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7. Game theory and replicator dynamics

Definition. *Game* (more precisely: two player game in normal form) is given by:

- finite sets S_1 and S_2 (*strategies* of the first and the second player)
- functions $\pi_1 : S_1 \times S_2 \to \mathbb{R}$ and $\pi_2 : S_1 \times S_2 \to \mathbb{R}$ (*payoff* of the first and the second player)

For simplicity, we will write $S_1 = \{1, \dots, m\}$ a $S_2 = \{1, \dots, n\}$ and introduce matrices A a B (type $m \times n$) as

$$a_{kl} = \pi_1(k,l), \qquad b_{kl} = \pi_2(k,l) \qquad k = 1, \dots, m,\ l = 1, \dots, n$$

Hence, game can be identified with a matrix couple (A, B). We thus also speak of (bi)-matrix games, and call the first and the second player row player and column player, respectively.
Special cases: $A^T = B$ …symmetric game, $A = A^T = B$ …doubly symmetric game, $A = -B$ (i.e. $\pi_1 = -\pi_2$) …zero sum game.

Definition. By the space of *mixed* strategies of the first and the second player, respectively, we mean

$$\Delta_1 = \left\{p \in \mathbb{R}^m;\ p_i \in [0,1],\ \sum_{i=1}^m p_i = 1\right\}$$
$$\Delta_2 = \left\{q \in \mathbb{R}^n;\ q_i \in [0,1],\ \sum_{i=1}^n q_i = 1\right\}$$

Elements of S_1 and S_2 respectively are called *pure strategies* and are naturally identified with the basis vectors $e^{(k)} = (0 \dots, 1, 0, \dots)$.
Mixed strategies can be understood either probabilistically (random choice of pure strategies) or statistically (large population of pure players). In either case, generalized payoff functions $\pi_{1,2} : \Delta_1 \times \Delta_2 \to \mathbb{R}$ are equal to

$$\pi_1(p,q) = \sum_{k,l} p_k q_l a_{kl} = p \cdot Aq$$
$$\pi_2(p,q) = \sum_{k,l} p_k q_l b_{kl} = p \cdot Bq$$

Definition. Strategy $p^* \in \Delta_1$ is called *best response* to the strategy $q \in \Delta_2$, if

$$\pi_1(p^*, q) = \max_{p \in \Delta_1} \pi_1(p,q)$$

We will write $p^* \in \beta_1(q)$. In other words, $\beta_1(q)$ is the set of best resonses to q. Analogously, we define $\beta_2(p) \subset \Delta_2$ for a given $p \in \Delta_1$.
Further, we defined *support* of the strategy p or q as

$$C(p) = \{k;\ p_k > 0\}, \qquad C(q) = \{l;\ q_l > 0\},$$

It corresponds to (the indices of) the pure strategies, that are present in the strategy p or q.

Remark. Note that $C(p)$, $C(q)$ are always non-empty. Another important observation is that $\beta_2(p)$, $\beta_1(q)$ are non-empty, convex and compact sets.

Lemma 7.1. [Characterisation of best response strategy.] One has $p \in \beta_1(q)$ if and only if $e^{(k)} \in \beta_1(q)$ for every $k \in C(p)$. In particular, there always exists best response among the pure strategies.

Definition. A couple of strategies $(p^*, q^*) \in \Delta_1 \times \Delta_2$ is called *Nash equilibrium* (in short N.e.), if $p^* \in \beta_1(q^*)$ and $q^* \in \beta_2(p^*)$.

Theorem 7.1. Every game has at least one Nash equilibrium.

Remarks. We only discuss *normal* form games. Other type are so called *extended* form games (described by a tree-like structure). Suitable for games like chess, bridge, ... They allow for random moves and incomplete information.

Simplification. In view of applications to population dynamics, we only consider symmetric games from now on. Payoff function is $\pi(x, y) = x \cdot Ay$, where $A \in \mathbb{R}^{n\times n}$. Vectors x, y belong to n-dimensional simplex

$$\Delta = \big\{x \in \mathbb{R}^n;\ x_i \in [0,1],\ \sum_{i=1}^{n} x_i = 1\big\}$$

We think of x representing some large population of pure players, where x_i is the percentage of i-th strategy. Previous definitions (support, best reply) apply here:

$$C(x) = \{i;\ x_i > 0\}$$
$$\beta(x) = \{y \in \Delta;\ \pi(y, x) = \sup_{y\in\Delta} \pi(y, x)\}$$

As a special case, we now have:

Definition. We say that $x \in \Delta$ is Nash equilibrium (NE), provided that $x \in \beta(x)$. This just means that $\pi(x, x) = \sup_{y\in\Delta} \pi(y, x)$.

Remarks. By Lemma 7.1, x is (NE) if and only if $\pi(e^{(i)}, x) \le \pi(x, x)$, with equality for $i \in C(x)$. In words: any pure (or random) strategy cannot do better than the average member of the population. Existence of (NE) follows by a simple modification of Theorem 7.1. The problem is that there can be more than one, so one looks for possible strengthening of the concept. An important example is:

Definition. We say that $x \in \Delta$ is *evolutionary stable (strategy)* (ESS), provided that

$$\big(\forall y \in \Delta, y \ne x\big)\big(\exists \bar\varepsilon = \bar\varepsilon_y > 0\big)\big(\forall \varepsilon \in (0, \bar\varepsilon)\big):\ \pi(x, (1-\varepsilon)x + \varepsilon y) > \pi(y, (1-\varepsilon)x + \varepsilon y)$$

The number $\bar\varepsilon_y$ is called *invasion barrier* and it can be in fact chosen indpendently of y.

Lemma 7.2. x is (ESS) $\iff$ x is (NE) and moreover for any $y \in \beta(x)$, $y \ne x$ one has $\pi(y, y) < \pi(x, y)$.

Remark. One also has another characterization: $x \in \Delta$ is (ESS) $\iff$ for any $y \in \Delta$ close to x, $y \ne x$ there holds $\pi(y, y) < \pi(x, y)$.

Plan. We will now assume $x = x(t)$ and want to write some differential equations, describing the populations dynamics - think of darwinian competition of (pure) strategies. Axiomatically, we expect something like

$$x_i' = x_i g_i(x) \tag{7.1}$$

where $g_i : \Delta \to \mathbb{R}$ should satisfy

1. $g_i(x) > 0$ (or < 0) iff $\pi_i(x) > \pi(x)$ (or $< \pi(x)$) *(payoff monotonicity)*
2. $\sum_i x_i g_i(x) = 0$ *(regularity)*

Simplest choice is $g_i(x) = \pi_i(x) - \pi(x)$, which leads to

$$x_i' = x_i\big(\pi_i(x) - \pi(x)\big) \tag{RD}$$

Here and in what follows, we write

$$\pi_i(x) = \pi(e^{(i)}, x) = (Ax)_i$$
$$\pi(x) = \pi(x, x) = x \cdot Ax$$

where $\pi_i(x)$ is the average payoff of the i-th pure strategy, and $\pi(x)$ is the average payoff of the whole population. Note that $\pi(x) = \sum_i x_i \pi_i(x)$. From now on, we will only study (RD), but many of the results hold for more general systems, as long as the properties 1. and 2. above hold.

Theorem 7.2. For arbitrary initial condition in Δ, there exists a unique $x(t)$ solution to (RD), defined and satisfying $x(t) \in \Delta$ for all $t \in \mathbb{R}$.
Moreover: the support $C(x(t))$ and in particular: the boundary of Δ, its interior, edges, and vertices, are invariant with respect to the equation.

Theorem 7.3. For replicator dynamics (RD) holds:

1. $\tilde{x}$ is N.e. $\implies$ $\tilde{x}$ is stationary point
2. $\tilde{x}$ is stable stationary point $\implies$ $\tilde{x}$ is N.e.
3. $\tilde{x}$ is interior stationary point $\implies$ $\tilde{x}$ is N.e.

Theorem 7.4. Let $\tilde{x}$ be ESS. Then $\tilde{x}$ is asymptotically stable stationary point for (RD).

Remark. The proof of the previous theorem is based on the Lyapunov function (Kullback-Leibler divergence)

$$H(x) = \sum_{i \in C(\tilde{x})} \tilde{x}_i \log\left(\frac{\tilde{x}_i}{x_i}\right), \qquad x \in Q_{\tilde{x}}$$

where $Q_{\tilde{x}} = \{x \in \Delta; C(x) \supset C(\tilde{x})\}$ is relative neighborhood of $\tilde{x}$ in Δ.

Example. Consider game with payoff matrix

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 5 \\ 5 & 0 & 4 \end{pmatrix}$$

The corresponding (RD) has a unique equilibrium $\tilde{x} \in \operatorname{int} \Delta$, where $\tilde{x} = (1/6, 4/9, 7/18)$. It is N.e. and is asymptotically stable (by linearization), but not an ESS (since $\pi_3(\tilde{x}) > \pi(\tilde{x})$).

Remark. Adding constant to an arbitrary column of A does not alter the value of $\pi(x - y, z) = \pi(x, z) - \pi(y, z)$, where x, y, $z \in \Delta$. In particular, this operation does not affect $\beta(\cdot)$, N.e., ESS, (RD), since their definitions only depend on expressions of the above type.
We use this for the so-called *game normalization* where a suitable constant added to each column make the diagonal zero. In the above example, the normalized game is

$$\begin{pmatrix} 0 & 4 & -4 \\ -1 & 0 & 1 \\ 4 & -1 & 0 \end{pmatrix}$$

Hence, the strategies cyclically defeat each other, which gives a sort of rock-scissor-paper game (though not a zero-sum indeed).

Theorem 7.5. [Fisher's fundamental theorem of natural selection.] Let A be *symmetric* matrix. Then the solutions of (RD) satisfy $\frac{d}{dt}\pi(x(t)) \geq 0$, with inequality in (and only in) stationary points.

Lemma 7.3. For replicator dynamics (RD) further holds:

1. $\tilde{x} \in \operatorname{int}\Delta$ is stationary point, if and only iff $\pi_i(\tilde{x})$ does not depend on i.
2. if $\tilde{x}$, $\tilde{y} \in \operatorname{int}\Delta$ are stationary points, then arbitrary convex combination $t\tilde{x} + (1-t)\tilde{y}$ is stationary point.
3. if $\operatorname{int}\Delta$ contains periodic orbit, it also containe stationary point.

Theorem 7.6. Set $u = (1, \ldots, 1) \in \mathbb{R}^n$. Assume that elements of $(\operatorname{adj} A)u$ are not of the same sign. Then replicator dynamics (RD) has no stationary points in $\operatorname{int}\Delta$.

Remarks. For the sake of previous theorem, we distinguish three different signs: $+1$, -1 and 0. Recall that $\operatorname{adj} A$ is the so-called adjugate matrix, with elements equal to $(-1)^{i+j}M_{ji}$, kde M_{ij} is the determinant obtained after deleting row i and column j from A. One has the formula $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)I$. In particular, for regular A we can write $A^{-1} = (\operatorname{adj} A)/\det A$.