HW 3.1 Let $P(\lambda) = \lambda^2 + p\lambda + q$ be a polynomial with *real* coefficients p, q. Prove that all (i.e. both) roots of $P(\lambda)$ have negative real part if and only if $p > 0$ and $q > 0$.
Deduce the following useful criterion: for a given real 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

all (i.e. both) eigenvalues have negative real part if and only if $\operatorname{tr} A < 0$ and $\det A > 0$.
By $\operatorname{tr} A$ we mean the *trace* of the matrix, i.e. the sum of the diagonal $a_{11} + a_{22}$.

HW 3.2 Consider again the system (same as in HW 2.3)

$$x' = x + y^3$$
$$y' = x - x^3$$

Focus on the behavior near the equilibrium points: $(0,0)$, $(1,-1)$ and $(-1,1)$; in particular:

- compute the linearization matrix and its spectrum
- if possible, apply Theorems P.5 and/or P.6
- using the above, sketch an improved picture of solutions near the equilibrium

(You can skip the parts you already answered while solving HW2.)

HW 3.3 Consider the system

$$x' = x^2 + y$$
$$y' = -x(y+1)$$

near the equilibrium point $E = (0,0)$. What can you say about its stability, in view of Theorem P.5?

Verify that $V = x^2(y+1)^2 + \frac{2}{3}y^3 + y^2$ is a prime integral in $\mathbb{R}^2$.
* Can you deduce V from the equation?
* What does V tell us about the stability of E?

See also hints on page 2.

1) Complete the square and discuss with respect to the sign of discriminant. Note how $\det A$ and trace of A appear in the characteristic polynomial A.

3) Divide the equations; and consider x as a function of y, which leads to Bernoulli's equation. – Stable, not asymptotically.