the phase portraits of this system for various values of $(\mu, \nu) \in \mathbb{R}^2$. What is the form of the degenerate singularity occurring at $(x, \dot{x}) = (0, 0)$ when $(\mu, \nu) = (0, 0)$? (The global aspects of this problem are difficult and require the use of techniques discussed in Chapters 4 and 7. See Section 7.3 in particular.)

EXERCISE 3.4.8 (For the computationally minded). For $\sigma = 10$, $\beta = \frac{8}{3}$, the nonzero equilibria of the Lorenz equation undergo a Hopf bifurcation with $24 < \rho < 25$. Compute the cubic coefficient which determines stability in this example. (Cf. Marsden and McCracken [1976] for a discussion of this problem, but beware, there are some mistakes in their derivation.)

We end this section by noting that Allwright [1977] and Mees [1981] have obtained Hopf bifurcation criteria by means of harmonic balance and the use of a Liapunov function approach.

Appendix to Section 3.4: Derivation of Stability Formula (3.4.11)

If the reduced (approximate) system has a purely imaginary pair of eigenvalues λ, $\bar{\lambda} = \pm i\omega$, then it can be conveniently represented as a single complex equation:

$$\dot{z} = \lambda z + h(z, \bar{z}), \tag{3.4.17}$$

where

$$z = x + iy \quad \text{and} \quad \lambda = i\omega.$$

The normal form (3.4.8) becomes, at $\mu = 0$,

$$\begin{aligned} \dot{w} &= \lambda w + c_1 w^2 \bar{w} + c_2 w^3 \bar{w}^2 + \cdots + c_k w^{k+1} \bar{w}^k + O(|w|^{2k+3}) \\ &\stackrel{\text{def}}{=} \lambda w + \hat{h}(w, \bar{w}), \end{aligned} \tag{3.4.18}$$

where the complex coefficients are of the form

$$c_j = a_j + ib_j, \tag{3.4.19}$$

and an overbar denotes complex conjugation.

EXERCISE 3.4.9. Check the assertions above.

Since in polar coordinates we have

$$\begin{aligned} \dot{r} &= a_1 r^3 + a_2 r^5 + \cdots, \\ \dot{\theta} &= \omega + b_1 r^2 + b_2 r^4 + \cdots, \end{aligned} \tag{3.4.20}$$

the first nonvanishing coefficients a_j, b_j determine the stability (and local amplitude growth) of the periodic orbit and the amplitude dependent modification to its period.

Thus far we have merely recast our system into complex form. Now, following Hassard and Wan [1978], we will show how this form enables us to calculate the leading coefficient, $a_1 = \text{Re}(c_1)$, relatively simply. The computations are considerably easier than those of Marsden and McCracken [1976]. To transform (3.4.17) to (3.4.18) we use the near identity transformation

$$z = w + \psi(w, \bar{w}), \qquad \psi = O(|w|^2). \tag{3.4.21}$$

Substituting (3.4.21) in (3.4.17) and using (3.4.18) we obtain

$$\lambda(w\psi_w - \psi) + \overline{\lambda w \psi_{\bar{w}}} = h(w + \psi, \bar{w} + \bar{\psi}) - \hat{h}(w, \bar{w})(1 + \psi_w) - \overline{\hat{h}(w, \bar{w})}\psi_{\bar{w}}, \tag{3.4.22}$$

where subscripts denote partial differentiation. We now express ψ as a Taylor series (with $\psi_{jk} = \partial\psi^{j+k}/\partial w^j \partial \bar{w}^k$):

$$\psi(w, \bar{w}) = \sum_{2 \le j+k \le 3} \psi_{jk} \frac{w^j \bar{w}^k}{j!\,k!} + O(|w|^4). \tag{3.4.23}$$

Next, using the fact that the normal form $\hat{h}(w, \bar{w}) = c_1 w^2 \bar{w} + O(|w|^5)$ and substituting (3.4.23) in (3.4.22), we obtain

$$\begin{aligned} \lambda\psi_{ww}\frac{w^2}{2} + \bar{\lambda}\psi_{w\bar{w}} w\bar{w} + (2\bar{\lambda} - \lambda)\psi_{\bar{w}\bar{w}}\frac{\bar{w}^2}{2} &= h_{ww}\frac{w^2}{2} + h_{w\bar{w}} w\bar{w} + h_{\bar{w}\bar{w}}\frac{\bar{w}^2}{2} \\ &\quad + O(|w|^3). \end{aligned} \tag{3.4.24}$$

Equating coefficients yields the leading terms in the transformation

$$\begin{aligned} \psi_{ww} &= \frac{h_{ww}}{\lambda} = -\frac{ih_{ww}}{\omega}, \qquad \psi_{w\bar{w}} = \frac{h_{w\bar{w}}}{\bar{\lambda}} = \frac{ih_{w\bar{w}}}{\omega}, \\ \psi_{\bar{w}\bar{w}} &= \frac{h_{\bar{w}\bar{w}}}{(2\bar{\lambda} - \lambda)} = \frac{ih_{\bar{w}\bar{w}}}{3\omega}. \end{aligned} \tag{3.4.25}$$

We now carry out the expansion to one higher order and equate the coefficients of the normal form term $w^2\bar{w}$. The reader can check that, for this term, the coefficient on the left-hand side of (3.4.22) vanishes identically and the right-hand side therefore becomes

$$h_{ww}\psi_{w\bar{w}} + h_{w\bar{w}}\left(\frac{\psi_{ww}}{2} + \bar{\psi}_{w\bar{w}}\right) + h_{\bar{w}\bar{w}}\frac{\overline{\psi_{\bar{w}\bar{w}}}}{2} + \frac{h_{ww\bar{w}}}{2} - c_1 = 0,$$

or, using (3.4.25)

$$c_1 = \frac{i}{2\omega}(h_{ww}h_{w\bar{w}} - 2|h_{w\bar{w}}|^2 - \tfrac{1}{3}|h_{ww}|^2) + \frac{h_{ww\bar{w}}}{2}. \tag{3.4.26}$$

Hence we have

$$2a_1 = 2\,\text{Re}(c_1) = h^R_{ww\bar{w}} - \frac{1}{\omega}(h^R_{ww}h^I_{w\bar{w}} + h^R_{w\bar{w}}h^I_{ww}), \tag{3.4.27}$$

where the superscripts R, I denote real and imaginary parts, respectively. A similar expression can be found for b_1. In this way we see precisely how the third-order terms are modified by our transformation $\phi = \mathrm{id} + \psi$, with which we have removed the second-order terms. In Hassard and Wan's [1978] paper the second (fifth-order) coefficient c_2 is calculated, and the system is embedded in a higher-dimensional problem, so that one also has additional terms arising from the center manifold approximation.

We stress that these calculations can be carried out for the original system in real variables, but that they are considerably more cumbersome in that form. However, since we are typically working with systems in real form, it is convenient to express a_1 in terms of the real functions f, g of Equation (3.4.8). Expanding f and g in Taylor series and taking real and imaginary parts of the complex valued function h (and its series) we find that the relevant terms in (3.4.27) may be expressed as

$$\left.\begin{aligned} h^R_{ww\bar{w}} &= \tfrac{1}{8}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}),\\ h^R_{ww} &= \tfrac{1}{4}(f_{xx} - f_{yy} + 2g_{xy}),\\ h^I_{ww} &= \tfrac{1}{4}(g_{xx} - g_{yy} - 2f_{xy}),\\ h^R_{w\bar{w}} &= \tfrac{1}{4}(f_{xx} + f_{yy}),\\ h^I_{w\bar{w}} &= \tfrac{1}{4}(g_{xx} + g_{yy}). \end{aligned}\right\} \tag{3.4.28}$$

EXERCISE 3.4.10. Verify (3.4.28) and find expressions for $|h_{w\bar{w}}|^2$, $|h_{\bar{w}\bar{w}}|^2$ and $h^I_{ww\bar{w}}$ in terms of f and g, so that you can calculate $b_1 = \mathrm{Im}(c_1)$.

The stability formula (3.4.11) can now be derived by substitution of the expressions of (3.4.28) into (3.4.27):

$$\begin{aligned} 16a_1 &= (f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy})\\ &\quad + \frac{1}{\omega}[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}]. \end{aligned} \tag{3.4.29}$$

We end by noting that the normal form for the parametrized Hopf bifurcation is neatly expressed in complex variables as

$$\dot{w} = \lambda w + c_1 w^2 \bar{w} + O(|w|^5), \tag{3.4.30}$$

when $\lambda = \mu + i\omega$; cf. Arnold [1972].

3.5. Codimension One Bifurcations of Maps and Periodic Orbits

In this section we consider the simplest bifurcations for periodic orbits. The strategy that we adopt involves computing Poincaré return maps and then trying to repeat the results of Section 3.4 for these discrete dynamical systems. There are some additional complications that introduce new subtleties to some of these problems. In practice, computations of the bifurcations of periodic orbits from a defining system of equations are substantially more difficult than those for equilibria because one must first integrate the equations near the periodic orbit to find the Poincaré return map before further analysis can proceed. Thus, the results obtained here have been most frequently applied:

(1) in comparison with numerical calculations;
(2) directly to discrete dynamical systems defined by a mapping; or
(3) in perturbation situations close to ones in which a system can be explicitly integrated.

The third category will form the subject of Chapter 4. In view of these computational difficulties, in this section we shall focus upon the geometric aspects of these bifurcations.

There are three ways in which a fixed point p of a discrete mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ may fail to be hyperbolic: $Df(p)$ may have an eigenvalue $+1$, an eigenvalue -1, or a pair of complex eigenvalues λ, $\bar{\lambda}$ with $|\lambda| = 1$. (If $Df(p)$ has an eigenvalue μ at the fixed point p, we say p has eigenvalue μ.) The bifurcation theory for fixed points with eigenvalue 1 is completely analogous to the bifurcation theory for equilibria with eigenvalue 0. The generic one-parameter family has a two-dimensional center manifold (including the parameter direction) on which it is topologically equivalent to the *saddle-node* family defined by the map

$$f_\mu(x) = x + \mu - x^2. \tag{3.5.1}$$

The same considerations of constraint and symmetry as discussed in Section 4 alter the generic picture, giving either *transcritical* or *pitchfork* bifurcations. Rather than working out examples in detail, we leave the computation of the following exercises to the reader:

EXERCISE 3.5.1. Show that the map $x \to \mu - x^2$ undergoes a saddle-node bifurcation at $(x, \mu) = (-\frac{1}{2}, -\frac{1}{4})$. On which side of the bifurcation value $\mu = -\frac{1}{4}$ do the fixed points lie?

EXERCISE 3.5.2. Show that the map $x \to \mu x(1 - x)$ undergoes a transcritical bifurcation at $(x, \mu) = (0, 1)$.

EXERCISE 3.5.3. Show that the map $(x, y) \to (y, -\frac{1}{2}x + \mu y - y^3)$ undergoes a pitchfork bifurcation at $(x, y, \mu) = (0, 0, \frac{3}{2})$. Is it sub- or supercritical? Approximate a suspended center manifold near $(0, 0, \frac{3}{2})$ and sketch the bifurcation diagram (cf. Exercise 3.2.8(c)).

Bifurcations with eigenvalue -1 do not have an analogue for equilibria, while the theory for complex eigenvalues is more subtle than that of the Hopf bifurcation for flows.

Eigenvalues with -1 are associated with *flip* bifurcations, also referred to as *period doubling* or *subharmonic* bifurcations. Using a center manifold