21th lesson

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Hints

$$x^{-a} = \frac{1}{x^a} \qquad \cos^2 x + \sin^2 x = 1$$

$$x^{a/b} = \sqrt[b]{x^a} \qquad a^b = e^{b \ln a} \qquad a^2 - b^2 = (a+b)(a-b)$$

Exercises

1. Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

   (a) $f(x) = x^{13}$
   Solution:
   $$\int x^{13}\,\mathrm{d}x = \frac{x^{14}}{14} + c$$
   $x \in \mathbb{R}$

   (b) $f(x) = \sqrt{x}$
   Solution:
   $$\int \sqrt{x}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x = \frac{x^{3/2}}{3/2} + c = \frac{2\sqrt{x^3}}{3} + c$$
   $x > 0$

   (c) $f(x) = \frac{1}{x^3}$
   Solution:
   $$\int \frac{1}{x^3}\,\mathrm{d}x = \int x^{-3}\,\mathrm{d}x = \frac{x^{-2}}{-2} + c = \frac{-1}{2x^2} + c$$
   $x \neq 0$

   (d) $f(x) = \frac{1}{x}$
   Solution:
   $$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$$
   $x \neq 0$

   (e) $f(x) = (1 + \sin x + \cos x)$
   Solution:
   $$\int (1 + \sin x + \cos x)\,dx = x - \cos x + \sin x + C.$$
   $x \in \mathbb{R}$

(f) $f(x) = 7\sqrt[3]{x^2} + \frac{1}{2}\sin x - \frac{2}{1+x^2}$

Solution:

$$\int 7\sqrt[3]{x^2} + \frac{1}{2}\sin x - \frac{2}{1+x^2}\,\mathrm{d}x = \frac{7x^{5/3}}{5/3} - \frac{1}{2}\cos x - 2\text{arctan } x + c$$

$x \in \mathbb{R}$

(g) $f(x) = \frac{2}{\cos^2 x} - e^x$

Solution:

$$\int \frac{2}{\cos^2 x} - e^x\,\mathrm{d}x = 2\text{tan } x - e^x + c$$

$x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

(h) $f(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{1+x^2} + 1 + x^2$

Solution:

$$\int \frac{1}{\sqrt{1-x^2}} + \frac{1}{1+x^2} + 1 + x^2 = \text{arcsin } x + \text{arctan } x + x + \frac{x^3}{3} + c$$

$x \in (-1, 1)$

(i) $f(x) = \sqrt{x^3} - \frac{1}{\sqrt{x}}$

Solution:

$$\int \sqrt{x^3} - \frac{1}{\sqrt{x}}\,\mathrm{d}x = \frac{x^{5/2}}{5/2} - \frac{x^{1/2}}{1/2} + c$$

$x > 0$

(j) $f(x) = \frac{3x^2+4x+2}{3x}$

Solution:

$$\int \frac{3x^2+4x+2}{3x}\,\mathrm{d}x = \int x + \frac{4}{3} + \frac{2}{3x}\,\mathrm{d}x = \frac{x^2}{2} + \frac{4x}{3} + \frac{2}{3}\ln|x| + c$$

$x \neq 0$

(k) $f(x) = (1-x)(1-2x)(1-3x)$

Solution: Let start with the multiplication

$$\int (1-x)(1-2x)(1-3x)\,dx = \int (1-6x+11x^2-6x^3)\,dx = x-3x^2+\frac{11}{3}x^3-\frac{3}{2}x^4+C.$$

$x \in \mathbb{R}$

(l) $f(x) = \dfrac{x+1}{\sqrt{x}}$

Solution:

$$\int \frac{x+1}{\sqrt{x}}\,dx = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\,dx = \int \left(x^{1/2} + x^{-1/2}\right)\,dx = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C.$$

$x > 0$

2. Prove that if $F'(x) = f(x)$, then $\left(\frac{1}{a}F(ax+b) + C\right)' = f(ax+b)$, for $a \neq 0$.

Solution:

It follows from the derivative rules.

$$\left(\frac{1}{a}F(ax+b) + C\right)' = \frac{1}{a}F'(ax+b)a + C' = f(ax+b).$$

3. Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

(a) $f(x) = \cos(3x)$

Solution:

$$\int \cos 3x\,\mathrm{d}x = \frac{1}{3}\sin 3x + c$$

$x \in \mathbb{R}$

(b) $f(x) = \sin(2x - \pi)$

Solution:

$$\int \sin(2x-\pi)\,\mathrm{d}x = -\frac{1}{2}\cos(2x-\pi) + c$$

$x \in \mathbb{R}$

(c) $f(x) = e^{5-3x}$

Solution:

$$\int e^{5-3x}\,\mathrm{d}x = \frac{-1}{3}e^{5-3x} + c$$

$x \in \mathbb{R}$

(d) $f(x) = \dfrac{1}{1+4x^2}$

Solution:

$$\int \frac{1}{1+4x^2}\,\mathrm{d}x = \int \frac{1}{1+(2x)^2}\,\mathrm{d}x = \frac{1}{2}\arctan 2x + c$$

$x \in \mathbb{R}$

(e) $f(x) = \dfrac{1}{1-4x}$
Solution:
$$\int \frac{1}{1-4x} = -\frac{1}{4}\ln|1-4x| + c$$
$x \neq \frac{1}{4}$

(f) $f(x) = (2x+1)^7$
Solution:
$$\int (2x+1)^7 \,\mathrm{d}x = \frac{1}{16}(2x+1)^8 + c$$
$x \in \mathbb{R}$

(g) $f(x) = e^{3x} + \dfrac{7}{x}$
Solution:
$$\int e^{3x} + \frac{7}{x} \,\mathrm{d}x = \frac{1}{3}e^{3x} + 7\ln|x| + c$$
$x \in \mathbb{R} \setminus \{0\}$

(h) $f(x) = (e^{-x} + e^{-2x})$
Solution: Let us use the linear substitution $y = -x$, resp. $y = -2x$
$$\int (e^{-x} + e^{-2x})\, dx \overset{C}{=} -e^{-x} - \frac{1}{2}e^{-2x}$$
$x \in \mathbb{R}$

(i) $f(x) = (3-x^2)^3$
Solution: There is no linear substituion, we need to start with the multiplying
$$\int (3-x^2)^3\, dx = \int (27 - 27x^2 + 9x^4 - x^6)\, dx = 27x - 9x^3 + \frac{9}{5}x^5 - \frac{x^7}{7} + C.$$
$x \in \mathbb{R}$

(j) $f(x) = (\sin 5x - \sin 5\alpha)$
Solution: Let us use the linear substitution $y = 5x$
$$\int (\sin 5x - \sin 5\alpha)\, dx = -\frac{1}{5}\cos 5x - x\sin 5\alpha + c,$$
$x \in \mathbb{R}$,
since $\sin 5\alpha$ is a constant (no dependency on x).

(k) $f(x) = \dfrac{1}{x-2} + (3x+7)^5$
Solution:
$$\int \frac{1}{x-2} + (3x+7)^5 \,\mathrm{d}x = \ln|x-2| + \frac{1}{18}(3x+7)^6 + c$$
$x \neq 2$

(l) $f(x) = \dfrac{1}{\sin^2\left(2x + \frac{\pi}{4}\right)}$

Solution: Let us use the linear substitution $y = 2x + \frac{\pi}{4}$

$$\int \frac{1}{\sin^2\left(2x + \frac{\pi}{4}\right)}\, dx \stackrel{C}{=} -\frac{1}{2}\text{cotg}\, \left(2x + \frac{\pi}{4}\right)$$

$2x + \frac{\pi}{4} \neq k\pi$, hence $x \neq k\frac{\pi}{2} - \frac{\pi}{8}$, $k \in \mathbb{Z}$

(m) $f(x) = \dfrac{-2}{\sqrt{1-2x^2}}$

Solution:

$$\int \frac{-2}{\sqrt{1-2x^2}}\,\mathrm{d}x = \int \frac{-2}{\sqrt{1-(\sqrt{2}x)^2}}\,\mathrm{d}x = \frac{-2}{\sqrt{2}}\text{arcsin}\,(\sqrt{2}x) + c$$

$\sqrt{2}x \in (-1, 1)$, hence $x \in (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(n) $f(x) = \dfrac{1}{x+A}$

Solution:

$$\int \frac{1}{x+A}\, dx = \ln|x+A| + C.$$

$x \neq -A$

4. Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

(a) $f(x) = \dfrac{e^{2x}-1}{e^x+1} + \dfrac{4}{1-\cos^2 x}$

Solution:

$$\int \frac{e^{2x}-1}{e^x+1} + \frac{4}{1-\cos^2 x}\,\mathrm{d}x = \int \frac{(e^x-1)(e^x+1)}{e^x+1} + \frac{4}{\sin^2 x}\,\mathrm{d}x = \int (e^x-1) + \frac{4}{\sin^2 x}\,\mathrm{d}x$$
$$= e^x - x - 4\cot x + c$$

$x \neq k\pi$, $k \in \mathbb{Z}$

(b) $f(x) = \dfrac{1}{\sqrt{4-(3x-1)^2}}$

Solution:

$$\int \frac{1}{\sqrt{4-(3x-1)^2}}\,\mathrm{d}x = \int \frac{1}{2\sqrt{1-(\frac{3x-1}{2})^2}}\,\mathrm{d}x = \frac{1}{3}\text{arcsin}\,\left(\frac{3x-1}{2}\right) + c$$

$x \in (-1/3, 1)$

(c) $f(x) = (1-\sqrt{x})^2$

Solution:

$$\int (1-\sqrt{x})^2 \,\mathrm{d}x = \int 1 - 2\sqrt{x} + x \,\mathrm{d}x = x - \frac{4}{3}\sqrt{x^3} + \frac{x^2}{2} + c$$

$x > 0$

(d) $f(x) = \tan^{\,2} x$

Solution:

$$\int \operatorname{tg}^{\,2} x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1-\cos^2 x}{\cos^2 x} \, dx = \int \left(\frac{1}{\cos^2 x} - 1 \right) \, dx = \operatorname{tg} x - x + C.$$

$x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

(e) $f(x) = \dfrac{x^2}{1+x^2}$

Solution:

$$\int \frac{x^2}{1+x^2} \, dx = \int \frac{(1+x^2)-1}{1+x^2} \, dx = \int \left(1 - \frac{1}{1+x^2} \right) \, dx = x - \arctan x + C.$$

$x \in \mathbb{R}$

(f) $f(x) = \dfrac{x^2+3}{x^2-1}$

Solution:

$$\int \frac{x^2+3}{x^2-1} \, dx = \int \frac{x^2-1+4}{x^2-1} \, dx = \int \left(1 + \frac{4}{x^2-1} \right) \, dx = \int \left(1 - \frac{4}{1-x^2} \right) \, dx$$
$$= x - 2\ln\left| \frac{1+x}{1-x} \right| + C.$$

$x \neq \pm 1$

(g) $f(x) = (2^x + 3^x)^2$

Solution:

$$\int (2^x + 3^x)^2 \, dx = \int (4^x + 2 \cdot 6^x + 9^x) \, dx = \frac{4^x}{\ln 4} + \frac{2 \cdot 6^x}{\ln 6} + \frac{9^x}{\ln 9} + C.$$

$x \in \mathbb{R}$

(h) $f(x) = \dfrac{1}{2+3x^2}$

Solution: We need to start with the expression $\frac{1}{1+c^2x^2}$. Then we can use the substitution $y = cx$.

$$\int \frac{1}{2+3x^2}\,dx = \frac{1}{2}\int \frac{1}{1+\left(\sqrt{\frac{3}{2}}x\right)^2}\,dx \overset{C}{=} \frac{1}{2}\sqrt{\frac{2}{3}}\text{arctan}\,\sqrt{\frac{3}{2}}x = \sqrt{\frac{1}{6}}\text{arctan}\,\sqrt{\frac{3}{2}}x$$

$x \in \mathbb{R}$

(i) $f(x) = \operatorname{cotg}{}^2 x$

Solution:

$$\int \operatorname{cotg}{}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int \left(\frac{1}{\sin^2 x} - 1\right)\,dx$$
$$= -\operatorname{cotg}\,x - x + C.$$

$x \neq k\pi$, $k \in \mathbb{Z}$

(j) $f(x) = \dfrac{1}{\sqrt{2-5x}}$

Solution: Since

$$\int \frac{1}{\sqrt{y}}\,dy = 2\sqrt{y},$$

then

$$\int \frac{1}{\sqrt{2-5x}}\,dx \overset{C}{=} \frac{1}{-5}\cdot 2\sqrt{2-5x} = -\frac{2}{5}\sqrt{2-5x}$$

$x < 2/5$

(k) $f(x) = \left(\dfrac{a}{x} + \dfrac{a^2}{x^2} + \dfrac{a^3}{x^3}\right)$, $a \in \mathbb{R}$

Solution:

$$\int \left(\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3}\right)\,dx = a\ln|x| - \frac{a^2}{x} - \frac{a^3}{2x^2} + C.$$

$x \neq 0$

5. Find a function f such that $f'(x) = 6x(1-x)$ and $f(0) = 1$.

   Solution: $\int 6x(1-x)\,\mathrm{d}x = -2x^3 + 3x^2 + c$. Since we have $f(0) = 1$, then $1 = -2\cdot 0^3 + 3\cdot 0^2 + c = c$. Hence, our wanted function is $f(x) = -2x^3 + 3x^2 + 1$, $x \in \mathbb{R}$.

6. Find mistakes

(a) $\int x^2 e^x \,\mathrm{d}x = \frac{1}{3}x^3 e^x + c$

Solution: The integral of product is **not** product of integrals. (The same problem derivatives have.)

(b) $\int \frac{x}{\sqrt{1-x^2}} \,\mathrm{d}x = x \int \frac{1}{\sqrt{1-x^2}} \,\mathrm{d}x = x \arcsin x + c$

Solution: x can not be factored out of the integral. Only constants can.