22nd lesson

`https://www2.karlin.mff.cuni.cz/~kuncova/en/teaching.php`,
kuncova@karlin.mff.cuni.cz

Exercises

Find F - the antiderivative of a function f at the maximal (open) set. (Specify the set.)

1. Substitution

   (a) $\displaystyle\int \sin^5 x \cos x \,\mathrm{d}x.$

   Solution: Let us use the substitution $y = \sin x$. Then $dy = \cos x\, dx$ and we have
   $$\int \sin^5 x \cos x \, dx = \int y^5 \, dy \stackrel{C}{=} \frac{y^6}{6} = \frac{\sin^6 x}{6}$$

   (b) $\displaystyle\int -2xe^{-x^2} \,\mathrm{d}x$

   Solution: Let us use the substitution $y = -x^2$. Then $dy = -2x\, dx$ and we have
   $$\int -2xe^{-x^2} \, dx = \int e^y \, dy \stackrel{C}{=} e^y = e^{-x^2}$$

   (c) $\displaystyle\int \frac{x}{(1+x^2)^2} \,\mathrm{d}x$

   Solution: Let us use the substitution $y = 1 + x^2$. Then $dy = 2x\, dx$ and we have
   $$\int \frac{x}{(1+x^2)^2} \, dx = \frac{1}{2} \int \frac{dy}{y^2} \stackrel{C}{=} -\frac{1}{2}\frac{1}{y} = -\frac{1}{2}\frac{1}{1+x^2}$$

   (d) $\displaystyle\int \frac{1}{(\arcsin x)^2 \sqrt{1-x^2}} \,\mathrm{d}x$

   Solution: Let us use the substitution $y = \arcsin x$, then $dy = \frac{1}{\sqrt{1-x^2}}\, dx$ and we have
   $$\int \frac{1}{(\arcsin x)^2 \sqrt{1-x^2}} \, dx = \int \frac{dy}{y^2} \stackrel{C}{=} -\frac{1}{y} = -\frac{1}{\arcsin x}$$

2. Per partes

   (a) $\displaystyle\int x \cos x \,\mathrm{d}x$

   Solution: Per partes: $u' = \cos x$, $u = \sin x$, $v = x$, $v' = 1$.
   $$\int x \cos x \, dx = [x \sin x] - \int \sin x \, dx \stackrel{C}{=} x \sin x + \cos x$$

(b) $\int xe^{-x}\,\mathrm{d}x$

Solution: Per partes: $u' = e^{-x}$, $u = -e^{-x}$, $v = x$, $v' = 1$.

$$\int xe^{-x}\,dx = \left[-xe^{-x}\right] - \int -e^{-x}\,dx \stackrel{C}{=} -xe^{-x} - e^{-x}$$

(c) $e^x \sin x\,\mathrm{d}x$

Solution:

Let us apply per partes twice. (In both per partes we will derive e^x and integrate the goniometric function.)

We have

$$\int e^x \sin x\ \mathrm{d}x = -e^x \cos x + \int e^x \cos x\ \mathrm{d}x = -e^x \cos x + e^x \sin x - \int e^x \sin x\ \mathrm{d}x.$$

Hence

$$2\int e^x \sin x\ \mathrm{d}x \stackrel{C}{=} -e^x \cos x + e^x \sin x$$

thus

$$\int e^x \sin x\ \mathrm{d}x \stackrel{C}{=} \frac{1}{2}(-e^x \cos x + e^x \sin x)$$

3. Mixture

(a) $\int \frac{1}{x^2} \sin \frac{1}{x}\,\mathrm{d}x$

Solution: Let us use the substitution $y = \frac{1}{x}$. Then $dy = -\frac{1}{x^2}\,dx$ and we have

$$\int \frac{1}{x^2} \sin \frac{1}{x}\,dx = -\int \sin y\,dy \stackrel{C}{=} \cos y = \cos \frac{1}{x}$$

(b) $\int \ln x\,\mathrm{d}x$

Solution:

Let $u' = 1$, $v = \ln x$. Then $u = \int 1\,dx = x$ a $v' = (\ln x)' = \frac{1}{x}$ and by per partes we have

$$\int \ln x\,dx = [x \ln x] - \int x \cdot \frac{1}{x}\,dx \stackrel{C}{=} x \ln x - x$$

(c) $\int \frac{e^x}{2 + e^x}\,\mathrm{d}x$

Solution: Let us use the substitution $y = e^x$. Then $dy = e^x\,dx$ and we have

$$\int \frac{e^x}{2 + e^x}\,dx = \int \frac{dy}{2 + y} \stackrel{C}{=} \ln|2 + y| = \ln(2 + e^x)$$

(d) $\displaystyle\int \frac{1}{x \ln x \ln(\ln x)} \, \mathrm{d}x$

Solution: Let us use the substitution $y = \ln(\ln x)$. Then $dy = \frac{1}{x \ln x} \, dx$ and we have

$$\int \frac{1}{x \ln x \ln(\ln x)} \, dx = \int \frac{1}{y} \, dy \stackrel{C}{=} \ln|y| = \ln(|\ln(\ln x)|)$$

(e) $\displaystyle\int \arcsin x \, \mathrm{d}x$

Solution:

Per partes: $u' = 1$, $u = x$, $v = \arcsin x$, $v' = \frac{1}{\sqrt{1-x^2}}$.

$$\int 1 \cdot \arcsin x \, dx = [x \arcsin x] - \int \frac{x}{\sqrt{1-x^2}} \, dx =$$

Substitution $y = 1 - x^2$.

$$= x \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{y}} \, dy \stackrel{C}{=} x \arcsin x + \sqrt{y} = x \arcsin x + \sqrt{1-x^2}$$

(f) $\displaystyle\int \frac{x}{3 - 2x^2} \, \mathrm{d}x$

Solution: Let us use the substitution $y = 3 - 2x^2$. Then $dy = -4x \, dx$ and we have

$$\int \frac{x}{3-2x^2} \, dx = -\frac{1}{4} \int \frac{dy}{y} \stackrel{C}{=} -\frac{1}{4} \ln|y| = -\frac{1}{4} \ln|3 - 2x^2|$$

(g) $\displaystyle\int x^2 \sin 2x \, \mathrm{d}x$

Solution:

First per partes: $u' = \sin 2x$, $u = -\frac{1}{2} \cos 2x$, $v = x^2$, $v' = 2x$.

$$\int x^2 \sin 2x \, dx = \left[-\frac{1}{2} x^2 \cos 2x \right] + \int x \cos 2x =$$

Second per partes: $u' = \cos 2x$, $u = \frac{1}{2} \sin 2x$, $v = x$, $v' = 1$.

$$= -\frac{1}{2} x^2 \cos 2x + \left[\frac{1}{2} x \sin 2x \right] - \frac{1}{2} \int \sin 2x \, dx \stackrel{C}{=} -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x$$

(h) $\displaystyle\int e^{ax} \cos bx \, \mathrm{d}x$

Solution:

For $a = b = 0$ we have $\int e^{0x} \cos(0x) \, \mathrm{d}x = \int 1 \, \mathrm{d}x \stackrel{C}{=} x$.

Now let $a \neq 0$, $b \neq 0$. Let us use per partes twice. Then we have

$$\int e^{ax}\cos bx \ \mathrm{d}x = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx \ \mathrm{d}x = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx \ \mathrm{d}x.$$

Hence

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx \ \mathrm{d}x \overset{C}{=} \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

$$\int e^{ax}\cos bx \ \mathrm{d}x \overset{C}{=} \frac{b}{a^2+b^2}e^{ax}\sin bx + \frac{a}{a^2+b^2}e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}\left(b\sin bx + a\cos bx\right)$$

It is easy to verify that the solution is correct also for $b=0$, $a \neq 0$, or for $a=0$, $b\neq 0$.

(i) $\displaystyle\int \frac{1}{\sin^2 x\sqrt[4]{\operatorname{cotg} x}}\,\mathrm{d}x$

Solution:

Let us use the substitution $y = \operatorname{cotg} x$. Then $dy = -\frac{1}{\sin^2 x}\,dx$ and we have

$$\int \frac{1}{\sin^2 x\sqrt[4]{\operatorname{cotg} x}}\,dx = -\int \frac{dy}{y^{1/4}} \overset{C}{=} \frac{4}{3}y^{3/4} = \frac{4}{3}\sqrt[4]{\operatorname{cotg}^3 x}$$

(j) $\displaystyle\int \cos(\ln x)\,\mathrm{d}x$

Solution: Per partes, let us set $v' = 1$, $u = \cos(\ln x)$. Then $v = x$ and $u' = -\sin(\ln x)\cdot\frac{1}{x}$. We obtain

$$\int 1\cdot\cos(\ln x) = x\cos(\ln x) + \int \sin(\ln x)\,dx =$$

Per partes again: $v' = 1$ and $u = \sin(\ln x)$ and we have

$$\int 1\cdot\cos(\ln x) = x\cos(\ln x) + \int \sin(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)$$

Let us move both integrals to the left side

$$2\int 1\cdot\cos(\ln x) \overset{C}{=} x\cos(\ln x) + x\sin(\ln x)$$

$$\int \cos(\ln x) \overset{C}{=} \frac{1}{2}x(\cos(\ln x) + \sin(\ln x))$$

(k) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$

Solution: Let us use the substitution $y = 1-x^2$, then $\mathrm{d}y = -2x\,\mathrm{d}x$. We obtain

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{-2}\frac{-2x}{\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{1}{\sqrt{y}}\,\mathrm{d}y = -\frac{1}{2}2\sqrt{y}+c = -\sqrt{1-x^2}+c$$

(l) $\int \sin x \ln(\operatorname{tg} x)\,\mathrm{d}x$

Solution: Per partes: $u' = \sin x$, $u = -\cos x$, $v = \ln(\operatorname{tg} x)$, $v' = \frac{1}{\operatorname{tg} x}\frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x}$.

$$\int \sin x \ln(\operatorname{tg} x)\,dx = -\cos x \ln(\operatorname{tg} x) + \int \frac{1}{\sin x}\,dx \overset{C}{=} -\cos x \ln(\operatorname{tg} x) + \ln\left|\operatorname{tg} \frac{x}{2}\right|$$

(m) $\int \frac{\arctan x}{1+x^2}\,\mathrm{d}x$

Solution: Let us use the substitution $y = \arctan x$, then $dy = \frac{1}{1+x^2}\,dx$ and we have

$$\int \frac{\arctan x}{1+x^2}\,dx = \int y\,dy \overset{C}{=} \frac{y^2}{2} = \frac{\arctan {}^2 x}{2}$$

(n) $\int x^2 \arccos x\,\mathrm{d}x$

Solution: Per partes: $u' = x^2$, $u = \frac{x^3}{3}$, $v = \arccos x$, $v' = -\frac{1}{\sqrt{1-x^2}}$.

$$\int x^2 \arccos x\,dx = \left[\frac{x^3}{3}\arccos x\right] + \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx =$$

Substitution $y = 1 - x^2$, gives $dy = -2x\,dx$ and $x^2 = 1 - y$.

$$= \frac{x^3}{3}\arccos x + \frac{1}{6}\int \frac{y-1}{\sqrt{y}}\,dy = \frac{x^3}{3}\arccos x + \frac{1}{6}\int \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)\,dy \overset{C}{=}$$

$$\overset{C}{=} \frac{x^3}{3}\arccos x + \frac{1}{9}y^{3/2} - \frac{1}{3}y^{1/2} = \frac{x^3}{3}\arccos x + \frac{1}{9}(1-x^2)^{3/2} - \frac{1}{3}(1-x^2)^{1/2}$$

(o) $\int \frac{\sin x}{\sqrt{\cos^3 x}}\,\mathrm{d}x$

Solution: Let us use the substitution $y = \cos x$. Then $dy = -\sin x\,dx$ and we have

$$\int \frac{\sin x}{\sqrt{\cos^3 x}}\,dx = -\int \frac{dy}{\sqrt{y^3}} \overset{C}{=} 2y^{-1/2} = \frac{2}{\sqrt{\cos x}}$$

(p) $\int \sqrt{x}\ln^2 x\,\mathrm{d}x$

Solution: First per partes: $u' = \sqrt{x}$, $u = \frac{2}{3}x^{3/2}$, $v = \ln^2 x$, $v' = 2\ln x \frac{1}{x}$.

$$\int \sqrt{x}\ln^2 x\,dx = \left[\frac{2}{3}x^{3/2}\ln^2 x\right] - \int \frac{4}{3}x^{1/2}\ln x\,dx =$$

Second per partes: $u' = \frac{4}{3}x^{1/2}$, $u = \frac{8}{9}x^{3/2}$, $v = \ln x$, $v' = 1/x$.

$$= \left[\frac{2}{3}x^{3/2}\ln^2 x\right] - \left[\frac{8}{9}x^{3/2}\ln x\right] + \int \frac{8}{9}x^{1/2}\,dx \overset{C}{=} \frac{2}{3}x^{3/2}\ln^2 x - \frac{8}{9}x^{3/2}\ln x + \frac{16}{27}x^{3/2}$$

(q) $\displaystyle\int \frac{\ln^2 x}{x}\,\mathrm{d}x$

Solution: Let us use the substitution $y = \ln x$. Then $dy = \frac{1}{x}\,dx$ and we have

$$\int \frac{\ln^2 x}{x}\,dx = \int y^2\,dy \overset{C}{=} \frac{y^3}{3} = \frac{\ln^3 x}{3}$$

(r) $\displaystyle\int x^3 e^{-x^2}\,\mathrm{d}x$

Solution:

Let us use the substitution $y = x^2$. Then $dy = 2x\,dx$ and we have

$$\int x^3 e^{-x^2}\,dx = \frac{1}{2}\int y e^{-y}\,dy =$$

Now per partes: $u' = e^{-y}$, $u = -e^{-y}$, $v = y$, $v' = 1$.

$$= \left[-ye^{-y}\right] + \int e^{-y}\,dy \overset{C}{=} -ye^{-y} - e^{-y} = -x^2 e^{-x^2} - e^{-x^2}$$

(s) $\displaystyle\int \mathrm{tg}\ x\,\mathrm{d}x$

Solution: Let us use the substitution $y = \cos x$. Then $dy = -\sin x\,dx$ and we have

$$\int \mathrm{tg}\ x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{dy}{y} \overset{C}{=} -\ln|y| = -\ln|\cos x|$$

(t) $\displaystyle\int \frac{1}{(1+x)\sqrt{x}}\,\mathrm{d}x$

Solution: Let's start with the conditions $x \in (0, \infty)$. Let us use the substitution $y = \sqrt{x}$. Then $y^2 = x$, $dy = \frac{1}{2\sqrt{x}}\,dx$ and we have

$$\int \frac{1}{(1+x)\sqrt{x}}\,dx = 2\int \frac{1}{1+x}\frac{dx}{2\sqrt{x}} = 2\int \frac{1}{1+y^2}\,dy \overset{C}{=} 2\mathrm{arctan}\ y = 2\mathrm{arctan}\ \sqrt{x}$$

(u) $\displaystyle\int \frac{1}{e^x + e^{-x}}\,\mathrm{d}x$

Solution: At first we modify the function. Then we can use the substitution $y = e^x$, $dy = e^x\,dx$

$$\int \frac{1}{e^x + e^{-x}}\,dx = \int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{dy}{1+y^2} \overset{C}{=} \arctan y = \arctan e^x$$

(v) $\displaystyle\int \frac{1}{\sin x}\,\mathrm{d}x$

Let us use the substitution $y = \cos x$. Then $dy = -\sin x\, dx$ and we have

$$\int \frac{1}{\sin x}\, dx = \int \frac{\sin x}{\sin^2 x}\, dx = \int \frac{\sin x}{1-\cos^2 x}\, dx = -\int \frac{dy}{1-y^2} \overset{C}{=} -\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| =$$

$$= \frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| = -\frac{1}{2}\ln\left|\frac{\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}}\right| = \ln\left|\operatorname{tg}\,\frac{x}{2}\right|$$

(w) $\int \cos^3 x\, \mathrm{d}x$

Solution: Let us use the substitution $y = \sin x$. Then $dy = \cos x\, dx$ and we have

$$\int \cos^3 x\, dx = \int (1-\sin^2 x)\cos x\, dx = \int (1-y^2)\, dy \overset{C}{=} y - \frac{y^3}{3} = \sin x - \frac{\sin^3 x}{3}$$

(x) $\int \frac{x}{4+x^4}\, \mathrm{d}x$

Solution: Let us use the substitution $y = x^2$. Then $dy = 2x\, dx$ and we have

$$\int \frac{x}{4+x^4}\, dx = \frac{1}{2}\int \frac{dy}{4+y^2} = \frac{1}{8}\int \frac{dy}{1+(y/2)^2} \overset{C}{=} \frac{1}{4}\arctan\,\frac{y}{2} = \frac{1}{4}\arctan\,\frac{x^2}{2}$$

(y) $\int \frac{1}{\sqrt{1+e^{2x}}}\, \mathrm{d}x$

Solution: At first let us use the substitution $y = e^x$, $dy = e^x\, dx$

$$\int \frac{1}{\sqrt{1+e^{2x}}}\, dx = \int \frac{e^x\, dx}{e^x\sqrt{1+e^{2x}}} = \int \frac{dy}{y\sqrt{1+y^2}}$$

Then we expand the expression and we use the substitution $t = 1 + y^2$, $\mathrm{d}t = 2y\, \mathrm{d}y$:

$$= \int \frac{dy}{y\sqrt{1+y^2}} = \int \frac{2y}{2y^2\sqrt{1+y^2}}\, \mathrm{d}y = \int \frac{1}{2(t-1)\sqrt{t}}\, \mathrm{d}t.$$

Another substitution $s = \sqrt{t}$, then $\mathrm{d}s = \frac{1}{2\sqrt{t}}\, \mathrm{d}t$. Finally we apply integration table

$$\int \frac{1}{s^2-1}\, \mathrm{d}t \overset{C}{=} -\frac{1}{2}\ln\left|\frac{1+s}{1-s}\right|.$$

Backward substitution

$$= -\frac{1}{2}\ln\left|\frac{1+\sqrt{t}}{1-\sqrt{t}}\right|. = -\frac{1}{2}\ln\frac{1+\sqrt{y^2+1}}{1-\sqrt{y^2+1}} = -\frac{1}{2}\ln\frac{1+\sqrt{e^{2x}+1}}{1-\sqrt{e^{2x}+1}}$$

(z) $\displaystyle\int \frac{\arcsin x}{x^2}\,\mathrm{d}x$

Solution: Per partes: $u' = \frac{1}{x^2}$, $u = -\frac{1}{x}$, $v = \arcsin x$, $v' = \frac{1}{\sqrt{1-x^2}}$.

$$\int \frac{\arcsin x}{x^2}\,dx = \left[-\frac{1}{x}\arcsin x\right] + \int \frac{1}{x}\frac{1}{\sqrt{1-x^2}}\,dx =$$

Substitution $y = \sqrt{1-x^2}$. Then $dy = \frac{x}{\sqrt{1-x^2}}\,dx$ a $x^2 = 1-y^2$.

$$= -\frac{1}{x}\arcsin x + \int \frac{1}{x}\frac{1}{\sqrt{1-x^2}}\,dx = -\frac{1}{x}\arcsin x + \int \frac{1}{1-y^2}\,dy \overset{C}{=}$$

$$\overset{C}{=} -\frac{1}{x}\arcsin x + \frac{1}{2}\ln\frac{1+y}{1-y} = -\frac{1}{x}\arcsin x + \frac{1}{2}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}$$