Chapter III

Differential Inequalities and Uniqueness

The most important techniques in the theory of differential equations involve the "integration" of differential inequalities. The first part of this chapter deals with basic results of this type which will be used throughout the book. In the second part of this chapter immediate applications are given, including the derivation of some uniqueness theorems.

In this chapter u, v, U, V are scalars; y, z, f, g are d-dimensional vectors.

1. Gronwall's Inequality

One of the simplest and most useful results involving an integral inequality is the following.

Theorem 1.1. *Let* $u(t)$, $v(t)$ *be non-negative, continuous functions on* $[a, b]$; $C \geqq 0$ *a constant; and*

$$v(t) \leqq C + \int_a^t v(s)u(s)\,ds \qquad \text{for } a \leqq t \leqq b. \tag{1.1}$$

Then

$$v(t) \leqq C \exp\int_a^t u(s)\,ds \qquad \text{for } a \leqq t \leqq b; \tag{1.2}$$

in particular, if $C = 0$, *then* $v(t) \equiv 0$.

For a generalization, see Corollary 4.4.

Proof. *Case* (i), $C > 0$. Let $V(t)$ denote the right side of (1.1), so that $v(t) \leqq V(t)$, $V(t) \geqq C > 0$ on $[a, b]$. Also, $V'(t) = u(t)v(t) \leqq u(t)V(t)$. Since $V > 0$, $V'/V \leqq u$, and $V(a) = C$, an integration over $[a, t]$ gives $V(t) \leqq C \exp \int_a^t u(s)\,ds$. Thus (1.2) follows from $v(t) \leqq V(t)$.

Case (ii), $C = 0$. If (1.1) holds with $C = 0$, then Case (i) implies (1.2) for every $C > 0$. The desired result follows by letting C tend to 0.

Exercise 1.1. Show that Theorem 1.1 implies the uniqueness assertion of Theorem II 1.1.

2. Maximal and Minimal Solutions

Let $U(t, u)$ be a continuous function on a plane (t, u)-set E. By a *maximal* solution $u = u^0(t)$ of

$$(2.1) \qquad u' = U(t, u), \qquad u(t_0) = u_0$$

is meant a solution of (2.1) on a maximal interval of existence such that if $u(t)$ is any solution of (2.1), then

$$(2.2) \qquad u(t) \leqq u^0(t)$$

holds on the common interval of existence of u, u^0. A minimal solution is similarly defined.

Lemma 2.1. *Let $U(t, u)$ be continuous on a rectangle $R: t_0 \leqq t \leqq t_0 + a$, $|y - y_0| \leqq b$; let $|U(t, u)| \leqq M$ and $\alpha = \min(a, b/M)$. Then* (2.1) *has a solution $u = u^0(t)$ on $[t_0, t_0 + \alpha]$ with the property that every solution $u = u(t)$ of $u' = U(t, u)$, $u(t_0) \leqq u_0$ satisfies* (2.2) *on $[t_0, t_0 + \alpha]$.*

In view of the proof of the Extension Theorem II 3.1, this lemma implies existence theorems for maximal and minimal solutions (which will be stated only for an open set E):

Theorem 2.1. *Let $U(t, u)$ be continuous on an open set E and $(t_0, u_0) \in E$. Then* (2.1) *has a maximal and a minimal solution.*

Proof of Lemma 2.1. Let $0 < \alpha' < \alpha$. Then, by Theorem II 2.1,

$$(2.3) \qquad u' = U(t, u) + 1/n, \qquad u(t_0) = u_0$$

has a solution $u = u_n(t)$ on an interval $[t_0, t_0 + \alpha']$ if n is sufficiently large. By Theorem I 2.4, there is a sequence $n(1) < n(2) < \cdots$ such that

$$(2.4) \qquad u^0(t) = \lim_{k \to \infty} u_{n(k)}(t)$$

exists uniformly on $[t_0, t_0 + \alpha']$ and is a solution of (2.1).

It will be verified that (2.2) holds on $[t_0, t_0 + \alpha']$. To this end, it is sufficient to verify

$$(2.5) \qquad u(t) \leqq u_n(t) \qquad \text{on } [t_0, t_0 + \alpha']$$

for all large fixed n. If (2.5) does not hold, there is a $t = t_1$, $t_0 < t_1 < t_0 + \alpha'$ such that $u(t_1) > u_n(t_1)$. Hence there is a largest t_2 on $[t_0, t_1)$, where $u(t_2) = u_n(t_2)$, so that $u(t) > u_n(t)$ on $(t_2, t_1]$. But (2.3) implies that $u_n'(t_2) = u'(t_2) + 1/n$, so that $u_n(t) > u(t)$ for $t(> t_2)$ near t_2. This contradication proves (2.5). Since $\alpha' < \alpha$ is arbitrary, the lemma follows.

Remark. The uniqueness of the solution $u = u^0(t)$ shows that $u_n(t) \to u^0(t)$ uniformly on $[t_0, t_0 + \alpha']$ as $n \to \infty$ continuously.

3. Right Derivatives

The following simple lemmas will be needed subsequently.

Lemma 3.1. *Let $u(t) \in C^1[a, b]$. Then $|u(t)|$ has a right derivative $D_R\,|u(t)|$ for $a \leqq t < b$, where*

$$(3.1) \qquad D_R\,|u(t)| = \lim h^{-1}(|u(t+h)| - |u(t)|) \qquad \text{as } 0 < h \to 0,$$

and $D_R\,|u(t)| = u'(t)\operatorname{sgn} u(t)$ if $u(t) \neq 0$ and $D_R u(t) = |u'(t)|$ if $u(t) = 0$. In particular, $\big|D_R\,|u(t)|\,\big| = |u'(t)|$.

The assertion concerning $D_R\,|u(t)|$ is clear if $u(t) \neq 0$. The case when $u(t) = 0$ follows from $u(t+h) = h(u'(t) + o(1))$ as $h \to 0$, so that $|u(t+h)| = h(|u'(t)| + o(1))$ as $0 < h \to 0$.

Lemma 3.2. *Let $y = y(t) \in C^1[a, b]$. Then $|y(t)|$ has a right derivative $D_R\,|y(t)|$ and $\big|D_R\,|y(t)|\,\big| \leqq |y'(t)|$ for $a \leqq t < b$.*

Since $|y(t)| = \max(|y^1(t)|, \ldots, |y^d(t)|)$, there are indices k such that $|y^k(t)| = |y(t)|$. In the following, k denotes any such index. By the last lemma, $|y^k(t)|$ has a right derivative, so that

$$|y^k(t+h)| = |y(t)| + h(D_R\,|y^k(t)| + o(1)) \qquad \text{as } 0 < h \to 0.$$

For small $h > 0$, $|y(t+h)| = \max_k |y^k(t+h)|$, so that by taking the $\max_k$ in the last formula line,

$$|y(t+h)| = |y(t)| + h(\max_k D_R\,|y^k(t)| + o(1)) \qquad \text{as} \quad 0 < h \to 0.$$

Thus $D_R\,|y(t)|$ exists and is $\max_k D_R\,|y^k(t)|$. Since $\big|D_R\,|y^k(t)|\,\big| = |y^{k\prime}(t)| \leqq |y'(t)|$, Lemma 3.2 follows.

Exercise 3.1. Show that Lemma 3.2 is correct if $|y|$ is replaced by the Euclidean length of y.

4. Differential Inequalities

The next theorem concerns the integration of a differential inequality. It is one of the results which is used most often in the theory of differential equations.

Theorem 4.1. *Let $U(t, u)$ be continuous on an open (t, u)-set E and $u = u^0(t)$ the maximal solution of* (2.1). *Let $v(t)$ be a continuous function on $[t_0, t_0 + a]$ satisfying the conditions $v(t_0) \leqq u_0$, $(t, v(t)) \in E$, and $v(t)$ has a right derivative $D_R v(t)$ on $t_0 \leqq t < t_0 + a$ such that*

$$(4.1) \qquad D_R v(t) \leqq U(t, v(t)).$$

Then, on a common interval of existence of $u^0(t)$ and $v(t)$,

$$(4.2) \qquad v(t) \leqq u^0(t).$$

Remark 1. If the inequality (4.1) is reversed and $v(t_0) \geqq u_0$, then the conclusion (4.2) must be replaced by $v(t) \geqq u_0(t)$, where $u = u_0(t)$ is the minimal solution of (2.1). Correspondingly, if in Theorem 4.1 the function $v(t)$ is continuous on an interval $t_0 - \alpha \leqq t \leqq t_0$ with a left derivative $D_L v(t)$ on $(t_0 - \alpha, t_0]$ satisfying $D_L v(t) \leqq U(t, v(t))$ and $v(t_0) \geqq u_0$, then again (4.2) must be replaced by $v(t) \geqq u_0(t)$.

Remark 2. It will be clear from the proof that Theorem 4.1 holds if the "right derivative" D_R is replaced by the "upper right derivative" where the latter is defined by replacing "lim" by "lim sup" in (3.1).

Proof of Theorem 4.1. It is sufficient to show that there exists a $\delta > 0$ such that (4.2) holds for $[t_0, t_0 + \delta]$. For if this is the case and $u^0(t), v(t)$ are defined on $[t_0, t_0 + \beta]$, it follows that the set of t-values where (4.2) holds cannot have an upper bound different from β.

Let $n > 0$ be large and let $\delta > 0$ be chosen independent of n such that (2.3) has a solution $u = u_n(t)$ on $[t_0, t_0 + \delta]$. In view of the proof of Lemma 2.1, it is sufficient to verify that $v(t) \leqq u_n(t)$ on $[t_0, t_0 + \delta]$, but the proof of this is identical to the proof of (2.5) in § 2.

Corollary 4.1. *Let $v(t)$ be continuous on $[a, b]$ and possess a right derivative $D_R v(t) \leqq 0$ on $[a, b]$. Then $v(t) \leqq v(a)$.*

Corollary 4.2. *Let $U(t, u)$, $u^0(t)$ be as in Theorem* 4.1. *Let $V(t, u)$ be continuous on E and satisfy*

(4.3) $$V(t, u) \leqq U(t, u).$$

Let $v = v(t)$ be a solution of

(4.4) $$v' = V(t, v), \qquad v(t_0) = v_0(\leqq u_0)$$

on an interval $[t_0, t_0 + a]$. Then (4.2) *holds on any common interval of existence of $v(t)$ and $u^0(t)$ to the right of $t = t_0$.*

It is clear from Remark 2 that if $v(t)$ is extended to an interval to the left of $t = t_0$, then, on such an interval, (4.2) must be replaced by $v(t) \geqq u_0(t)$ where $u_0(t)$ is a minimal solution of (2.1) with $u_0 \geqq v(t_0)$.

Corollary 4.3. *Let $U(t, u) \geqq 0$, $u^0(t)$ be as in Theorem* 4.1; *$u = u_0(t)$ the minimal solution of*

(4.5) $$u' = -U(t, u), \qquad u(t_0) = u_0(\geqq 0).$$

Let $y = y(t)$ be a C^1 vector-valued function on $[t_0, t_0 + \alpha]$ such that $u^0 \leqq |y(t_0)| \leqq u_0$, $(t, |y(t)|) \in E$ and

(4.6) $$|y'(t)| \leqq U(t, |y(t)|)$$

on $[t, t_0 + \alpha]$. Then the first [second] of the two inequalities

(4.7) $$u_0(t) \leqq |y(t)| \leqq u^0(t)$$

holds on any common interval of existence of $u_0(t)$ and y [$u^0(t)$ and y].

This is an immediate consequence of Theorem 4.1 and Remark 1 following it, since $|y(t)|$ has a right derivative satisfying $-|y'(t)| \leqq D_R\,|y(t)| \leqq |y'(t)|$ by Lemma 3.2. (In view of Exercise 3.1, this corollary remains valid if $|y|$ denotes the Euclidean norm.)

Exercise 4.1. (*a*) Let $f(t, y)$ be continuous on the strip $S: a \leqq t \leqq b$, y arbitrary, and let $f^k(t, y^1, \ldots, y^d)$ be nondecreasing with respect to each of the components y^i, $i \neq k$, of y. Assume that the solution of the initial value problem $y' = f(t, y)$, $y(a) = y_0$ is unique for a fixed y_0, and that this solution $y = y(t)$ exists on $[a, b]$. Let $z(t) = (z^1(t), \ldots, z^d(t))$ be continuous on $[a, b]$ such that $z^k(t)$ has a right derivative for $k = 1, \ldots, d$, $z^k(a) \leqq y_0^k$ and $D_R z^k(t) \leqq f^k(t, z(t))$ for $a \leqq t \leqq b$ [or $z^k(a) \geqq y_0^k$ and $D_R z^k(t) \geqq f^k(t, z(t))$ for $a \leqq t \leqq b$]. Then $z^k(t) \leqq y^k(t)$ [or $z^k(t) \geqq y^k(t)$] for $a \leqq t \leqq b$. (This is applicable if $g(t, y)$ is continuous on S, $z(t)$ is a solution of $z' = g(t, z)$ and $z^k(a) \leqq y_0^k$, $g^k(t, y) \leqq f^k(t, y)$ on S [or $z^k(a) \geqq y_0^k, g^k(t, y) \geqq f^k(t, y)$ on S].) See Remark in Exercise 4.3.

(*b*) If, in part (*a*), all initial value problems associated with $y' = f(t, y)$ have unique solutions, $f^k(t, y)$ is increasing with respect to y^i, $i \neq k$ and $k = 1, \ldots, d$, and $z^j(a) < y_0^j$ [or $z^j(a) > y_0^j$] for at least one index j, then $z^k(t) < y_0^k(t)$ [or $z^k(t) > y_0^k(t)$] for $a < t \leqq b$, $k = 1, \ldots, d$.

(*c*) If, in addition to the assumptions of (*a*), there is an index h such that $f^h(t, y)$ is nondecreasing with respect to y^h, then $y_0^h(t) - z^h(t)$ is nondecreasing [or nonincreasing] on $a \leqq t \leqq b$.

(*d*) If the assumptions of (*b*) and (*c*) hold, then $y_0^h(t) - z^h(t)$ is increasing [or decreasing] on $a \leqq t \leqq b$.

(*e*) Let u, U denote real-valued scalars and $y = (y^1, \ldots, y^d)$ a real d-dimensional vector. Let $U(t, y)$ be continuous for $a \leqq t \leqq b$ and arbitrary y such that solutions of $u^{(d)} = U(t, u, u', \ldots, u^{(d-1)})$ are uniquely determined by initial conditions and that $U(t, y^1, \ldots, y^d)$ is nondecreasing with respect to each of the first $d - 1$ components y^j, $j = 1, \ldots, d - 1$, of y. Let $u_1(t)$, $u_2(t)$ be two solutions of $u^{(d)} = U$ on $[a, b]$ satisfying $u_1^{(j)}(a) \leqq u_2^{(j)}(a)$ for $j = 0, \ldots, d - 1$. Then $u_1^{(j)}(t) \leqq u_2^{(j)}(t)$ for $j = 0, \ldots, d - 1$ and $a \leqq t \leqq b$; furthermore, $u_2^{(j)}(t) - u_1^{(j)}(t)$ is nondecreasing for $j = 0, \ldots, d - 2$ and $a \leqq t \leqq b$.

Exercise 4.2. Let $f(t, y)$, $g(t, y)$ be continuous on a strip, $a \leqq t \leqq b$, and y arbitrary, such that $f^k(t, y) < g^k(t, y)$ for $k = 1, \ldots, d$ and that, for each $k = 1, \ldots, d$, either $f^k(t, y^1, \ldots, y^d)$ or $g^k(t, y^1, \ldots, y^d)$ is nondecreasing with respect to y^i, $i \neq k$. On $a \leqq t \leqq b$, let $y = y(t)$ be a solution of $y' = f(t, y)$, $y(a) = y_0$ and $z = z(t)$ a solution of $z' = g(t, z)$, $z(a) = z_0$, where $y_0^k \leqq z_0^k$ for $k = 1, \ldots, d$. Then $y^k(t) \leqq z^k(t)$ for $a \leqq t \leqq b$.

Exercise 4.3. Let $f(t, y)$ be continuous for $t_0 \leqq t \leqq t_0 + a$, $|y - y_0| \leqq b$ such that $f^k(t, y^1, \ldots, y^d)$ is nondecreasing with respect to each y^i,

$i \neq k$. Show that $y' = f(t, y), y(t_0) = y_0$ has a maximal [minimal] solution $y_0(t)$ with the property that if $y = y(t)$ is any other solution, then $y^k(t) \leqq y_0^k(t)$ [$y^k(t) \geqq y_0^k(t)$] holds on the common interval of existence. Remark: The assumption in Exercise 4.1(a) that the solution of the initial value problem $y' = f(t, y), y(a) = y_0$, is unique can be dropped if $y(t)$ is replaced by the maximal solution [or minimal solution] $y_0(t)$.

Exercise 4.4. Let $f(t, y), g(t, y)$ be linear in y, say $f^k(t, y) = \Sigma a_{kj}(t)y^j + f^k(t)$ and $g^k(t, y) = \Sigma b_{kj}(t)y^j + g^k(t)$, where $a_{kj}(t), b_{kj}(t), f^k(t), g^k(t)$ are continuous for $a \leqq t \leqq b$. Let $y(t)$, $z(t)$ be solutions of $y' = f(t, y)$, $y(a) = y_0$ and $z' = g(t, z), z(a) = z_0$, respectively. (These solutions exist on $[a, b]$; cf. Corollary 5.1.) What conditions on $a_{jk}(t), b_{jk}(t), f^k(t), g^k(t)$, y_0, z_0 imply that $|z^k(t)| \leqq y^k(t)$ on $[a, b]$ for $k = 1, \ldots, d$?

Theorem 4.1 has an "integrated" analogue which, however, requires the monotony of U with respect to u. This theorem is a generalization of Theorem 1.1:

Corollary 4.4. *Let $U(t, u)$ be continuous and nondecreasing with respect to u for $t_0 \leqq t \leqq t_0 + a$, u arbitrary. Let the maximal solution $u = u^0(t)$ of (2.1) exist on $[t_0, t_0 + a]$. On $[t_0, t_0 + a]$, let $v(t)$ be a continuous function satisfying*

$$(4.8) \qquad v(t) \leqq v_0 + \int_{t_0}^{t} U(s, v(s))\, ds,$$

where $v_0 \leqq u_0$. Then $v(t) \leqq u^0(t)$ holds on $[t_0, t_0 + a]$.

Proof. Let $V(t)$ be the right side of (4.8), so that $v(t) \leqq V(t)$, and $V'(t) = U(t, v(t))$. By the monotony of U, $V'(t) \leqq U(t, V(t))$. Hence Theorem 4.1 implies that $V(t) \leqq u^0(t)$ on $[t_0, t_0 + a]$; thus $v(t) \leqq u^0(t)$ holds.

Exercise 4.5. State the analogue of Corollary 4.4 for the case that the constant v_0 in (4.8) is replaced by a continuous function $v_0(t)$.

Exercise 4.6. Let y, f, z be d-dimensional vectors; $f(t, y)$ continuous for $t_0 \leqq t \leqq t_0 + a$ and y arbitrary such that $f^k(t, y^1, \ldots, y^d)$ is nondecreasing with respect to each y^j, $j = 1, \ldots, d$. Let the maximal solution $y_0(t)$ of $y' = f(t, y), y(t_0) = y_0$ exist on $[t_0, t_0 + a]$; cf. Exercise 4.3. Let $z(t)$ be a continuous vector-valued function such that $z^k(t) \leqq y_0^k + \int_{t_0}^{t} f^k(s, z(s))\, ds$ for $t_0 \leqq t \leqq t_0 + a$. Then $z^k(t) \leqq y_0^k(t)$ on $[t_0, t_0 + a]$.

5. A Theorem of Wintner

Theorem 4.1 and its corollaries can be used to help find intervals of existence of solutions of some differential equations.

Theorem 5.1. *Let $U(t, u)$ be continuous for $t_0 \leqq t \leqq t_0 + a$, $u \geqq 0$, and let the maximal solution of (2.1), where $u_0 \geqq 0$, exist on $[t_0, t_0 + a]$, e.g.,* *let* $U(t, u) = \psi(u)$, *where* $\psi(u)$ *is a positive, continuous function on* $u \geqq 0$ *such that*

$$\int^{\infty} du/\psi(u) = \infty. \tag{5.1}$$

Let $f(t, y)$ *be continuous on the strip* $t_0 \leqq t \leqq t_0 + a$, y *arbitrary, and satisfy*

$$|f(t, y)| \leqq U(t, |y|). \tag{5.2}$$

Then the maximal interval of existence of solutions of

$$y' = f(t, y), \qquad y(t_0) = y_0, \tag{5.3}$$

where $|y_0| \leqq u_0$, is $[t_0, t_0 + a]$.

Remark 1. It is clear that (5.2) is only required for large $|y|$. Admissible choices of $\psi(u)$ are, for example, $\psi(u) = Cu$, $Cu \log u, \ldots$ for large u and a constant C.

Proof. (5.2) implies the inequality (4.6) on any interval on which $y(t)$ exists. Hence, by Corollary 4.3, the second inequality in (4.7) holds on such an interval and so the main assertion follows from Corollary II 3.1.

In order to complete the proof, it has to be shown that the function $U(t, u) = \psi(u)$ satisfies the condition that the maximal solution of

$$u' = \psi(u), \qquad u(t_0) = u_0 (\geqq 0) \tag{5.4}$$

exists on $[t_0, t_0 + a]$ by virtue of (5.1). Since $\psi > 0$, (5.4) implies that for any solution $u = u(t)$,

$$t - t_0 = \int_{t_0}^{t} u'(t)\, dt/\psi(u(t)) = \int_{u_0}^{u(t)} du/\psi(u). \tag{5.5}$$

Note that $\psi > 0$ implies that $u'(t) > 0$ and $u(t) > 0$ for $t > t_0$. By Corollary II 3.1, the solution $u(t)$ can fail to exist on $[t_0, t_0 + a]$ only if it exists on some interval $[t_0, \delta)$ and satisfies $u(t) \to \infty$ as $t \to \delta$ $(< a)$. If this is the case, however, $t \to \delta$ in (5.5) gives a contradiction for the left side tends to $\delta - t_0$ and the right side to ∞ by (5.1). This completes the proof.

Remark 2. The type of argument in the proof of Theorem 5.1 supplies a priori estimates for solutions $y(t)$ of (5.3). For example, if $\psi(u)$ is the same as in the last part of Theorem 5.1, let

$$\Psi(u) = \int_{u_0}^{u} ds/\psi(s) \qquad \text{for } u \geqq u_0$$

and let $u = \Phi(v)$ be the function inverse to $v = \Psi(u)$. Then $|f(t, y)| \leqq \psi(|y|)$ implies that a solution $y(t)$ of (5.3) satisfies

$$|y(t)| \leqq \Phi(t - t_0) \qquad \text{for } t_0 \leqq t \leqq t_0 + a;$$

cf. (5.5).

Exercise 5.1. Let $f(t, y)$ be continuous on the strip $t_0 \leqq t \leqq t_0 + a$, y arbitrary. Let $|f(t, y)| \leqq \varphi(t)\psi(|y|)$, where $\varphi(t) \geqq 0$ is integrable on $[t_0, t_0 + a]$ and $\psi(u)$ is a positive continuous function on $u \geqq 0$ satisfying (5.1). Show that the assertion of Theorem 5.1 and an analogue of Remark 2 are valid.

Corollary 5.1. *If $A(t)$ is a continuous $d \times d$ matrix function and $g(t)$ a continuous vector function for $t_0 \leqq t \leqq t_0 + a$, then the (linear) initial value problem*

$$y' = A(t)y + g(t), \qquad y(t_0) = y_0 \tag{5.6}$$

has a unique solution $y = y(t)$, and $y(t)$ exists on $t_0 \leqq t \leqq t_0 + a$.

This is a consequence of Theorem II 1.1 and Theorem 5.1 with the choice of $\psi(u) = C(1 + u)$ for some large C.

In a scalar case, Theorem 5.1 can be "read backwards":

Corollary 5.2. *Let $U(t, u)$, $V(t, u)$ be continuous functions satisfying* (4.3) *on $t_0 \leqq t \leqq t_0 + a$, u arbitrary. Let some solution $v = v(t)$ of* (4.4) *on $[t_0, \delta)$, $\delta \leqq t_0 + a$, satisfy $v(t) \to \infty$ as $t \to \delta$. Then the maximal solution $u = u^0(t)$ of* (2.1) *has a maximal interval of existence $[a, \omega_+)$, where $\omega_+ \leqq \delta$, and $u^0(t) \to \infty$ as $t \to \omega_+$.*

6. Uniqueness Theorems

One of the principal uses of Theorem 4.1 and its corollaries is to obtain uniqueness theorems. The following result is often called Kamke's general uniqueness theorem.

Theorem 6.1. *Let $f(t, y)$ be continuous on the parallelepiped R: $t_0 \leqq t \leqq t_0 + a$, $|y - y_0| \leqq b$. Let $\omega(t, u)$ be a continuous (scalar) function on R_0: $t_0 < t \leqq t_0 + a$, $0 \leqq u \leqq 2b$, with the properties that $\omega(t, 0) = 0$ and that the only solution $u = u(t)$ of the differential equation*

$$u' = \omega(t, u) \tag{6.1}$$

on any interval $(t_0, t_0 + \epsilon]$ satisfying

$$u(t) \to 0 \quad \text{and} \quad \frac{u(t)}{t - t_0} \to 0 \qquad \text{as } t \to t_0 + 0 \tag{6.2}$$

is $u(t) \equiv 0$. For $(t, y_1), (t, y_2) \in R$ with $t > t_0$, let

$$|f(t, y_1) - f(t, y_2)| \leqq \omega(t, |y_1 - y_2|). \tag{6.3}$$

Then the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{6.4}$$

has at most one solution on any interval $[t_0, t_0 + \epsilon]$.

In Theorem 6.1, we can also conclude uniqueness for initial value problems $y' = f(t, y)$, $y(t_1) = y_1$ for $t_1 \neq t_0$. Theorem 6.1 remains valid if Euclidean norms are employed.

Exercise 6.1. Show that Theorem 6.1 is false if (6.2) is replaced by $u(t), u'(t) \to 0$ as $t \to t_0 + 0$.

Proof. The fact that

$$\omega(t, 0) = 0 \qquad \text{for } t_0 < t \leqq t_0 + a \tag{6.5}$$

implies of course that $u(t) \equiv 0$ is a solution of (6.1).

Suppose that, for some $\epsilon > 0$, (6.4) has two distinct solutions $y = y_1(t)$, $y_2(t)$ on $t_0 \leqq t \leqq t_0 + \epsilon$. Let $y(t) = y_1(t) - y_2(t)$. By decreasing ϵ, if necessary, it can be supposed that $y(t_0 + \epsilon) \neq 0$ and $|y(t_0 + \epsilon)| < 2b$. Also $y(t_0) = y'(t_0) = 0$. By (6.3), $|y'(t)| \leqq \omega(t, |y(t)|)$ on $(t_0, t_0 + \epsilon]$. It follows from Corollary 4.3 (and the Remark 1 following Theorem 4.1) that if $u = u_0(t)$ is the minimal solution of the initial value problem $u' = \omega(t, u)$, $u(t_0 + \epsilon) = |y(t_0 + \epsilon)|$, where $0 < |y(t_0 + \epsilon)| < 2b$, then

$$|y(t)| \geqq u_0(t) \tag{6.6}$$

on any subinterval of $(t_0, t_0 + \epsilon]$ on which $u_0(t)$ exists; see Figure 1.

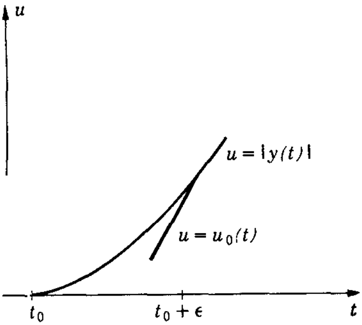

Figure 1.

By the proofs of the Extension Theorem II 3.1 and Lemma 2.1, $u_0(t)$ can be extended, as the minimal solution, to the left until $(t, u_0(t))$ approaches arbitrarily close to a point of ∂R_0 for some t-values. During the extension (6.6) holds, so that $(t, u^0(t))$ comes arbitrarily close to some point $(\delta, 0) \in \partial R_0$ for certain t-values, where $\delta \geqq t_0$. If $\delta > t_0$, then (6.5) shows that $u_0(t)$ has an extension over $(t_0, t_0 + \epsilon]$ with $u_0(t) = 0$ for $(t_0, \delta]$. Thus, in any case, the left maximum interval of existence of $u_0(t)$ is $(t_0, t_0 + \epsilon]$. It follows from (6.5) and (6.6) that $u_0(t) \to 0$ and $u_0(t)/(t - t_0) \to 0$ as $t \to t_0 + 0$. By the assumption concerning (6.1), $u_0(t) \equiv 0$. Since this contradicts $u_0(t_0 + \epsilon) = |y(t_0 + \epsilon)| \neq 0$, the theorem follows.

Corollary 6.1 (Nagumo's Criterion). *If* $t_0 = 0$, *then* $\omega(t, u) = u/t$ *is admissible in Theorem* 6.1 (*i.e., the conclusion of Theorem* 6.1 *holds if* (6.3) *is replaced by*

$$|f(t, y_1) - f(t, y_2)| \leqq \frac{|y_1 - y_2|}{t - t_0} \tag{6.7}$$

for $(t, y_1), (t, y_2) \in R$ *with* $t > t_0$).

Exercise 6.2. The function $\omega(t, u) = u/t$ in Corollary 6.1 cannot be replaced by $\omega(t, u) = Cu/t$ for any constant $C > 1$. Show that if $C > 1$, then there exist continuous real-valued functions $f(t, y)$ on $0 \leqq t \leqq 1$, $|y| \leqq 1$ with the properties that

$$|f(t, y_1) - f(t, y_2)| \leqq \frac{C\,|y_1 - y_2|}{t} \qquad \text{for } t > 0,$$

but that $y' = f(t, y)$, $y(0) = 0$ has more than one solution.

Corollary 6.2 (Osgood's Criterion). *If* $t_0 = 0$, *then* $\omega(t, u) = \varphi(t)\psi(u)$ *is admissible in Theorem* 6.1 *if* $\varphi(t) \geqq 0$ *is continuous for* $0 < t \leqq a$; $\psi(u)$ *is continuous for* $u \geqq 0$ *and* $\psi(0) = 0$, $\psi(u) > 0$ *if* $u > 0$; *and* $\int_{+0} \varphi(t)\,dt < \infty$, $\int_{+0} du/\psi(u) = \infty$.

Actually, the continuity condition on $\varphi(t)$ in this corollary can be weakened. The analogous uniqueness theorem can be proved directly if $\varphi(t)$ is only assumed to be integrable over $0 < t \leqq a$.

Exercise 6.3 [*Generalization of Corollaries* 6.1 *and* (6.2)]. Let $t_0 = 0$. (*a*) If $\varphi(t) \geqq 0$ is continuous for $0 < t \leqq a$, show that $\omega(t, u) = \varphi(t)u$ is admissible in Theorem 6.1 if and only if $\liminf \left[\int_t^a \varphi(s)\,ds + \log t\right] < \infty$ as $t \to +0$. (*b*) Let $\varphi(t) \geqq 0$ be continuous for $0 < t \leqq a$; $\psi(u)$ continuous for $0 \leqq u \leqq 2b$, $\psi(0) = 0$, $\psi(u) > 0$ for $0 < u \leqq b$, and $\int_{+0} du/\psi(u) = \infty$. Show that $\omega(t, u) = \varphi(t)\psi(u)$ is admissible in Theorem 6.1 if, for every $C > 0$, $\limsup t^{-1}\Phi\left(C + \int_t^a \varphi(s)\,ds\right) > 0$ as $t \to 0$, where $u = \Phi(v)$ is the function inverse to $\Psi(u) = \int_u^{2b} ds/\psi(s)$.

Exercise 6.4. Let $\psi(u)$ be continuous for $|u| \leqq 1$, $\psi(0) = 0$. Show that the initial value problem $u' = \psi(u)$, $u(0) = 0$ has a unique solution $u(t) \equiv 0$ unless there exists an ϵ, $0 < \epsilon \leqq 1$, such that either $\psi(u) \geqq 0$ for $0 \leqq u \leqq \epsilon$ and $1/\psi(u)$ is (Lebesgue) integrable over $[0, \epsilon]$ or $\psi(u) \leqq 0$ for $-\epsilon \leqq u \leqq 0$ and $1/\psi(u)$ is (Lebesgue) integrable over $[-\epsilon, 0]$.

Exercise 6.5. Let f, ω be as in Theorem 6.1. Show that there exists a function $\omega_0(t, u)$ which is continuous on the *closure* of R_0, is nondecreasing with respect to u for fixed t, and satisfies the conditions on $\omega(t, u)$; thus $\omega_0(t, 0) \equiv 0$; the only solution of $u' = \omega_0(t, u)$ and $u(t_0) = 0$ on any interval $[t_0, t_0 + \epsilon]$ is $u(t) \equiv 0$; and $|f(t, y_1) - f(t, y_2)| \leqq \omega_0(t, |y_1 - y_2|)$. (Note that, since ω_0 is continuous on the closure of R_0, any solution of $u' = \omega_0(t, u)$ on $(t_0, t_0 + \epsilon]$ satisfying (6.2) is necessarily continuously differentiable and is the usual type of solution on $[t_0, t_0 + \epsilon]$.)

Exercise 6.6. (*a*) Let $\epsilon_0, \ldots, \epsilon_{d-1}$ be non-negative constants such that $\epsilon_0 + \cdots + \epsilon_{d-1} = 1$. Let $U(t, y) = U(t, y^1, \ldots, y^d)$ be a real-valued continuous function on $R: 0 \leqq t \leqq a$ and $|y^k| \leqq b$ for $k = 1, \ldots, d$

such that $|U(t, y_1) - U(t, y_2)| \leqq \sum_{k=1}^{d} \epsilon_{k-1}(d-k+1)!t^{-(d-k+1)}|y_1^k - y_2^k|$ if $t > 0$. Show that the dth order (scalar) equation $u^{(d)} = U(t, u, u', \ldots, u^{(d-1)})$ has at most one solution (on any interval $0 \leqq t \leqq \epsilon \leqq a$) satisfying given initial conditions $u(0) = u_0$, $u' = u_0'$, ..., $u^{(d-1)}(0) = u_0^{(d-1)}$, where $u_0, u_0', \ldots, u^{(d-1)}$ are d given numbers on the range $|u| \leqq b$. (*b*) Note that part (*a*) remains correct if the constants $\epsilon_0, \ldots, \epsilon_{d-1}$ are replaced by continuous non-negative functions $\epsilon_0(t), \ldots, \epsilon_{d-1}(t)$ such that $\epsilon_0(t) + \cdots + \epsilon_{d-1}(t) \leqq 1$.

Exercise 6.7. (*a*) Let $f(t, y)$ be continuous for $R: 0 \leqq t \leqq a$, $|y| \leqq b$. On $R_0: 0 < t \leqq a$, $|u| \leqq 2b$, let $\omega_1(t, u)$, $\omega_2(t, u)$ be continuous non-negative functions which are nondecreasing in u for fixed t, satisfy $\omega_j(t, 0) = 0$, and

$$|f(t, y_1) - f(t, y_2)| \leqq \omega_j(t, |y_1 - y_2|) \qquad \text{for } j = 1, 2.$$

Let there exist continuous non-negative functions $\alpha(t)$, $\beta(t)$ for $0 \leqq t \leqq a$ satisfying $\alpha(0) = \beta(0) = 0$, $\beta(t) > 0$ for $0 < t < a$, and $\alpha(t)/\beta(t) \to 0$ as $t \to 0$. Suppose that each solution $u(t)$ of $u' = \omega_1(t, u)$ for small $t > 0$ with the property that $u(t) \to 0$ as $t \to 0$ satisfies $u(t) \leqq \alpha(t)$ on its interval of existence. Finally, suppose that the only solution of $v' = \omega_2(t, v)$ for small $t > 0$ satisfying $v(t)/\beta(t) \to 0$ as $t \to 0$ is $v(t) \equiv 0$. Then the initial value problem $y' = f(t, y)$, $y(0) = 0$ has exactly one solution. (*b*) Prove that $\omega_1(t, u) = Cu^\lambda$, $\omega_2(t, u) = ku/t$ are admissible if $k > 0$, $0 < \lambda < 1$, $k(1-\lambda) < 1$ with $\alpha(t) = C(1-\lambda)t^{1/(1-\lambda)}$, $\beta(t) = t^k$.

The following involves a "one-sided inequality" and gives "one-sided uniqueness."

Theorem 6.2 *Let $f(t, y)$ be continuous for $t_0 \leqq t \leqq t_0 + a$, $|y - y_0| \leqq b$. Considering y, f to be Euclidean vectors, suppose that*

$$(6.8) \qquad [f(t, y_2) - f(t, y_1)] \cdot (y_2 - y_1) \leqq 0$$

for $t_0 \leqq t \leqq t_0 + a$ and $|y_i - y_0| \leqq b$, $i = 1, 2$, where the dot denotes scalar multiplication. Then (6.4) *has at most one solution on any interval* $[t_0, t_0 + \epsilon]$, $\epsilon > 0$.

When it is desired to obtain uniqueness theorems for intervals $[t_0 - \epsilon, t_0]$. it is necessary to assume the reverse inequality in (6.8).

Corollary 6.3. *Let $U(t, u)$ be a continuous real-valued function for $t_0 \leqq t \leqq t_0 + a$, $|u - u_0| \leqq b$ which is nonincreasing with respect to u (for fixed t). Then the initial value problem $u' = U(t, u), u(t_0) = u_0$ has at most one solution on any interval* $[t_0, t_0 + \epsilon]$, $\epsilon > 0$.

Proof of Theorem 6.2. Let $y = y_1(t)$, $y_2(t)$ be solutions of (6.4) on $[t_0, t_0 + \epsilon]$. Let $\delta(t) = \|y_2(t) - y_1(t)\|^2 = (y_2 - y_1) \cdot (y_2 - y_1)$ be the square of the Euclidean length of $y_2(t) - y_1(t)$, so that $\delta(t_0) = 0$, $\delta(t) \geqq 0$.

But $\delta'(t) = 2(y_2' - y_1') \cdot (y_2 - y_1) \leqq 0$ by (6.8). Hence $\delta(t) = 0$ on $[t_0, t_0 + \epsilon]$ as was to be proved.

Exercise 6.8 (*One-sided Generalization of Nagumo's Criterion and of Theorem* 6.2). Theorem 6.2 remains valid if condition (6.8) is relaxed to

$$[f(t, y_2) - f(t, y_1)] \cdot (y_2 - y_1) \leqq \frac{\|y_2 - y_1\|^2}{t - t_0}$$

for $t_0 < t \leqq t_0 + a$.

7. van Kampen's Uniqueness Theorem

In the following uniqueness theorem, conditions are imposed on a family of solutions rather than on $f(t, y)$ in

$$(7.1) \qquad y' = f(t, y), \qquad y(t_0) = y_0.$$

Theorem 7.1. *Let $f(t, y)$ be continuous on a parallelepiped $R: t_0 \leqq t \leqq t_0 + a, |y - y_0| \leqq b$. Let there exist a function $\eta(t, t_1, y_1)$ on $t_0 \leqq t, t_1 \leqq t_0 + a, |y_1 - y_0| \leqq \beta(< b)$ with the properties* (i) *that, for a fixed (t_1, y_1), $y = \eta(t, t_1, y_1)$ is a solution of*

$$(7.2) \qquad y' = f(t, y), \qquad y(t_1) = y_1;$$

(ii) *that $\eta(t, t_1, y_1)$ is uniformly Lipschitz continuous with respect to y_1; finally,* (iii) *that no two solution arcs $y = \eta(t, t_1, y_1)$, $y = \eta(t, t_2, y_2)$ pass through the same point (t, y) unless $\eta(t, t_1, y_1) \equiv \eta(t, t_2, y_2)$ for $t_0 \leqq t \leqq t_0 + a$. Then $y = \eta(t, t_0, y_0)$ is the only solution of* (7.1) *for $t_0 \leqq t_1 \leqq t_0 + a, |y_1 - y_0| \leqq \beta$.*

Exercise 7.1. Show that the existence of a continuous $\eta(t, t_1, y_1)$ satisfying (i) and (iii) [but not (ii)] does not imply the uniqueness of the solution of (7.1).

Exercise 7.2. When $f(t, y)$ is uniformly Lipschitz continuous with respect to y, it can be shown that a function $y = \eta(t, t_1, y_1)$ satisfying the conditions of the theorem exists (for small $\beta > 0$); e.g., cf. Exercise II 1.2. Show that the converse is not correct, i.e., the existence of $\eta(t, t_1, y_1)$ satisfying (i)–(iii) does not imply that $f(t, y)$ is uniformly Lipschitz continuous with respect to y (for y near y_0).

Proof. Let $y(t)$ be any solution of (7.1). It will be shown that $y(t) = \eta(t, t_0, y_0)$ for small $t - t_0 \geqq 0$.

Condition (ii) means that there exists a constant K such that

$$(7.3) \qquad |\eta(t, t_1, y_1) - \eta(t, t_1, y_2)| \leqq K |y_1 - y_2|$$

for $t_0 \leqq t, t_1 \leqq t_0 + a$ and $|y_1 - y_0| \leqq \beta$, $|y_2 - y_0| \leqq \beta$.

Let $|f(t, y)| \leqq M$ on R. Then any solution $y = y(t)$ of (7.1) satisfies $|y(t) - y_0| \leqq M(t - t_0) \leqq \frac{1}{2}\beta$ if $t_0 \leqq t \leqq t_0 + \beta/2M$. Thus $\eta(t, s, y(s))$ is defined and $|\eta(t, s, y(s)) - y(s)| \leqq M\,|t - s| \leqq \frac{1}{2}\beta$ if $t_0 \leqq t,\ s \leqq t_0 + \beta/2M$. Hence

$$(7.4) \qquad |\eta(t, s, y(s)) - y_0| \leqq \beta \qquad \text{if } t_0 \leqq t, s \leqq t_0 + \gamma,$$

where $\gamma = \min(a, \beta/2M)$. Condition (iii) means that any point on any of the arcs $y = \eta(t, t_1, y_1)$ can be used to determine this arc. Thus (7.3),

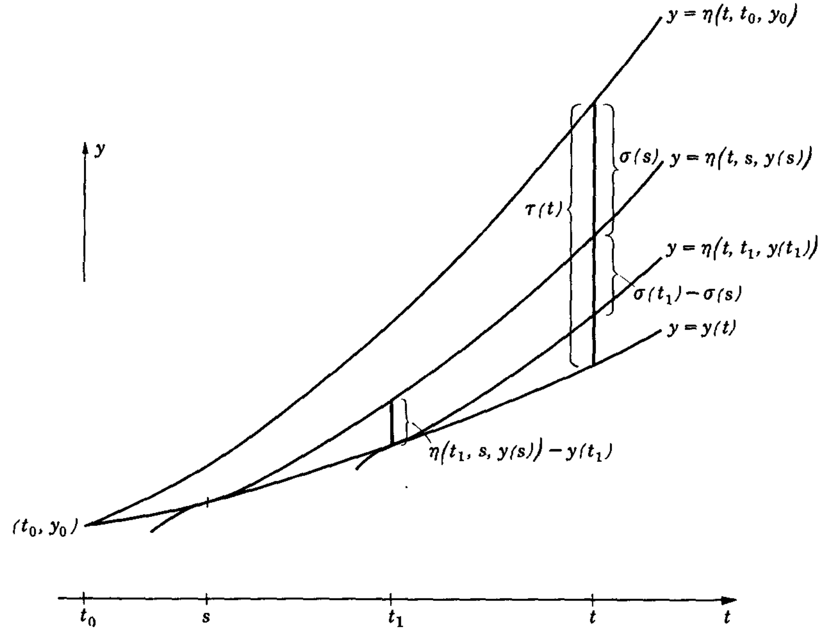

Figure 2. The case $d = \dim y$ is 1.

with $y_1 = y(t_1)$ and $y_2 = \eta(t_1, s, y(s))$, implies that

$$(7.5) \qquad |\eta(t, t_1, y(t_1)) - \eta(t, s, y(s))| \leqq K\,|y(t_1) - \eta(t_1, s, y(s))|$$

if $t_0 \leqq t, t_1, s \leqq t_0 + \gamma$; cf. Figure 2.

Let t be fixed on $t_0 \leqq t \leqq t_0 + \gamma$. It will be shown that

$$(7.6) \qquad \tau(t) \equiv \eta(t, t_0, y_0) - y(t) = 0.$$

To this end, put

$$(7.7) \qquad \sigma(s) = \eta(t, t_0, y_0) - \eta(t, s, y(s)) \qquad \text{for } t_0 \leqq s \leqq t\ (\leqq t_0 + \gamma),$$

so that $\sigma(t_0) = 0$ and $\sigma(t) = \tau(t)$. Then (7.5) and (7.7) imply that

$$(7.8) \qquad |\sigma(t_1) - \sigma(s)| \leqq K\,|y(t_1) - \eta(t_1, s, y(s))|.$$

Since $y = \eta(t, s, y(s))$ is a solution of $y' = f$ through the point $(s, y(s))$, it is seen that $\eta(t_1, s, y(s)) = y(s) + (t_1 - s)\,[f(s, y(s)) + o(1)]$ as $t_1 \to s$. Also, $y(t_1) = y(s) + (t_1 - s)\,[f(s, y(s)) + o(1)]$ as $t_1 \to s$. Hence (7.8) gives $\sigma(t_1) - \sigma(s) = Ko(1)\,|t_1 - s|$ as $t_1 \to s$; i.e., $d\sigma/ds$ exists and is 0. Thus $\sigma(s)$ is the constant $\sigma(t_0) = 0$ for $t_0 \leqq s \leqq t$. In particular, $\tau(t) = \sigma(t)$ satisfies (7.6), as was to be proved.

Exercise 7.3 (*One-sided Analogue of Theorem* 7.1). Let $f(t, y)$ be continuous on $R: t_0 \leqq t \leqq t_0 + a$, $|y - y_0| \leqq b$. Let there exist a function $\eta(t, t_1, y_1)$ on $t_0 \leqq t_1 \leqq t \leqq t_0 + a$, $|y_1 - y_0| \leqq \beta(< b)$ with the properties (i) that, for fixed (t_1, y_1), $y = \eta(t, t_1, y_1)$ is a solution of (7.2) and (ii) that there exists a constant K such that for $\max(t_1, t_2) \leqq t^* \leqq t \leqq t_0 + a$.

$$|\eta(t, t_1, y_1) - \eta(t, t_2, y_2)| \leqq K\,|\eta(t^*, t_1, y_1) - \eta(t^*, t_2, y_2)|.$$

Then $y = \eta(t, t_1, y_1)$ is the only solution of (7.2) for sufficiently small intervals $[t_1, t_1 + \epsilon]$, $\epsilon > 0$, to the right of t_1 (but not necessarily to the left of t_1).

8. Egress Points and Lyapunov Functions

Let $f(t, y)$ be continuous on an open (t, y)-set Ω and let Ω_0 be an open subset of Ω. Let $\partial\Omega_0$ and $\bar{\Omega}_0$ denote the boundary and closure of Ω_0, respectively. A point $(t_0, y_0) \in \partial\Omega_0 \cap \Omega$ is called an *egress point* [or an *ingress point*] of Ω_0 with respect to the system

$$y' = f(t, y) \tag{8.1}$$

if, for every solution $y = y(t)$ of (8.1) satisfying $y(t_0) = y_0$, there exists an $\epsilon > 0$ such that $(t, y(t)) \in \Omega_0$ for $t_0 - \epsilon < t < t_0$ [or for $t_0 < t < t_0 + \epsilon$]. If, in addition, $(t, y(t)) \notin \bar{\Omega}_0$ for $t_0 < t < t_0 + \epsilon$ [or for $t_0 - \epsilon < t < t_0$] for a small $\epsilon > 0$, then (t_0, y_0) is called a *strict egress point* [or *strict ingress point*]. A point $(t_0, y_0) \in \partial\Omega_0 \cap \Omega$ will be referred to as a nonegress point if it is not an egress point.

Lemma 8.1. *Let $f(t, y)$ be continuous on an open set Ω and Ω_0 an open subset of Ω such that $\partial\Omega_0 \cap \Omega$ is either empty or consists of nonegress points. Let $y(t)$ be a solution of* (8.1) *satisfying $(t^0, y(t^0)) \in \Omega_0$ for some t^0. Then $(t, y(t)) \in \Omega_0$ on a right maximal interval of existence $[t^0, \omega_+)$.*

If the conclusion is false, there is a least value $t_0 (> t^0)$ of t, where $(t_0, y(t_0)) \in \partial\Omega_0 \cap \Omega$. But then $(t_0, y(t_0))$ is an egress point, which contradicts the assumption and proves the lemma.

Let $u(t, y)$ be a real-valued function defined in a vicinity of a point $(t_1, y_1) \in \Omega$. Let $y(t)$ be a solution of (8.1) satisfying $y(t_1) = y_1$. If $u(t, y(t))$ is differentiable at $t = t_1$, this derivative is called the *trajectory derivative* of u at (t_1, y_1) along $y = y(t)$ and is denoted by $\dot{u}(t_1, y_1)$. When $u(t, y)$ has continuous partial derivatives, its trajectory derivative exists and can be calculated without finding solutions of (8.1). In fact,

$$\dot{u}(t, y) = \partial u/\partial t + (\operatorname{grad} u) \cdot f(t, y), \tag{8.2}$$

where the dot denotes scalar multiplication and grad $u = (\partial u/\partial y^1, \ldots, \partial u/\partial y^d)$ is the gradient of u with respect to y.

Let $(t_0, y_0) \in \partial\Omega_0 \cap \Omega$ and let $u(t, y)$ be a function of class C^1 on a neighborhood N of (t_0, y_0) in Ω such that $(t, y) \in \Omega_0 \cap N$ if and only if $u(t, y) < 0$. Then a necessary condition for (t_0, y_0) to be an egress point is that $\dot{u}(t_0, y_0) \geqq 0$ and a sufficient condition for (t_0, y_0) to be a strict egress point is that $\dot{u}(t_0, y_0) > 0$. Further, a sufficient condition for (t_0, y_0) to be a nonegress point is that $\dot{u}(t, y) \leqq 0$ for $(t, y) \in \Omega_0$.

When the system under consideration

$$y' = f(y), \tag{8.3}$$

is autonomous (i.e., when the right side does not depend on t), definitions are similar. For example, let $f(y)$ be continuous on an open y-set Ω, Ω_0 an open subset of Ω, and $y_0 \in \partial\Omega_0 \cap \Omega$. The point y_0 is called an egress point of Ω_0 with respect to (8.3) if, for every solution $y(t)$ of (8.3) satisfying $y(0) = y_0$, there exists an $\epsilon > 0$ such that $y(t) \in \Omega_0$ for $-\epsilon < t < 0$. If, in addition, $y(t) \notin \bar{\Omega}_0$ for $0 < t < \epsilon$ for some $\epsilon > 0$, then y_0 is called a strict egress point. A lemma analogous to Lemma 8.1 is clearly valid here.

For an application of these notions, consider a function $f(y)$ defined on an open set containing $y = 0$. A function $V(y)$ defined on a neighborhood of $y = 0$ is called a *Lyapunov function* if (i) it has continuous partial derivatives; (ii) $V(y) \gtreqless 0$ according as $|y| \gtreqless 0$; and (iii) the trajectory derivative of V satisfies $\dot{V}(y) \leqq 0$.

Theorem 8.1. *Let $f(y)$ be continuous on an open set containing $y = 0$, $f(0) = 0$, and let there exist a Lyapunov function $V(y)$. Then the solution $y \equiv 0$ of (8.3) is stable (in the sense of Lyapunov).*

Lyapunov stability of the solution $y \equiv 0$ means that if $\epsilon > 0$ is arbitrary, then there exists a $\delta_\epsilon > 0$ such that if $|y_0| < \delta_\epsilon$, then a solution $y(t)$ of (8.3) satisfying the initial condition $y(0) = y_0$ exists and satisfies $|y(t)| < \epsilon$ for $t \geqq 0$. If in addition, $y(t) \to 0$ as $t \to \infty$, then the solution $y \equiv 0$ of (8.3) is called *asymptotically stable* (in the sense of Lyapunov). Roughly speaking, Lyapunov stability of $y \equiv 0$ means that if a solution $y(t)$ starts near $y = 0$ it remains near $y = 0$ in the future ($t \geqq 0$); and Lyapunov asymptotic stability of $y \equiv 0$ means that, in addition, $y(t) \to 0$ as $t \to \infty$.

Proof. Let $\epsilon > 0$ be any number such that the set $|y| \leqq \epsilon$ is in the open set on which f and V are defined. For any $\eta > 0$, let $\delta(\eta)$ be chosen so that $0 < \delta(\eta) < \epsilon$ and $V(y) < \eta$ if $|y| < \delta(\eta)$.

Reference to Figure 3 will clarify the following arguments. Since $V(y)$ is continuous and positive on $|y| = \epsilon$, there is an $\eta = \eta_\epsilon > 0$ such that

$V(y) > \eta$ for $|y| = \epsilon$. Let Ω_0 be the open set $\{y: |y| < \epsilon, V(y) < \eta\}$. The boundary $\partial\Omega_0$ is contained in the set $\{y: |y| < \epsilon, V(y) = \eta\}$. The function $u(y) = V(y) - \eta$ satisfies $u(y) < 0$ at a point y, $|y| < \epsilon$, if and only if $y \in \Omega_0$. Clearly $\dot{u} = \dot{V} \leqq 0$. Hence, no point of $\partial\Omega_0$ is an egress point. Consequently, by the analogue of Lemma 8.1, a solution $y(t)$ of (8.3) satisfying $y(0) \in \Omega_0$ remains in Ω_0 on its right maximal interval of existence $[0, \omega_+)$. Since Ω_0 is contained in the sphere $|y| \leqq \epsilon$ in Ω, it follows that $\omega_+ = \infty$; Corollary II 3.2.

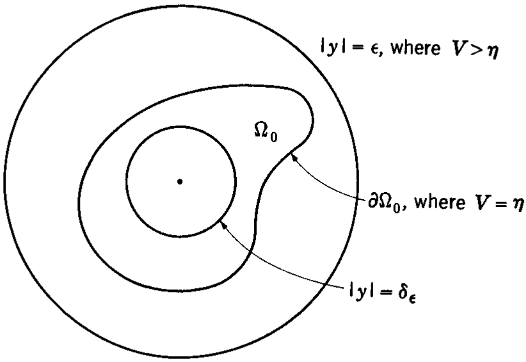

Figure 3.

Finally, put $\delta_\epsilon = \delta(\eta_\epsilon) > 0$, so that $V(y) < \eta$ if $|y| < \delta_\epsilon < \epsilon$. Thus $|y(0)| < \delta_\epsilon$ implies that $y(0) \in \Omega_0$, hence $y(t)$ exists and $y(t) \in \Omega_0$ for $t \geqq 0$. In particular, $|y(t)| < \epsilon$ for $t \geqq 0$. This proves the theorem.

Exercise 8.1. Let $f(y)$ be continuous on an open set containing $y = 0$ and let $f(0) = 0$. Let (8.3) possess a continuous first integral $V(y)$ [i.e., a function which is constant along solutions $y = y(t)$ of (8.3)] such that $V(y)$ has a strict extremum (maximum or minimum) at $y = 0$. Then the solution $y \equiv 0$ of (8.3) is stable.

Theorem 8.2. *If, in Theorem* 8.1, $\dot{V}(y) \lesseqgtr 0$ *according as* $|y| \gtreqless 0$, *then the solution* $y \equiv 0$ *of* (8.3) *is asymptotically stable* (*in the sense of Lyapunov*).

Proof. Use the notation of the last proof. Let $y(t)$ be a solution of (8.3) with $|y(0)| < \delta_\epsilon$. Since $\dot{V} \leqq 0$, it follows that $V(y(t))$ is nonincreasing and tends monotonically to a limit, say $c \geqq 0$, as $t \to \infty$.

Suppose first that $c = 0$. Then $y(t) \to 0$ as $t \to \infty$. For otherwise, there is an $\epsilon_0 > 0$ such that $\epsilon_0 \leqq |y(t)| \leqq \epsilon$ for certain large t-values. But there exists a constant $m_0 > 0$ such that $V(y) > m_0$ for $\epsilon_0 \leqq |y| \leqq \epsilon$; thus $V(y(t)) > m_0 > 0$ for certain large t-values. This is impossible; hence, $y(t) \to 0$ as $t \to \infty$.

Suppose, if possible, that $c > 0$, so that $0 < c < \eta$ and $V(y) < \frac{1}{2}c$ if $|y| < \delta(\frac{1}{2}c) < \epsilon$. Hence $|y(t)| \geqq \delta(\frac{1}{2}c)$ for large t. But the assumption on

$\dot{V}$ implies that there exists an $m > 0$ such that $\dot{V}(y) \leqq -m < 0$ if $\delta(\frac{1}{2}c) \leqq |y| \leqq \epsilon$. In particular, $\dot{V}(y(t)) \leqq -m < 0$, for all large t. This is impossible. Hence $c = 0$ and $y(t) \to 0$ as $t \to \infty$. This proves the theorem.

A result analogous to Theorem 8.1 in which the conclusion is that the solution $y = 0$ is not stable is given by the following:

Exercise 8.2. Let $f(y)$ be continuous on an open set E containing $y = 0$ and let $f(0) = 0$. Let there exist a function $V(y)$ on E satisfying $V(0) = 0$, having continuous partial derivatives and a trajectory derivative such that $\dot{V}(y) \lesseqgtr 0$ according as $|y| \gtreqless 0$ on E. Let $V(y)$ assume negative values for some y arbitrarily near $y = 0$. Then the solution $y \equiv 0$ is not (Lyapunov) stable.

Theorems 8.1 and 8.2 have analogues for nonautonomous systems which depend on a suitable modification of the definition of Lyapunov function: Let $f(t, y)$ be continuous for $t \geqq T$, $|y| \leqq b$ and satisfy

$$f(t, 0) = 0 \qquad \text{for } t \geqq T. \tag{8.4}$$

A function $V(t, y)$ defined for $t \geqq T$, $|y| \leqq b$ is called a *Lyapunov function* if (i) $V(t, y)$ has continuous partial derivatives; (ii) $V(t, 0) = 0$ for $t \geqq T$ and there exists a continuous function $W(y)$ on $|y| \leqq b$ such that $W(y) \gtreqless 0$ according as $|y| \gtreqless 0$, and $V(t, y) \geqq W(y)$ for $t \geqq T$; (iii) the trajectory derivative of V satisfies $\dot{V}(t, y) \leqq 0$.

Theorem 8.3. *Let $f(t, y)$ be continuous for $t \geqq T$, $|y| \leqq b$ and satisfy* (8.4). *Let there exist a Lyapunov function $V(t, y)$. Then the solution $y \equiv 0$ of* (8.1) *is uniformly stable (in the sense of Lyapunov).*

Here, *Lyapunov stability* means that if $\epsilon > 0$ is arbitrary, then there exists a $\delta_\epsilon > 0$ and a $t_\epsilon \geqq T$ such that if $y(t)$ is a solution of (8.1) satisfying $|y(t^0)| < \delta_\epsilon$ for some $t^0 \geqq t_\epsilon$, then $y(t)$ exists and $|y(t)| < \epsilon$ for all $t \geqq t^0$. If, in addition, $y(t) \to 0$ as $t \to \infty$, then the solution $y \equiv 0$ is called *Lyapunov asymptotically stable.* The modifier "uniform" for "stability" or "asymptotic stability" means that t_ϵ can be chosen to be T for all $\epsilon > 0$.

Theorem 8.4. *Let $f(t, y)$, $V(t, y)$ be as in Theorem* 8.3. *In addition, assume that there exists a continuous $W_1(y)$ for $|y| \leqq b$ such that $W_1(y) \gtreqless 0$ according as $|y| \gtreqless 0$ and that $\dot{V}(t, y) \leqq -W_1(y)$ for $t \geqq T$. Then the solution $y \equiv 0$ of* (8.1) *is uniformly asymptotic stable (in the sense of Lyapunov).*

Exercise 8.3. (*a*) Prove Theorem 8.3. (*b*) Prove Theorem 8.4.

9. Successive Approximations

The proof of Theorem II 1.1 suggests the question as to whether or not a solution of

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{9.1}$$

can always be obtained as the limit of the sequence (or a subsequence) of the successive approximations defined in § II 1. That the answer is in the negative is shown by the following example for a scalar initial value problem

$$(9.2) \qquad u' = U(t, u), \qquad u(0) = 0,$$

where $U(t, u)$ will be defined for $t \geqq 0$ and all u.

Consider the approximations $u_0(t) \equiv 0$ and

$$u_{n+1}(t) = \int_0^t U(s, u_n(s))\, ds \qquad \text{if } n \geqq 0.$$

Let $U(t, 0) = 2t$, hence $u_1(t) = t^2$; put $U(t, t^2) = -2t$, hence $u_2(t) = -t^2$. Finally, put $U(t, -t^2) = 2t$, so that $u_3(t) = t^2$. Then $u_{2n}(t) = -t^2$ for $n > 0$ and $u_{2n+1}(t) = t^2$ for $n \geqq 0$. It only remains to complete the definition of $U(t, u)$ as a continuous function to obtain the desired example.

One possible completion of this definition is to let $U(t, u) = 2t$ if $u \leqq 0$, $U(t, u) = -2t$ if $u \geqq t^2$, and to be a linear function of u when $0 \leqq u \leqq t^2$, $t > 0$ fixed. In this way, we obtain an example in which $U(t, u)$ is nonincreasing with respect to u (for fixed $t \geqq 0$). In this case, the solution of (9.2) is unique (Corollary 6.3) although no subsequence of the successive approximations converge to a solution.

It turns out, however, that if the solutions of (9.1) are unique by virtue of Theorem 6.1, then successive approximations converge to a solution.

Theorem 9.1. *Let R, R_0, f, ω be as in Theorem* 6.1. *Let* $|f(t, y)| \leqq M$ *on R and* $\alpha = \min(a, b/M)$. *Then the functions* $y_0(t) = y_0$,

$$(9.3) \qquad y_n(t) = y_0 + \int_{t_0}^t f(s, y_{n-1}(s))\, ds \qquad \text{if } n \geqq 1,$$

are defined and converge uniformly on $[t_0, t_0 + \alpha]$ *to the solution* $y = y(t)$ *of* (9.1).

Proof. By Exercise 6.5, it can be supposed that $\omega(t, u)$ is continuous on the closure of R_0 and is nondecreasing with respect to u for fixed t.

The sequence of approximations (9.3) are uniformly bounded and equicontinuous on $[t_0, t_0 + \alpha]$ and hence possesses uniformly convergent subsequences. If it is known that $y_n(t) - y_{n-1}(t) \to 0$ as $n \to \infty$, then (9.3) implies that the limit of any such subsequence is the unique solution $y(t)$ of (9.1). It then follows that the full sequence $y_0, y_1, \ldots$ converges uniformly to $y(t)$; cf. Remark 2 following Theorem I 2.3. Thus, in order to prove Theorem 9.1, it suffices to verify that $\lambda(t) \equiv 0$, where

$$(9.4) \qquad \lambda(t) = \limsup |y_n(t) - y_{n-1}(t)| \qquad \text{as } n \to \infty.$$

Since $|f| \leqq M$ on R,

$$|y_n(t_1) - y_{n-1}(t_1)| \leqq |y_n(t_2) - y_{n-1}(t_2)| + 2M\,|t_1 - t_2|.$$

The right side is at most $\lambda(t_2) + \epsilon + 2M\,|t_1 - t_2|$ for large n if $\epsilon > 0$. Hence $\lambda(t_1) \leqq \lambda(t_2) + \epsilon + 2M\,|t_1 - t_2|$. Since $\epsilon > 0$ is arbitrary and t_1, t_2 can be interchanged, $|\lambda(t_1) - \lambda(t_2)| \leqq 2M\,|t_1 - t_2|$. In particular, $\lambda(t)$ is continuous for $t_0 \leqq t \leqq t_0 + \alpha$.

By the relation (9.3),

$$y_{n+1}(t) - y_n(t) = \int_{t_0}^{t} [f(s, y_n(s)) - f(s, y_{n-1}(s))]\, ds.$$

Hence, by (6.3),

$$|y_{n+1}(t) - y_n(t)| \leqq \int_{t_0}^{t} \omega(s, |y_n(s) - y_{n-1}(s)|)\, ds.$$

For a fixed t on the range $t_0 < t \leqq t_0 + \alpha$, there is a sequence of integers $n(1) < n(2) < \ldots$ such that $|y_{n+1}(t) - y_n(t)| \to \lambda(t)$ as $n = n(k) \to \infty$ and that $\lambda_1(s) = \lim |y_n(s) - y_{n-1}(s)|$ exists uniformly on $t_0 \leqq s \leqq t_0 + \alpha$ as $n = n(k) \to \infty$. Thus,

$$\lambda(t) \leqq \int_{t_0}^{t} \omega(s, \lambda_1(s))\, ds.$$

Since $\lambda_1(s) \leqq \limsup |y_n(s) - y_{n-1}(s)| = \lambda(s)$ and $\omega(t, u)$ is monotone in u,

$$\lambda(t) \leqq \int_{t_0}^{t} \omega(s, \lambda(s))\, ds.$$

By Corollary 4.4, $\lambda(t) \leqq u_0(t)$, where $u_0(t)$ is the maximal solution of

$$u' = \omega(t, u), \qquad u(t_0) = 0.$$

Since this initial value problem has the unique solution $u_0(t) \equiv 0$, it follows that $\lambda(t) \equiv 0$. This proves the theorem.

Exercise 9.1. Show that under the conditions of Exercise 6.7(a), the successive approximations $y_0(t) = 0$ and (9.3), where $t_0 = 0$ and $y_0 = 0$, converge uniformly on $0 \leqq t \leqq \min(a, b/M)$ to the solution of $y' = f(t, y)$, $y(0) = 0$.

Exercise 9.2. For two vectors, $y = (y^1, \ldots, y^d)$ and $z = (z^1, \ldots, z^d)$, use the notation $y \geqq z$ if $y^k \geqq z^k$ for $k = 1, \ldots, d$. Let $f = (f^1, \ldots, f^d)$ and $y = (y^1, \ldots, y^d)$. Assume that $f(t, y)$ is continuous on $R: 0 \leqq t \leqq a$, $|y| \leqq b$ and that $f(t, y_1) \leqq f(t, y_2)$ if $y_1 \leqq y_2$. (a) Define two sequences of successive approximations $y_{0\pm}(t), y_{1\pm}(t), \ldots$ on $0 \leqq t \leqq \alpha = \min(a, b/M)$, where $y_{0\pm}(t) = \pm M(1, \ldots, 1)t$ and

$$y_{n\pm}(t) = \int_0^t f(s, y_{n-1\pm}(s))\, ds \text{ for } n = 1, 2, \ldots.$$

Show that $y_{0+}(t) \geqq y_{1+}(t) \geqq \ldots$ and $y_{0-}(t) \leqq y_{1-}(t) \leqq \ldots$ and that both sequences converge uniformly to solutions of $y' = f(t, y)$, $y(0) = 0$. (*b*) Show that $y_{0\pm}(t)$ can be replaced by continuous functions $y_{0\pm}(t)$ on $0 \leqq t \leqq \alpha$ satisfying $|y_{0\pm}(t)| \leqq b$ and

$$y_{0+}(t) \geqq \int_0^t f(s, y_{0+}(s))\, ds, \qquad y_{0-}(t) \leqq \int_0^t f(s, y_{0-}(s))\, ds$$

(e.g., $y_{0-}(t) \equiv y_0$ is admissible if $f(t, y_0) \geqq 0$).

Exercise 9.3. (*a*) Using the notation $y \geqq z$ introduced in Exercise 9.2, let $f(t, y)$ be continuous for $t \geqq 0$ and all y and satisfy $f(t, y_1) \leqq f(t, y_2)$ if $y_1 \leqq y_2$. Let $y(t)$ be a solution of $y' = -f(t, y)$ satisfying $y(t) \leqq y(0)$ for $t \geqq 0$; cf., e.g., § XIV 2. Consider the successive approximations $y_0(t), y_1(t), \ldots$ defined by $y_0(t) \equiv y(0)$, $y_n(t) = y(0) - \int_0^t f(s, y_{n-1}(s))\, ds$ for $n = 1, 2, \ldots$. Let $z_n(t)$ denote the "error" $z_n(t) = y_n(t) - y(t)$. Show that $(-1)^n z_n(t) \geqq 0$ for $n = 0, 1, \ldots$ and $(-1)^n z_n'(t) \geqq 0$ for $n = 1, 2, \ldots$ and $t \geqq 0$. (Convergence of the successive approximations is not asserted.) (*b*) Let $E_n(t) = \sum_{m=0}^{n} (-1)^m t^m/m!$ be the nth partial sum of the MacLaurin series for e^{-t}. Show that $(-1)^n(E_n(t) - e^{-t}) \geqq 0$ for $n = 0, 1, \ldots$ and $t \geqq 0$.

Exercise 9.4. Let $U(t, u)$ be real-valued and continuous for $t \geqq 0$ and arbitrary u and $U(t, u)$ nondecreasing with respect to u for fixed t. Let u_0, u_0' be fixed numbers and $u(t)$ a solution of $u'' = -U(t, u)$. Define successive approximations for $u(t)$ by putting $u_0(t) = u_0 + u_0't$ and

$$u_n(t) = u_0(t) - \int_0^t (t - s)U(s, u_{n-1}(s))\, ds \qquad \text{for } n = 1, 2, \ldots .$$

Then $u_0(t), u_1(t), \ldots$ are defined for $t \geqq 0$. (*a*) Suppose that $u(t)$ satisfies $u(t) \leqq u_0 + u_0't$ on its right maximal interval of existence $[0, \omega_+)$. Show that $\omega_+ = \infty$ and that the "error" $v_n(t) = u_n(t) - u(t)$ satisfies $(-1)^n v_n(t) \geqq 0$, $(-1)^n v_n'(t) \geqq 0$ for $n = 1, 2, \ldots$ and $t \geqq 0$. (Convergence of the successive approximations is not asserted.) (*b*) Let $C_n(t) = \sum_{m=0}^{n} (-1)^m t^{2m}/(2m)!$ and $S_n(t) = \sum_{m=0}^{n} (-1)^m t^{2m+1}/(2m + 1)!$ be the nth partial sums of the Maclaurin series for $\cos t$ and $\sin t$, respectively. Show that $(-1)^n [C_n(t) - \cos t] \geqq 0$ and $(-1)^n [S_n(t) - \sin t] \geqq 0$ for $n = 0, 1, \ldots$ and $t \geqq 0$. (*c*) Let $U(t, u) = q(t)u$, where $q(t) \geqq 0$ is continuous and nondecreasing for $t \geqq 0$. Using Theorem XIV 3.1_∞ and the remarks following it, show that (*a*) is applicable if $u_0 \geqq 0$ and $u_0' \geqq 0$ [i.e., show that $u(t) \leqq u_0 + u_0't$ for $t \geqq 0$].

Notes

SECTION 1. Theorem 1.1 goes back essentially to Peano [1]. A special case was stated and proved by Gronwall [1]; a slightly more general form of the theorem (which is contained in Corollary 4.4) is given by Reid [1, p. 290]. The proof in the text is that of Titchmarsh [1, pp. 97–98].

SECTION 2. Maximal and minimal solutions were considered by Peano [1]; see Perron [4].

SECTION 4. Differential inequalities of the type (4.1) occur in the work of Peano [1] and of Perron [4]. Theorem 4.1 and its proof are taken from Kamke [1] and are essentially due to Peano. Exercises 4.2 and 4.3 are results of Kamke [2]; see Ważewski [7]. A special case of Corollary 4.4 is given by Bihari [1]. Exercise 4.6 is a result of Opial [1].

SECTION 5. Results of the type in Theorem 5.1 and Exercise 5.1 were first given by Wintner [1], [4].

SECTION 6. Theorem 6.1 is due to Kamke [1]. An earlier version, in which it is assumed that $\omega(t, u)$ is continuous also for $t = 0$, was given by Perron [6]. (Exercise 6.5, due to Olech [2], shows that, in a certain sense, Perron's theorem is not less general than Kamke's.) For the case $d = 1$, earlier results of the type of Perron's were given by Bompiani [1] and Iyanaga [1]. For Exercise 6.1, see Szarski [1]. For Corollary 6.1, see Nagumo [1]; a less sharp form was first proved by Rosenblatt [1] with $\omega(t, u) = Cu/t$ and $0 < C < 1$. An example of the type required in Exercise 6.2 was given by Perron [8]. For Corollary 6.2, see Osgood [1]. For Exercise 6.3(*a*), see Lévy [1, pp. 46–47]. For Exercise 6.4, see Wallach [1]. For Exercise 6.5, see Olech [2]. For a particular case of Exercise 6.6, see Wintner [22]. For Exercise 6.8, part (*a*), see F. Brauer [1], who generalized the result of part (*b*) due to Krasnosel'skiĭ and S. G. Krein [1].

For other uniqueness theorems related to those of this section, see F. Brauer and S. Sternberg [1]. These involve estimates for a function $V(t, |y_2(t) - y_1(t)|)$ instead of $|y_1(t) - y_2(t)|$. For earlier references on the subject of uniqueness theorems, see Müller [3] and Kamke [4, pp. 2 and 33].

SECTION 7. Theorem 7.1 is a result of van Kampen [2].

SECTION 8. The terminology "egress point" and "ingress point" is that of Ważewski [5]. Exercise 8.1 is due to Dirichlet [1]; it was first given by Lagrange [1, pp. 36–44] under the assumption that $V(y)$ is analytic and that the Hessian matrix $(\partial^2 V/\partial y^i\, \partial y^j)$ of V at $y = 0$ is definite. This result is the forerunner of Lyapunov's Theorem 8.1. Theorems 8.1 and 8.2, Exercise 8.2, and Theorems 8.3 and 8.4 are due to Lyapunov [2] (and constitute the basis for his "direct" or "second" method); cf. LaSalle and Lefschetz [1]. For references and recent developments on this subject, see W. Hahn [1], Antosiewicz [1], Massera [2], and Krasovskiĭ [4].

SECTION 9. The example of nonconvergent successive approximations is due to Müller [1]. Theorem 9.1, as stated, is due to Olech [2] and avoids an assumption of monotony on $\omega(t, u)$ occurring in earlier versions of this result. Earlier versions and special cases are to be found in Rosenblatt [1], van Kampen [3] (cf. also Haviland [1]), Dieudonné [1], Wintner [2], LaSalle [1], Coddington and Levinson [1], Viswanatham [1], and Ważewski [8]. The reduction of the proof of Theorem 9.1 to the verification that $\lambda(t) \equiv 0$ is due to Dieudonné (and independently to Wintner) and is used by the authors following them. Exercise 9.1 is a result of F. Brauer [1] and generalizes Luxemburg [1]. Exercise 9.2(*a*) is due to Müller [1]; cf. LaSalle [1] for part (*b*). For Exercise 9.3(*a*), cf. Hartman and Wintner [16]. For Exercise 9.4, cf. Wintner [16].