Př.
$$x' = 1 - x^2 - y^2$$
$$y' = z^2 - x - y$$
$$z' = z^2 - 1$$

? stac. body – stabilita

1) výpočet stac. bodů:
$$\left.\begin{aligned} 1 - x^2 - y^2 &= 0 \\ z^2 - x - y &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x^2 + y^2 &= 1 \\ x + y &= 1 \end{aligned} \quad *)$$
$$z^2 - 1 = 0 \Rightarrow z = \pm 1$$

$$\boxed{\begin{matrix}[0, 1, \pm 1] \\ [1, 0, \pm 1]\end{matrix}}$$

*) 2. rce: $y = 1 - x$

1. rce: $x + (1-x)^2 = 1$
$$2x^2 - 2x + 1 = 1$$
$$x^2 - x = 0 \Rightarrow x = 0, 1$$

2) bod $[1, 0, -1]$: $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $F(X) = \begin{pmatrix} 1 - x^2 - y^2 \\ z^2 - x - y \\ z^2 - 1 \end{pmatrix}$

$$\nabla F = \begin{pmatrix} \frac{\partial F_1}{\partial x}, & \frac{\partial F_1}{\partial y}, & \\ & \ddots & \\ & & \frac{\partial F_3}{\partial z} \end{pmatrix} = \begin{pmatrix} -2x, & -2y, & 0 \\ -1, & -1, & 2z \\ 0, & 0, & 2z \end{pmatrix}$$

$$\nabla F \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2, & 0, & 0 \\ -1, & -1, & -2 \\ 0, & 0, & -2 \end{pmatrix} =: A \quad \dots \quad p(\lambda) = \det(\lambda I - A)$$

$$= \begin{vmatrix} \lambda+2, & 0, & 0 \\ 1 & \lambda+1, & 2 \\ 0 & 0, & \lambda+2 \end{vmatrix} =$$

$$= (\lambda+2) \begin{vmatrix} \lambda+1, & 2 \\ 0, & \lambda+2 \end{vmatrix} = (\lambda+2)^2(\lambda+1)$$

$\Rightarrow \sigma(A)=\{-2,-1\} \Rightarrow [1,0,-1] \ldots$ asympt. stabilní

3) bod $[0,1,-1]$: $A=\begin{pmatrix}0, & -2, & 0\\ -1, & -1, & -2\\ 0, & 0, & -2\end{pmatrix}$

char. pol. $\chi(\lambda)=\begin{vmatrix}\lambda, & 2, & 0\\ 1, & \lambda+1, & 2\\ 0, & 0, & \lambda+2\end{vmatrix}=(\lambda+2)\begin{vmatrix}\lambda, & 2\\ 1, & \lambda+1\end{vmatrix}$

$=(\lambda+2)\left(\lambda(\lambda+1)-2\right)$

$=(\lambda+2)(\underbrace{\lambda^2+\lambda-2}_{(\lambda+2)(\lambda-1)})$

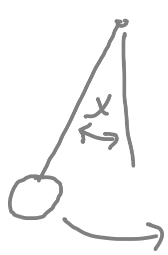
$\chi(\lambda)=(\lambda+2)^2(\lambda-1)$

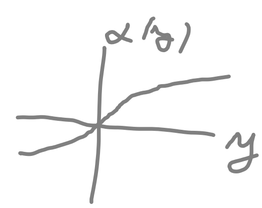
$\Rightarrow \sigma(A)=\{-2,+1\} \Rightarrow [0,1,-1]$ nestabilní

4) $[0,1,1]$

$[1,0,1]$ ... $A=\begin{pmatrix} & \cdots & \\ 0, & 0, & 2\end{pmatrix} \Rightarrow 2\in\sigma(A)$ nestabilita

Př: kyvadlo s tlumením: $x''+\underline{\alpha(x')}+\sin x=0$





nechť: $\alpha(0)=0$, $\alpha\in C^1$

$\alpha'(0)=a>0$

$$\Rightarrow \quad x' = y$$
$$y' = -\sin x - \alpha(y)$$

? stabilita $[0,0]$

$$\nabla F = \begin{pmatrix} 0, & 1 \\ -\cos x, & -\alpha'(y) \end{pmatrix}$$

$$A = \nabla F\binom{0}{0} = \begin{pmatrix} 0, & 1 \\ -1, & -a \end{pmatrix}$$

char. poly.

$$\chi(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda, & -1 \\ 1, & \lambda + a \end{vmatrix} = \lambda(\lambda + a) + 1$$
$$= \lambda^2 + a\lambda + 1$$

$$\Rightarrow \quad \lambda_{1,2} = \frac{-a \pm \sqrt{D}}{2}, \quad D = a^2 - 4$$

diskuse:

(i) $a^2 - 4 < 0 \Rightarrow \sqrt{D} = i\omega$

leč: $\mathrm{Re}\,\lambda_{1,2} = \frac{-a}{2} < 0$

(ii) $a^2 - 4 \geq 0 \Rightarrow \sqrt{D} \leq \sqrt{a^2} = a$

$\Rightarrow \lambda_{1,2} < 0$

CELKEM: vždy asympt. stab.

Pozn. ad **)

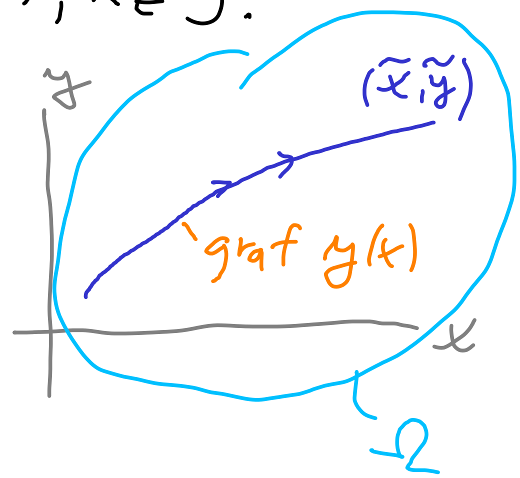
(1) $x' = f(x,y)$, $y' = g(x,y)$ $\xrightarrow[\text{podělením}]{\text{formálně}}$ (2) $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$

Tvrzení: Nechť $(\tilde{x}(t), \tilde{y}(t))$, $t \in I$ řeší (1) v $\Omega \subset \mathbb{R}^2$, nechť $f(x,y) \neq 0$ v Ω.

Pak: $t \mapsto \tilde{x}(t)$ je prosté $I \to J$, a označíme-li $y(x) := \tilde{y}(\tilde{x}_{-1}(x))$, $x \in J$, potom $y(x)$ je řešení (2), $x \in J$.

Dk.:

$$\underbrace{\frac{dy}{dx}(x)}_{} = \underbrace{\tilde{y}'(\tilde{x}_{-1}(x))}_{(i)} \underbrace{(\tilde{x}_{-1})'(x)}_{(ii)}$$



ad (i) víme:

$$\left.\begin{aligned} \tilde{x}'(t) &= f(\tilde{x}(t), \tilde{y}(t)) \\ \tilde{y}'(t) &= g(\tilde{x}(t), \tilde{y}(t)) \end{aligned}\right\} , \ t \in I$$

$$\Rightarrow \ (i) = g\big(\tilde{x}(\tilde{x}_{-1}(x)), \tilde{y}(\tilde{x}_{-1}(x))\big) = \underbrace{g(x, y(x))}_{}$$

ad (ii):

$$(\tilde{x}_{-1})'(x) = \frac{1}{\tilde{x}'(\tilde{x}_{-1}(x))} = \frac{1}{\underbrace{f(x, y(x))}_{}}$$

CELKEM: (2) platí

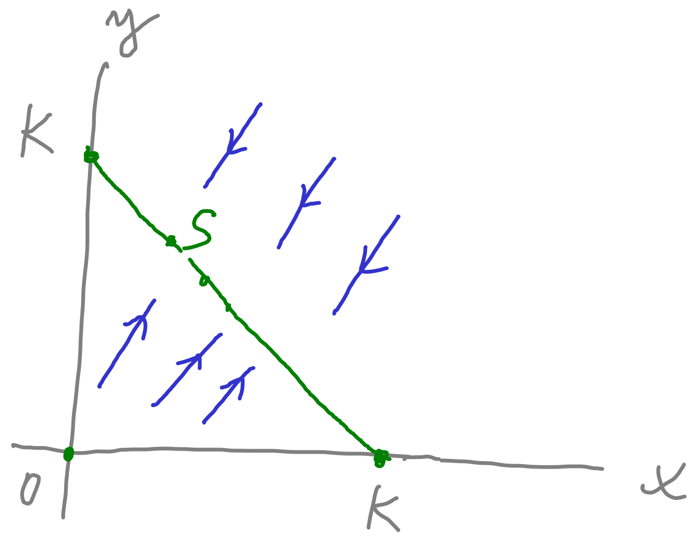
Pozn: $f \neq 0$ v Ω $\Rightarrow \tilde{x}'(t) \neq 0$, tedy prosté

Př. model konkurence

$$x' = r\left(1 - \frac{x+ay}{K}\right)x$$
$$y' = s\left(1 - \frac{y+bx}{L}\right)y$$

spec. případ (degenerovaný)

$$x' = r\left(1 - \frac{x+y}{K}\right)x$$
$$y' = s\left(1 - \frac{x+y}{K}\right)y$$



? stabilita $S = (x_0, y_0)$

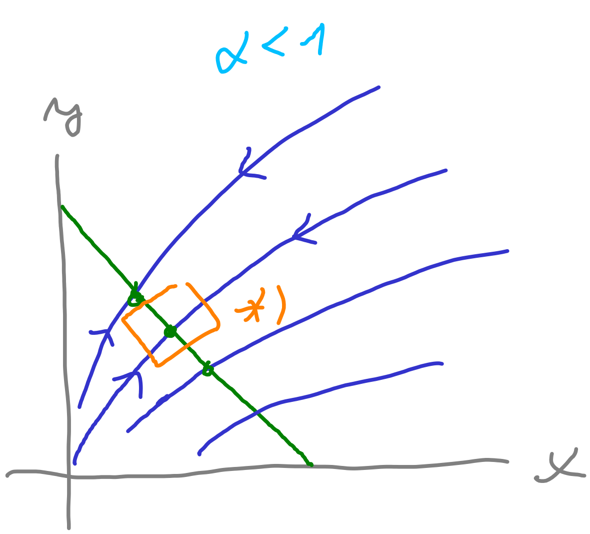
$$\nabla F = \begin{pmatrix} r\left(1 - \frac{x+y}{K}\right) - \frac{rx}{K}, & -\frac{rx}{K} \\ -\frac{sy}{K}, & s\left(1 - \frac{x+y}{K}\right) - \frac{sy}{K} \end{pmatrix}$$

$$A = \nabla F\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -\frac{1}{K}\begin{pmatrix} rx_0, & rx_0 \\ sy_0, & sy_0 \end{pmatrix} \Rightarrow \lambda_1 = 0, \; v^{(1)} = (-1, 1)$$

$$\lambda_2 = -\frac{1}{K}(rx_0 + sy_0) < 0$$

$\Rightarrow$ stabilita nelze určit z $\underline{\underline{A}}$

TRIK: podělení rovnic



$$\Downarrow$$

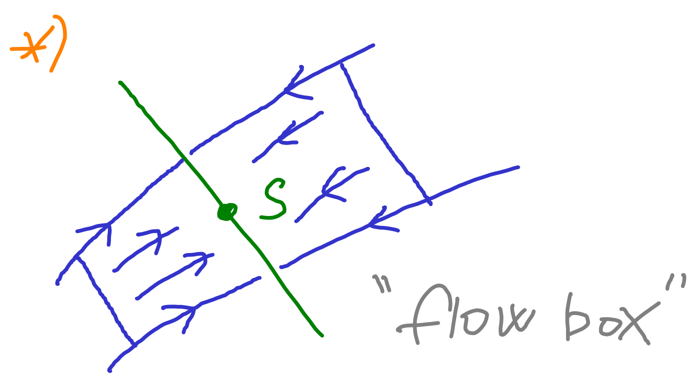
$$\frac{dy}{dx} = \frac{sy}{rx}$$

$$\frac{dy}{y} = \overset{\alpha}{\left(\frac{s}{r}\right)} \frac{dx}{x}$$

$$\ln y = \alpha \cdot \ln x + C$$

$$\boxed{y = d \cdot x^{\alpha}}$$

Tvrdíme: stac. body na Γ jsou stabilní, ne asympt. stab.

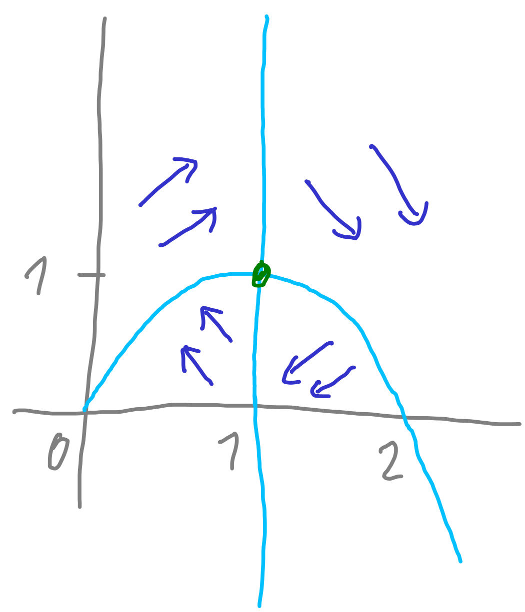


Př.: $x' = x^2 - 2x + y$

$y' = y(1-x)$

$x' = 0 \iff y = 2x - x^2$

$y' = 0 \iff y = 0$ nebo $x = 1$



?? stabilita [1,1] .... ?? linearizace:

$$DF = \begin{pmatrix} 2x-2, & 1 \\ -y, & 1-x \end{pmatrix}$$

$$A = DF\binom{1}{1} = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}$$

$\Rightarrow$ $\sigma(A) = \{\pm i\}$

věty nelze použít !!

TRIK: vydělení:

posun souřadnic

$x(t) = 1 + \tilde{x}(t)$
$y(t) = 1 + \tilde{y}(t)$

$\Rightarrow$
$$\tilde{x}' = x' = (x-1)^2 + y - 1 = \tilde{x}^2 + \tilde{y}$$
$$\tilde{y}' = y' = -\tilde{x}(\tilde{y}+1)$$

$(\tilde{x},\tilde{y})$ v okolí $(0,0)$

$\xRightarrow[\text{vlnek}]{\text{bez}}$
$$x' = x^2 + y$$
$$y' = -x(y+1)$$

podělení $\Rightarrow$
$$\frac{dx}{dy} = \frac{x^2+y}{-x(y+1)}, \quad \boxed{x = x(y)}$$

Bernoulliho rce

$\alpha = -1$

$$x' + \frac{x}{y+1} = -\frac{1}{x}\cdot\frac{y}{y+1} \quad \Big/ \cdot 2x(y+1)^2$$

$$2xx'(y+1)^2 + x^2\,2(y+1) = -2y(y+1) \quad \Big/ \int dy$$

$$x^2(y+1)^2 = C - \frac{2y^3}{3} - y^2$$

$\Rightarrow$ $V(x,y) = C$, kde

$$V(x,y) = x^2(y+1)^2 + y^2 + \frac{2y^3}{3}$$

?? chování v okolí $(0,0)$

pozoruji: $V(x,y) = x^2(y^2+2y+1) + y^2 + \dots$

$$= x^2 + y^2 + \underbrace{2yx^2 + \dots}_{3.\ \text{řád a výše}}$$

$\Rightarrow$ $\nabla V(0,0) = (0,0)$

$$\nabla^2 V(0,0) = \begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}$$

$\Rightarrow$ V ryze konvexní v okolí $(0,0)$

ostré lok. minim.