int_{-\pi}^{\pi}f(h\cosh\xi\cos\eta\cos\theta+h\sinh\xi\sin\eta\sin\theta+iz,\ \theta)\,d\theta,$$

given in § 18·3, it is natural to expect that*

$$f(v,\theta)\equiv F(0)e^{mv}\phi(\theta),$$

where $\phi(\theta)$ is a function of θ to be determined. Thus

$$F(\xi)G(\eta)e^{miz}=\int_{-\pi}^{\pi}F(0)\phi(\theta)\exp\{mh\cosh\xi\cos\eta\cos\theta + mh\sinh\xi\sin\eta\sin\theta+miz\}\,d\theta.$$

Since ξ and η are independent, we may put $\xi=0$; and we are thus led to consider the possibility of Mathieu's equation possessing a solution of the form

$$G(\eta)=\int_{-\pi}^{\pi}e^{mh\cos\eta\cos\theta}\phi(\theta)\,d\theta.$$

19·22. *Proof that the even Mathieu functions satisfy the integral equation.*

It is readily verified (§ 5·31) that, if $\phi(\theta)$ be analytic in the range $(-\pi,\pi)$ and if $G(\eta)$ be *defined* by the equation

$$G(\eta)=\int_{-\pi}^{\pi}e^{mh\cos\eta\cos\theta}\phi(\theta)\,d\theta,$$

then $G(\eta)$ is an even periodic integral function of η and

$$\frac{d^2G(\eta)}{d\eta^2}+(A+m^2h^2\cos^2\eta)G(\eta)$$

$$=\int_{-\pi}^{\pi}\{m^2h^2(\sin^2\eta\cos^2\theta+\cos^2\eta)-mh\cos\eta\cos\theta+A\}e^{mh\cos\eta\cos\theta}\phi(\theta)\,d\theta$$

$$=-\left[\{mh\sin\theta\cos\eta\,\phi(\theta)+\phi'(\theta)\}e^{mh\cos\eta\cos\theta}\right]_{-\pi}^{\pi}$$

$$+\int_{-\pi}^{\pi}\{\phi''(\theta)+(A+m^2h^2\cos^2\theta)\phi(\theta)\}e^{mh\cos\eta\cos\theta}\,d\theta,$$

on integrating by parts.

* The constant $F(0)$ is inserted to simplify the algebra.

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But if $\phi(\theta)$ *be a* periodic *function (with period* 2π*) such that*

$$\phi''(\theta) + (A + m^2h^2\cos^2\theta)\,\phi(\theta) = 0,$$

both the integral and the integrated part vanish; that is to say, $G(\eta)$, defined by the integral, is a periodic solution of Mathieu's equation.

Consequently $G(\eta)$ is an even periodic solution of Mathieu's equation if $\phi(\theta)$ is a periodic solution of Mathieu's equation formed with the same constants; and therefore $\phi(\theta)$ is a constant multiple of $G(\theta)$; let it be $\lambda G(\theta)$.

[In the case when the Mathieu equation has two periodic solutions, if this case exist, we have $\phi(\theta) = \lambda G(\theta) + G_1(\theta)$ where $G_1(\theta)$ is an odd periodic function; but

$$\int_{-\pi}^{\pi} e^{mh\cos\eta\cos\theta}\, G_1(\theta)\, d\theta$$

vanishes, so the subsequent work is unaffected.]

If we take a and q as the parameters of the Mathieu equation instead of A and mh, it is obvious that $mh = \sqrt{(32q)} = k$.

We have thus proved that, if $G(\eta)$ be an even periodic solution of Mathieu's equation, then

$$G(\eta) = \lambda \int_{-\pi}^{\pi} e^{k\cos\eta\cos\theta}\, G(\theta)\, d\theta,$$

which is the result stated in § 19·21.

From § 11·23, it is known that this integral equation has a solution only when λ has one of the 'characteristic values.' It will be shewn in § 19·3 that for such values of λ, the integral equation affords a simple means of constructing the even Mathieu functions.

Example 1. Shew that the odd Mathieu functions satisfy the integral equation

$$G(\eta) = \lambda \int_{-\pi}^{\pi} \sin(k\sin\eta\sin\theta)\, G(\theta)\, d\theta.$$

Example 2. Shew that both the even and the odd Mathieu functions satisfy the integral equation

$$G(\eta) = \lambda \int_{-\pi}^{\pi} e^{ik\sin\eta\sin\theta}\, G(\theta)\, d\theta.$$

Example 3. Shew that when the eccentricity of the fundamental ellipse tends to zero, the confluent form of the integral equation for the even Mathieu functions is

$$J_n(x) = \frac{1}{2\pi i^n} \int_{-\pi}^{\pi} e^{ix\cos\theta} \cos n\theta\, d\theta.$$

19·3. *The construction of Mathieu functions.*

We shall now make use of the integral equation of § 19·21 to construct Mathieu functions; the canonical form of Mathieu's equation will be taken as

$$\frac{d^2u}{dz^2} + (a + 16q\cos 2z)\,u = 0.$$

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In the special case when q is zero, the periodic solutions are obtained by taking $a = n^2$, where n is any integer; the solutions are then

$$\begin{array}{lll} 1, & \cos z, & \cos 2z, \ldots, \\ & \sin z, & \sin 2z, \ldots. \end{array}$$

The Mathieu functions, which reduce to these when $q \to 0$, will be called

$$\begin{array}{lll} ce_0(z, q), & ce_1(z, q), & ce_2(z, q), \ldots, \\ & se_1(z, q), & se_2(z, q), \ldots. \end{array}$$

To make the functions precise, we take the coefficients of $\cos nz$ and $\sin nz$ in the respective Fourier series for $ce_n(z, q)$ and $se_n(z, q)$ to be unity. The functions $ce_n(z, q)$, $se_n(z, q)$ will be called *Mathieu functions of order n*.

Let us now construct $ce_0(z, q)$.

Since $ce_0(z, 0) = 1$, we see that $\lambda \to (2\pi)^{-1}$ as $q \to 0$. Accordingly we suppose that, for general values of q, the characteristic value of λ which gives rise to $ce_0(z, q)$ can be expanded in the form

$$(2\pi\lambda)^{-1} = 1 + \alpha_1 q + \alpha_2 q^2 + \ldots,$$

and that

$$ce_0(z, q) = 1 + q\beta_1(z) + q^2\beta_2(z) + \ldots,$$

where $\alpha_1, \alpha_2, \ldots$ are numerical constants and $\beta_1(z), \beta_2(z), \ldots$ are periodic functions of z which are independent of q and which contain no constant term.

On substituting in the integral equation, we find that

$$\begin{aligned}(1 + \alpha_1 q + \alpha_2 q^2 + \ldots)\{1 + q\beta_1(z) + q^2\beta_2(z) + \ldots\} \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{1 + \sqrt{(32q)}\,.\cos z \cos\theta + 16q\cos^2 z\cos^2\theta + \ldots\} \\ \times \{1 + q\beta_1(\theta) + q^2\beta_2(\theta) + \ldots\}\, d\theta.\end{aligned}$$

Equating coefficients of successive powers of q in this result and making use of the fact that $\beta_1(z), \beta_2(z), \ldots$ contain no constant term, we find in succession

$$\begin{array}{ll} \alpha_1 = 4, & \beta_1(z) = 4\cos 2z, \\ \alpha_2 = 14, & \beta_2(z) = 2\cos 4z, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \end{array}$$

and we thus obtain the following expansion:

$$\begin{aligned} ce_0(z, q) = 1 &+ \left(4q - 28q^3 + \frac{2^7.29}{9}q^5 - \ldots\right)\cos 2z + \left(2q^2 - \frac{160}{9}q^4 + \ldots\right)\cos 4z \\ &+ \left(\frac{4}{9}q^3 - \frac{13}{3}q^5 + \ldots\right)\cos 6z + \left(\frac{1}{18}q^4 - \ldots\right)\cos 8z \\ &+ \left(\frac{1}{225}q^5 - \ldots\right)\cos 10z + \ldots, \end{aligned}$$

the terms not written down being $O(q^6)$ as $q \to 0$.

The value of a is $-32q^2 + 224q^4 - \frac{2^{10}.29}{9}q^6 + O(q^8)$; it will be observed that the coefficient of $\cos 2z$ in the series for $ce_0(z, q)$ is $-a/(8q)$.

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The Mathieu functions of higher order may be obtained in a similar manner from the same integral equation and from the integral equation of § 19·22 example 1. The consideration of the convergence of the series thus obtained is postponed to § 19·61.

Example 1. Obtain the following expansions*:

(i) $$ce_0(z, q) = 1 + \sum_{r=1}^{\infty} \left\{ \frac{2^{r+1} q^r}{r!\, r!} - \frac{2^{r+3} r (3r+4) q^{r+2}}{(r+1)!\,(r+1)!} + O(q^{r+4}) \right\} \cos 2rz,$$

(ii) $$ce_1(z, q) = \cos z + \sum_{r=1}^{\infty} \left\{ \frac{2^r q^r}{(r+1)!\, r!} - \frac{2^{r+1} r q^{r+1}}{(r+1)!\,(r+1)!} + \frac{2^r q^{r+2}}{(r-1)!\,(r+2)!} + O(q^{r+3}) \right\} \cos(2r+1) z,$$

(iii) $$se_1(z, q) = \sin z + \sum_{r=1}^{\infty} \left\{ \frac{2^r q^r}{(r+1)!\, r!} + \frac{2^{r+1} r q^{r+1}}{(r+1)!\,(r+1)!} + \frac{2^r q^{r+2}}{(r-1)!\,(r+2)!} + O(q^{r+3}) \right\} \sin(2r+1) z,$$

(iv) $$ce_2(z, q) = \left\{ -2q + \frac{40}{3} q^3 + O(q^5) \right\} + \cos 2z + \sum_{r=1}^{\infty} \left\{ \frac{2^{r+1} q^r}{r!\,(r+2)!} + \frac{2^{r+1} r (47r^2 + 222r + 247) q^{r+2}}{3^2 . (r+2)!\,(r+3)!} + O(q^{r+4}) \right\} \cos(2r+2) z,$$

where, in each case, the constant implied in the symbol O depends on r but not on z.

(Whittaker.)

Example 2. Shew that the values of a associated with (i) $ce_0(z, q)$, (ii) $ce_1(z, q)$, (iii) $se_1(z, q)$, (iv) $ce_2(z, q)$ are respectively:

(i) $$-32q^2 + 224q^4 - \frac{2^{10} . 29}{9} q^6 + O(q^8),$$

(ii) $$1 - 8q - 8q^2 + 8q^3 - \frac{8}{3} q^4 + O(q^5),$$

(iii) $$1 + 8q - 8q^2 - 8q^3 - \frac{8}{3} q^4 + O(q^5),$$

(iv) $$4 + \frac{80}{3} q^2 - \frac{6104}{27} q^4 + O(q^6).$$

(Mathieu.)

Example 3. Shew that, if n be an integer,

$$ce_{2n+1}(z, q) = (-)^n se_{2n+1}(z + \tfrac{1}{2}\pi, -q).$$

19·31. *The integral formulae for the Mathieu functions.*

Since all the Mathieu functions satisfy a homogeneous integral equation with a symmetrical nucleus (§ 19·22 example 3), it follows (§ 11·61) that

$$\int_{-\pi}^{\pi} ce_m(z, q)\, ce_n(z, q)\, dz = 0 \qquad (m \neq n),$$

$$\int_{-\pi}^{\pi} se_m(z, q)\, se_n(z, q)\, dz = 0 \qquad (m \neq n),$$

$$\int_{-\pi}^{\pi} ce_m(z, q)\, se_n(z, q)\, dz = 0.$$

* The leading terms of these series, as given in example 4 at the end of the chapter (p. 427), were obtained by Mathieu.

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Example 1. Obtain expansions of the form:

$$\text{(i)} \qquad e^{k\cos z\cos\theta} = \sum_{n=0}^{\infty} A_n ce_n(z, q)\, ce_n(\theta, q),$$

$$\text{(ii)} \qquad \cos(k\sin z\sin\theta) = \sum_{n=0}^{\infty} B_n ce_n(z, q)\, ce_n(\theta, q),$$

$$\text{(iii)} \qquad \sin(k\sin z\sin\theta) = \sum_{n=0}^{\infty} C_n se_n(z, q)\, se_n(\theta, q),$$

where $k = \sqrt{(32q)}$.

Example 2. Obtain the expansion

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{ni\phi}$$

as a confluent form of expansions (ii) and (iii) of example 1.

19·4. *The nature of the solution of Mathieu's general equation; Floquet's theory.*

We shall now discuss the nature of the solution of Mathieu's equation when the parameter a is no longer restricted so as to give rise to periodic solutions; this is the case which is of importance in astronomical problems, as distinguished from other physical applications of the theory.

The method is applicable to any linear equation with *periodic* coefficients which are one-valued functions of the independent variable; the nature of the general solution of particular equations of this type has long been perceived by astronomers, by inference from the circumstances in which the equations arise. These inferences have been confirmed by the following analytical investigation which was published in 1883 by Floquet*.

Let $g(z)$, $h(z)$ be a fundamental system of solutions of Mathieu's equation (or, indeed, of any linear equation in which the coefficients have period 2π); then, if $F(z)$ be any other integral of such an equation, we must have

$$F(z) = Ag(z) + Bh(z),$$

where A and B are definite constants.

Since $g(z+2\pi)$, $h(z+2\pi)$ are obviously solutions of the equation†, they can be expressed in terms of the continuations of $g(z)$ and $h(z)$ by equations of the type

$$g(z+2\pi) = \alpha_1 g(z) + \alpha_2 h(z), \quad h(z+2\pi) = \beta_1 g(z) + \beta_2 h(z),$$

where α_1, α_2, β_1, β_2 are definite constants; and then

$$F(z+2\pi) = (A\alpha_1 + B\beta_1)\, g(z) + (A\alpha_2 + B\beta_2)\, h(z).$$

* *Ann. de l'École norm. sup.* (2), XII. (1883), p. 47. Floquet's analysis is a natural sequel to Picard's theory of differential equations with doubly-periodic coefficients (§ 20·1), and to the theory of the fundamental equation due to Fuchs and Hamburger.

† These solutions may not be identical with $g(z)$, $h(z)$ respectively, as the solution of an equation with periodic coefficients is not necessarily periodic. To take a simple case, $u = e^z \sin z$ is a solution of $\frac{du}{dz} - (1+\cot z)\, u = 0$.

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Consequently $F(z+2\pi)=kF(z)$, *where* k *is a constant**, *if* A *and* B *are chosen so that*

$$A\alpha_1+B\beta_1=kA, \quad A\alpha_2+B\beta_2=kB.$$

These equations will have a solution, other than $A=B=0$, if, and only if,

$$\begin{vmatrix} \alpha_1-k, & \beta_1 \\ \alpha_2\ \ , & \beta_2-k \end{vmatrix}=0;$$

and if k be taken to be either root of this equation, the function $F(z)$ can be constructed so as to be a solution of the differential equation such that

$$F(z+2\pi)=kF(z).$$

Defining μ by the equation $k=e^{2\pi\mu}$ and writing $\phi(z)$ for $e^{-\mu z}F(z)$, we see that

$$\phi(z+2\pi)=e^{-\mu(z+2\pi)}F(z+2\pi)=\phi(z).$$

Hence the differential equation has a particular solution of the form $e^{\mu z}\phi(z)$, *where* $\phi(z)$ *is a periodic function with period* 2π.

We have seen that in physical problems, the parameters involved in the differential equation have to be so chosen that $k=1$ is a root of the quadratic, and a solution is periodic. In general, however, in astronomical problems, in which the parameters are given, $k\neq 1$ and there is no periodic solution.

In the particular case of Mathieu's general equation or Hill's equation, a fundamental system of solutions† is then $e^{\mu z}\phi(z)$, $e^{-\mu z}\phi(-z)$, since the equation is unaltered by writing $-z$ for z; so that the complete solution of Mathieu's general equation is then

$$u=c_1e^{\mu z}\phi(z)+c_2e^{-\mu z}\phi(-z),$$

where c_1, c_2 are arbitrary constants, and μ is a definite function of a and q.

Example. Shew that the roots of the equation

$$\begin{vmatrix} \alpha_1-k, & \beta_1 \\ \alpha_2\ \ , & \beta_2-k \end{vmatrix}=0$$

are independent of the particular pair of solutions, $g(z)$ and $h(z)$, chosen.

19·41. *Hill's method of solution.*

Now that the general functional character of the solution of equations with periodic coefficients has been found by Floquet's theory, it might be expected that the determination of an explicit expression for the solutions of Mathieu's and Hill's equations would be a comparatively easy matter; this however is not the case. For example, in the particular case of Mathieu's general equation, a solution has to be obtained in the form

$$y=e^{\mu z}\phi(z),$$

* The symbol k is used in this particular sense only in this section. It must not be confused with the constant k of § 19·21, which was associated with the parameter q of Mathieu's equation.

† The ratio of these solutions is not even periodic; still less is it a constant.

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where $\phi(z)$ is periodic and μ is a function of the parameters a and q. The crux of the problem is to determine μ; when this is done, the determination of $\phi(z)$ presents comparatively little difficulty.

The first successful method of attacking the problem was published by Hill in the memoir cited in § 19·12; since the method for Hill's equation is no more difficult than for the special case of Mathieu's general equation, we shall discuss the case of Hill's equation, viz.

$$\frac{d^2u}{dz^2} + J(z)\,u = 0,$$

where $J(z)$ is an even function of z with period π. Two cases are of interest, the analysis being the same in each:

(I) The astronomical case when z is real and, for real values of z, $J(z)$ can be expanded in the form

$$J(z) = \theta_0 + 2\theta_1 \cos 2z + 2\theta_2 \cos 4z + 2\theta_3 \cos 6z + \dots;$$

the coefficients θ_n are known constants and $\sum_{n=0}^{\infty} \theta_n$ converges absolutely.

(II) The case when z is a complex variable and $J(z)$ is analytic in a strip of the plane (containing the real axis), whose sides are parallel to the real axis. The expansion of $J(z)$ in the Fourier series $\theta_0 + 2\sum_{n=1}^{\infty} \theta_n \cos 2nz$ is then valid (§ 9·11) throughout the interior of the strip, and, as before, $\sum_{n=0}^{\infty} \theta_n$ converges absolutely.

Defining θ_{-n} to be equal to θ_n, we assume

$$u = e^{\mu z} \sum_{n=-\infty}^{\infty} b_n e^{2niz}$$

as a solution of Hill's equation.

[In case (II) this is the solution analytic in the strip (§§ 10·2, 19·4); in case (I) it will have to be shewn ultimately (see the note at the end of § 19·42) that the values of b_n which will be determined are such as to make $\sum_{n=-\infty}^{\infty} n^2 b_n$ absolutely convergent, in order to justify the processes which we shall now carry out.]

On substitution in the equation, we find

$$\sum_{n=-\infty}^{\infty} (\mu + 2ni)^2 b_n e^{(\mu+2ni)z} + \left(\sum_{n=-\infty}^{\infty} \theta_n e^{2niz}\right)\left(\sum_{n=-\infty}^{\infty} b_n e^{(\mu+2ni)z}\right) = 0.$$

Multiplying out the absolutely convergent series and equating coefficients of powers of e^{2iz} to zero (§§ 9·6–9·632), we obtain the system of equations

$$(\mu + 2ni)^2 b_n + \sum_{m=-\infty}^{\infty} \theta_m b_{n-m} = 0 \qquad (n = \dots, -2, -1, 0, 1, 2, \dots).$$

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If we eliminate the coefficients b_n determinantally (after dividing the typical equation by $\theta_0 - 4n^2$ to secure convergence) we obtain* Hill's determinantal equation:

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \frac{(i\mu+4)^2-\theta_0}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_4}{4^2-\theta_0} & \cdots \\ \cdots & \frac{-\theta_1}{2^2-\theta_0} & \frac{(i\mu+2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_3}{2^2-\theta_0} & \cdots \\ \cdots & \frac{-\theta_2}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & \frac{(i\mu)^2-\theta_0}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & \frac{-\theta_2}{0^2-\theta_0} & \cdots \\ \cdots & \frac{-\theta_3}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \frac{(i\mu-2)^2-\theta_0}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & \cdots \\ \cdots & \frac{\theta_4}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & \frac{(i\mu-4)^2-\theta_0}{4^2-\theta_0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

We write $\Delta(i\mu)$ for the determinant, so the equation determining μ is

$$\Delta(i\mu) = 0.$$

19·42. *The evaluation of Hill's determinant.*

We shall now obtain an extremely simple expression for Hill's determinant, namely

$$\Delta(i\mu) \equiv \Delta(0) - \sin^2\left(\tfrac{1}{2}\pi i\mu\right) \operatorname{cosec}^2\left(\tfrac{1}{2}\pi\sqrt{\theta_0}\right).$$

Adopting the notation of § 2·8, we write

$$\Delta(i\mu) \equiv [A_{m,n}],$$

where $$A_{m,m} = \frac{(i\mu - 2m)^2 - \theta_0}{4m^2 - \theta_0}, \qquad A_{m,n} = \frac{-\theta_{m-n}}{4m^2 - \theta_0} \qquad (m \neq n).$$

The determinant $[A_{m,n}]$ is only *conditionally* convergent, since the product of the principal diagonal elements does not converge absolutely (§§ 2·81, 2·7). We can, however, obtain an *absolutely* convergent determinant, $\Delta_1(i\mu)$, by dividing the linear equations of § 19·41 by $\theta_0 - (i\mu - 2n)^2$ instead of dividing by $\theta_0 - 4n^2$. We write this determinant $\Delta_1(i\mu)$ in the form $[B_{m,n}]$, where

$$B_{m,m} = 1, \quad B_{m,n} = \frac{-\theta_{m-n}}{(2m - i\mu)^2 - \theta_0} \qquad (m \neq n).$$

The absolute convergence of $\sum_{n=0}^{\infty} \theta_n$ secures the convergence of the determinant $[B_{m,n}]$, except when μ has such a value that the denominator of one of the expressions $B_{m,n}$ vanishes.

* Since the coefficients b_n are not all zero, we may obtain the infinite determinant as the eliminant of the system of linear equations by multiplying these equations by suitably chosen cofactors and adding up.

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From the definition of an infinite determinant (§ 2·8) it follows that

$$\Delta(i\mu) = \Delta_1(i\mu) \lim_{p\to\infty} \prod_{n=-p}^{p} \left\{\frac{\theta_0 - (i\mu - 2n)^2}{\theta_0 - 4n^2}\right\},$$

and so
$$\Delta(i\mu) = -\Delta_1(i\mu) \frac{\sin\frac{1}{2}\pi(i\mu - \sqrt{\theta_0})\sin\frac{1}{2}\pi(i\mu + \sqrt{\theta_0})}{\sin^2(\frac{1}{2}\pi\sqrt{\theta_0})}.$$

Now, if the determinant $\Delta_1(i\mu)$ be written out in full, it is easy to see (i) that $\Delta_1(i\mu)$ is an even periodic function of μ with period $2i$, (ii) that $\Delta_1(i\mu)$ is an analytic function (cf. §§ 2·81, 3·34, 5·3) of μ (except at its obvious simple poles), which tends to unity as the real part of μ tends to $\pm\infty$.

If now we choose the constant K so that the function $D(\mu)$, defined by the equation

$$D(\mu) \equiv \Delta_1(i\mu) - K\{\cot\tfrac{1}{2}\pi(i\mu + \sqrt{\theta_0}) - \cot\tfrac{1}{2}\pi(i\mu - \sqrt{\theta_0})\},$$

has no pole at the point $\mu = i\sqrt{\theta_0}$, then, since $D(\mu)$ is an even periodic function of μ, it follows that $D(\mu)$ has no pole at any of the points

$$2ni \pm i\sqrt{\theta_0},$$

where n is any integer.

The function $D(\mu)$ is therefore a periodic function of μ (with period $2i$) which has no poles, and which is obviously bounded as $R(\mu) \to \pm\infty$. The conditions postulated in Liouville's theorem (§ 5·63) are satisfied, and so $D(\mu)$ is a constant; making $\mu \to +\infty$, we see that this constant is unity.

Therefore

$$\Delta_1(i\mu) = 1 + K\{\cot\tfrac{1}{2}\pi(i\mu + \sqrt{\theta_0}) - \cot\tfrac{1}{2}\pi(i\mu - \sqrt{\theta_0})\},$$

and so

$$\Delta(i\mu) = -\frac{\sin\frac{1}{2}\pi(i\mu - \sqrt{\theta_0})\sin\frac{1}{2}\pi(i\mu + \sqrt{\theta_0})}{\sin^2(\frac{1}{2}\pi\sqrt{\theta_0})} + 2K\cot(\tfrac{1}{2}\pi\sqrt{\theta_0}).$$

To determine K, put $\mu = 0$; then

$$\Delta(0) = 1 + 2K\cot(\tfrac{1}{2}\pi\sqrt{\theta_0}).$$

Hence, on subtraction,

$$\Delta(i\mu) = \Delta(0) - \frac{\sin^2(\frac{1}{2}\pi i\mu)}{\sin^2(\frac{1}{2}\pi\sqrt{\theta_0})},$$

which is the result stated.

The roots of Hill's determinantal equation are therefore the roots of the equation

$$\sin^2(\tfrac{1}{2}\pi i\mu) = \Delta(0) \,.\, \sin^2(\tfrac{1}{2}\pi\sqrt{\theta_0}).$$

When μ has thus been determined, the coefficients b_n can be determined in terms of b_0 and cofactors of $\Delta(i\mu)$; and the solution of Hill's differential equation is complete.

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[In case (I) of § 19·41, the convergence of $\Sigma |b_n|$ follows from the rearrangement theorem of § 2·82; for $\Sigma n^2 |b_n|$ is equal to $|b_0| \sum_{m=-\infty}^{\infty} |C_{m,0}| \div |C_{0,0}|$, where $C_{m,n}$ is the cofactor of $B_{m,n}$ in $\Delta_1(i\mu)$; and $\Sigma |C_{m,0}|$ is the determinant obtained by replacing the elements of the row through the origin by numbers whose moduli are bounded.]

It was shewn by Hill that, for the purposes of his astronomical problem, a remarkably good approximation to the value of μ could be obtained by considering only the three central rows and columns of his determinant.

19·5. *The Lindemann-Stieltjes' theory of Mathieu's general equation.*

Up to the present, Mathieu's equation has been treated as a linear differential equation with periodic coefficients. Some extremely interesting properties of the equation have been obtained by Lindemann* by the substitution $\zeta = \cos^2 z$, which transforms the equation into an equation with rational coefficients, namely

$$4\zeta(1-\zeta)\frac{d^2u}{d\zeta^2} + 2(1-2\zeta)\frac{du}{d\zeta} + (a - 16q + 32q\zeta)\,u = 0.$$

This equation, though it somewhat resembles the hypergeometric equation, is of higher type than the equations dealt with in Chapters XIV and XVI, inasmuch as it has two regular singularities at 0 and 1 and an irregular singularity at ∞; whereas the three singularities of the hypergeometric equation are all regular, while the equation for $W_{k,m}(z)$ has one irregular singularity and only one regular singularity.

We shall now give a short account of Lindemann's analysis, with some modifications due to Stieltjes†.

19·51. *Lindemann's form of Floquet's theorem.*

Since Mathieu's equation (in Lindemann's form) has singularities at $\zeta = 0$ and $\zeta = 1$, the exponents at each being $0, \frac{1}{2}$, there exist solutions of the form

$$y_{00} = \sum_{n=0}^{\infty} a_n \zeta^n, \qquad y_{01} = \zeta^{\frac{1}{2}} \sum_{n=0}^{\infty} b_n \zeta^n,$$

$$y_{10} = \sum_{n=0}^{\infty} a_n'(1-\zeta)^n, \quad y_{11} = (1-\zeta)^{\frac{1}{2}} \sum_{n=0}^{\infty} b_n'(1-\zeta)^n;$$

the first two series converge when $|\zeta| < 1$, the last two when $|1-\zeta| < 1$.

When the ζ-plane is cut along the real axis from 1 to $+\infty$ and from 0 to $-\infty$, the four functions defined by these series are one-valued in the cut plane; and so relations of the form

$$y_{10} = \alpha y_{00} + \beta y_{01}, \quad y_{11} = \gamma y_{00} + \delta y_{01}$$

will exist throughout the cut plane.

Now suppose that ζ describes a closed circuit round the origin, so that the circuit crosses the cut from $-\infty$ to 0; the analytic continuation of y_{10} is

* *Math. Ann.* XXII. (1883), p. 117.

† *Astr. Nach.* CIX. (1884), cols. 145–152, 261–266. The analysis is very similar to that employed by Hermite in his lectures at the École Polytechnique in 1872–1873 [*Oeuvres*, III. (Paris, 1912), pp. 118–122] in connexion with Lamé's equation. See § 23·7.

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$\alpha y_{00} - \beta y_{01}$ (since y_{00} is unaffected by the description of the circuit, but y_{01} changes sign) and the continuation of y_{11} is $\gamma y_{00} - \delta y_{01}$; *and so* $Ay_{10}^2 + By_{11}^2$ *will be unaffected by the description of the circuit if*

$$A(\alpha y_{00} + \beta y_{01})^2 + B(\gamma y_{00} + \delta y_{01})^2 \equiv A(\alpha y_{00} - \beta y_{01})^2 + B(\gamma y_{00} - \delta y_{01})^2,$$

i.e. if $$A\alpha\beta + B\gamma\delta = 0.$$

Also $Ay_{10}^2 + By_{11}^2$ obviously has not a branch-point at $\zeta = 1$, and so, if $A\alpha\beta + B\gamma\delta = 0$, this function has no branch-points at 0 or 1, and, as it has no other possible singularities in the finite part of the plane, *it must be an integral function of* ζ.

The two expressions

$$A^{\frac{1}{2}} y_{10} + iB^{\frac{1}{2}} y_{11}, \quad A^{\frac{1}{2}} y_{10} - iB^{\frac{1}{2}} y_{11}$$

are consequently two solutions of Mathieu's equation whose product is an integral function of ζ.

[This amounts to the fact (§ 19·4) that the product of $e^{\mu z}\phi(z)$ and $e^{-\mu z}\phi(-z)$ is a *periodic integral* function of z.]

19·52. *The determination of the integral function associated with Mathieu's general equation.*

The integral function $F(z) \equiv Ay_{10}^2 + By_{11}^2$, just introduced, can be determined without difficulty; for, if y_{10} and y_{11} are any solutions of

$$\frac{d^2u}{d\zeta^2} + P(\zeta)\frac{du}{d\zeta} + Q(\zeta)u = 0,$$

their squares (and consequently any linear combination of their squares) satisfy the equation*

$$\frac{d^3y}{d\zeta^3} + 3P(\zeta)\frac{d^2y}{d\zeta^2} + [P'(\zeta) + 4Q(\zeta) + 2\{P(\zeta)\}^2]\frac{dy}{d\zeta}$$
$$+ 2[Q'(\zeta) + 2P(\zeta)Q(\zeta)]y = 0;$$

in the case under consideration, this result reduces to

$$\zeta(1-\zeta)\frac{d^3F(\zeta)}{d\zeta^3} + \tfrac{3}{2}(1-2\zeta)\frac{d^2F(\zeta)}{d\zeta^2}$$
$$+ (a - 1 - 16q + 32q\zeta)\frac{dF(\zeta)}{d\zeta} + 16qF(\zeta) = 0.$$

Let the Maclaurin series for $F(\zeta)$ be $\sum_{n=0}^{\infty} c_n \zeta^n$; on substitution, we easily obtain the recurrence formula for the coefficients c_n, namely

$$v_{n+1}c_{n+2} = u_n c_{n+1} + c_n,$$

where

$$u_n = -\frac{(n+1)\{(n+1)^2 - a + 16q\}}{16q(2n+1)}, \quad v_n = -\frac{n(n+1)(2n+1)}{32q(2n-1)}.$$

* Appell, *Comptes Rendus*, XCI. (1880), pp. 211–214; cf. example 10, p. 298 *supra*.

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At first sight, it appears from the recurrence formula that c_0 and c_1 can be chosen arbitrarily, and the remaining coefficients c_2, c_3, ... calculated in terms of them; but the third order equation has a singularity at $\zeta = 1$, and the series thus obtained would have only unit radius of convergence. It is necessary to choose the value of the ratio c_1/c_0 so that the series may converge for all values of ζ.

The recurrence formula, when written in the form

$$(c_n/c_{n+1}) = u_n + \frac{v_{n+1}}{(c_{n+1}/c_{n+2})},$$

suggests the consideration of the infinite continued fraction

$$u_n + \frac{v_{n+1}}{u_{n+1}+}\,\frac{v_{n+2}}{u_{n+2}+}\cdots = \lim_{m\to\infty}\left\{u_n + \frac{v_{n+1}}{u_{n+1}+}\cdots\frac{v_{n+m}}{+u_{n+m}}\right\}.$$

The continued fraction on the right can be written*

$$u_n K(n, n+m)/K(n+1, n+m),$$

where
$$K(n, n+m) = \begin{vmatrix} 1, & v_{n+1}/u_n, & 0, & \cdots\cdots \\ -u_{n+1}^{-1}, & 1, & v_{n+2}/u_{n+1}, & \cdots\cdots \\ 0, & -u_{n+2}^{-1}, & 1, & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & -u_{n+m}^{-1}, & 1 \end{vmatrix}$$

The limit of this, as $m \to \infty$, is a convergent determinant of von Koch's type (by the example of § 2·82); and since

$$\sum_{r=n}^{\infty}\left|\frac{v_{r+1}}{u_r u_{r+1}}\right| \to 0 \text{ as } n \to \infty,$$

it is easily seen that $K(n, \infty) \to 1$ as $n \to \infty$.

Therefore, if
$$\frac{c_n}{c_{n+1}} = \frac{u_n K(n, \infty)}{K(n+1, \infty)},$$

then c_n satisfies the recurrence formula and, since $c_{n+1}/c_n \to 0$ as $n \to \infty$, the resulting series for $F(\zeta)$ is an integral function. From the recurrence formula it is obvious that all the coefficients c_n are finite, since they are finite when n is sufficiently large. The construction of the integral function $F(\zeta)$ has therefore been effected.

19·53. *The solution of Mathieu's equation in terms of $F(\zeta)$.*

If w_1 and w_2 be two particular solutions of

$$\frac{d^2u}{d\zeta^2} + P(\zeta)\frac{du}{d\zeta} + Q(\zeta)\,u = 0,$$

then†
$$w_2 w_1' - w_1 w_2' = C \exp\left\{-\int_0^{\zeta} P(\zeta)\,d\zeta\right\},$$

* Sylvester, *Phil. Mag.* (4), v. (1853), p. 446 [*Math. Papers*, I. p. 609].

† Abel, *Journal für Math.* II. (1827), p. 22. Primes denote differentiations with regard to ζ.

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where C is a definite constant. Taking w_1 and w_2 to be those two solutions of Mathieu's general equation whose product is $F(\zeta)$, we have

$$\frac{w_1'}{w_1} - \frac{w_2'}{w_2} = \frac{C}{\zeta^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}} F(\zeta)}, \quad \frac{w_1'}{w_1} + \frac{w_2'}{w_2} = \frac{F'(\zeta)}{F(\zeta)},$$

the latter following at once from the equation $w_1 w_2 = F(\zeta)$.

Solving these equations for w_1'/w_1 and w_2'/w_2, and then integrating, we at once get

$$w_1 = \gamma_1 \{F(\zeta)\}^{\frac{1}{2}} \exp \left\{ \tfrac{1}{2} C \int_0^\zeta \frac{d\zeta}{\zeta^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}} F(\zeta)} \right\},$$

$$w_2 = \gamma_2 \{F(\zeta)\}^{\frac{1}{2}} \exp \left\{ -\tfrac{1}{2} C \int_0^\zeta \frac{d\zeta}{\zeta^{\frac{1}{2}}(1-\zeta)^{\frac{1}{2}} F(\zeta)} \right\},$$

where γ_1, γ_2 are constants of integration; obviously no real generality is lost by taking $c_0 = \gamma_1 = \gamma_2 = 1$.

From the former result we have, for small values of $|\zeta|$,

$$w_1 = 1 + C\zeta^{\frac{1}{2}} + \tfrac{1}{2}(c_1 + C^2)\zeta + O(\zeta^{\frac{3}{2}}),$$

while, in the notation of § 19·51, we have $a_1/a_0 = -\tfrac{1}{2}a + 8q$.

Hence $$C^2 = 16q - a - c_1.$$

This equation determines C in terms of a, q and c_1, the value of c_1 being

$$K(1, \infty) \div \{u_0 K(0, \infty)\}.$$

Example 1. If the solutions of Mathieu's equation be $e^{\pm\mu z}\phi(\pm z)$, where $\phi(z)$ is periodic, shew that

$$\pi\mu = \pm C \int_0^\pi \frac{dz}{F(\cos^2 z)}.$$

Example 2. Shew that the zeros of $F(\zeta)$ are all simple, unless $C = 0$.

(Stieltjes.)

[If $F(\zeta)$ could have a repeated zero, w_1 and w_2 would then have an essential singularity.]

19·6. *A second method of constructing the Mathieu function.*

So far, it has been assumed that all the various series of § 19·3 involved in the expressions for $ce_N(z, q)$ and $se_N(z, q)$ are convergent. *It will now be shewn that $ce_N(z, q)$ and $se_N(z, q)$ are integral functions of z and that the coefficients in their expansions as Fourier series are power series in q which converge absolutely when $|q|$ is sufficiently small**.

To obtain this result for the functions $ce_N(z, q)$, we shall shew how to determine a particular integral of the equation

$$\frac{d^2u}{dz^2} + (a + 16q\cos 2z)u = \psi(a, q)\cos Nz$$

* The essential part of this theorem is the proof of the *convergence of the series which occur in the coefficients*; it is already known (§§ 10·2, 10·21) that solutions of Mathieu's equation are integral functions of z, and (in the case of *periodic* solutions) the existence of the Fourier expansion follows from § 9·11.

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in the form of a Fourier series converging over the whole z-plane, where $\psi(a, q)$ is a function of the parameters a and q. The equation $\psi(a, q) = 0$ then determines a relation between a and q which gives rise to a Mathieu function. The reader who is acquainted with the method of Frobenius* as applied to the solution of linear differential equations in power series will recognise the resemblance of the following analysis to his work.

Write $a = N^2 + 8p$, where N is zero or a positive or negative integer.

Mathieu's equation becomes

$$\frac{d^2u}{dz^2} + N^2 u = -8(p + 2q\cos 2z)\, u.$$

If p and q are neglected, a solution of this equation is $u = \cos Nz = U_0(z)$, say.

To obtain a closer approximation, write $-8(p + 2q\cos 2z)\, U_0(z)$ as a sum of cosines, i.e. in the form

$$-8\{q\cos(N-2)z + p\cos Nz + q\cos(N+2)z\} = V_1(z), \text{ say.}$$

Then, instead of solving $\dfrac{d^2u}{dz^2} + N^2 u = V_1(z)$, suppress the terms† in $V_1(z)$ which involve $\cos Nz$; i.e. consider the function $W_1(z)$ where‡

$$W_1(z) = V_1(z) + 8p\cos Nz.$$

A particular integral of

$$\frac{d^2u}{dz^2} + N^2 u = W_1(z)$$

is

$$u = 2\left\{\frac{q}{1(1-N)}\cos(N-2)z + \frac{q}{1(1+N)}\cos(N+2)z\right\} = U_1(z), \text{ say.}$$

Now express $-8(p + 2q\cos 2z)\, U_1(z)$ as a sum of cosines; calling this sum $V_2(z)$, choose α_2 to be such a function of p and q that $V_2(z) + \alpha_2\cos Nz$ contains no term in $\cos Nz$; and let $V_2(z) + \alpha_2\cos Nz = W_2(z)$.

Solve the equation $\qquad \dfrac{d^2u}{dz^2} + N^2 u = W_2(z),$

and continue the process. Three sets of functions $U_m(z)$, $V_m(z)$, $W_m(z)$ are thus obtained, such that $U_m(z)$ and $W_m(z)$ contain no term in $\cos Nz$ when $m \neq 0$, and

$$W_m(z) = V_m(z) + \alpha_m \cos Nz, \quad V_m(z) = -8(p + 2q\cos 2z)\, U_{m-1}(z),$$

$$\frac{d^2 U_m(z)}{dz^2} + N^2 U_m(z) = W_m(z),$$

where α_m is a function of p and q but not of z.

* *Journal für Math.* LXXVI. (1873), pp. 214–224.

† The reason for this suppression is that the particular integral of $\dfrac{d^2u}{dz^2} + N^2u = \cos Nz$ contains non-periodic terms.

‡ Unless $N = 1$, in which case $W_1(z) = V_1(z) + 8(p+q)\cos z$.

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It follows that

$$\left\{\frac{d^2}{dz^2}+N^2\right\}\sum_{m=0}^{n} U_m(z)=\sum_{m=1}^{n} W_m(z)$$
$$=\sum_{m=1}^{n} V_m(z)+\left(\sum_{m=1}^{n}\alpha_m\right)\cos Nz$$
$$=-8\,(p+2q\cos 2z)\sum_{m=0}^{n-1} U_{m-1}(z)+\left(\sum_{m=1}^{n}\alpha_m\right)\cos Nz.$$

Therefore, if $U(z)=\sum_{m=0}^{\infty} U_m(z)$ be a uniformly convergent series of analytic functions throughout a two-dimensional region in the z-plane, we have (§ 5·3)

$$\frac{d^2U(z)}{dz^2}+(a+16q\cos 2z)\,U(z)=\psi(a,q)\cos Nz,$$

where
$$\psi(a,q)=\sum_{m=1}^{\infty}\alpha_m.$$

It is obvious that, if a be so chosen that $\psi(a,q)=0$, then $U(z)$ reduces to $ce_N(z)$.

A similar process can obviously be carried out for the functions $se_N(z,q)$ by making use of sines of multiples of z.

19·61. *The convergence of the series defining Mathieu functions.*

We shall now examine the expansion of § 19·6 more closely, with a view to investigating the convergence of the series involved.

When $n\geqslant 1$, we may obviously write

$$U_n(z)=\sum_{r=1}^{n}{}^*\beta_{n,r}\cos(N-2r)z+\sum_{r=1}^{n}a_{n,r}\cos(N+2r)z,$$

the asterisk denoting that the first summation ceases at the greatest value of r for which $r\leqslant\frac{1}{2}N$.

Since
$$\left\{\frac{d^2}{dz^2}+N^2\right\}U_{n+1}(z)=a_{n+1}\cos Nz-8\,(p+2q\cos 2z)\,U_n(z),$$

it follows on equating coefficients of $\cos(N\pm 2r)z$ on each side of the equation† that

$$a_{n+1}=8q\,(a_{n,1}+\beta_{n,1}),$$
$$r(r+N)\,a_{n+1,r}=2\,\{pa_{n,r}+q\,(a_{n,r-1}+a_{n,r+1})\}\qquad(r=1,2,\ldots),$$
$$r(r-N)\,\beta_{n+1,r}=2\,\{p\beta_{n,r}+q\,(\beta_{n,r-1}+\beta_{n,r+1})\}\qquad(r\leqslant\tfrac{1}{2}N).$$

These formulae hold universally with the following conventions‡:

(i) $a_{n,0}=\beta_{n,0}=0\quad(n=1,2,\ldots);\qquad a_{n,r}=\beta_{n,r}=0\quad(r>n)$,

(ii) $\beta_{n,\frac{1}{2}N+1}=\beta_{n,\frac{1}{2}N-1}$ when N is even and $r=\frac{1}{2}N$,

(iii) $\beta_{n,\frac{1}{2}(N+1)}=\beta_{n,\frac{1}{2}(N-1)}$ when N is odd and $r=\frac{1}{2}(N-1)$.

† When $N=0$ or 1 these equations must be modified by the suppression of all the coefficients $\beta_{n,r}$.

‡ The conventions (ii) and (iii) are due to the fact that $\cos z=\cos(-z)$, $\cos 2z=\cos(-2z)$.

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The reader will easily obtain the following special formulae:

(I) $a_1 = 8p, \quad (N \neq 1); \qquad a_1 = 8(p+q), \quad (N=1),$

(II) $a_{n,n} = \dfrac{(2q)^n . N!}{n!\,(N+n)!}, \quad (N \neq 0); \qquad a_{n,n} = \dfrac{2^{n+1} q^n}{(n!)^2}, \quad (N=0).$

(III) $a_{n,r}$ and $\beta_{n,r}$ are homogeneous polynomials of degree n in p and q.

If
$$\sum_{n=r}^{\infty} a_{n,r} = A_r, \quad \sum_{n=r}^{\infty} \beta_{n,r} = B_r,$$
we have
$$\psi(a, q) = 8p + 8q(A_1 + B_1) \qquad (N \neq 1),$$
$$r(r+N) A_r = 2\{pA_r + q(A_{r-1} + A_{r+1})\} \quad \text{..........(A)},$$
$$r(r-N) B_r = 2\{pB_r + q(B_{r-1} + B_{r+1})\} \quad \text{..........(B)},$$
where $A_0 = B_0 = 1$ and B_r is subject to conventions due to (ii) and (iii) above.

Now write $\quad w_r = -q\{r(r+N) - 2p\}^{-1}, \quad w_r' = -q\{r(r-N) - 2p\}^{-1}.$

The result of eliminating $A_1, A_2, \dots A_{r-1}, A_{r+1}, \dots$ from the set of equations (A) is
$$A_r \Delta_0 = (-)^r w_1 w_2 \dots w_r \Delta_r,$$
where Δ_r is the infinite determinant of von Koch's type (§ 2·82)
$$\Delta_r = \begin{vmatrix} 1, & w_{r+1}, & 0, & 0, & \dots \\ w_{r+2}, & 1, & w_{r+2}, & 0, & \dots \\ 0, & w_{r+3}, & 1, & w_{r+3}, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

The determinant converges absolutely (§ 2·82 example) if no denominator vanishes; and $\Delta_r \to 1$ as $r \to \infty$ (cf. § 19·52). If p and q be given such values that $\Delta_0 \neq 0$, $2p \neq r(r+N)$, where $r = 1, 2, 3, \dots$, the series
$$\sum_{r=1}^{\infty} (-)^r w_1 w_2 \dots w_r \Delta_r \Delta_0^{-1} \cos(N+2r) z$$
represents an integral function of z.

In like manner $B_r D_0 = (-)^r w_1' w_2' \dots w_r' D_r$, where D_r is the finite determinant
$$\begin{vmatrix} 1, & w'_{r+1}, & 0, & \dots \\ w'_{r+2}, & 1, & w'_{r+2}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$
the last row being $0, 0, \dots 0, 2w'_{\frac{1}{2}N}, 1$ or $0, 0, \dots 0, w'_{\frac{1}{2}(N-1)}, 1 + w'_{\frac{1}{2}(N-1)}$ according as N is even or odd.

The series $\sum_{n=0}^{\infty} U_n(z)$ is therefore
$$\cos Nz + \Delta_0^{-1} \sum_{r=1}^{\infty} (-)^r w_1 w_2 \dots w_r \Delta_r \cos(N+2r) z$$
$$+ D_0^{-1} \sum_{r=1}^{r < \frac{1}{2}N} (-)^r w_1' w_2' \dots w_r' D_r \cos(N-2r) z,$$
these series converging uniformly in any bounded domain of values of z, so that term-by-term differentiations are permissible.

Further, the condition $\psi(a, q) = 0$ is equivalent to
$$p = q\left(\frac{w_1 \Delta_1}{\Delta_0} + \frac{w_1' D_1}{D_0}\right),$$
i.e.
$$p \Delta_0 D_0 - q(w_1 \Delta_1 D_0 + w_1' D_1 \Delta_0) = 0.$$

If we multiply by
$$\prod_{r=1}^{\infty} \left\{1 - \frac{2p}{r(r+N)}\right\} \prod_{r=1}^{r < \frac{1}{2}N} \left\{1 - \frac{2p}{r(r-N)}\right\},$$

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the expression on the left becomes an integral function of both p and q, $\Psi(a, q)$, say; the terms of $\Psi(a, q)$, which are of lowest degrees in p and q, are respectively p and

$$q^2\left\{\frac{1}{N-1}-\frac{1}{N+1}\right\}.$$

Now expand
$$\frac{1}{2\pi i}\int\frac{p}{\Psi(N^2+8p, q)}\frac{\partial\Psi(N^2+8p, q)}{\partial p}\,dp$$
in ascending powers of q (cf. § 7·31), the contour being a small circle in the p-plane, with centre at the origin, and $|q|$ being so small that $\Psi(N^2+8p, q)$ has only one zero inside the contour. Then it follows, just as in § 7·31, that, for sufficiently small values of $|q|$, we may expand p as a power series in q commencing* with a term in q^2; and if $|q|$ be sufficiently small D_0 and Δ_0 will not vanish, since both are equal to 1 when $q=0$.

On substituting for p in terms of q throughout the series for $U(z)$, we see that the series involved in $ce_N(z, q)$ are absolutely convergent when $|q|$ is sufficiently small.

The series involved in $se_N(z, q)$ may obviously be investigated in a similar manner.

19·7. *The method of change of parameter†.*

The methods of Hill and of Lindemann-Stieltjes are effective in determining μ, but only after elaborate analysis. Such analysis is inevitable, as μ is by no means a simple function of q; this may be seen by giving q an assigned real value and making a vary from $-\infty$ to $+\infty$; then μ alternates between real and complex values, the changes taking place when, with the Hill-Mathieu notation, $\Delta(0)\sin^2(\frac{1}{2}\pi\sqrt{a})$ passes through the values 0 and 1; the complicated nature of this condition is due to the fact that $\Delta(0)$ is an elaborate expression involving both a and q.

It is, however, possible to express μ and a in terms of q and of a new parameter σ, and the results are very well adapted for purposes of numerical computation when $|q|$ is small‡.

The introduction of the parameter σ is suggested by the series for $ce_1(z, q)$ and $se_1(z, q)$ given in § 19·3 example 1; a consideration of these series leads us to investigate the potentialities of a solution of Mathieu's general equation in the form $y=e^{\mu z}\phi(z)$, where

$$\phi(z)=\sin(z-\sigma)+a_3\cos(3z-\sigma)+b_3\sin(3z-\sigma)+a_5\cos(5z-\sigma)+b_5\sin(5z-\sigma)+\dots,$$

the parameter σ being rendered definite by the fact that no term in $\cos(z-\sigma)$ is to appear in $\phi(z)$; the special functions $se_1(z, q)$, $ce_1(z, q)$ are the cases of this solution in which σ is 0 or $\frac{1}{2}\pi$.

On substituting this expression in Mathieu's equation, the reader will have no difficulty in obtaining the following approximations, valid for§ small values of q and real values of σ:

$$\begin{aligned}
\mu &= 4q\sin 2\sigma-12q^3\sin 2\sigma-12q^4\sin 4\sigma+O(q^5),\\
a &= 1+8q\cos 2\sigma+(-16+8\cos 4\sigma)q^2-8q^3\cos 2\sigma+(\tfrac{256}{3}-88\cos 4\sigma)q^4+O(q^5),\\
a_3 &= 3q^2\sin 2\sigma+3q^3\sin 4\sigma+(-\tfrac{274}{9}\sin 2\sigma+9\sin 6\sigma)q^4+O(q^5),\\
b_3 &= q+q^2\cos 2\sigma+(-\tfrac{14}{3}+5\cos 4\sigma)q^3+(-\tfrac{74}{9}\cos 2\sigma+7\cos 6\sigma)q^4+O(q^5),\\
a_5 &= \tfrac{14}{9}q^3\sin 2\sigma+\tfrac{44}{27}q^4\sin 4\sigma+O(q^5),\\
b_5 &= \tfrac{1}{3}q^2+\tfrac{4}{9}q^3\cos 2\sigma+(-\tfrac{155}{54}+\tfrac{82}{27}\cos 4\sigma)q^4+O(q^5),\\
a_7 &= \tfrac{35}{108}q^4\sin 2\sigma+O(q^5),\quad b_7=\tfrac{1}{18}q^3+\tfrac{1}{12}q^4\cos 2\sigma+O(q^5),\\
a_9 &= O(q^5),\quad b_9=\tfrac{1}{180}q^4+O(q^5),
\end{aligned}$$

the constants involved in the various functions $O(q^5)$ depending on σ.

* If $N=1$ this result has to be modified, since there is an additional term q on the right and the term $q^2/(N-1)$ does not appear.

† Whittaker, *Proc. Edinburgh Math. Soc.* XXXII. (1914), pp. 75–80.

‡ They have been applied to Hill's problem by Ince, *Monthly Notices of the R. A. S.* LXXV. (1915), pp. 436–448.

§ The parameters q and σ are to be regarded as fundamental in this analysis, instead of a and q as hitherto.

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The domains of values of q and σ for which these series converge have not yet been determined*.

If the solution thus obtained be called $\Lambda(z, \sigma, q)$, then $\Lambda(z, \sigma, q)$ and $\Lambda(z, -\sigma, q)$ form a fundamental system of solutions of Mathieu's general equation if $\mu \neq 0$.

Example 1. Shew that, if $\sigma = i \times 0{\cdot}5$ and $q = 0{\cdot}01$, then

$$a = 1{\cdot}124{,}841{,}4\ldots, \qquad \mu = i \times 0{\cdot}046{,}993{,}5\ldots;$$

shew also that, if $\sigma = i$ and $q = 0{\cdot}01$, then

$$a = 1{\cdot}321{,}169{,}3\ldots, \qquad \mu = i \times 0{\cdot}145{,}027{,}6\ldots.$$

Example 2. Obtain the equations

$$\mu = 4q \sin 2\sigma - 4qa_3,$$
$$a = 1 + 8q \cos 2\sigma - \mu^2 - 8qb_3,$$

expressing μ and a in finite terms as functions of q, σ, a_3 and b_3.

Example 3. Obtain the recurrence formulae

$$\{-4n(n+1) + 8q \cos 2\sigma - 8qb_3 \pm 8qi(2n+1)(a_3 - \sin 2\sigma)\} z_{2n+1} + 8q(z_{2n-1} + z_{2n+3}) = 0,$$

where z_{2n+1} denotes $b_{2n+1} + ia_{2n+1}$ or $b_{2n+1} - ia_{2n+1}$, according as the upper or lower sign is taken.

19·8. *The asymptotic solution of Mathieu's equation.*

If in Mathieu's equation

$$\frac{d^2u}{dz^2} + \left(a + \frac{1}{2}k^2 \cos 2z\right) u = 0$$

we write $k \sin z = \xi$, we get

$$(\xi^2 - k^2)\frac{d^2u}{d\xi^2} + \xi \frac{du}{d\xi} + (\xi^2 - M^2)u = 0,$$

where $M^2 \equiv a + \frac{1}{2}k^2$.

This equation has an irregular singularity at infinity. From its resemblance to Bessel's equation, we are led to write $u = e^{i\xi}\xi^{-\frac{1}{2}}v$, and substitute

$$v = 1 + (a_1/\xi) + (a_2/\xi^2) + \ldots$$

in the resulting equation for v; we then find that

$$a_1 = -\tfrac{1}{2}i\left(\tfrac{1}{4} - M^2 + k^2\right), \quad a_2 = -\tfrac{1}{8}\left(\tfrac{1}{4} - M^2 + k^2\right)\left(\tfrac{9}{4} - M^2 + k^2\right) + \tfrac{1}{4}k^2,$$

the general coefficient being given by the recurrence formula

$$2i(r+1)a_{r+1} = \{\tfrac{1}{4} - M^2 + k^2 + r(r+1)\} + (2r-1)ik^2 a_{r-1} - (r^2 - 2r + \tfrac{3}{4})k^2 a_{r-2}.$$

The two series

$$e^{i\xi}\xi^{-\frac{1}{2}}\left(1 + \frac{a_1}{\xi} + \frac{a_2}{\xi^2} + \ldots\right), \quad e^{-i\xi}\xi^{-\frac{1}{2}}\left(1 - \frac{a_1}{\xi} + \frac{a_2}{\xi^2} - \ldots\right)$$

are formal solutions of Mathieu's equation, reducing to the well-known asymptotic solutions of Bessel's equation (§ 17·5) when $k \to 0$. The complete formulae which connect them with the solutions $e^{\pm\mu z}\phi(\pm z)$ have not yet been published, though some steps towards obtaining them have been made by Dougall, *Proc. Edinburgh Math. Soc.* XXXIV. (1916), pp. 176–196.

* It seems highly probable that, if $|q|$ is sufficiently small, the series converge for all real values of σ, and also for complex values of σ for which $|I(\sigma)|$ is sufficiently small. It may be noticed that, when q is real, real and purely imaginary values of σ correspond respectively to real and purely imaginary values of μ.

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MISCELLANEOUS EXAMPLES.

1. Shew that, if $k=\sqrt{(32q)}$,

$$2\pi ce_0(z, q)=ce_0(0, q)\int_{-\pi}^{\pi}\cos(k\sin z\sin\theta)\,ce_0(\theta, q)\,d\theta.$$

2. Shew that the even Mathieu functions satisfy the integral equation

$$G(z)=\lambda\int_{-\pi}^{\pi}J_0\{ik(\cos z+\cos\theta)\}\,G(\theta)\,d\theta.$$

3. Shew that the equation

$$(az^2+c)\frac{d^2u}{dz^2}+2az\frac{du}{dz}+(\lambda^2cz^2+m)\,u=0$$

(where a, c, λ, m are constants) is satisfied by

$$u=\int e^{\lambda zs}\,v(s)\,ds$$

taken round an appropriate contour, provided that $v(s)$ satisfies

$$(as^2+c)\frac{d^2v(s)}{ds^2}+2as\frac{dv(s)}{ds}+(\lambda^2cs^2+m)\,v(s)=0,$$

which is the same as the equation for u.

Derive the integral equations satisfied by the Mathieu functions as particular cases of this result.

* A complete bibliography is given by Humbert, *Fonctions de Mathieu et fonctions de Lamé* (Paris, 1926).

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4. Shew that, if powers of q above the fourth are neglected, then

$$\begin{aligned}ce_1(z, q) = \cos z + q\cos 3z &+ q^2\left(\tfrac{1}{3}\cos 5z - \cos 3z\right)\\ &+ q^3\left(\tfrac{1}{18}\cos 7z - \tfrac{4}{9}\cos 5z + \tfrac{1}{3}\cos 3z\right)\\ &+ q^4\left(\tfrac{1}{180}\cos 9z - \tfrac{1}{12}\cos 7z + \tfrac{1}{6}\cos 5z + \tfrac{11}{9}\cos 3z\right),\end{aligned}$$

$$\begin{aligned}se_1(z, q) = \sin z + q\sin 3z &+ q^2\left(\tfrac{1}{3}\sin 5z + \sin 3z\right)\\ &+ q^3\left(\tfrac{1}{18}\sin 7z + \tfrac{4}{9}\sin 5z + \tfrac{1}{3}\sin 3z\right)\\ &+ q^4\left(\tfrac{1}{180}\sin 9z + \tfrac{1}{12}\sin 7z + \tfrac{1}{6}\sin 5z - \tfrac{11}{9}\sin 3z\right),\end{aligned}$$

$$\begin{aligned}ce_2(z, q) = \cos 2z + q\left(\tfrac{2}{3}\cos 4z - 2\right) &+ \tfrac{1}{6}q^2\cos 6z\\ &+ q^3\left(\tfrac{1}{45}\cos 8z + \tfrac{43}{27}\cos 4z + \tfrac{40}{3}\right)\\ &+ q^4\left(\tfrac{1}{540}\cos 10z + \tfrac{293}{540}\cos 6z\right).\end{aligned}$$

(Mathieu.)

5. Shew that

$$\begin{aligned}ce_3(z, q) = \cos 3z &+ q\left(-\cos z + \tfrac{1}{2}\cos 5z\right)\\ &+ q^2\left(\cos z + \tfrac{1}{10}\cos 7z\right) + q^3\left(-\tfrac{1}{2}\cos z + \tfrac{7}{40}\cos 5z + \tfrac{1}{90}\cos 9z\right) + O(q^4),\end{aligned}$$

and that, in the case of this function

$$a = 9 + 4q^2 - 8q^3 + O(q^4).$$

(Mathieu.)

6. Shew that, if $y(z)$ be a Mathieu function, then a second solution of the corresponding differential equation is

$$y(z)\int^z \{y(t)\}^{-2}\,dt.$$

Shew that a second solution* of the equation for $ce_0(z, q)$ is

$$z\,ce_0(z, q) - 4q\sin 2z - 3q^2\sin 4z - \dots.$$

7. If $y(z)$ be a solution of Mathieu's general equation, shew that

$$\{y(z+2\pi) + y(z-2\pi)\}/y(z)$$

is constant.

8. Express the Mathieu functions as series of Bessel functions in which the coefficients are multiples of the coefficients in the Fourier series for the Mathieu functions.

[Substitute the Fourier series under the integral sign in the integral equations of § 19·22.]

9. Shew that the confluent form of the equations for $ce_n(z, q)$ and $se_n(z, q)$, when the eccentricity of the fundamental ellipse tends to zero, is, in each case, the equation satisfied by $J_n(ik\cos z)$.

10. Obtain the parabolic cylinder functions of Chapter XVI as confluent forms of the Mathieu functions, by making the eccentricity of the fundamental ellipse tend to unity.

11. Shew that $ce_n(z, q)$ can be expanded in series of the form

$$\sum_{m=0}^{\infty} A_m \cos^{2m} z \quad\text{or}\quad \sum_{m=0}^{\infty} B_m \cos^{2m+1} z,$$

according as n is even or odd; and that these series converge when $|\cos z| < 1$.

* This solution is called $in_0(z, q)$; the second solutions of the equations satisfied by Mathieu functions have been investigated by Ince, *Proc. Edinburgh Math. Soc.* XXXIII. (1915), pp. 2–15. See also § 19·2.

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12. With the notation of example 11, shew that, if

$$ce_n(z, q) = \lambda_n \int_{-\pi}^{\pi} e^{k \cos z \cos \theta} \, ce_n(\theta, q) \, d\theta,$$

then λ_n is given by one or other of the series

$$A_0 = 2\pi\lambda_n \sum_{m=0}^{\infty} \frac{2m}{2^{2m}(m\,!)^2} A_m, \quad B_0 = 2\pi\lambda_n k \sum_{m=0}^{\infty} \frac{(2m+1)\,!}{2^{2m+1}\, m\,!\,(m+1)\,!} B_m,$$

provided that these series converge.

13. Shew that the differential equation satisfied by the product of any two solutions of Bessel's equation for functions of order n is

$$\vartheta(\vartheta - 2n)(\vartheta + 2n)\, u + 4z^2(\vartheta + 1)\, u = 0,$$

where ϑ denotes $z\dfrac{d}{dz}$.

Shew that one solution of this equation is an integral function of z; and thence, by the methods of §§ 19·5–19·53, obtain the Bessel functions, discussing particularly the case in which n is an integer.

14. Shew that an approximate solution of the equation

$$\frac{d^2u}{dz^2} + (A + k^2 \sinh^2 z)\, u = 0$$

is

$$u = C(\operatorname{cosech} z)^{\frac{1}{2}} \sin(k \cosh z + \epsilon),$$

where C and ϵ are constants of integration; it is to be assumed that k is large, A is not very large and z is not small.

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