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Definition. By solution to an ODE $X' = F(t, X)$ we mean a function $X(t) : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an open interval, such that, for any $t \in I$:

1. there exists a finite derivative $X'(t)$
2. and $X'(t) = F(t, X(t))$

Solution is called *maximal* if it cannot be extended to a strictly larger interval $\tilde{I} \supset I$.

Definition. [Stability.] Let X_0 be an equilibrium to $X' = F(X)$. We say that X_0 is:

1. *stable*, if for any $\varepsilon > 0$ there is $\delta > 0$ such that any (maximal) solution $X(t)$, satisfying $|X(t_0) - X_0| < \delta$, is defined and satisfies $|X(t) - X_0| < \varepsilon$ for all $t \geq t_0$;
2. *unstable*, if it is not stable;
3. *asymptotically stable*, if it is stable and moreover, there exists $\eta > 0$ such that any (maximal) solution $X(t)$, satisfying $|X(t_0) - X_0| < \eta$, is defined for all $t \geq t_0$ and moreover, $X(t) \to X_0$ as $t \to +\infty$.

Definition. [Prime integral.] Function $V(X) : \Omega \to \mathbb{R}$ is called *prime integral* to the equation $X' = F(X)$ in Ω, provided that:

1. $V(X)$ is not constant in Ω
2. for any solution $X(t) : I \to \Omega$, the function $t \mapsto V(X(t))$ is constant in I

Definition. [Orbital derivative.] Let $V(X) : \Omega \to \mathbb{R}$ be a C^1 function. By *orbital derivative* of V with respect to solutions of $X' = F(X)$ we mean

$$\dot{V}_F(X) := \nabla V(X) \cdot F(X)$$

where the right-hand side can also be written as $\sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(X) F_i(X)$.

Definition. [Lyapunov function.] Let X_0 be an equilibrium to $X' = F(X)$, let $\mathcal{U}$ be a neighborhood of X_0. We call $V(X) : \mathcal{U} \to \mathbb{R}$ a *Lyapunov function* (to the equation at point X_0), provided that:

1. $V(X_0) = 0$ and $V(X) > 0$ for all $X \in \mathcal{U} \setminus \{X_0\}$
2. the orbital derivative $\dot{V}_F(X) \leq 0$ for all $X \in \mathcal{U}$

Lyapunov function is called *strict*, if moreover $\dot{V}_F(X) < 0$ for all $X \in \mathcal{U} \setminus \{X_0\}$.