(1) Určete derivaci $F(a)=\int_0^{\pi} \ln\left(1+\cos^2 x + a\sin^2 x\right)dx$, $a>0$

(2) $\int_P z^2\,dS$; $P=\{x^2+y^2+z^2=1\}\cap\{z>\sqrt{x^2+y^2}\}$

plošný 1. druhu

(3) $\phi(y)=\int_0^1 (1+x)e^x y + \frac{1}{2}e^x (y')^2\,dx$; $y(0)=1$; $y(1)=\frac{3}{2}$

a) extremály

b) jsou to lokální / globální extrémy

(4) $f(x)=(\pi-|x|)^2$; $x\in[-\pi,\pi]$;

2π - periodické.

a) $F_f(x)=?$ a její součet

b) Parsevalova rovnost

(1) $F(a) = \int_0^{\pi} \ln(1+\cos^2 x + a\sin^2 x)\,dx$; $a>0$.

$$F'(a) = \int_0^{\pi} \frac{\sin^2 x}{1+\cos^2 x + a\sin^2 x}\,dx \; ; \quad \text{majoranta:}$$ form. 1

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| t = \operatorname{tg} x \right| = \int_{\mathbb{R}} \frac{t^2}{(1+t^2)((a+1)t^2+2)}\,dt \; ;$$ rac. fce. 2

par. zlomky: $\frac{1}{a-1}\left(\frac{1}{t^2+1} - \frac{2}{(a+1)t^2+2}\right)$ rozklad 2

$$F'(a) = \frac{2\pi}{\sqrt{2+2a}\,(\sqrt{2+2a}+2)}.$$ dopočet 1

hint: $\int_{-\infty}^{\infty} \frac{dy}{\lambda y^2+1} = \frac{\pi}{\sqrt{\lambda}}$

$$F'(1) = \frac{2\pi}{2\cdot(2+2)} = \frac{\pi}{4}.$$

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předpoklady: $f(a,x) = \ln(1+\cos^2 x + a\sin^2 x)$; 2

$a \in J = (0,\infty)$;

$x \in I = (0,\pi)$.

(i) $f(a,\cdot)$ měřitelná $\Leftarrow$ spojitá v I

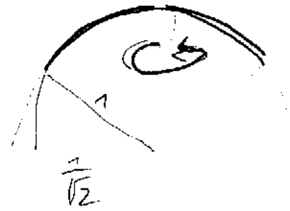
(ii) $\frac{\partial f}{\partial a}(\cdot,x)$ $\exists$ vlastní: viz výpočet výše

(iii) $\left|\frac{\partial f}{\partial a}(a,x)\right| \leq 1$ $\forall x \forall a$; $1 \in \mathcal{L}^1(0,\pi)$.

(iv) $f(1,x) = \ln 2 \in L^1(0,\pi)$

$a_0 = 1$.

② $\displaystyle\int_P z^2\, dS$; $\quad P = \{x^2+y^2+z^2 = 1\} \cap \{z > \sqrt{x^2+y^2}\}$.



$$\varphi:\ \begin{aligned} x &= u \\ y &= v \\ z &= \sqrt{1-u^2-v^2} \end{aligned} \qquad (u,v) \in \Omega; \qquad \Omega = \{u^2+v^2<2\}.$$

param: 3

$$\partial_u \varphi = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right)$$

$$\partial_v \varphi = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right);$$

$$\partial_u\varphi \times \partial_v\varphi = \left(\frac{u}{\sqrt{\quad}}, \frac{v}{\sqrt{\quad}}, 1\right)$$

$$\|\partial_u\varphi \times \partial_v\varphi\|^2 = \frac{u^2}{1-u^2-v^2} + \frac{v^2}{1-u^2-v^2} + 1 = \frac{1}{1-u^2-v^2};$$

$$\boxed{\int_\Omega \frac{1-u^2-v^2}{\sqrt{1-u^2-v^2}}\, du\, dv = \int_\Omega \sqrt{1-u^2-v^2}\, du\, dv} \qquad \left|\ \begin{aligned} u &= r\cos\omega & r &\in (0, \tfrac{1}{\sqrt{2}}) \\ v &= r\cdot\sin\omega & \omega &\in (0, 2\pi) \\ J &= r \end{aligned}\right.$$

2

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}\, dr = 2\pi \left[-\frac{1}{3}(1-r^2)^{3/2}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{2\pi}{3}\left(1 - \left(\frac{1}{2}\right)^{3/2}\right).$$

3

varianta : $\longrightarrow$ sférické: $\quad \left.\begin{aligned} x &= \cos u \sin v & v &\in (0, \tfrac{\pi}{4}) \\ y &= \sin u \sin v & u &\in (0, 2\pi) \end{aligned}\right\}\Omega$

$$z = \cos v$$

$$\boxed{dS = \sin v}$$

$$\int_\Omega \cos^2 v \cdot \sin v\, du\, dv = 2\pi \cdot \left(\frac{1}{3} - \frac{\sqrt{2}}{12}\right).$$

$\cancel{1}$ 2b

$$dx = dr \cosh t + r \sinh t \, dt$$

$$dy = dr \, \cancel{\cos} \sinh t + r \cosh t \, dt$$

$$d\left(x^2 - y^2\right) = 2x\,dx - 2y\,dy$$

$$\phi^*(d\omega) = 2r\cosh t\left(dr\cosh t + r\sinh t\,dt\right)$$

$$-2r\sinh t\left(dr\sinh t + r\cosh t\,dt\right)$$

$$= 2r\,dr\left(\cosh^2 t - \sinh^2 t\right) = \underline{\underline{2r\,dr}}.$$

$$\phi^*(\omega) = r^2\left(\cosh^2 t - \sinh^2 t\right) = r^2$$

$$d\,\phi^*(\omega) = \underline{\underline{2r\,dr}}$$

(3) $\Phi(y) = \int_0^1 (1+x)e^x y + \frac{1}{2}e^x (y')^2\,dx$; $y(0)=1$; $y(1)=\frac{3}{2}$

$f_R = e^x R$; $f_y = (1+x)e^x$

E.L. $\quad -(e^x y')' + (1+x)e^x = 0$

$$y'' + y' = 1+x;$$

$$F.S.: \{1, e^{-x}\};$$

$$y_p = Ax^2 + Bx + C$$

extremála: $\quad y_0 = 1 + \frac{x^2}{2}$

Diskuse: $\quad f_{RR}(x, y_0(x), y_0'(x)) = e^x > 0$:

podezření na minimum

Jacobi: $\quad f_{yy} = f_{yz} = 0$; $\quad -(e^x u')' = 0$

$$e^x u' = K$$

$$u' = Ke^{-x}$$

$$u = -Ke^{-x} + L$$

$u(x)$.. monotónní: neexistuje konj. bod: y_0 lok. min.

$\eta \in C_0^1([0,1])$ libovolné;

$$\Phi(y_0+\eta) - \Phi(y_0) = \int_0^1 (1+x)e^x\eta + \frac{1}{2}e^x\Big[(\eta')^2 + \underbrace{2y_0'\eta'}_{p.p.}\Big]dx$$

$\Rightarrow$ globální minimum.

$$= \underbrace{\int_0^1 \frac{1}{2}e^x(\eta')^2}_{\geq 0} + \underbrace{\int_0^1 \Big[(1+x)e^x - (e^x y_0')'\Big]\eta\,dx}_{=0\ (E.L.!!)}$$

(4) $f(x) = (\pi - |x|)^2$, sudá. $b_k = 0$.

$$\pi a_0 = \int_{-\pi}^{\pi} (\pi - |x|)^2 \, dx = 2 \int_0^{\pi} (\pi - x)^2 \, dx = \left[-\frac{2}{3} (\pi - x)^3 \right]_0^{\pi} = \frac{2}{3} \pi^3 ;$$

$$a_0 = \frac{2\pi^2}{3} .$$ 1

$$\frac{\pi}{2} a_k = \int_0^{\pi} (\pi - x)^2 \cos kx \, dx = \left[(\pi - x)^2 \frac{\sin kx}{k} \right]_0^{\pi} + \frac{2}{k} \underbrace{\int_0^{\pi} (\pi - x) \sin kx \, dx}_{I} ;$$

$$I = \left[-(\pi - x) \frac{\cos kx}{k} \right]_0^{\pi} - \underbrace{\frac{1}{k} \int_0^{\pi} \cos kx \, dx}_{=0} = \pi \cdot \frac{1}{k} ;$$

tj. $\frac{\pi}{2} a_k = \frac{2}{k} \cdot \frac{\pi}{k}$; $a_k = \frac{4}{k^2}$; 2

$$F_f(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2} \cos kx ;$$ 1

$f(x)$ spojitá, po částech C^1: $\boxed{F_f(x) = f(x) \text{ všude!!}}$

spec.: $f(0) = \pi^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}$; $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ 2

Parseval: $\int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2 \int_0^{\pi} (\pi - x)^4 = 2 \int_0^{\pi} y^4 \, dy = 2 \left[\frac{y^5}{5} \right]_0^{\pi} = \frac{2}{5} \pi^5 .$

$$\boxed{\frac{2}{5} \cdot \pi^4 = \frac{4}{9} \pi^4 \cdot \frac{1}{2} + \sum_{k=1}^{\infty} \frac{16}{k^4}} ; \quad \sum_{k=1}^{\infty} \frac{1}{k^4} = \pi^4 \left(\frac{2}{5} - \frac{2}{9} \right) \cdot \frac{1}{16} = \frac{\pi^4}{90} .$$

2

$$\frac{18 - 10}{45} \cdot \frac{1}{16} = \frac{1}{90}$$