22nd lesson

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Theory

Theorem 1 (substitution)**.**

1. Let F be an antiderivative of f on (a,b). Let $\varphi\colon (\alpha,\beta)\to(a,b)$ have a finite derivative at each point of (α,β). Then

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{C}{=} F\big(\varphi(x)\big) \quad \text{on } (\alpha,\beta).$$

2. Let φ be a function with a finite derivative in each point of (α,β) such that the derivative is either everywhere positive or everywhere negative, and such that $\varphi\big((\alpha,\beta)\big)=(a,b)$. Let f be a function defined on (a,b) and suppose that

$$\int f\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t \overset{C}{=} G(t) \quad \text{on } (\alpha,\beta).$$

Then

$$\int f(x)\,\mathrm{d}x \overset{C}{=} G\big(\varphi^{-1}(x)\big) \quad \text{on } (a,b).$$

Theorem 2 (integration by parts)**.** Let I be an open interval and let the functions f and g be continuous on I. Let F be an antiderivative of f on I and G an antiderivative of g on I. Then

$$\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x \quad \text{on } I.$$

Remarks 3. Per partes can be expressed also as

$$\int u'(x)v(x)\,\mathrm{d}x = u(x)v(x) - \int u(x)v'(x)\,\mathrm{d}x \text{ na } I.$$

Remarks 4. Let $P(x)$ be a polynomial. The following table can help with choosing u' and v.

	$v(x)$	$u'(x)$
$P(x)\cdot e^{kx}$	$P(x)$	e^{kx}
$P(x)\cdot a^{kx}$	$P(x)$	a^{kx}
$P(x)\cdot \sin(kx)$	$P(x)$	$\sin(kx)$
$P(x)\cdot \cos(kx)$	$P(x)$	$\cos(kx)$

	$v(x)$	$u'(x)$
$P(x)\cdot \ln^n x$	$\ln^n x$	$P(x)$
$P(x)\cdot \arcsin\,(kx)$	$\arcsin\,(kx)$	$P(x)$
$P(x)\cdot \arccos\,(kx)$	$\arccos\,(kx)$	$P(x)$
$P(x)\cdot \arctan\,(kx)$	$\arctan\,(kx)$	$P(x)$
$P(x)\cdot \mathrm{arcctg}\,(kx)$	$\mathrm{arcctg}\,(kx)$	$P(x)$

Hints

$x^3 = x \cdot x^2$

$\tan x = \dfrac{\sin x}{\cos x}$

$\dfrac{1}{\sin x} = \dfrac{\sin x}{\sin^2 x} = \dfrac{\sin x}{1-\cos^2 x}$

$\cos^3 x = \cos x \cdot \cos^2 x = \cos x(1-\sin^2 x)$

$x^4 = (x^2)^2$

Exercises

Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

1. Substitution

   (a) $\int \sin^5 x \cos x \, \mathrm{d}x.$

   (b) $\int -2xe^{-x^2} \, \mathrm{d}x$

   (c) $\int \frac{x}{(1+x^2)^2} \, \mathrm{d}x$

   (d) $\int \frac{1}{(\arcsin x)^2\sqrt{1-x^2}} \, \mathrm{d}x$

2. Per partes

   (a) $\int x \cos x \, \mathrm{d}x$

   (b) $\int xe^{-x} \, \mathrm{d}x$

   (c) $\int e^x \sin x \, \mathrm{d}x$

3. Mixture

   (a) $\int \frac{1}{x^2} \sin \frac{1}{x} \, \mathrm{d}x$

   (b) $\int \ln x \, \mathrm{d}x$

   (c) $\int \frac{e^x}{2+e^x} \, \mathrm{d}x$

   (d) $\int \frac{1}{x \ln x \ln(\ln x)} \, \mathrm{d}x$

   (e) $\int \arcsin x \, \mathrm{d}x$

   (f) $\int \frac{x}{3-2x^2} \, \mathrm{d}x$

   (g) $\int x^2 \sin 2x \, \mathrm{d}x$

   (h) $\int e^{ax} \cos bx \, \mathrm{d}x$

   (i) $\int \frac{1}{\sin^2 x \sqrt[4]{\operatorname{cotg} x}} \, \mathrm{d}x$

   (j) $\int \cos(\ln x) \, \mathrm{d}x$

   (k) $\int \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x$

   (l) $\int \sin x \ln(\operatorname{tg} x) \, \mathrm{d}x$

   (m) $\int \frac{\arctan x}{1+x^2} \, \mathrm{d}x$

   (n) $\int x^2 \arccos x \, \mathrm{d}x$

   (o) $\int \frac{\sin x}{\sqrt{\cos^3 x}} \, \mathrm{d}x$

   (p) $\int \sqrt{x} \ln^2 x \, \mathrm{d}x$

   (q) $\int \frac{\ln^2 x}{x} \, \mathrm{d}x$

   (r) $\int x^3 e^{-x^2} \, \mathrm{d}x$

   (s) $\int \operatorname{tg} x \, \mathrm{d}x$

   (t) $\int \frac{1}{(1+x)\sqrt{x}} \, \mathrm{d}x$

   (u) $\int \frac{1}{e^x + e^{-x}} \, \mathrm{d}x$

   (v) $\int \frac{1}{\sin x} \, \mathrm{d}x$

   (w) $\int \cos^3 x \, \mathrm{d}x$

   (x) $\int \frac{x}{4+x^4} \, \mathrm{d}x$

   (y) $\int \frac{1}{\sqrt{1+e^{2x}}} \, \mathrm{d}x$

   (z) $\int \frac{\arcsin x}{x^2} \, \mathrm{d}x$